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DIOPHANTINE GEOMETRY OVER GROUPS IV: AN ITERATIVE PROCEDURE FOR VALIDATION OF A SENTENCE

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ABSTRACT

This paper is the fourth in a series on the structure of sets of solutions to systems of equations in a free group, projections of such sets, and the structure of elementary sets defined over a free group. In the fourth paper we present an iterative procedure that validates the correctness of an AE sentence defined over a free group. The terminating procedure presented in this paper is the basis for our analysis of elementary sets defined over a free group presented in the next papers in the series.

Introduction

In the first three papers in the sequence on Diophantine geometry over groups we studied sets of solutions to systems of equations in a free group, and developed basic techniques and objects required for the analysis of sentences and elementary sets defined over a free group. In the first paper in this sequence we studied sets of solutions to systems of equations defined over a free group and parametric families of such sets, and associated a canonical Makanin–Razborov diagram that encodes the entire set of solutions to the system. Later on we studied systems of equations with parameters, and with each such system we associated a (canonical) graded Makanin–Razborov diagram that encodes the Makanin–Razborov diagrams of the systems of equations associated with each

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specialization of the defining parameters. In the second paper we generalized Merzlyakov's theorem on the existence of a formal solution associated with a positive sentence [Me]. We first constructed a formal solution to a general AE sentence which is known to be true over some variety, and then presented formal limit groups and graded formal limit groups that enable us to collect and analyze the collection of all such formal solutions.

In this paper we apply the structural results obtained in the first two papers in the sequence, to analyze AE sentences. As we have seen in the first section of the second paper, given a true sentence of the form

$$\forall y \quad \exists x \quad \Sigma(x,y,a)=1 \wedge \Psi(x,y,a) \neq 1$$

it is impossible to find a finite collection of formulas of the form $x = x(y,a)$ that prove the validity of the sentence. To analyze an AE sentence we associate with it an iterative procedure that produces a sequence of varieties and formal solutions defined over them.

The general form of Merzlyakov's theorem ([Se2], 1.18) implies that if an AE sentence is true over some variety, then there exist formal solutions defined over closures of completions of the given variety (i.e., the formal solutions constructed by the general Merzlyakov theorem are defined not on the variety itself, but rather on Diophantine subsets of the variety). Since the definition of completions and closures associated with a given variety ([Se2], 1.12) require the introduction of additional variables, the varieties produced along the iterative procedure for the validation of an AE sentence we present are determined by larger and larger sets of variables, and so are the formal solutions defined over them. Still, by carefully analyzing these varieties and properly measuring the complexity of Diophantine sets associated with them, we are able to show that certain complexity of the varieties produced along the procedure strictly decreases, which finally forces the iterative procedure to terminate after finitely many steps.

The outcome of the terminating iterative procedure is a collection of varieties, together with a collection of formal solutions defined over them. The varieties are determined by the original universal variables y, and extra (auxiliary) variables. The collection of varieties gives a partition of the initial domain of the universal variables y, which is a power of the original free group of coefficients, into sets which are in the Boolean algebra of universal sets, so that on each such set the sentence can be validated using a finite family of formal solutions. Hence, the outcome of the iterative procedure can be viewed as a *stratification theorem* that generalizes Merzlyakov's theorem from positive sentences to general AE ones.

Since the iterative procedure is rather involved, we prefered to present it first in two special cases, which are conceptually and technically simpler, but they already demonstrate some of the principles used in the general case. In the first section we present an iterative procedure in the minimal rank case, i.e., for sentences for which the limit groups involved in their analysis are all of minimal possible rank (rank 0), i.e., limit groups that do not admit an epimorphism onto a free group so that the coefficient group is mapped onto a proper factor. In the second section we analyze a generalization of the minimal rank case, the case of maximal rank homomorphisms of general limit groups. In both cases all maps in question are shown to be "geometric", which enables one to use "purely" geometric concepts in analyzing the varieties and the resolutions constructed along the procedure. In the third section we construct tools needed for generalizing the iterative procedure. These involve resolutions inherited by a Diophantine set from a resolution of a variety. The fourth section finitely presents the iterative procedure for validation of a sentence in the general case and proves its finite termination.

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1. The minimal rank case

Our "trial and error" procedure for quantifier elimination is constructed iteratively. As we will see in the next papers the existence of formal solutions, and graded formal limit groups presented in [Se2], are the basic tools needed for constructing the procedure. This paper is devoted to the remaining tools needed for constructing our iterative procedure for quantifier elimination. In order to develop the needed tools and demonstrate our approach to quantifier elimination, we present a simplified version of the iterative procedure in this section.

Given a true sentence of the form

$$\forall y \quad \exists x \quad \Sigma(x,y,a) = 1 \wedge \Psi(x,y,a) \neq 1$$

the iterative procedure presented in this section constructs a finite sequence of limit groups (defined in terms of the coefficients a, the variables y, and some additional auxiliary variables), some well-structured resolutions of these limit groups, and for each well-structured resolution a set of formal solutions defined over a covering closure of the resolution. Using the resolutions, closures, and formal solutions constructed by the procedure, we will be able to divide the domain of y's into a finite collection of sets, so that on each set from this finite union, the validity of the sentence can be verified using a unique formula.

We start our warm-up procedure with the following rather straightforward proposition.

PROPOSITION 1.1: *Let $F_k =< a_1, \dots, a_k >$ be a free group and let*

$$\forall y \quad \exists x \quad \Sigma(x,y,a) = 1 \wedge \Psi(x,y,a) \neq 1$$

be a true sentence defined over F_k. Let $F_y =< y_1, \dots, y_\ell >$ be the free group defined over the variables $y_1, \dots, y_\ell$.

Then there exists a formal solution $x = x(y,a)$, and a finite set of restricted limit groups $Rlim_1(y,a), \dots, Rlim_m(y,a)$ for which:

(i) *The words corresponding to the equations in the system $\Sigma(x(y,a),y,a) = 1$ represent the trivial word in the free group $F_k * F_y$.*
(ii) *For every index i, $Rlim_i(y,a)$ is a proper quotient of the free group $F_k * F_y$.*
(iii) *Let $B_1(y), \dots, B_m(y)$ be the basic sets (varieties) corresponding to the restricted limit groups $Rlim_1(y,a), \dots, Rlim_m(y,a)$. If*

$$y \notin B_1(y) \cup \dots \cup B_m(y),$$

and $\psi_j(x,y,a)$ is a word corresponding to one of the inequalities in the system $\Psi(x,y,a) \neq 1$, then $\psi_j(x(y,a),y,a) \neq 1$.

Proof: Let $\psi_1(x,y,a) = 1, \dots, \psi_r(x,y,a) = 1$ be the defining equations of the system $\Psi(x,y,a) = 1$. By theorem 1.2 of [Se2] there exists a formal solution $x = x(y,a)$ for which condition (i) holds, and there exists some specialization y_0 of the variables y, so that for every index j, $1 \leq j \leq r$, $\psi_j(x(y_0,a),y_0,a) \neq 1$.

Let $Rlim_1(y,a), \dots, Rlim_m(y,a)$ be the union of the maximal restricted limit groups corresponding to the equations $\psi_j(x(y,a),y,a) = 1$, for $1 \leq j \leq r$. Since there exists a specialization y_0 for which for every index j, $1 \leq j \leq r$, $\psi_j(x(y_0,a),y_0,a) \neq 1$, each restricted limit group $Rlim_i(y,a)$ is a proper quotient of the free group $F_k * F_y$, and we get part (ii). Let $B_1(y), \dots, B_m(y)$ be the basic sets corresponding to the maximal restricted limit groups $Rlim_1(y,a), \dots, Rlim_m(y,a)$. By the construction of these maximal restricted limit groups, if $y \notin B_1(y) \cup \cdots \cup B_m(y)$ then $\psi_j(x(y,a),y,a) \neq 1$ for every $1 \leq j \leq r$, and we get part (iii) of the proposition. ∎

Proposition 1.1 gives a formal solution that proves the validity of the given sentence on a co-basic set $(F_k)^\ell \setminus (B_1(y) \cup \cdots \cup B_m(y))$, hence the rest of the procedure needs to construct formal solutions that prove the validity of the sentence on the remaining basic sets (varieties) $B_1(y),\dots,B_m(y)$. Since our treatment of the restricted limit groups $Rlim_1(y,a),\dots,Rlim_m(y,a)$ that define the basic sets $B_1(y),\dots,B_m(y)$ is independent, we will continue with one of these limit groups, which for brevity we denote $Rlim(y,a)$. Note that $Rlim(y,a)$ is a proper quotient of the free group $F_k * F_y$.

Let $Rlim(y,a)$ be a restricted limit group. By proposition 1.10 of [Se2] with $Rlim(y,a)$ one can associate its strict Makanin–Razborov diagram. Let

$$Res_1(y,a),\dots,Res_s(y,a)$$

be the resolutions in this strict diagram. Each resolution in the diagram is a strict Makanin–Razborov resolution either of the restricted limit group $Rlim(y,a)$ or of a proper quotient of it, and every specialization of the restricted limit group $Rlim(y,a)$ factors through one of the resolutions

$$Res_1,(y,a),\dots,Res_s(y,a).$$

Hence, with the strict Makanin–Razborov diagram of $Rlim(y,a)$ we can associate a tuple of couples $(Rlim_i(y,a), Res_i(y,a))$, where $Rlim_i(y,a)$ is either $Rlim(y,a)$ or a proper quotient of it, and $Res_i(y,a)$ is a strict Makanin–Razborov resolution of $Rlim_i(y,a)$.

Since every strict Makanin–Razborov resolution is, in particular, a well structured resolution (definition 1.11 in [Se2]), with each tuple

$$(Rlim_i(y,a), Res_i(y,a))$$

we can associate its completed resolution $Comp(Res)_i(z,y,a)$ and a completed limit group $Comp(Rlim)_i(z,y,a)$. By lemma 1.14 of [Se2], every specialization y_0 that factors through the resolution $Res_i(y,a)$ factors through the completed resolution $Comp(Res)_i(z,y,a)$ and through the completed limit group $Comp(Rlim)_i(z,y,a)$. Therefore, if $D_i(y)$ denotes the Diophantine set corresponding to the completed limit group $Comp(Rlim)_i(z,y,a)$, and $B(y)$ is the basic set corresponding to the original limit group $Rlim(y,a)$, then $B(y) = D_1(y) \cup \cdots \cup D_s(y)$.

Since the continuation of the procedure is conducted independently for the tuples $(Rlim_i(y,a), Res_i(y,a))$, we will continue with one of them, which we denote $(Rlim(y,a), Res(y,a))$ for brevity, and we denote its completed resolution by $Comp(Res)(y,a)$ and its completed limit group by $Comp(Rlim)(y,a)$.

PROPOSITION 1.2: *Let $F_k =< a_1, \dots, a_k >$ be a free group, let $Rlim(y,a)$ be a limit group, and let $Res(y,a)$ be a well-structured resolution of $Rlim(y,a)$. There exists a covering closure $Cl(Res)_1(s,z,y,a), \dots, Cl(Res)_q(s,z,y,a)$ of the resolution $Res(y,a)$, and for each index n, $1 \leq n \leq q$ there exists a formal solution $x_n(s,z,y,a)$, and a finite set of restricted limit groups*

$$QRlim_1^n(s,z,y,a), \dots, QRlim_{m(n)}^n(s,z,y,a)$$

for which:

(i) *For each index n, $1 \leq n \leq q$, the words corresponding to the equations in the system $\Sigma(x_n(s,z,y,a),y,a)$ represent the trivial word in the closure $Cl(Res)_n(s,z,y,a)$.*

(ii) *For every index n, the restricted limit groups*

$$QRlim_1^n(s,z,y,a), \dots, QRlim_{m(n)}^n(s,z,y,a)$$

are proper quotients of the n-th closure $Cl(Res)_n(s,z,y,a)$.

(iii) *For every index n, $1 \leq n \leq q$, let $D^n(y)$ be the Diophantine set corresponding to the closure $Cl(Res)_n(s,z,y,a)$, and let $D_1^n(y), \dots, D_{m(n)}^n(y)$ be the Diophantine sets corresponding to the restricted limit groups*

$$QRlim_1^n(s,z,y,a), \dots, QRlim_{m(n)}^n(s,z,y,a).$$

Let $\psi_j(x,y,a)$ be a word corresponding to one of the inequalities in the system $\Psi(x,y,a)$. If for some index n, $1 \leq n \leq q$, $y_0 \in D^n(y)$ and there exists some specialization (s_0,z_0,y_0,a) that factors through the closure $Cl(Res)_n(s,z,y,a)$, and (s_0,z_0,y_0,a) does not factor through any of the quotient limit groups $QRlim_i^n(s,z,y,a)$, then for every index j, $1 \leq j \leq r$,

$$\psi_j(x_n(s_0,z_0,y_0,a),y_0,a) \neq 1.$$

Proof: Let $\psi_1(x,y,a) = 1, \dots, \psi_r(x,y,a) = 1$ be the defining equations of the system $\Psi(x,y,a) = 1$. By theorem 1.18 of [Se2], there exists a covering closure $Cl(Res)_1(s,z,y,a), \dots, Cl(Res)_q(s,z,y,a)$ of the resolution $Res(y,a)$, and formal solutions $x_1(s,z,y,a), \dots, x_q(s,z,y,a)$ for which (i) holds, and for each index n, $1 \leq n \leq q$, there exists some specialization (s_0^n, z_0^n, y_0^n, a) of the variables (s, z, y, a) that factors through the n-th closure $Cl(Res)_n(s, z, y, a)$, so that for every index j, $1 \leq j \leq r$, $\psi_j(x_n(s_0^n, z_0^n, y_0^n, a), y_0^n, a) \neq 1$.

Hence, if for each index n, $1 \leq n \leq q$, we set

$$QRlim_1^n(s, z, y, a), \ldots, QRlim_{m(n)}^n(s, z, y, a)$$

to be the maximal restricted limit groups, which are the maximal quotients of the n-th closure $Cl(Res)_n(s, z, y, a)$, that correspond to the collection of all the specializations of the closure $Cl(Res)_n(s, z, y, a)$, that satisfy

$$\psi_j(x_n(s, z, y, a), y, a) = 1$$

for some index j, $1 \leq j \leq r$, then these maximal limit groups $QRlim_t^n$ are proper quotients of the limit group associated with the n-th closure, $Cl(Res)_n(s, z, y, a)$, and part (ii) follows. Part (iii) follows from the way the limit groups $QRlim_t^n$ are defined. ∎

The formal solutions constructed by Propositions 1.1 and 1.2 prove the validity of the given sentence on the co-Diophantine set

$$(F_k)^\ell \setminus (D_1^1(y) \cup \cdots \cup D_{m(q)}^q(y)),$$

hence the rest of the procedure needs to construct formal solutions that prove the validity of the sentence on the remaining Diophantine sets $D_1^1(y), \ldots, D_{m(q)}^q(y)$. Note that for each index n, $1 \leq n \leq q$, the Diophantine sets $D_1^n(y), \ldots, D_{m(n)}^n(y)$ are defined using the tuple of variables (s, z, y, a) of the closure $Cl(Res)_n(s, z, y, a)$ of the strict Makanin–Razborov resolution $Res(y, a)$ (of the restricted limit group $Rlim(y, a)$). This appearance of additional variables emerging from closures of resolutions, a by-product of the use of formal solutions and the generalized Merzlyakov theorem ([Se2], 1.18), is unavoidable and is going to be an "obstacle" throughout our iterative procedure. Naturally, these additional variables make the iterative procedure more involved.

Because of the technical complications, we prefer to present the iterative procedure assuming that all the restricted limit groups $Rlim_1(y, a), \ldots, Rlim_m(y, a)$ obtained in the initial step of the procedure (Proposition 1.1) are of minimal rank (rank 0), i.e., we will assume that for every restricted homomorphism from $Rlim_i(y, a)$ onto a free group $F_k * F$, F must be the trivial group. This assumption simplifies the technicalities considerably, and allows better demonstration of the geometric considerations that will eventually guarantee the termination of the procedure. In the next sections of this paper, we gradually modify the procedure to the general case.

Since the continuation of the procedure is conducted independently for each tuple $(Rlim_i(y,a), Res_i(y,a))$, and a proper quotient $QRlim_t^n(s,z,y,a)$ of some closure $Cl(Res)_n(s,z,y,a)$ of the resolution $Res_i(y,a)$, we will continue with one tuple which we denote $(Rlim(y,a), Res(y,a))$, and a proper quotient $QRlim(s,z,y,a)$ of a closure $Cl(Res)(s,z,y,a)$ of the resolution $Res(y,a)$, and omit the relevant indices.

The iterative procedure we present produces a sequence of well-structured resolutions and their completions. We will associate a *complexity* with each of the resolutions produced by the procedure, and show that the complexity decreases with each step of the procedure, a decrease that eventually forces the iterative procedure to terminate.

As we already pointed out, for presentation purposes we will start by restricting ourselves to the minimal rank case. We start describing the procedure by presenting the first step of the procedure, and define the resolutions it produces and their complexity. Next, we present the general step of the procedure, and prove that the complexity is strictly decreasing. By the way we define the complexity, the termination of the procedure is a straightforward consequence of the decrease in complexity.

The first step of the iterative procedure. Recall that the completion of a resolution ([Se2], 1.12) is an ω-residually free tower ([Se1], 6). In case the resolution is of minimal rank, the abelian decomposition associated with each level of the completion of the resolution contains a single distinguished vertex, all the other vertices are connected only to the distinguished one, and the stabilizers of the other vertices are either abelian or QH subgroups.

Let z be (a finite set of) elements that generate the completed resolution $Comp(Res)(z,y,a)$, and let z_{base} be a subset of the elements z that generate the subgroups associated with all the levels of the completed resolution $Comp(Res)(z,y,a)$ except the top level (i.e., the subgroup generated by z_{base} is the distinguished vertex group in the abelian decomposition associated with the top level of the completion, $Comp(Res)(z,y,a)$). We will call this subset of elements the **basis** of the completed resolution $Comp(Res)(z,y,a)$.

Following the construction of the strict Makanin–Razborov diagram ([Se2], 1.10), we use the entire collection of specializations that factor through the quotient limit group $QRlim(s,z,y,a)$ to construct the corresponding (canonical) strict graded Makanin–Razborov diagram of the limit group $QRlim(s,z,y,a)$, viewed as a graded limit group with respect to the parameters z_{base}. Let

$$GRes_1(s,z,y,z_{base},a),\ldots,GRes_m(s,z,y,z_{base},a)$$

be the strict graded Makanin–Razborov resolutions that appear in the strict graded Makanin–Razborov diagram of the (graded) limit group $QRlim(s,z,y,z_{base},a)$ with respect to the parameters z_{base}, where each graded resolution is terminating in either a rigid or a solid graded limit group (with respect to the defining parameters z_{base}). Note that by definition, a homomorphism factors through a graded resolution if it can be expressed as a composition of graded modular automorphisms of the different levels, the canonical epimorphisms between consecutive levels of the graded resolution, and a rigid (strictly solid) homomorphism from the terminal rigid (solid) graded limit group of the resolution to the free group F_k.

We will treat the graded resolutions $GRes_i(s,z,y,z_{base},a)$ in parallel, so for the continuation we will restrict ourselves to one of them which we denote $GRes(s,z,y,z_{base},a)$ for brevity. Let $GRlim(s,z,y,z_{base},a)$ be the graded limit group corresponding to the graded resolution $GRes(s,z,y,z_{base},a)$.

Let $Glim_j(s,z,y,z_{base},a)$ be a graded limit group that appears in the j-th level of the graded resolution $GRes(s,z,y,z_{base},a)$. Naturally, there exists a canonical map τ_j: $Rlim(y,a) \to Glim_j(s,z,y,z_{base},a)$. Let Λ_j be the graded quadratic decomposition of $Glim_j(s,z,y,z_{base},a)$, i.e., the graded abelian decomposition of $Glim_j$ obtained from the graded JSJ decomposition of $Glim_j$ by collapsing all the edges connecting two non-QH subgroups. Let Q be a quadratically hanging subgroup in the JSJ decomposition of $Rlim(y,a)$, and let S be the corresponding (punctured) surface. Since the boundary elements of Q are mapped by τ_j to either the trivial element or elliptic elements in Λ_j (because they are in $< z_{base}, a >$), the (possibly trivial) cyclic decomposition inherited by $\tau_j(Q)$ from the cyclic decomposition Λ_j can be lifted to a (possibly trivial) maximal cyclic decomposition of the QH subgroup Q of $Rlim(y,a)$, in which every cyclic edge group is mapped by τ_j to either the trivial element or to an elliptic element in Λ_j, which corresponds to some decomposition of the (punctured) surface S along a (possibly trivial) maximal collection of disjoint non-homotopic s.c.c. Let $\Gamma_j(Q)$ be the corresponding cyclic decomposition of the QH subgroup Q, and let $\Gamma_j(S)$ be the associated maximal collection of non-homotopic disjoint essential s.c.c. on S, that are mapped by τ_j either to the trivial element or into an edge group in Λ_j.

Lemma 1.3:

(i) *Every non-separating s.c.c. on the surface S is mapped to a non-trivial element by the homomorphism τ_j.*

(ii) *Let Q' be a quadratically hanging subgroup in Λ_j, and let S' be the* *corresponding (punctured) surface. If τ_j maps non-trivially a connected subsurface of $S \setminus \Gamma_j(S)$ into Q', then $genus(S') \leq genus(S)$ and $|\chi(S')| \leq |\chi(S)|$. Furthermore, in this case τ_j maps the fundamental group of a subsurface of S into a finite index subgroup of Q'.*

Proof: If a non-separating s.c.c. on the surface S is mapped to the trivial element by τ_j, then the limit group $Rlim(y,a)$ admits an epimorphism onto the free group $F_k * Z$, where Z is the infinite cyclic group, a contradiction to $Rlim(y,a)$ being of minimal rank, and we get part (i).

Let S_1 be the connected subsurface of $S \setminus \Gamma_j(S)$ that is mapped non-trivially into Q' by the homomorphism τ_j. Let Q_1 be the fundamental group of S_1. By construction, the homomorphism τ_j maps the boundary components of S_1 to either the trivial element or to non-trivial elliptic elements in Λ_j. By part (i) no s.c.c. on S_1 which is a non-separating s.c.c. in S is mapped to the trivial element by the homomorphism τ_j. Clearly, $genus(S_1) \leq genus(S)$ and $|\chi(S_1)| \leq |\chi(S)|$.

If all the boundary components of the subsurface S_1 are mapped to the trivial element by the homomorphism τ_j, and S_1 itself is mapped non-trivially into $Glim_j(s,z,y,z_{base},a)$, then $Rlim(y,a)$ admits an epimorphism onto a free group $F_k * F$ for some non-trivial free group F, a contradiction to $Rlim(y,a)$ being of minimal rank. Hence, some of the boundary components of S_1 are mapped by τ_j to non-trivial elliptic elements in the cyclic decomposition Λ_j of $Glim_j(s,z,y,z_{base},a)$. Let S_2 be the punctured surface obtained from the punctured surface S_1 by adjoining disks along each of the boundary components of S_1 that is mapped to the trivial element by τ_j. Let Q_2 be the fundamental group of S_2.

By the maximality of the collection $\Gamma_j(S)$, no s.c.c. on the surface S_2 is mapped to the trivial element in τ_j, and by the way the punctured surface S_2 was constructed, $genus(S_2) \leq genus(S_1) \leq genus(S)$ and $|\chi(S_2)| \leq |\chi(S_1)| \leq |\chi(S)|$.

To complete the proof of part (ii) of the lemma, we use the following basic fact which is used often in the next sections.

LEMMA 1.4: *Let M be either a punctured torus or a punctured surface with $\chi(M) \leq -2$, and let B be a f.g. subgroup of infinite index in $\pi_1(M)$. Then either:*

(i) *B contains no conjugate of a boundary element in M; or*

(ii) *$B = < c_1 > * \cdots * < c_\ell > * B'$. Each of the elements c_i is conjugate to some boundary element in M, and every element $b \in B$ which is conjugate to a boundary element in $\pi_1(M)$, is conjugate to one of the elements c_j.*

Proof: By P. Scott [Sc] every subgroup of a hyperbolic surface group is almost geometric, i.e., there exists a finite cover $\hat{M}$ of the punctured surface M, for which B is a subgroup of the image of $\pi_1(\hat{M})$ in M, and the subgroup associated with B in $\pi_1(\hat{M})$ is the fundamental group of a punctured subsurface T in $\hat{M}$. Since B is of infinite index in $\pi_1(M)$, T is a proper subsurface of $\hat{M}$, i.e., it contains a boundary component that is not a boundary component of $\hat{M}$. This structure of the subsurface T implies the conclusion of the lemma. ∎

To complete the proof of Lemma 1.3, note that if $\tau_j(\pi_1(S_2)) = \tau_j(Q_2)$ is of infinite index in the QH subgroup Q', then Lemma 1.4 implies that $Rlim(y,a)$ admits an epimorphism onto a free group $F_k * F$ for some non-trivial free group F, a contradiction to $Rlim(y,a)$ being of minimal rank. Hence, $\tau_j(Q_2)$ is of finite index in the QH subgroup Q'. Hence, $genus(S') \leq genus(S_2) \leq genus(S)$. Now, since $\tau_j(Q_2)$ is a finite index subgroup of Q', the double of the punctured surface S_2 along its boundary components is mapped non-trivially onto a finite cover of the double of the punctured surface S'. Hence $|\chi(S')| \leq |\chi(S_2)| \leq |\chi(S)|$, and we get part (ii) of the lemma. ∎

Part (ii) of Lemma 1.3 bounds the topological complexity of those QH subgroups Q' that appear in the graded abelian JSJ decompositions associated with the various levels of the graded resolution $GRes(s,z,y,z_{base},a)$ into which a QH subgroup Q that appears in the cyclic JSJ decomposition of $Rlim(y,a)$ is mapped non-trivially. To show that Lemma 1.3 can be applied to bound the topological complexity of all the QH subgroups Q' that appear in the graded abelian decompositions associated with the various levels of $GRes(s,z,y,z_{base},a)$, we need the following proposition.

Proposition 1.5: *Let $Glim_j(s,z,y,z_{base},a)$ be the graded limit group that appears in the j-th level of the graded resolution $GRes(s,z,y,z_{base},a)$, and let Q' be a QH subgroup that appears in the graded JSJ decomposition associated with $Glim_j(s,z,y,z_{base},a)$, and S' be its corresponding punctured surface. Let τ_j be the natural map τ_j: $Rlim(y,a) \to Glim_j(s,z,y,z_{base},a)$.*

Then there exists a QH subgroup Q in the JSJ decomposition of $Rlim(y,a)$ with corresponding punctured surface S, so that a subsurface S_1 of the punctured surface S is mapped by τ_j into a finite index subgroup of a conjugate of Q'. In particular, $genus(S') \leq genus(S)$ and $|\chi(S')| \leq |\chi(S)|$.

Proof: Let $\Lambda_{Q'}$ be the decomposition obtained from the graded JSJ decomposition of $Glim_j(s,z,y,z_{base},a)$ by collapsing all the edges that are not connected to the QH subgroup Q'. The cyclic decomposition $\Lambda_{Q'}$ contains a (distinguished) vertex connected to Q' and stabilized by a vertex group containing the subgroup $< z_{base}, a >$, and perhaps few other vertices connected to disjoint boundary components of the QH subgroup Q'.

For each QH subgroup Q in the JSJ decomposition of $Rlim(y,a)$, let G_Q be the subgroup of $Comp(Rlim)(z,y,a)$, which is the fundamental group of the subgraph of groups containing two vertices, the vertex stabilized by the subgroup $< z_{base}, a >$ and the vertex stabilized by the subgroup Q. There is a natural map ζ: $Comp(Rlim)(z,y,a) \to Glim_j(s,z,y,z_{base},a)$. For each QH subgroup Q in the JSJ decomposition of $Rlim(y,a)$, the image group $\zeta(G_Q)$ inherits a decomposition Δ_Q from the cyclic decomposition $\Lambda_{Q'}$ of $Glim_j(s,z,y,z_{base},a)$.

Let S be the punctured surface corresponding to Q, and suppose that no subsurface of S is mapped non-trivially into a conjugate of Q'. Since the subgroup $\zeta(< z_{base}, a >)$ is contained in a vertex group in Δ_Q, and the fundamental group of any subsurface of S is not mapped non-trivially into Q, Δ_Q is either a trivial graph of groups, or it gives rise to a free decomposition of G_Q in which the subgroup $\zeta(< z_{base}, a >)$ is contained in one of the factors.

If the decomposition Δ_Q gives rise to a free decomposition of the group G_Q in which the subgroup $\zeta(< z_{base}, a >)$ is contained in one of the factors, then the restricted limit group $Rlim(y,a)$ can be mapped onto a limit group H_Q which obtains a free decomposition in which the subgroup $F_k < H_Q$ is contained in one of the factors. Hence, in case Δ_Q gives rise to a free product of G_Q, the restricted limit group $Rlim(y,a)$ admits an epimorphism onto a free group $F_k * F$ for some non-trivial free group F, a contradiction to $Rlim(y,a)$ being of minimal rank.

Therefore, for every QH subgroup Q in the JSJ decomposition of $Rlim(y,a)$, the decomposition Δ_Q is the trivial decomposition of G_Q, so the homomorphism τ_j: $Rlim(y,a) \to Glim_j(s,z,y,z_{base},a)$ maps the restricted limit group $Rlim(y,a)$ into the vertex group stabilized by the subgroup $< z_{base}, a >$ in the cyclic decomposition $\Lambda_{Q'}$. Since the quotient limit group $QRlim(s,z,y,a)$ was obtained as the limit of a sequence of specializations $\{(s_n,z_n,y_n,a)\}$, if $Rlim(y,a)$ is mapped to the vertex stabilized by $< z_{base}, a >$ in $\Lambda_{Q'}$, the completed limit group $Comp(Rlim)(z,y,a)$ is mapped to the vertex group stabilized by $< z_{base}, a >$, so the entire graded limit group $Glim_j(s,z,y,z_{base},a)$ stabilizes that vertex, and the decomposition $\Lambda_{Q'}$ is trivial, which contradicts the way it was constructed. Therefore, there exists some QH subgroup Q in the cyclic JSJ decomposition of $Rlim(y,a)$ with corresponding punctured surface S, so that a subsurface S_1 of the punctured surface S is mapped non-trivially by τ_j into some conjugate of Q'. By Lemma 1.3, the fundamental group of S_1 is mapped into a finite index subgroup of a conjugate of Q'. In particular, $genus(S') \leq genus(S)$ and $|\chi(S')| \leq |\chi(S)|$. ∎

Definition 1.6: Let Q be a quadratically hanging subgroup in the JSJ decomposition of $Rlim(y,a)$ and let S be its corresponding (punctured) surface. The QH subgroup Q (and the corresponding surface S) is called **surviving** if for some level j, there exists some quadratically hanging subgroup Q' in Λ_j, the JSJ decomposition of $Glim_j(s,z,y,z_{base},a)$, with corresponding surface S', so that τ_j maps Q non-trivially into Q', $genus(S') = genus(S)$ and $\chi(S') = \chi(S)$.

By definition, if Q is a non-surviving QH subgroup in the JSJ decomposition of $Rlim(y,a)$, then every QH subgroup Q' in any level of the graded resolution $GRes(s,z,y,z_{base},a)$ into which a subsurface of Q is mapped non-trivially, has either a strictly lower genus or a strictly smaller absolute value of Euler characteristic than that of the QH subgroup Q. This would eventually "force" the complexity of the graded resolution $GRes(s,z,y,z_{base},a)$ to be smaller than that of the resolution $Res(y,a)$ once one is able to "isolate" the surviving surfaces. This is the purpose of the following theorem.

THEOREM 1.7: *The graded resolution $GRes(s,z,y,z_{base},a)$ can be replaced by finitely many graded resolutions that we denote $GRes(u,s,z,y,z_{base},a)$, each composed from two consecutive parts.*

In the first part all the surviving surfaces are mapped into the distinguished vertex, i.e., the vertex stabilized by the subgroup $< z_{base}, a >$. The second part is a one-step resolution in which all the images of the surviving surfaces appear as QH vertex groups, i.e., if $Q_1,\ldots,Q_r$ are the surviving surfaces in the abelian decomposition of $Rlim(y,a)$ with respect to the graded resolution $GRes$, then the graded decomposition corresponding to the second part of the resolution $GRes$ contains a vertex stabilized by the terminal rigid (solid) graded limit group of the resolution $GRes(u,s,z,y,z_{base},a)$ connected to r' surviving QH subgroups $Q'_{i_1},\ldots,Q'_{i_{r'}}$, for some $r' \leq r$, and $1 \leq i_1 < i_2 < \cdots < i_{r'} \leq r$.

Proof: We replace the graded resolution $GRes(s,z,y,z_{base},a)$ with finitely many graded resolutions, so that every homomorphism that factors through the original graded resolution $GRes$ factors through at least one of the new ones.

Let $Q_1,\ldots,Q_t$ be the surviving surfaces in the graded JSJ decomposition of $Rlim(y,a)$ with respect to the given graded resolution $GRes(s,z,y,z_{base},a)$. By definition, for each surviving surface Q_i there exists a QH subgroup Q'_i that appears in one of the graded JSJ decompositions associated with the various levels of the graded resolution $GRes(s, z, y, z_{base}, a)$, so that Q_i is mapped isomorphically onto Q'_i. For each Q_i we pick Q'_i to be the QH subgroup that appears in the highest possible level graded abelian decomposition so that Q_i is mapped isomorphically onto Q'_i. Up to a change of order, we may assume that the QH subgroups $Q'_1, \dots, Q'_t$ are ordered according to the level of their appearance in the graded resolution $GRes(s, z, y, z_{base}, a)$, from top to bottom. Note that it may be that some of the QH subgroups $Q'_1, \dots, Q'_t$ are conjugate in the corresponding graded limit groups. Also, note that if the fundamental group of a subsurface of Q_i is mapped non-trivially into a QH vertex group $\hat{Q}$ that lies above the QH vertex group Q'_i in the graded resolution $GRes$, then the fundamental group of the corresponding subsurface of Q_i is mapped isomorphically onto $\hat{Q}$.

To replace the graded resolution $GRes(s, z, y, z_{base}, a)$ with finitely many graded resolutions that have all their surviving surfaces in the bottom level, we do the following. Let $Q'_1, \dots, Q'_{t'}$ be the subgroups that appear in the highest level among the entire collection $Q'_1, \dots, Q'_t$. Let $Q'_{i_1}, \dots, Q'_{i_m}$ be the representatives of the conjugacy classes of the QH subgroups $Q'_1, \dots, Q'_{t'}$, and let $Q_{i_1}, \dots, Q_{i_m}$ be the corresponding QH vertex groups in the abelian JSJ decomposition of $Rlim(y, a)$ that are mapped onto $Q'_{i_1}, \dots, Q'_{i_m}$.

For every QH subgroup Q_{i_j} from the collection of the QH subgroups $Q_{i_1}, \dots, Q_{i_m}$, for which no QH subgroup from the collection $Q_1, \dots, Q_t$ other than Q_{i_j} is mapped non-trivially into a conjugate of Q'_{i_j} we do the following.

The QH subgroup Q_{i_j} is naturally mapped into each of the limit groups associated with the various levels of the graded resolution $GRes(s, z, y, z_{base}, a)$. If for some level of the graded resolution $GRes(s, z, y, z_{base}, a)$ that lies above the level in which the QH vertex group Q'_{i_j} appears, the QH subgroup Q_{i_j} is mapped onto a subgroup that is not contained in the distinguished vertex group in the graded abelian decomposition associated with that level, then there exists a minimal subgraph Λ of that graded abelian decomposition that contains its distinguished vertex group and the image of the subgroup Q_{i_j}. Furthermore, for every QH vertex group Q' in Λ there exists a subsurface in Q_{i_j}, so that the fundamental group of that subsurface of Q_{i_j} is mapped isomorphically onto Q'. For every edge group in Λ there exists a boundary component or a non-boundary parallel s.c.c. in Q_{i_j} that is mapped onto it.

We change the graded decomposition in such a level (that lies above the level in which the subgroup Q'_{i_j} appears) by collapsing the subgraph of groups Λ into the (distinguished) vertex stabilized by $< z_{base}, a >$. Other than that, we leave the levels that lie above the one that contains the QH subgroup Q'_{i_j} unchanged.

We continue the modified graded resolution by mapping each of the QH subgroups Q'_{i_j} from the set $Q'_{i_1}, \dots, Q'_{i_m}$ to the image of the corresponding surviving surface Q_{i_j} in the subgroup $< z_{base}, a >$ under the quotient map of the completed resolution $Comp(Res)(z, y, a)$.

So far we have modified the levels of the resolution $GRes$ we have started with, from the top level until the level that contains the QH vertex group Q'_1. By construction, all the specializations that factor through the original resolution $GRes$ factor through the (graded) resolution consists of the modified abelian decompositions associated with the levels of $GRes$ from the one containing Q'_1 and above.

Starting with the entire collection of specializations that factor through the original resolution $GRes$, and applying the modular groups associated with the resolution obtained from the modified top levels, we get a set of specializations, with which we associate a (canonical) finite collection of (graded) limit groups that are all proper quotients of the original limit group $QRlim$. We continue the construction of the modified graded resolutions, by starting with each of these limit groups (in parallel), constructing the strict graded Makanin–Razborov diagrams associated with them, and modify each of the graded resolutions that appear in these diagrams in exactly the same way we have modified the original resolution.

The modification described above gives rise to finitely many modified resolutions. Every specialization (s_0, z_0, y_0, a) with respect to the closure $Cl(Res)(s, z, y, a)$ that factors through the graded resolution $GRes(s, z, y, z_{base}, a)$ factors through at least one of the modified resolutions constructed by the above procedure.

Let $Q'_1, \dots, Q'_{r'}$ be representatives for the conjugacy classes of the (images of the) surviving QH subgroups $Q'_1, \dots, Q'_r$ that appear in the various level of one of the obtained resolutions. Note that each of the surviving subgroups $Q_1, \dots, Q_r$ is mapped to a conjugate of one of the subgroups $Q'_1, \dots, Q'_{r'}$. Therefore, in each of the modified graded resolutions obtained by the above procedure, we can push all the QH subgroups $Q'_1, \dots, Q'_{r'}$ to the bottom level, and the obtained (finitely many) graded resolutions satisfy the properties listed in the statement of the theorem. ∎

We continue the analysis of the limit group $QRlim(s, z, y, a)$ by reducing the set of defining parameters sequentially. Recall that in order to obtain the graded resolution $GRes(s,z,y,z_{base},a)$, we used the variables z_{base} as parameters, where z_{base} were the subset of the variables z that appear in all the levels of the completed resolution, $Comp(Res)(z,y,a)$, except those defining the highest level. For the next step of the analysis we take z^2_{base} as the defining parameters, where z^2_{base} generate the subgroup associated with all the levels of the completed resolution $Comp(Res)(z,y,a)$ except for the two highest levels.

Let $T_1(s,z,y,a)$ be the terminal rigid or solid graded limit group in the graded resolution $GRes(s,z,y,z_{base},a)$ with respect to the parameters z_{base}. From the collection of rigid (solid) specializations of $T_1(s,z,y,z_{base},a)$, that are obtained (using our shortening procedure) from specializations that factor through the graded resolution $GRes(s,z,y,z_{base},a)$, we construct the graded strict Makanin–Razborov diagram of $T_1(s,z,y,a)$, viewed as a graded limit group with respect to the parameters z^2_{base}. Let

$$GRes_1(s,z,y,z^2_{base},a),\ldots,GRes_m(s,z,y,z^2_{base},a)$$

be the strict graded Makanin–Razborov resolutions that appear in the strict graded Makanin–Razborov diagram of the (graded) limit group $T_1(s,z,y,z^2_{base},a)$ with respect to the parameters z^2_{base}, where each graded resolution is terminating in either a rigid or a solid graded limit group (with respect to the defining parameters z^2_{base}).

We will treat the graded resolutions $GRes_i(s,z,y,z^2_{base},a)$ in parallel, so for the continuation we will restrict ourselves to one of them which we denote $GRes(s,z,y,z^2_{base},a)$ for brevity. Let $Glim(s,z,y,z^2_{base},a)$ be the graded limit group corresponding to the graded resolution $GRes(s,z,y,z^2_{base},a)$. If the subgroup $Glim(s,z,y,z^2_{base},a)$ is a proper quotient of the subgroup $T_1(s,z,y,a)$, we need to modify the graded resolution $GRes(s,z,y,z_{base},a)$ so that it becomes a strict graded Makanin–Razborov resolution terminating with the limit group $Glim(s,z,y,z^2_{base},a)$ (see proposition 1.10 in [Se2] for the procedure that replaces a given resolution with finitely many strict ones). Note that by modifying the graded resolution $GRes(s,z,y,z_{base},a)$, we may need to replace the quotient limit group $QRlim(s,z,y,a)$ or one of the groups $Glim_j(s,z,y,z_{base},a)$ that appear in the graded resolution $GRes(s,z,y,z_{base},a)$ by a proper quotient of itself.

Let $Glim_j(s,z,y,z^2_{base},a)$ be a graded limit group that appears in the j-th level of the graded resolution $GRes(s,z,y,z^2_{base},a)$. Let $Zlim(z_{base},a)$ be the limit group generated by $< z_{base},a >$ in the completed limit group

$Comp(Rlim)(z,y,a)$. Naturally, there exists a canonical map

$$\tau_j\colon Zlim(z_{base},a) \to Glim_j(s,z,y,z^2_{base},a).$$

Let Λ_j be the graded quadratic decomposition of $Glim_j(s,z,y,z^2_{base},a)$, i.e., the graded cyclic decomposition of $Glim_j(s,z,y,z^2_{base},a)$ obtained from the graded JSJ decomposition of $Glim_j(s,z,y,z^2_{base},a)$ by collapsing all the edges connecting two non-QH subgroups. Let Q be a quadratically hanging subgroup in the graded JSJ decomposition of $Zlim(z_{base},a)$ and let S be the corresponding (punctured) surface. Since the boundary elements of Q are mapped by τ_j to either the trivial element or to elliptic elements in Λ_j, the (possibly trivial) cyclic decomposition inherited by $\tau_j(Q)$ from the cyclic decomposition Λ_j can be lifted to a (possibly trivial) cyclic decomposition of the QH subgroup Q of $Glim_j(s,z,y,z^2_{base},a)$, which corresponds to some decomposition of the (punctured) surface S along a (possibly trivial) collection of disjoint non-homotopic s.c.c. Let $\Gamma_j(Q)$ be the corresponding cyclic decomposition of the QH subgroup Q, and let $\Gamma_j(S)$ be a maximal associated collection of non-homotopic disjoint essential s.c.c. on S. Note that by construction every s.c.c. from the defining collection of $\Gamma_j(S)$ is mapped by τ_j to either a trivial element or to an elliptic element in Λ_j.

Lemma 1.8:
(i) *Every non-separating s.c.c. on the surface S is mapped to a non-trivial element by the homomorphism τ_j.*
(ii) *Let Q' be a quadratically hanging subgroup in Λ_j, and let S' be the corresponding (punctured) surface. If τ_j maps non-trivially a connected subsurface of $S \setminus \Gamma_j(S)$ into Q', then $genus(S') \leq genus(S)$ and $|\chi(S')| \leq |\chi(S)|$. Furthermore, in this case τ_j maps the fundamental group of a subsurface of S into a finite index subgroup of Q'.*

Proof: Identical to the proof of Lemma 1.3. ∎

The precise statement of Proposition 1.5 is not valid for the graded resolution $GRes(s,z,y,z^2_{base},a)$. If Q' is a QH subgroup that appears in an abelian decomposition associated with one of the levels of the graded resolution $GRes(s,z,y,z^2_{base},a)$, then it is not true that there exists some QH subgroup Q in the graded JSJ decomposition of $Zlim(z_{base},a)$, so that the fundamental group of a subsurface of the punctured surface corresponding to Q is mapped non-trivially into Q'. However, we can still define *surviving surfaces*.

Definition 1.9: Let Q be a quadratically hanging subgroup in the JSJ decomposition of $Zlim(z_{base}, a)$ and let S be its corresponding (punctured) surface. The QH subgroup Q (and the corresponding surface S) is called **surviving** if for some level j, there exists some quadratically hanging subgroup Q' in Λ_j, the JSJ decomposition of $Glim_j(s, z, y, z^2_{base}, a)$, with corresponding surface S', so that τ_j maps Q non-trivially into Q', $genus(S') = genus(S)$ and $\chi(S') = \chi(S)$.

To control the "complexity" of the graded resolution $GRes(s, z, y, z^2_{base}, a)$ we need to "isolate" the surviving QH subgroups. This can be done in a similar way to Theorem 1.7.

THEOREM 1.10: *The graded resolution $GRes(s, z, y, z^2_{base}, a)$ can be replaced by finitely many graded resolutions, each composed from two consecutive parts.*

The second part is a one-step resolution in which all the images of the surviving surfaces appear, i.e., if $Q_1, \dots, Q_r$ are the surviving surfaces in the abelian decomposition of $Zlim(z_{base}, a)$ with respect to the graded resolution $GRes$, then the graded decomposition corresponding to the second part of the resolution $GRes$ contains a vertex stabilized by the terminal rigid (solid) graded limit group of the resolution $GRes(s, z, y, z^2_{base}, a)$ connected to r' surviving QH subgroups $Q'_{i_1}, \dots, Q'_{i_{r'}}$, for some $r' \leq r$, and $1 \leq i_1 < i_2 < \dots < i_{r'} \leq r$.

Proof: Identical to the proof of Theorem 1.7. ∎

Given Lemma 1.8 and Theorem 1.10, to complete the analysis of the structure of the graded resolution $GRes(s, z, y, z^2_{base}, a)$ we still need an appropriate analogue of Proposition 1.5, i.e., we need to associate with every QH subgroup that appears in one of the graded JSJ decompositions of the different levels of the graded resolution $GRes(s, z, y, z^2_{base}, a)$, a QH subgroup of either the graded JSJ decomposition of $Zlim(z_{base}, a)$ or a QH subgroup that appears in the graded JSJ decomposition of the terminal subgroup $T_1(s, z, y, z_{base}, a)$ of the graded resolution $GRes(s, z, y, z_{base}, a)$ in case $T_1(s, z, y, z_{base}, a)$ is solid. We divide the final analysis of $GRes(s, z, y, z^2_{base}, a)$ into two cases depending on $T_1(s, z, y, a)$ being rigid or solid.

PROPOSITION 1.11: *Suppose that the terminal subgroup $T_1(s, z, y, z_{base}, a)$ of the graded resolution $GRes(s, z, y, z_{base}, a)$ is rigid. Let Q' be a QH subgroup that appears in the j-th level graded JSJ decomposition of the graded resolution $GRes(s, z, y, z^2_{base}, a)$, and let S' be its corresponding surface. Then:*

(i) *There exists a QH subgroup Q in the JSJ decomposition of $Zlim(z_{base}, a)$, with an associated surface S, so that the map*

$$\tau_j \colon Zlim(z_{base}, a) \to Glim_j(u, s, z, y, z^2_{base}, a)$$

maps the fundamental group of a subsurface of S onto a subgroup of finite index of Q'.

(ii) $genus(S') \leq genus(S)$ *and* $|\chi(S')| \leq |\chi(S)|$.

(iii) $genus(S') = genus(S)$ *and* $|\chi(S')| = |\chi(S)|$ *if and only if Q is a surviving QH subgroup. In this last case, j is the bottom level of the graded resolution $GRes(u, s, z, y, z^2_{base}, a)$, and Q is mapped to the vertex stabilized by $< z^2_{base}, a >$ by the maps $\tau_j \colon Zlim(z_{base}, a) \to Glim_j(u, s, z, y, z^2_{base}, a)$ in all the levels above the bottom one.*

Proof: If, under the map $\tau_j \colon Zlim(z_{base}, a) \to Glim_j(u, s, z, y, z^2_{base}, a)$, none of the QH subgroups Q in the JSJ decomposition of $Zlim(z_{base}, a)$ is mapped non-trivially into (a conjugate of) Q', then the entire image of $Zlim(z_{base}, a)$, $\tau_j(Zlim(z_{base}, a))$, is contained in a proper subgraph of groups of the graded JSJ decomposition of $Glim_j(u, s, z, y, z^2_{base}, a)$, a subgraph that does not contain the vertex stabilized by the QH subgroup Q'. Hence, if no QH subgroup Q is mapped non-trivially by τ_j into the QH subgroup Q', the graded limit groups that appear along the graded resolution $GRes(u, s, z, y, z^2_{base}, a)$ from $Glim_j(u, s, z, y, z^2_{base}, a)$ and above,

$$Glim_j(u, s, z, y, z^2_{base}, a), Glim_{j-1}(u, s, z, y, z^2_{base}, a), \ldots, \\ Glim_1(u, s, z, y, z^2_{base}, a),$$

are all flexible quotients of the rigid limit group $T_1(s, z, y, z_{base}, a)$. But the graded resolution $GRes(u, s, z, y, z^2_{base}, a)$ was constructed from a collection of rigid specializations of $T_1(s, z, y, z^2_{base}, a)$, a contradiction.

Therefore, there exists some QH subgroup Q in the JSJ decomposition of $Zlim(z_{base}, a)$ that is mapped non-trivially into Q' and we get part (i). Parts (ii) and (iii) follow from (i) by applying Lemma 1.8 and Theorem 1.10, respectively. ∎

PROPOSITION 1.12: *With the notation of Proposition 1.11, suppose that the terminal subgroup $T_1(s, z, y, z_{base}, a)$ of the graded resolution $GRes(s, z, y, z_{base}, a)$ is solid. Let Q' be a QH subgroup that appears in the j-th level graded JSJ decomposition of the graded resolution $GRes(s, z, y, z^2_{base}, a)$. Then:*

(i) *There exists a QH subgroup Q with an associated surface S, which is either:*
   (1) *a QH subgroup in the (graded) JSJ decomposition of $Zlim(z_{base}, a)$, so that the map τ_j: $Zlim(z_{base}, a) \to Glim_j(u, s, z, y, z^2_{base}, a)$ maps the fundamental group of a subsurface of S onto a subgroup of finite index of Q'; or*
   (2) *a QH subgroup in the (graded) JSJ decomposition of (the solid graded limit group) $T_1(s, z, y, z_{base}, a)$, so that the map*

$$\eta_j\colon T_1(s, z, y, z_{base}, a) \to Glim_j(u, s, z, y, z^2_{base}, a)$$

   maps the fundamental group of a subsurface of S onto a subgroup of finite index of Q'.

(ii) $genus(S') \leq genus(S)$ *and* $|\chi(S')| \leq |\chi(S)|$.

(iii) *If Q is a QH subgroup in the JSJ decomposition of $Zlim(z_{base}, a)$ and $genus(S') = genus(S)$ and $|\chi(S')| = |\chi(S)|$, then Q is a surviving QH subgroup. In this last case, j is the bottom level of the graded resolution $GRes(u, s, z, y, z^2_{base}, a)$, and Q is mapped to the vertex stabilized by $< z^2_{base}, a >$ by the maps τ_j: $Rlim(z_{base}, a) \to Glim_j(u, s, z, y, z^2_{base}, a)$ in all the levels above the bottom one.*

Proof: The proof is essentially similar to the proof of Proposition 1.11. If, under the map τ_j: $Zlim(z_{base}, a) \to Glim_j(s, z, y, z^2_{base}, a)$, none of the QH subgroups Q in the JSJ decomposition of $Zlim(z_{base}, a)$ is mapped non-trivially into (a conjugate of) Q', then the entire image of $Zlim(z_{base}, a)$, $\tau_j(Zlim(z_{base}, a))$, is contained in a proper connected subgraph of groups of the graded JSJ decomposition of $Glim_j(s, z, y, z^2_{base}, a)$, a subgraph that does not contain the vertex stabilized by the QH subgroup Q'. Let Θ_j be the graph of groups obtained from the graded JSJ decomposition of $Glim_j(s, z, y, z^2_{base}, a)$ by collapsing the maximal connected subgraph of groups of the graded JSJ decomposition that does not contain the QH vertex group Q', and for which τ_j maps $Zlim(z_{base}, a)$, into a point stabilizer.

If the map: η_j: $T_1(s, z, y, z_{base}, a) \to Glim_j(u, s, z, y, z_{base}, a)$ maps a non-QH vertex group in the graded JSJ decomposition of $T_1(s, z, y, z_{base}, a)$ into a non-elliptic or a QH subgroup in the graph of groups Θ_j, then the graded limit groups that appear along the graded resolution $GRes(u, s, z, y, z^2_{base}, a)$ from $Glim_j(s, z, y, z^2_{base}, a)$ and above,

$$Glim_j(u, s, z, y, z^2_{base}, a), Glim_{j-1}(u, s, z, y, z^2_{base}, a), \ldots,$$
$$Glim_1(u, s, z, y, z^2_{base}, a),$$

are all flexible quotients of the solid limit group $T_1(s,z,y,z_{base},a)$. But the graded resolution $GRes(u,s,z,y,z^2_{base},a)$ was constructed from a collection of solid specializations of $T_1(s,z,y,z^2_{base},a)$, a contradiction.

Hence, every non-QH vertex group in the graded JSJ decomposition of $T_1(s,z,y,z^2_{base},a)$ is mapped by the quotient map η_j into a non-QH vertex group in Θ_j. Since the restricted limit group we have started with, $Rlim(y,a)$, is assumed to be of minimal rank, if, in addition, no QH vertex group in the graded JSJ decomposition of $T_1(s,z,y,z_{base},a)$ is mapped non-trivially into the QH subgroup Q' of $Glim_j(u,s,z,y,z^2_{base},a)$, then the entire group $T_1(s,z,y,z^2_{base},a)$ is mapped into the vertex group that contains the subgroup $\eta_j(Zlim(z_{base},a))$ in the graph of groups Θ_j, a contradiction to η_j being a quotient map.

Therefore, there exists some QH subgroup Q either in the JSJ decomposition of $Zlim(z_{base},a)$ or in the graded JSJ decomposition of $T_1(s,z,y,z^2_{base},a)$ that is mapped non-trivially into Q' and we get part (i). Parts (ii) and (iii) follow from (i) by applying (the proof of) Lemma 1.8 and Theorem 1.10, respectively. ∎

We continue the analysis of the limit group $QRlim(s,z,y,a)$ by further reducing the set of defining parameters sequentially. Recall that in order to obtain the graded resolution $GRes(s,z,y,z^2_{base},a)$ we first used the variables z_{base} as parameters, where z_{base} were the subset of the variables z that generate the subgroups that appear in all the levels of the completed resolution $Comp(Res)(z,y,a)$ except the one associated with the highest level, and then used the variables z^2_{base} as defining parameters, where z^2_{base} generate the subgroups that appear in all the levels of the completed resolution $Comp(Res)(z,y,a)$ except the ones associated with the two highest levels. To continue the analysis of the limit group $QRlim(s,z,y,a)$, we set the variables z^{ℓ}_{base} to be the subset of the variables z that generate the subgroups that appear in all levels of the completed resolution $Comp(Res)(z,y,a)$ except those associated with the ℓ highest levels.

Let $T_2(s,z,y,a)$ be the terminal rigid or solid graded limit group in the graded resolution $GRes(s,z,y,z^2_{base},a)$ with respect to the parameters z^2_{base}. We continue the graded resolution $GRes(s,z,y,z^2_{base},a)$ by viewing $T_2(s,z,y,a)$ as a graded limit group with respect to the defining parameters z^3_{base}, the obtained terminal rigid or solid graded limit group as a graded limit group with respect to the defining parameters z^4_{base} and so on, until we exclude all the z variables that appear in the different levels of the completed resolution $Comp(Res)(z,y,a)$.

Clearly, Lemma 1.8 and Theorem 1.10 remain valid for all the steps of the procedure as well as (appropriate versions of) Propositions 1.2 and 1.12. Note that the final resolution we obtain is an ungraded resolution of the limit group $QRlim(s,z,y,a)$, or of some quotient of it. Also, note that the described procedure produces (canonically) finitely many such (ungraded) resolutions of the limit group $QRlim(s,z,y,a)$. We will denote an (ungraded) resolution obtained by our procedure $QRes(s,z,y,a)$.

PROPOSITION 1.13: *The resolution $QRes(s,z,y,a)$ is a well-structured resolution.*

Proof: Since the restricted limit group is assumed to be of minimal rank, so is the quotient limit group $QRlim(s,z,y,a)$. By the definition of a well-structured resolution ([Se2], 1.11), if the quotient limit group $QRlim(s,z,y,a)$ is of minimal rank then any strict resolution of it is a well-structured resolution. By construction, a quotient resolution $QRes(s,z,y,a)$ is a strict resolution of $QRlim(s,z,y,a)$ or of a quotient of it, hence $QRes(s,z,y,a)$ is a well-structured resolution. ∎

At this point we are finally ready to define the complexity of a resolution in the minimal rank case, and to show that the complexity of a resolution $QRes(s,z,y,a)$ obtained by the above procedure, $Cmplx(QRes(s,z,y,a))$, is strictly smaller than the complexity of the original resolution $Cmplx(Res(y,a))$. This sequential reduction in complexity in each of the next steps of the iterative procedure will finally lead to its termination.

Definition 1.14: Let $Rlim(t,a)$ be a limit group of minimal rank and let $Res(t,a)$ be a well-structured resolution of $Rlim(t,a)$. Let $Q_1,\dots,Q_m$ be the QH subgroups that appear in the resolution $Res(t,a)$ and let $S_1,\dots,S_m$ be the corresponding punctured surfaces. To each punctured surface S_j we associate an ordered couple $(genus(S_j), |\chi(S_j)|)$ (for a closed non-orientable surface S, we set $genus(S)$ to be its first betti number, $b_1(S)$). We will assume that the QH subgroups $Q_1,\dots,Q_m$ are ordered according to the lexicographically (decreasing) order of the ordered couples associated with their corresponding surfaces.

Let $A_1,\dots,A_n$ be the (pegged) abelian groups that appear as vertex groups in the completed resolution $Comp(Res)(u,t,a)$. Let $eA_j < A_j$ be the subgroup generated by the edge groups connected to the vertex stabilized by the abelian group A_j in $Comp(Res)(u,t,a)$. We set the *abelian rank* of the resolution

$Res(t,a)$, denoted $Abrk(Res(t,a))$, to be

$$Abrk(Res(t,a)) = \Sigma(rk(A_j) - rk(eA_j)).$$

We set the complexity of the resolution $Res(t,a)$, denoted $Cmplx(Res(t,a))$, to be

$$Cmplx(Res(t,a)) \\ = ((genus(S_1), |\chi(S_1)|), \ldots, (genus(S_m), |\chi(S_m)|), Abrk(Res(t,a))).$$

On the set of resolutions of minimal rank limit groups we can define a linear order. Let $Rlim(t_1,a)$ and $Rlim(t_2,a)$ be two minimal rank limit groups with well-structured resolutions $Res(t_1,a)$, $Res(t_2,a)$ in correspondence. We say that $Cmplx(Res(t_1,a)) = Cmplx(Res(t_2,a))$ if the tuples defining the two complexities are identical. We say that $Cmplx(Res(t_1,a)) < Cmplx(Res(t_2,a))$ if:

(1) The tuple $((genus(S_1^1), |\chi(S_1^1)|), \ldots, (genus(S_{m_1}^1), |\chi(S_{m_1}^1|))$ is smaller in the lexicographical order than the tuple

$$((genus(S_1^2), |\chi(S_1^2)|), \ldots, (genus(S_{m_2}^2), |\chi(S_{m_2}^2|)).$$

(2) If the above tuples are equal then $Abrk(Res(t_1,a)) < Abrk(Res(t_2,a))$.

In order to ensure the termination of the iterative procedure we need the complexities of the resolutions obtained in its different steps to decrease. Indeed, the complexity of a (well-structured) resolution $QRes(s,z,y,a)$, obtained after the first step of our procedure, is strictly smaller than the complexity of the resolution $Res(y,a)$ we started with.

THEOREM 1.15: *The complexity of the resolution $QRes(s,z,y,a)$ is strictly smaller than the complexity of the resolution $Res(y,a)$, i.e.,*

$$Cmplx(QRes(s,z,y,a)) < Cmplx(Res(y,a)).$$

Proof: Let Q' be a QH subgroup that appears in a graded abelian decomposition associated with one of the levels of the quotient resolution $QRes(s,z,y,a)$, and let S' be its associated punctured surface. By Propositions 1.5, 1.11 and 1.12 there exists some QH subgroup Q with associated punctured surface S that appears in a cyclic JSJ decomposition associated with one of the levels of the original resolution $Res(y,a)$, so that the fundamental group of a subsurface of S is mapped into a finite index subgroup of S' and, in particular, $genus(S') \leq genus(S)$ and $|\chi(S')| \leq |\chi(S)|$. Furthermore, if $genus(S') = genus(S)$ and

$|\chi(S')| = |\chi(S)|$ then Q is by definition a surviving surface, so by Theorems 1.7 and 1.10 the QH subgroup Q is mapped isomorphically onto Q'.

In this case of a surviving surface, either Q is mapped into the (distinguished) vertex group containing F_k in all levels above the one in which Q' appears, and in all the levels below the one containing Q', Q is identified with its image in the completed resolution $Comp(Res)(z,y,a)$, or there is another QH subgroup $\hat{Q}$ that appears along the original resolution $Res(y,a)$ and is mapped non-trivially into a conjugate of Q'. In the second possibility, the topological complexity of the surface $\hat{S}$, associated with the QH subgroup $\hat{Q}$, is strictly bigger than the topological complexity of all the QH vertex groups in the resolution $QRes(s,z,y,a)$ into which Q is mapped, except perhaps Q' itself.

Hence, to any surviving QH subgroup Q with associated punctured surface S, there corresponds a unique QH subgroup Q' in the resolution $QRes(s,z,y,a)$ with associated punctured surface S', so that S is homeomorphic to S'. If Q' is a QH subgroup in the quotient resolution $QRes(s,z,y,a)$ with an associated punctured surface S', and no surviving surface Q is mapped isomorphically onto a conjugate of Q', then for any QH subgroup Q with an associated surface S in $Res(y,a)$, for which the fundamental group of some subsurface of S is mapped into a finite index subgroup of a conjugate of Q', $genus(S') \leq genus(S)$, $|\chi(S')| \leq |\chi(S)|$, and either $genus(S') < genus(S)$ or $|\chi(S')| < |\chi(S)|$. Therefore, if $Q_1,\ldots,Q_m$ are the QH subgroups that appear in the original resolution $Res(y,a)$ and $S_1,\ldots,S_m$ are their associated punctured surfaces, and $Q'_1,\ldots,Q'_{m'}$ are the QH subgroups that appear in the quotient resolution $QRes(s,z,y,a)$ and $S'_1,\ldots,S'_{m'}$ are their associated punctured surfaces, then the tuple $((genus(S_1),|\chi(S_1)|),\ldots,(genus(S_m),|\chi(S_m|))$ is less than or equal to, in the lexicographical order, the tuple

$$((genus(S'_1),|\chi(S'_1)|),\ldots,(genus(S'_{m'}),|\chi(S'_{m'}|)),$$

and the two tuples are equal if and only if all the QH subgroups $Q_1,\ldots,Q_m$ are surviving QH subgroups.

Since the quotient limit group $QRlim(s,z,y,a)$ is a proper quotient of the closure $Cl(Res)(s,z,y,a)$, if all the QH subgroups $Q_1,\ldots,Q_m$ are surviving, then necessarily $Abrk(QRes(s,z,y,a)) < Abrk(Res(y,a))$, and we finally get $Cmplx(QRes(s,z,y,a)) < Cmplx(Res(y,a))$. ∎

Having constructed the (finitely many) well-structured resolutions $QRes(s,z,y,a)$ and their corresponding limit groups $QRlim(s,z,y,a)$, to prove the validity of our given sentence we only need to prove its validity on the set of specializations of the variables y for which there exist specializations for the variables s and z so that the specialization (s,z,y,a) factors through one of the resolutions $QRes(s,z,y,a)$.

The final part of the first step of the procedure for the validation of a sentence (in the minimal rank case) is the construction of formal solutions defined over a covering closure of the resolution $QRes(s,z,y,a)$ in a similar way to the ones described in Proposition 1.2.

PROPOSITION 1.16: *Let $QRes(s,z,y,a)$ be one of the (well-structured) resolutions constructed in the first step of our iterative procedure for validation of a sentence, and let $QRlim(s,z,y,a)$ be its corresponding limit group. For brevity, we will denote the resolution by $QRes(t,y,a)$ and the corresponding limit group by $QRlim(t,y,a)$. There exists a covering closure*

$$Cl(QRes)_1(\hat{s},\hat{z},t,y,a),\ldots,Cl(QRes)_q(\hat{s},\hat{z},t,y,a)$$

of the resolution $QRes(t,y,a)$, and for each index n, $1\leq n\leq q$, there exists a formal solution $x_n(\hat{s},\hat{z},t,y,a)$, and a finite set of restricted limit groups

$$QRlim_1^n(\hat{s},\hat{z},t,y,a),\ldots,QRlim_{m(n)}^n(\hat{s},\hat{z},t,y,a)$$

for which:

(i) *For every index n, $1\leq n\leq q$, the words corresponding to the equations in the system $\Sigma(x_n(\hat{s},\hat{z},t,y,a),y,a)=1$ represent the trivial word in the closure $Cl(Res)_n(\hat{s},\hat{z},t,y,a)$.*

(ii) *For every index n, the restricted limit groups*

$$QRlim_1^n(\hat{s},\hat{z},t,y,a),\ldots,QRlim_{m(n)}^n(\hat{s},\hat{z},t,y,a)$$

are proper quotients of the n-th closure $Cl(Res)_n(\hat{s},\hat{z},t,y,a)$.

(iii) *For every index n, $1\leq n\leq q$, let $D^n(y)$ be the Diophantine set corresponding to the closure $Cl(QRes)_n(\hat{s},\hat{z},t,y,a)$, and let $D_1^n(y),\ldots,D_{m(n)}^n(y)$ be the Diophantine sets corresponding to the restricted limit groups*

$$QRlim_1^n(\hat{s},\hat{z},t,y,a),\ldots,QRlim_{m(n)}^n(\hat{s},\hat{z},t,y,a).$$

Let $\psi_j(x,y,a)$ be a word corresponding to one of the equations in the system $\Psi(x,y,a)\neq 1$. If for some index n, $1\leq n\leq q$, $y_0\in D^n(y)$ and there exists a specialization $(\hat{s}_0,\hat{z}_0,t_0,y_0,a)$ that factors through the closure $Cl(Res)_n(\hat{s},\hat{z},t,y,a)$, and $y_0\notin D_1^n(y)\cup\cdots\cup D_{m(n)}^n(y)$, then for every index j, $1\leq j\leq r$:

$$\psi_j(x_n(\hat{s}_0,\hat{z}_0,t_0,y_0,a),y_0,a)\neq 1.$$

Proof: Identical to the proof of Proposition 1.2. ∎

The restricted limit groups $QRlim_i^n(\hat{s},\hat{z},t,y,a)$ are the input for the second step of the iterative procedure for validation of a sentence in the minimal rank case.

The general step of the iterative procedure. In a similar way to the first step of the procedure, the input to the general (m-th) step of the iterative procedure consists of finitely many well-structured resolutions $QRes(t_m,y,a)$ with corresponding limit groups $QRlim(t_m,y,a)$, and to each resolution $QRes(t_m,y,a)$ there are finitely many associated limit groups of the form $QRlim(s_m,z_m,t_m,y,a)$, where each limit group $QRlim(s_m,z_m,t_m,y,a)$ is a proper quotient of a closure of the resolution $QRes(t_m,y,a)$,

$$Cl(QRes)(s_m,z_m,t_m,y,a).$$

Applying the same procedure used to construct the quotient resolutions $QRes(s,z,y,a)$ in the first step, with each given proper quotient $QRlim(s_m,z_m,t_m,y,a)$ of a closure of the resolution $QRes(t_m,y,a)$, $Cl(QRes)(s_m,z_m,t_m,y,a)$, we can associate finitely many (ungraded) resolutions $\{Res_j(s_m,z_m,t_m,y,a)\}$ so that:

(1) Every specialization of $QRlim(s_m,z_m,t_m,y,a)$ which is a specialization of the closure $Cl(QRes)(s_m,z_m,t_m,y,a)$ factors through at least one of the resolutions $\{Res_j(s_m,z_m,t_m,y,a)\}$.
(2) By a similar argument to the one used to prove Theorem 1.15, for every possible index j

$$Cmplx(Res_j(s_m,z_m,t_m,y,a)) < Cmplx(QRes(t_m,y,a)).$$

The final part of the general step of the procedure for validation of a sentence in the minimal rank case is the construction of formal solutions in a similar way to the one described in Propositions 1.2 and 1.16.

Proposition 1.17: *Let $Res(s_m,z_m,t_m,y,a)$ be one of the resolutions constructed above and let $Rlim(s_m,z_m,t_m,y,a)$ be its corresponding limit group. For brevity, we will denote the resolution by $Res(t_{m+1},y,a)$ and the corresponding limit group by $Rlim(t_{m+1},y,a)$. There exists a covering closure*

$$Cl(Res)_1(s_{m+1},z_{m+1},t_{m+1},y,a),\dots,Cl(Res)_q(s_{m+1},z_{m+1},t_{m+1},y,a)$$

of the resolution $Res(t_{m+1},y,a)$, and for each index $1\leq n\leq q$ there exists a formal solution $x_n(s_{m+1},z_{m+1},t_{m+1},y,a)$, and a finite set of restricted limit *groups* $QRlim_1^n(s_{m+1}, z_{m+1}, t_{m+1}, y, a), \ldots, QRlim_{u(n)}^n(s_{m+1}, z_{m+1}, t_{m+1}, y, a)$ *for which:*

(i) *For each index* n, $1 \leq n \leq q$, *the words corresponding to the equations in the system* $\Sigma(x_n(s_{m+1}, z_{m+1}, t_{m+1}, y, a), y, a)$ *represent the trivial word in the closure* $Cl(Res)_n(s_{m+1}, z_{m+1}, t_{m+1}, y, a)$.

(ii) *For every index* n, *the restricted limit groups*

$$QRlim_1^n(s_{m+1}, z_{m+1}, t_{m+1}, y, a), \ldots, QRlim_{u(n)}^n(s_{m+1}, z_{m+1}, t_{m+1}, y, a)$$

are proper quotients of the n*-th closure* $Cl(Res)_n(s_{m+1}, z_{m+1}, t_{m+1}, y, a)$.

(iii) *For every index* n, $1 \leq n \leq q$, *let* $D^n(y)$ *be the Diophantine set corresponding to the closure* $Cl(QRes)_n(s_{m+1}, z_{m+1}, t_{m+1}, y, a)$, *and let*

$$D_1^n(y), \ldots, D_{u(n)}^n(y)$$

be the Diophantine sets corresponding to the restricted limit groups

$$QRlim_1^n(s_{m+1}, z_{m+1}, t_{m+1}, y, a), \ldots, QRlim_{u(n)}^n(s_{m+1}, z_{m+1}, t_{m+1}, y, a).$$

Let $\psi_j(x, y, a)$ *be a word corresponding to one of the equations in the system* $\Psi(x, y, a)$. *If for some index* n, $1 \leq n \leq q$, $y_0 \in D^n(y)$ *and there exists a specialization* $(s_{m+1_0}, z_{m+1_0}, t_{m+1_0}, y_0, a)$ *that factors through the closure* $Cl(Res)_n(s_{m+1}, z_{m+1}, t_{m+1}, y, a)$, *and*

$$y_0 \notin D_1^n(y) \cup \cdots \cup D_{u(n)}^n(y),$$

then for every index j, $1 \leq j \leq r$,

$$\psi_j(x_n(s_{m+1_1}, z_{m+1_1}, t_{m+1_0}, y_0, a), y_0, a) \neq 1.$$

Proof: Identical to the proof of Proposition 1.2. ∎

The resolutions $Res(t_{m+1}, y, a)$, their corresponding limit groups $Rlim(t_{m+1}, y, a)$, and the restricted limit groups $QRlim_i^n(s_{m+1}, z_{m+1}, t_{m+1}, y, a)$ which are proper quotients of their corresponding closures, are the input for the next step of the iterative procedure for validation of a sentence in the minimal rank case.

The successive reduction in the complexity of the restricted resolutions constructed in the different steps of the iterative procedure for validation of a sentence in the minimal rank case, finally leads to its termination at a finite step. This termination makes it useful for the purposes of quantifier elimination.

THEOREM 1.18: *If the initial restricted limit group $Rlim(y,a)$ is of minimal rank, then the iterative procedure for validation of a sentence terminates after finitely many steps.*

Proof: Let $Res(t_m,y,a)$ be a resolution obtained in the m-th step of the iterative procedure, and let $Res(t_{m+1},y,a)$ be a resolution obtained in the $m+1$-th step from the resolution $Res(t_m,y,a)$. By applying Theorem 1.15 to the $m+1$-th step of the iterative procedure, $Cmplx(Res(t_{m+1},y,a)) < Cmplx(Res(t_m,y,a))$. Hence, if our iterative procedure does not terminate in a finite time there exists an infinite sequence of strictly decreasing finite tuples of (decreasing) positive integers. But the lexicographical order on finite tuples of (decreasing) positive integers guarantees that every sequence of strictly decreasing finite tuples of (decreasing) positive integers terminates, a contradiction. ∎

2. Taut homomorphisms of maximal rank

In the previous section we used an iterative procedure for validation of a sentence. Given a true sentence of the form

$$\forall y \quad \exists x \quad \Sigma(x,y,a)=1 \wedge \Psi(x,y,a) \neq 1$$

the procedure uses a sequence of formal solutions, whose existence is guaranteed by theorem 1.18 in [Se2], to iteratively construct resolutions containing the decreasing sets of the "remaining" y's. To each resolution we have assigned its complexity, and the iterative procedure guarantees a decrease in the complexity of the resolutions constructed in consecutive steps of the procedure. This complexity decrease forces the iterative procedure to terminate in a finite time.

The approach presented in the previous section for validation of a sentence is the basis for our "trial and error" procedure for quantifier elimination presented in the next paper. However, the procedure described in the previous section is bounded to the minimal rank case (rank 0 case). In this section we generalize the notion of complexity to general resolutions, and start modifying the iterative procedure presented in the previous section to guarantee the decrease in the complexity of resolutions constructed by the procedure, that finally implies the termination in a finite time of the iterative procedure for validation of a general sentence, omitting the minimal rank assumption.

Since some of the intermediate claims made in the previous section remain true in the general case, we will continue using the notation used in the previous section. Before starting with the procedure itself, we need to introduce some further notions associated with restricted limit groups and their resolutions. We start with the natural notion of the **rank** of a resolution $Res(y,a)$, which we denote $rk(Res(y,a))$.

Definition 2.1: Let $Res(y,a)$ be a resolution of a restricted limit group $Rlim(y,a)$. The **rank** of the resolution $Res(y,a)$, denoted $rk(Res(y,a))$, is the maximal integer n for which there exists an epimorphism $h\colon Rlim(y,a) \to F_k * F_n$ that factors through the resolution $Res(y,a)$. Note that $rk(Res(y,a))$ is precisely the sum of the ranks of the free factors dropped along the resolution $Res(y,a)$. Also, note that the minimal rank resolutions analyzed in the previous section are rank 0 resolutions.

By definition, if $h\colon Rlim(y,a) \to F_k * F_n$ is an epimorphism that factors through the resolution $Res(y,a)$, then $n \leq rk(Res(y,a))$. Hence, the rank of a resolution is a basic measure of its complexity that was not needed in the minimal rank (rank 0) case.

In section 1 of [Se2], we have defined the completion of a well-structured resolution (definitions 1.11 and 1.12 in [Se2]). To get a terminating iterative procedure in the minimal rank case, analyzing well-structured resolutions and their completions is sufficient, as was shown in the previous section. However, to get a terminating iterative procedure in the general case, we need to restrict ourselves to a special class of well-structured resolutions which we call **well-separated resolutions**.

We start by defining well-separated resolutions (Definition 2.2), and associate with such a resolution the collection of homomorphisms that factor through a well-separated resolution and are "compatible" with its structure, that we call **taut** with respect to the given well-separated resolution (Definition 2.4). Finally, given a (restricted, graded) limit group we modify the construction of the Makanin–Razborov diagram presented in section 5 of [Se1] and construct its (canonical) taut Makanin–Razborov diagram, which contains finitely many well-separated resolutions, so that every homomorphism from the given limit group into a free group is taut with respect to at least one of the resolutions in the taut MR diagram.

Definition 2.2: Let $Res(y,a)$ be a well-structured resolution of a restricted limit group $Rlim(y,a)$. Suppose that the resolution $Res(y,a)$ is given by a decreasing sequence of restricted limit groups

$$Rlim(y,a) = Rlim_0(y,a) \to Rlim_1(y,a) \to \cdots \to Rlim_\ell(y,a) =< a,f > *H^\ell$$

where $< a, f >$ and H^ℓ are free groups. For each index i, $0 \le i \le \ell - 1$, let $\eta_i: Rlim_i(y,a) \to Rlim_{i+1}(y,a)$ be the canonical quotient map.

According to the definition of a well-structured resolution (definition 1.11 in [Se2]), with each restricted limit group $Rlim_i(y,a)$ that appears along the well-structured resolution $Res(y,a)$, there is an associated free decomposition

$$Rlim_i(y,a) = R_1^i * \cdots * R_{q(i)}^i * F_{rk(i)}^i * H^i$$

where $F_{rk(i)}^i$ is a (possibly trivial) free group of rank $rk(i)$ and H^i is a (possibly trivial) free group. With each factor R_j^i there is an associated (restricted, possibly trivial) abelian decomposition (graph of groups), and a corresponding restricted modular group.

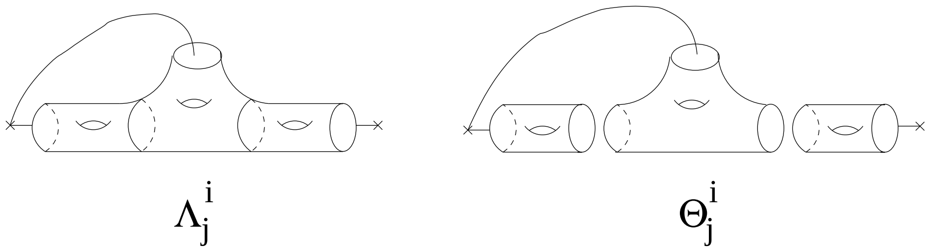
Let R_j^i be a factor in the free decomposition of $Rlim_i(y,a)$ which is not a surface or an abelian factor, and let Λ_j^i be the abelian decomposition associated with the factor R_j^i in the resolution $Res(y,a)$. Let Δ_j^i be the cyclic decomposition of R_j^i obtained from Λ_j^i by collapsing all the edge groups connecting two non-QH vertex groups. In Δ_j^i each non-QH vertex group is connected only to QH vertex groups.

Let $Q_1, \dots, Q_s$ be the QH vertex groups in the cyclic decomposition Δ_j^i, and for each QH subgroup Q_h let S_h be its corresponding surface, and let $bd_{h,1}, \dots, bd_{h,c}$ be its boundary components. Let

$$\eta_i(Q_h) = V_1 * \cdots * V_{m(h)} * H_h$$

be the maximal (most refined) free decomposition inherited by $\eta_i(Q_h)$ from the given free decomposition of $Rlim_{i+1}(y,a)$ in which each of the images of the boundary elements $\eta_i(bd_{h,n})$ can be conjugated into one of the factors V_r, and each factor V_r contains a conjugate of an image of a boundary element, $\eta_i(bd_{h,n})$.

The free product inherited by the image $\eta_i(Q_h)$ lifts to a decomposition of the surface S_h. Let $\Gamma_h(Q_h)$ be a maximal decomposition of S_h along a collection of disjoint, non-homotopic s.c.c. which is a lift of the free decomposition inherited by $\eta_i(Q_h)$. Let α_j^i be the graph of groups obtained from the graph of groups Δ_j^i by further cutting the surfaces $S_1, \dots, S_s$, corresponding to the QH subgroups $Q_1, \dots, Q_s$, along the collections of s.c.c. that appear in the decompositions $\Gamma_h(Q_h)$, and connecting the obtained (punctured) surfaces with edges stabilized by cyclic edge groups that correspond to the s.c.c. along which the original surfaces were cut. Clearly, cutting surfaces along the given collection of s.c.c. does not change the fundamental group of the obtained graph of groups, so the fundamental group of α^i_j is (still) the factor R^i_j.



With α^i_j we associate a graph of groups Θ^i_j, obtained from α^i_j by erasing the edges that correspond to s.c.c. along which the original surfaces in Δ^i_j were cut. With each connected component of Θ^i_j that contains a non-QH vertex group, we associate its fundamental group that we denote W_u. Let $W_1, \dots, W_m$ be the subgroups associated with the various connected components (that contain a non-QH vertex group) in the graph of groups Θ^i_j. We say that the well-structured resolution $Res(y,a)$ is a **well-separated resolution** if the following conditions hold:

(i) For each subgroup associated with a connected component of Θ^i_j that contains a non-QH vertex group W_u, the image $\eta_i(W_u)$ is a factor in the given free decomposition of $Rlim_{i+1}(y,a)$, which is a free product of some (possibly trivial) factors $\{R^{i+1}_{j'}\}$, and a (possibly trivial) factor of the free group $F^{i+1}_{rk(i+1)}$. Furthermore

$$\eta_i(R^i_j) = \eta_i(< W_1, \dots, W_m >) * H^i_j = \eta_i(W_1) * \dots * \eta_i(W_m) * H^i_j.$$

(ii) The subgroup H^i_j is free, and its rank is equal to the difference between the first betti number of the graph α^i_j and the first betti number of the graph Θ^i_j plus the number of once punctured Klein bottles among the QH vertex groups in Θ^i_j.

(iii) Let S' be a punctured surface obtained from a punctured surface in the decomposition Θ^i_j that is connected to a non-QH vertex group in the decomposition Θ^i_j, by adjoining disks to the boundary components that are mapped to the trivial element by η_i. Then no s.c.c. on S' is mapped to the trivial element by the map η_i.

(iv) Let Q be one of the QH subgroups in the graph of groups Δ^i_j, and let S_Q be the punctured surface associated with Q. Let $\hat{S}_1, \dots, \hat{S}_q$ be all the punctured surfaces in Θ^i_j which are subsurfaces of the surface S_Q, so that each of the subsurfaces $\hat{S}_b$ is connected to a non-QH vertex group in the graph of groups Θ^i_j, and each subsurface of S_Q that is connected to a non-QH vertex group in Θ^i_j is one of the surfaces $\hat{S}_1, \dots, \hat{S}_q$. Let $\hat{S}$ be the punctured surface $\hat{S} = S_Q \setminus \{\hat{S}_1 \cup \dots \cup \hat{S}_q\}$ and let S' be the closed surface obtained from $\hat{S}$ by adding disks to all its boundary components. Then $\eta_i(\pi_1(\hat{S})) = \eta_i(\pi_1(S'))$, is a maximal possible rank free quotient of $\pi_1(S')$.

(v) Let R^i_j, a factor in the given free decomposition of $Rlim_i(y,a)$, be a closed surface group. Then $\eta_i(R^i_j)$ is a free group, which is a maximal possible rank free quotient of the surface group R^i_j, and $\eta_i(R^i_j)$ is a free factor of H^{i+1}.

By lemma 1.14 of [Se2], if $Res(y,a)$ is a well-structured resolution, and $Comp(Res)(z,y,a)$ is its completion, then if we replace each of the completed limit groups $Comp(Rlim)_i(z,y,a)$ by $Comp(Rlim)_i(z,y,a) * H^i$, where H^i is the corresponding free factor that appears in the given free decomposition of $Rlim_i(y,a)$, then the modified completed resolution is a well-structured resolution. By construction, the same is true for well-separated resolutions.

Lemma 2.3: *Let $Res(y,a)$ be a well-separated resolution, and $Comp(Res)(z,y,a)$ be its completion. If we replace each of the completed limit groups $Comp(Rlim)_i(z,y,a)$ by $Comp(Rlim)_i(z,y,a) * H^i$, where H^i is the corresponding free factor that appears in the given free decomposition of $Rlim_i(y,a)$, then the modified completed resolution is a well-separated resolution.*

In the minimal rank case, we have analyzed the entire collection of homomorphisms that factor through a given well-structured resolution in each step of our iterative procedure. We start the analysis of general sentences by imposing a further restriction on the homomorphisms associated with a given well-separated resolution. Later on, we show that there exists a canonical Makanin–Razborov diagram, called the **taut Makanin–Razborov diagram**, containing only well-separated resolutions, so that every specialization of the limit group $Rlim(y,a)$ satisfies the additional (*taut*) condition with respect to at least one of the well-separated resolutions $Res(y,a)$ from the taut Makanin–Razborov diagram.

Definition 2.4: Let $Res(y,a)$ be a well-separated resolution of a restricted limit group $Rlim(y,a)$ and let $Comp(Res)(z,y,a)$ be its completion. Let $h: Rlim(y,a) \to F_k$ be a homomorphism that factors through the resolution $Res(y,a)$. Since the homomorphism h factors through the resolution $Res(y,a)$, it can be extended to a homomorphism $\hat{h}$: $Comp(Rlim)(z,y,a) \to F_k$ that factors through the completed resolution $Comp(Res)(z, y, a)$.

We say that the homomorphism h: $Rlim(y, a) \to F_k$ is *taut* with respect to the well-separated resolution $Res(y, a)$ if the following conditions hold:

(i) The image under the homomorphism $\hat{h}$ of an abelian edge group in any of the abelian decompositions associated with the various levels of the completed resolution $Comp(Res)(z, y, a)$ is non-trivial.

(ii) The image under the homomorphism $\hat{h}$ of any QH subgroup Q in any of the abelian decompositions associated with the various levels of the completed resolution $Comp(Res)(z, y, a)$ is non-abelian.

(iii) Let Λ^i_j be the abelian decomposition associated with a factor R^i_j of the restricted limit group $Rlim_i(y, a)$, and let Θ^i_j be the abelian decomposition associated with Λ^i_j and obtained by cutting the (punctured) surfaces associated with QH vertex groups in Λ^i_j along the specified collections of s.c.c. (see Definition 2.2). Let Q be a QH subgroup in the abelian decomposition Λ^i_j, and let S_Q be its corresponding punctured surface. The homomorphism $\hat{h}$ factors through the completion $Comp(Res)(z, y, a)$, hence, with the homomorphism $\hat{h}$ and the QH vertex group Q, there is an associated modular automorphism φ_Q of S_Q.

Let $\hat{S}$ be a connected punctured subsurface of the surface S_Q connected to some non-QH vertex group in the decomposition Θ^i_j. Let S' be the punctured surface obtained from $\hat{S}$ by adjoining disks to the boundary components of $\hat{S}$ that are mapped trivially by the quotient map η_i. Then no s.c.c. on the punctured surface S' is mapped to the trivial element by the composition $\hat{h} \circ \varphi_Q$.

Proposition 2.5: *Given a limit group $Rlim(y, a)$, there exists a (canonical)* **taut Makanin–Razborov diagram** *in which every resolution is well-separated, and every homomorphism h: $Rlim(y, a) \to F_k$ is a taut homomorphism with respect to at least one of the Makanin–Razborov resolutions in the (canonical) taut Makanin–Razborov diagram.*

Proof: We start with the restricted limit group $Rlim(y, a)$. If $Rlim(y, a)$ is freely decomposable we continue with each factor separately. Hence, w.l.o.g. we assume that $Rlim(y, a)$ is freely indecomposable. Therefore, we may associate with $Rlim(y, a)$ its abelian JSJ decomposition which we denote Λ_0, and the collection of homomorphisms h: $Rlim(y, a) \to F_k$. First we look at degenerate homomorphisms h: $Rlim(y, a) \to F_k$, i.e., those homomorphisms that either map an edge group in Λ_0 into the trivial element in F_k, or a non-abelian vertex group in Λ_0 into a cyclic (or trivial) subgroup in F_k. By the arguments used in the construction of the Makanin–Razborov diagram of a limit group ([Se1], section 5), the collection of degenerate homomorphisms h: $Rlim(y,a) \to F_k$ factor through finitely many maximal limit groups $DRlim_1(y,a), \dots, DRlim_s(y,a)$, where each of the restricted limit groups $DRlim_i(y,a)$ is a proper quotient of the original limit group $Rlim(y,a)$. Hence, to analyze the entire set of homomorphisms h: $Rlim(y,a) \to F_k$, we can restrict ourselves to the set of all the non-degenerate homomorphisms h: $Rlim(y,a) \to F_k$, and later analyze the collections of homomorphisms from the proper quotients of $Rlim(y,a)$: $DRlim_1(y,a), \dots, DRlim_s(y,a)$ to the free group F_k, in the same way we have analyzed the collection of all homomorphisms from $Rlim(y,a)$ to F_k.

Given the graph of groups Λ_0, up to automorphisms of the (punctured) surfaces associated with its QH subgroups, there are only finitely many possibilities to decompose these (punctured) surfaces along collections of disjoint non-homotopic s.c.c., hence, up to automorphisms of these (punctured) surfaces, there are only finitely many possibilities for the graph Θ_1^0 (see Definition 2.2).

With each possibility for the graph Θ_1^0 we associate the collection of non-degenerate homomorphisms h: $Rlim(y,a) \to F_k$ which are taut with respect to the single level well-separated resolution corresponding to the given abelian decomposition of $Rlim(y,a)$ and the collection of s.c.c. on each of the surfaces associated with its QH subgroups. Note that this collection may be empty, in which case we exclude the corresponding possibility for the graph Θ_1^0 from our list of possibilities. We continue with each of the (finitely many) possibilities for the graph Θ_1^0 (except the excluded ones) in parallel. Given the graph Θ_1^0, we apply the shortening procedure for all the non-degenerate homomorphisms h: $Rlim(y,a) \to F_k$ that are taut with respect to the (resolution associated with the) graph Θ_1^0. From the collection of shortened homomorphisms we construct a (canonical) collection of finitely many maximal shortening quotients of $Rlim(y,a)$, which we denote $Rlim_1^1(y,a), \dots, Rlim_{q_1}^1(y,a)$. By ([Se1], 5.3) every shortening quotient, $Rlim_d^1(y,a)$, is a proper quotient of the restricted limit group $Rlim(y,a)$.

We continue with each of the maximal shortening quotients $Rlim_d^1(y,a)$ and its associated collection of homomorphisms h_1: $Rlim_d^1(y,a) \to F_k$ in parallel. If $Rlim_d^1(y,a)$ is freely decomposable we continue with each of its factors in parallel, so w.l.o.g. we may assume that $Rlim_d^1(y,a)$ is freely indecomposable. Hence, we may associate with $Rlim_d^1(y,a)$ its abelian JSJ decomposition which we denote Λ_d^1. With $Rlim_d^1(y,a)$ we canonically associate its canonical finite collection of proper quotients obtained from the set of degenerate homomorphisms $h_1: Rlim^1_d(y,a) \to F_k$. We further continue the analysis of all homomorphisms $h_1: Rlim(y,a)^1_d(y,a) \to F_k$ that factor through one of the degenerate quotients, and the set of non-degenerate homomorphisms $h_1: Rlim^1_d(y,a) \to F_k$ in parallel. To analyze the set of non-degenerate homomorphisms $h_1: Rlim^1_d(y,a) \to F_k$ we again look for the finitely many possibilities (up to automorphisms of the relevant surfaces) for the graph Θ^1_1 associated with $Rlim^1_d$.

With each possibility for the graph Θ^1_1 we associate the collection of non-degenerate homomorphisms $h_1: Rlim^1_d(y,a) \to F_k$ which are taut with respect to (the well-separated resolution associated with) Θ^1_1, and from this collection we obtain a canonical finite collection of maximal shortening (proper) quotients, and continue to the next level.

Since at each step in our construction we obtain (canonical) collections of proper quotients of limit groups obtained in previous steps, the descending chain condition for decreasing sequences of limit groups proved in ([Se1], 5.1) guarantees that the first part for the construction of the taut Makanin–Razborov diagram terminates. Once it terminates, we have obtained a (canonical) finite set of resolutions $Res(y,a)$ of the restricted limit group $Rlim(y,a)$, and with each resolution $Res(y,a)$ a collection of homomorphisms that factor through the resolution $Res(y,a)$ and are non-degenerate with respect to the abelian decompositions associated with all its levels. By construction, the union of the collections of homomorphisms associated with the resolutions $Res(y,a)$ constructed by the above procedure includes every homomorphism $h: Rlim(y,a) \to F_k$.

Since the resolutions we get are obtained by successive application of the shortening procedure, if a resolution in our final collection is a strict resolution, then it is necessarily a well-separated resolution, so we can take it to be a resolution in the taut Makanin–Razborov diagram. If a resolution $Res(y,a)$ is not a strict resolution, we modify it as we did in the construction of the strict Makanin–Razborov diagram (proposition 1.10 in [Se2]). With the limit groups associated with the various levels of a (non-strict) resolution $Res(y,a)$ we associate the limit groups obtained from the set of homomorphisms associated with the resolution $Res(y,a)$, i.e., from the collection of homomorphisms that factor and that are (non-degenerate and) taut with respect to the resolution $Res(y,a)$. Since $Res(y,a)$ is a non-strict resolution, at least one of the obtained limit groups is a proper quotient of the corresponding limit group that is associated with one of the levels of $Res(y,a)$. Let i be the highest such level in $Res(y,a)$.

We replace the limit group $Rlim_i(y,a)$ associated with the i-th level in $Res(y,a)$ by the proper quotient obtained from the set of homomorphisms that factor through the resolution $Res(y,a)$ and are taut with respect to it, and continue our construction of resolutions with the newly obtained limit group. Since each time we replace a resolution, we replace a limit group associated with one of its levels by its proper quotient, and a decreasing sequence of limit groups must terminate ([Se1], 5.1), the iterative process for the construction of the taut Makanin–Razborov diagram associated with the original limit group $Rlim(y,a)$ terminates after finitely many steps. Each of the resolutions obtained by the process is a strict resolution, and since they were all obtained using shortening quotients, all the obtained resolutions are well-separated resolutions.

We set the taut Makanin–Razborov diagram to be the collection of resolutions obtained by the above iterative procedure. By construction, every homomorphism $h\colon Rlim(y,a) \to F_k$ factors through and is taut with respect to at least one of the resolutions in the taut diagram. ∎

In each of the steps of the iterative procedure for the validation of an AE sentence we present, we analyze all the taut homomorphisms that factor through the resolutions constructed in the previous step of the procedure (and satisfy some additional non-trivial relation). By Proposition 2.5, every homomorphism is taut with respect to at least one of the resolutions in the taut Makanin–Razborov diagram. Hence, we are able to use the taut Makanin–Razborov diagram to further impose the taut restriction on the homomorphisms and the well-separated restriction on the resolutions we study at each step of the procedure.

Let
$$\forall y \quad \exists x \quad \Sigma(x,y,a) = 1 \wedge \Psi(x,y,a) \neq 1$$
be a true sentence defined over F_k. Let $F_y = < y_1, \dots, y_\ell >$ be the free group defined over the variables $y_1, \dots, y_\ell$. By Proposition 1.1 there exists a formal solution $x = x(y,a)$, and a finite set of restricted limit groups $Rlim_1(y,a), \dots, Rlim_m(y,a)$ for which:

(i) The words corresponding to the equations in the system $\Sigma(x(y,a),y,a) = 1$ represent the trivial word in the free group $F_k * F_y$.

(ii) For every index i, $Rlim_i(y,a)$ is a proper quotient of the free group $F_k * F_y$.

(iii) Let $B_1(y), \dots, B_m(y)$ be the basic sets corresponding to the restricted limit groups $Rlim_1(y,a), \dots, Rlim_m(y,a)$. If $y \notin B_1(y) \cup \dots \cup B_m(y)$, and $\psi_j(x,y,a)$ is a word corresponding to one of the equations in the system $\Psi(x,y,a)$, then $\psi_j(x(y,a),y,a) \neq 1$.

Proposition 1.1 gives a formal solution that proves the validity of the given sentence on a co-basic set $(F_k)^\ell \setminus (B_1(y) \cup \cdots \cup B_m(y))$, hence the rest of the procedure needs to construct formal solutions that prove the validity of the sentence on the remaining basic sets $B_1(y), \ldots, B_m(y)$. Since our treatment of the restricted limit groups $Rlim_1(y,a), \ldots, Rlim_m(y,a)$ that define the basic sets $B_1(y), \ldots, B_m(y)$ is independent, we will continue with one of these limit groups, which for brevity we denote $Rlim(y,a)$. Note that $Rlim(y,a)$ is a proper quotient of the free group $F_k * F_y$.

Let $Rlim(y,a)$ be a restricted limit group. By Proposition 2.5 with $Rlim(y,a)$ one can associate the (canonical) taut Makanin–Razborov diagram. Let $Res_1(y,a), \ldots, Res_r(y,a)$ be the resolutions in this taut diagram. Each resolution in the diagram is a well-separated Makanin–Razborov resolution of either the restricted limit group $Rlim(y,a)$ or of a proper quotient of it, and every specialization of the restricted limit group $Rlim(y,a)$ is taut with respect to at least one of the resolutions

$$Res_1, (y,a), \ldots, Res_r(y,a).$$

Since every resolution in the taut Makanin–Razborov diagram is, in particular, a well-separated resolution, with each resolution $Res_i(y,a)$ we can associate its completed resolution $Comp(Res)_i(z,y,a)$ with a completed limit group $Comp(Rlim)_i(z,y,a)$. By lemma 1.14 of [Se2] for every specialization (y_0,a) that factors through the resolution $Res_i(y,a)$, there exists some specialization (z_0,y_0,a) that factors through the completed resolution $Comp(Res)_i(z,y,a)$. Therefore, if $D_i(y)$ denotes the Diophantine set corresponding to the completed limit group $Comp(Rlim)_i(z,y,a)$, and $B(y)$ is the basic set corresponding to the original limit group $Rlim(y,a)$, then $B(y) = D_1(y) \cup \cdots \cup D_r(y)$.

As in the minimal rank case, the continuation of the iterative procedure in the general case is conducted independently for the different limit groups and each of their resolutions. Hence, for the rest of the procedure, we will denote the restricted limit group in question $Rlim(y,a)$, and its resolution $Res(y,a)$ omitting their indices.

By theorem 1.18 of [Se2], from the validity of our given sentence we get the existence of a covering closure $Cl(Res)_1(s,z,y,a), \ldots, Cl(Res)_q(s,z,y,a)$ of the resolution $Res(y,a)$, and for each index n, $1 \leq n \leq q$, there exists a formal solution $x_n(s,z,y,a)$ for which:

(i) The words corresponding to the equations in the system $\Sigma(x_n(s,z,y,a),y,a) = 1$ represent the trivial word in the closure $Cl(Res)_n(s,z,y,a)$.

(ii) There exists some specialization (s_0^n, z_0^n, y_0^n) for which

$$\Psi(x_n(s_0^n, z_0^n, y_0^n, a), y_0^n, a) \neq 1.$$

The formal solutions described above prove the validity of the given sentence on a certain co-Diophantine set. As in the minimal rank case, to analyze the "remaining" set of y's we construct an iterative procedure that produces a sequence of well-separated resolutions and their completions. The iterative procedure for analyzing general resolutions is a modification of the procedure presented in the previous section. To get such a generalization, the analysis of the set of the remaining y's in each step of the procedure needs to be modified, as well as the *complexity* presented for the minimal rank case. As in the minimal rank case, the decrease in the modified complexity will finally force the iterative procedure to terminate.

In each of the steps of the iterative procedure, we analyze all the taut homomorphisms that factor through the resolution $Res(y, a)$ and the closure $Cl(res)(s, z, y, a)$, and satisfy the equation $\psi_j(x(s, z, y, a), y, a) = 1$ for some index j. By Proposition 2.5, every homomorphism is taut with respect to at least one of the resolutions in the taut Makanin–Razborov diagram, hence we do not lose any homomorphism by imposing the taut condition.

In this section we present a modification of the procedure for validation of a sentence in the minimal rank case to a procedure for validation of a sentence for taut maximal rank homomorphisms. This is still much simpler than the iterative procedure for validation of a sentence in the general case which is presented in the next sections. In this section we analyze all the epimorphisms h: $Rlim(y, a) \to F_k * F_{rk(Res(y,a))}$ that are taut with respect to the resolution $Res(y, a)$, where $F_{rk(Res(y,a))}$ is a free group of rank $rk(Res)$, $Cl(Res)(s, z, y, a)$, and satisfy the equality $\psi_j(x(s, z, y, a), y, a) = 1$ in the free group $F_k * F_{rk(Res(y,a))}$, for some index j. In the next sections we make use of this analysis to get an iterative procedure for validation of a general sentence of the form given above.

A taut maximal rank homomorphism that factors through the closure $Cl(Res)(s, z, y, a)$ can be extended to a taut homomorphism $h: Cl(Res)(s, z, y, a) \to F_k * F_{rk(Res(y,a))}$. Since these homomorphisms are the maximal rank homomorphisms that factor through the resolution $Res(y, a)$, the rank of the image under the homomorphism h of each quotient of the limit group $Rlim(y, a)$ that appears along the resolution is determined. In particular, whenever a free factor which is a free group is dropped along the resolution $Res(y, a)$, its image under the homomorphism h is a free factor of the subgroup $F_{rk(Res(y,a))}$. For analyzing the entire collection of maximal rank homomorphisms we may further assume that the image of a free factor that drops along the resolution is a fixed free factor in the free group F. Hence, we may assume that the image of the corresponding z generators of a free factor that drops along the resolution are fixed generators f of the free group F. Furthermore, the image of each circumference of a QH subgroup that appears along the resolution $Res(y,a)$ under the homomorphism h is contained in a free factor $F_k * F_1 * F_2$ of the free group $F_k * F_{rk(Res(y,a))}$, where the factor $F_k * F_1$ contains the image of the boundary components of the QH subgroup, and F_2 is the (possibly trivial) free factor generated by some new generators corresponding to the image of the QH subgroup and (some of) the Bass–Serre generators connecting its boundary components. The rank of the free factor F_2 depends on the resolution $Res(y,a)$, and is fixed for all maximal rank homomorphisms. Hence, we may assume that for each QH subgroup, the factor F_2 is a fixed free factor of the free group $F_{rk(Res(y,a))}$.

Fixing the image of the "dropped" free factors along the resolution $Res(y,a)$, we can modify the procedure given in Section 1 for analyzing the quotients of a minimal rank resolution, to analyze the quotients obtained as limits of taut maximal rank homomorphisms. First, the entire set of taut maximal rank homomorphisms (that fix the image of "dropped" free factors) from the closure $Cl(Res)(s,z,y,a)$ onto the free group $F_k * F$,

$$h\colon Cl(Res)(s,z,y,a) \to F_k * F_{rk(Res(y,a))},$$

that satisfy

$$\psi_j(x(h(s),h(z),h(y),a),h(y),a) = 1$$

(in the free group $F_k * F_{rk(Res(y,a))}$) for some index j, is contained in a finite set of maximal limit groups (over the free group $F_k * F_{rk(Res(y,a))}$),

$$QRlim_1(s,z,y,a),\dots,QRlim_m(s,z,y,a).$$

By the construction of the formal solution $x(s,z,y,a)$ each of the quotient limit groups $QRlim_t(s,z,y,a)$ is a proper quotient of the closure $Cl(Res)(s,z,y,a)$. Since our analysis of the quotient groups

$$QRlim_1(s,z,y,a),\dots,QRlim_m(s,z,y,a)$$

is conducted in parallel, we will continue with only one of them, which we denote $QRlim(s,z,y,a)$.

At this point we can modify the procedure given in Section 1 to analyze the well-separated resolutions containing all the taut maximal rank specializations that factor through the quotient group $QRlim(s,z,y,a)$. We will keep the notation and notions used in analyzing minimal rank resolutions in the previous section.

Suppose that the free decomposition associated with the top level of the resolution $Res(y,a)$, i.e., with the restricted limit group $Rlim_0(y,a) = Rlim(y,a)$, is $Rlim(y,a) = L_1 * \cdots * L_s * F$ (where F is a free group). Then the closure $Cl(Res)(s,z,y,a)$ admits a natural free decomposition $Cl(Res)(s,z,y,a) = \hat{L}_1 * \cdots * \hat{L}_s * F$ and the canonical embedding ν_0: $Rlim(y,a) \to Cl(Res)(s,z,y,a)$ sends the factor L_i to the factor $\hat{L}_i$. By our assumptions on the maximal rank homomorphisms h: $Cl(Res)(s,z,y,a) \to F_k * F_{rk(Res(y,a))}$, any maximal limit group $QRlim(s,z,y,a)$ admits a free decomposition

$$QRlim(s,z,y,a) = M_1 * \cdots * M_s * F,$$

and the canonical quotient map $Cl(Res)(s,z,y,a) \to QRlim(s,z,y,a)$ sends each of the factors $\hat{L}_i$ onto the factor M_i. Hence, in the continuation of the analysis of the structure of the quotient limit group $QRlim(s,z,y,a)$ and its resolutions, we can analyze the factors $\hat{L}_i$ and their quotients in parallel, so w.l.o.g., we may assume that the free decomposition associated with the restricted limit group $Rlim(y,a)$ in the resolution $Res(y,a)$ is trivial.

Let z be a generating set for the completed resolution $Comp(Res)(z,y,a)$, and let z_{base} be a subset of the variables z that generate the subgroups that appear in all the levels of the completed resolution $Comp(Res)(z,y,a)$ except its top level. These variables are called the **basis** of the completed resolution $Comp(Res)(z,y,a)$.

With the top level of the well-separated resolution $Res(y,a)$ we have associated a graph of groups Θ_1^0. Let $W_1,\dots,W_\mu$ be the fundamental groups of those connected components of Θ_1^0 that contain a non-QH vertex group. Since the resolution $Res(y,a)$ is well-separated,

$$\eta_0(Rlim(y,a)) = Rlim_1(y,a) = \eta_0(W_1) * \cdots * \eta_0(W_\mu) * H^1$$

where each factor $\eta_0(W_d)$ is a free product of some (possibly none) of the factors R_j^1, and a (possibly trivial) factor of the free group $F^1_{rk(1)}$. Since the resolution $Res(y,a)$ is well-separated, it is in particular a well-structured resolution, so the free decomposition of $Rlim_1(y,a)$ is inherited by all the restricted limit groups that lie below $Rlim_1(y,a)$ in the resolution $Res(y,a)$ and, hence, by the part of the completed resolution $Comp(Res)(z,y,a)$ that contains all its levels except the top one. Therefore, the subgroup of the completed limit group,

$Comp(Rlim)_1(z,y,a)$, admits the free decomposition $Comp(Rlim)_1(z,y,a) = U_1 * \cdots * U_\mu$ where $\eta_0(W_d) < U_d$ and $F_k < U_1$. We will denote the z generators that generate the factor U_d by z^d_{base}. Since the homomorphisms in question are taut maximal rank ones, the subgroup $< z_{base}, a >< QRlim(s,z,y,a)$ admits the free decomposition $< z_{base}, a >=< z^1_{base}, a > * < z^2_{base} > * \cdots * < z^\mu_{base} >$.

In the sequel we will set the subgroup $P < QRlim(s,z,y,a)$ to be $P = < z^1_{base}, a >$ and for each index d, $2 \leq d \leq \mu$, we set $R_d < QRlim(s,z,y,a)$ to be $R_d =< z^d_{base} >$. Following the construction of the taut Makanin–Razborov diagram (Proposition 2.5), we construct the (canonical) taut multi-graded Makanin–Razborov diagram of the limit group $QRlim(s,z,y,a)$, containing all the taut homomorphisms h: $Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$ that factor through the closure $Cl(Res)(s,z,y,a)$ and the quotient limit group $QRlim(s,z,y,a)$, viewed as a multi-graded limit group with respect to the subgroups $P, R_2, \ldots, R_\mu$. Let $MGRes_1(s,z,y,r,p,a), \ldots, MGRes_m(s,z,y,r,p,a)$ be the well-separated multi-graded Makanin–Razborov resolutions that appear in the taut multi-graded Makanin–Razborov diagram of the (multi-graded) limit group $QRlim(s,z,y,r,p,a)$ with respect to the subgroups $P, R_2, \ldots, R_\mu$, where each multi-graded resolution is terminating in either a rigid or a solid multi-graded limit group (with respect to the defining subgroups $P, R_2, \ldots, R_\mu$). With each multi-graded resolution $MGRes_i(s,z,y,r,p,a)$ we associate its collection of taut maximal rank epimorphisms h: $QRlim(s,z,y,a) \to F_k * F_{rk(Res(y,a))}$ that are taut with respect to that given resolution and with respect to the closure $Cl(Res)(s,z,y,a)$. Note that by Proposition 2.5 every taut maximal rank epimorphism h: $Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$, that factors through the closure $Cl(Res)(s,z,y,a)$ and the quotient limit group $QRlim(s,z,y,a)$, is taut with respect to at least one of the multi-graded resolutions

$$MGRes_1(s,z,y,r,p,a), \ldots, MGRes_m(s,z,y,r,p,a).$$

We will treat the multi-graded resolutions $MGRes_i(s,z,y,r,p,a)$ in parallel, so for the continuation we will restrict ourselves to one of them, which we denote $MGRes(s,z,y,r,p,a)$ for brevity. Let $MGlim(s,z,y,r,p,a)$ be the multi-graded limit group corresponding to the multi-graded resolution $MGRes(s,z,y,r,p,a)$.

For each j, let $MGlim_j(s,z,y,r,p,a)$ be the multi-graded limit group that appears in the j-th level of the multi-graded resolution $MGRes(s,z,y,r,p,a)$. To analyze the complexity of the multi-graded resolution $MGRes(s,z,y,r,p,a)$ we start with the top level multi-graded limit group $MGlim_1(s,z,y,r,p,a)$, and go down through the levels of the multi-graded resolution. $MGlim_1(s,z,y,r,p,a)$ admits a canonical free decomposition

$$MGlim_1(s,z,y,r,p,a) = L_0^1 * L_1^1 * \cdots * L_{n(1)}^1 * F$$

(where F is a (possibly trivial) free group), which is the most refined free decomposition in which $AP =< z_{base}^1, a >< L_0^1$ and each of the subgroups $R_d =< z_{base}^d >$ can be conjugated into one of the factors L_i^1. Let Λ_i be the multi-graded abelian JSJ decomposition of L_0^i with respect to the subgroups $P, R_2, \dots, R_\mu$. Naturally, there exists a canonical map

$$\tau_1\colon Rlim(y,a) \to MGlim_1(s,z,y,r,p,a).$$

$Rlim(y,a)$ was assumed freely indecomposable so it admits an abelian JSJ decomposition. Let Q be a quadratically hanging subgroup in the JSJ decomposition of $Rlim(y,a)$, and let S_Q be the corresponding (punctured) surface. The boundary elements of Q are mapped by τ_1 to non-trivial elements that can be conjugated to the various factors $L_0^1, \dots, L_{n(1)}^1$ in $MGlim_1(s,z,y,r,p,a)$. $\tau_1(Q)$ is a subgroup of $MGlim_1(s,z,y,r,p,a)$, so it inherits a (possibly trivial) free decomposition from the canonical free decomposition $MGlim_1(s,z,y,r,p,a) = L_0^1 * \cdots * L_{n(1)}^1 * F$, a free decomposition in which all the boundary elements in Q are mapped to the various factors. This free decomposition naturally lifts to a cyclic decomposition of the QH subgroup Q and its corresponding surface S_Q.

THEOREM 2.6: *There exists a maximal (possibly trivial) collection of non-homotopic, non-boundary parallel s.c.c. $\zeta(S_Q)$ on the surface S_Q that separates S_Q into (punctured) surfaces $S_1^1, \dots, S_{m(1)}^1$ with fundamental groups $Q_1^1, \dots, Q_{m(1)}^1$ which have the following properties:*

(i) *Every s.c.c. from the defining collection $\zeta(S_Q)$ is in the kernel of the map τ_1.*

(ii) *The fundamental group of each of the subsurfaces S_c^1 that are connected to non-QH subgroups in the abelian JSJ decomposition of $Rlim(y,a)$ is mapped by τ_1 into a non-trivial subgroup of a conjugate of one of the factors L_i^1 of $MGlim_1(s,z,y,r,p,a)$ that either contains the subgroup AP or a conjugate of one of the subgroups R_d.*

(iii) *Let S_c^1 be a punctured subsurface of S_Q that is not connected to a non-QH subgroup in the abelian JSJ decomposition of $Rlim(y,a)$, and let $\hat{S}_c^1$ be the closed surface obtained from S_c^1 by adding disks to its boundary components. Given a taut maximal rank homomorphism, $h\colon Cl(Res)(s,z,y,a) \to$*

$F_k * F_{rk(Res(y,a))}$, which is taut with respect to the multi-graded resolution $MGRes(s,z,y,r,p,a)$, the subgroup Q_c^1 of Q inherits an h-rank, $rk_h(Q_c^1)$, which is the rank of $h(Q_c^1)$ in the free group $F_k * F_{rk(Res(y,a))}$. Then for all taut maximal rank homomorphisms h that can be extended to a homomorphism that factors through the given multi-graded resolution, $MGRes(s,z,y,r,p,a)$, the ranks $rk_h(Q_c^1)$ are identical, and equal to a maximal free quotient of $\pi_1(\hat{S}_c^1)$.

(iv) *Let c, $1 \leq c \leq m(1)$, be an index for which the punctured surface S_c^1 is not connected to any non-QH vertex group in the abelian JSJ decomposition of $Rlim(y,a)$. Then $\tau_1(Q_c^1)$ is a factor in a free decomposition of $MGlim_1(s,z,y,r,p,a)$ in which all the subgroups $P, R_2, \dots, R_\mu$ can be conjugated into other factors. Furthermore, if $S_{c_1}^1, \dots, S_{c_v}^1$ are all the punctured surfaces that are not connected to any non-QH vertex groups in the abelian JSJ decomposition of $Rlim(y,a)$, then*

$$MGlim_1(s,z,y,r,p,a) = M * \tau_1(Q_{c_1}^1) * \cdots * \tau_1(Q_{c_v}^1) * \hat{H}$$

where all the subgroups $P, R_2, \dots, R_\mu$ can be conjugated into the factor M, and $\hat{H}$ is a (possibly trivial) free group.

(v) *$rk(\hat{H})$ is equal to the difference between the first betti number of the graph Δ obtained from the given abelian decomposition of $Rlim(y,a)$ by collapsing all the edges that connect between non-QH vertex groups, and the first betti number of the graph obtained from Δ by cutting the surfaces associated with the QH vertex groups in Δ along the collection of s.c.c. associated with them.*

(vi) *Let h: $Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$ be a taut maximal rank homomorphism with respect to the resolution $Res(y,a)$, and let*

$$\hat{h}\colon Comp(Rlim)(z,y,a) \to F_k * F_{rk(Res(y,a))}$$

be the corresponding (completed) homomorphism, and suppose that $\hat{h}$ factors through the closure $Cl(Res)(s,z,y,a)$ and the multi-graded limit group $MGlim_1(s,z,y,r,p,a)$. Then with the notation of part (iv)

$$\begin{aligned}\hat{h}(MGlim_1(s,z,y,r,p,a)) &= F_k * F_{rk(Res(y,a))} \\ &= \hat{h}(M) * \hat{h}(\tau_1(Q_{c_1}^1)) * \cdots * \hat{h}(\tau_1(Q_{c_v}^1)) * \hat{h}(\hat{H}).\end{aligned}$$

Proof: (i) follows from the construction of the decomposition $\zeta(S_Q)$. The boundary components of the surface S_Q are mapped by τ_1 to elements that can be conjugated into either P or one of the subgroups R_d. Since the homomorphisms $\hat{h}$: $Comp(Rlim)(z,y,a) \to F_k * F_{rk(Res(y,a))}$, from which the multi-graded limit group $MGlim_1(s,z,y,r,p,a)$ is constructed, correspond to taut homomorphisms with respect to the resolution $Res(y,a)$, h: $Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$, all the boundary components of the surface S_Q are mapped by τ_1 into non-trivial elements in $MGlim_1(s,z,y,r,p,a)$. Since in the free decomposition, $MGlim_1(s,z,y,r,p,a) = L^1_0 * \cdots * L^1_{n(1)} * F$, the subgroups $P, R_2, \ldots, R_\mu$ can be conjugated into the various factors $L^1_0, \ldots, L^1_{n(1)}$, each of the images under the map τ_1 of the boundary components of the surface S_Q can be conjugated into one of the factors $L^1_0, \ldots, L^1_{n(1)}$.

Let S^1_c be one of the (punctured) surfaces separated by the collection $\zeta(S_Q)$, and suppose that some of the boundary components of S^1_c are the boundary components of S_Q. Let $\hat{S}^1_c$ be the punctured surface obtained by adjoining disks to those boundary components of S^1_c that are not boundary components of the ambient surface S_Q. All the boundary components of $\hat{S}^1_c$ are boundary components of S_Q, so their images can all be conjugated into the various factors $L^1_0, \ldots, L^1_{n(1)}$. Hence, if the image $\tau_1(\pi_1(\hat{S}^1_c))$ cannot be conjugated into a factor L^1_i it inherits a non-trivial free decomposition from the decomposition, $MGlim_1(s,z,y,r,p,a) = L^1_0 * \cdots * L^1_{n(1)} * F$, a decomposition in which all the boundary components of S^1_Q are mapped to one of the factors. Since the collection of s.c.c. $\zeta(S_Q)$ is assumed to be a maximal collection of s.c.c. inherited by the ambient surface S_Q from the free decomposition of the image $\tau_1(S_Q)$, the free decomposition inherited by $\tau_1(\pi_1(\hat{S}^1_c))$ from the free decomposition of $MGlim_1(s,z,y,r,p,a)$ has to be trivial. Therefore, $\tau_1(\pi_1(\hat{S}^1_c)) = \tau_1(\pi_1(S^1_c))$ can be conjugated into one of the factors L^1_i, and we get part (ii) of the theorem.

Let S^1_c be one of the punctured surfaces that does not contain any of the boundary components of the ambient surface S_Q. Since the resolution $Res(y,a)$ is well-separated, and since the homomorphisms

$$h\colon Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$$

are taut maximal rank homomorphisms with respect to the resolution $Res(y,a)$, the rank of the image $h(Q^1_c)$ has to be identical for all the homomorphisms h which can be extended to a homomorphism that factors through the multi-graded resolution $MGRes(s,z,y,r,p,a)$. Furthermore, since the homomorphisms h are maximal rank, $rk(h(Q^1_c))$ has the rank $rk(h(\pi_1(\hat{S}^1_c)))$, components in the decomposition and we get part (iii).

If $S^1_{c_1}, \ldots, S^1_{c_v}$ are the punctured surfaces separated by the collection of curves $\zeta(S_{Q_1}), \ldots, \zeta(S_{Q_s})$ which do not contain any boundary components of the ambient surfaces $S_{Q_1}, \ldots, S_{Q_s}$, then for every homomorphism h: $Rlim(y,a) \to F_k * F_{rk(res(y,a))}$ that is taut and maximal rank with respect to the resolution $Res(y,a)$,

$$\begin{aligned} F_k * F_{rk(Res(y,a))} &= h(Rlim(y,a)) \\ &= \hat{h}(QRlim(s,z,y,a)) = F_h * h(Q^1_{c_1}) * \cdots * h(Q^1_{c_v}) * h(\hat{H}) \end{aligned}$$

where $h(P), h(R_2), \ldots, h(R_\mu)$ can all be conjugated into the factor F_h, and $h|_{\hat{H}}$ is an isomorphism. Since the (multi-graded) limit group $MGlim_1(s,z,y,r,p,a)$ is obtained from a sequence of homomorphisms that admit the free decomposition given above, $MGlim_1(s,z,y,r,p,a)$ admits a free decomposition of the form

$$MGlim_1(s,z,y,r,p,a) = M * \tau_1(Q^1_{c_1}) * \cdots * \tau_1(Q^1_{c_v}) * \hat{H}$$

in which the subgroups $P, R_2, \ldots, R_\mu$ can be conjugated into the factor M, and we finally get parts (iv)–(vi) of the theorem. ∎

At this point we can apply the relevant part of the argument used in the minimal rank case, for analyzing the complexity of the graded resolution $MGRes(s,z,y,r,p,a)$. If the image of a QH subgroup Q^1_i, obtained by cutting a QH subgroup Q_t in the abelian JSJ decomposition of $Rlim(y,a)$ along the collection of s.c.c. $\zeta(S_{Q_t})$, is in the factor M, i.e., $\tau_1(Q^1_i) < M$, then Q^1_i inherits a (possibly trivial) decomposition from the multi-graded abelian JSJ decomposition of one of the factors in the most refined free decomposition of the factor M in which the subgroups $P, R_2, \ldots, R_\mu$ can be conjugated into the different factors. Otherwise, Q^1_i inherits a decomposition from the abelian JSJ decomposition of the factor $\tau_1(Q^1_i)$. We will denote these decompositions $\Gamma^1_i(Q^1_i)$. Such a decomposition corresponds to some decomposition of the surface S^1_i, corresponding to the QH subgroup Q^1_i, along a (possibly trivial) collection of disjoint non-homotopic s.c.c. which we denote $\Gamma^1_i(S^1_i)$. Note that by construction, every s.c.c. from the defining collection of $\Gamma^1_i(S^1_i)$ is mapped by τ_1 to either a trivial or an elliptic element in the abelian JSJ decomposition of either a factor of M or of $\tau_1(Q^1_i)$.

Lemma 2.7: *Let Q' be a quadratically hanging subgroup in the multi-graded abelian JSJ decomposition of either a factor of M, in a free decomposition in which all the subgroups $P, R_1, \ldots, R_\mu$ are elliptic, or a factor $\tau_1(Q^1_i)$. Suppose that either*

(i) *Q' is contained in a factor of M, and for some subsurface Q^1_i, obtained by cutting a QH vertex group Q_t in the abelian JSJ decomposition of*

$Rlim(y,a)$ along its associated collection of s.c.c. $\zeta(S_{Q_t})$, $\tau_1(Q_i^1)$ is contained in the same factor of M, but $\tau_1(Q_i^1)$ cannot be conjugated into the fundamental group of a subgraph of the multi-graded abelian decomposition of the corresponding factor of M that does not contain the vertex stabilized by Q', or

(ii) *Q' is a QH vertex group in the abelian JSJ decomposition of a factor $\tau_1(Q_i^1)$.*

Then some subsurface of S_i^1 is mapped into a finite index subgroup of Q', $genus(S') \leq genus(S_i^1)$ and $|\chi(S')| \leq |\chi(S_i^1)|$.

Proof: Let S_i^1 be the surface corresponding to Q_i^1, and suppose that Q_i^1 is a subgroup of the QH subgroup Q in the abelian JSJ decomposition of $Rlim(y,a)$. Let S_Q be the surface associated with the QH subgroup Q, and let $\hat{S}_i^1$ be the surface obtained from the punctured surface S_i^1 by adding disks to those of its boundary components which are not boundary components of the ambient surface S_Q.

Since the resolution $Res(y,a)$ is well-separated, and the homomorphisms h are taut and maximal rank, if there exists a s.c.c. on the closed surface $\hat{S}_i^1$ that is mapped to the identity by the map τ_1, then this s.c.c. can be added to the collection $\zeta(S_Q)$, which contradicts the maximality of the collection $\zeta(S_Q)$. Hence, no s.c.c. on the surface $\hat{S}_i^1$ is mapped to the identity by the homomorphism τ_1. Since there is no s.c.c. on $\hat{S}_i^1$ that is mapped to the trivial element by the homomorphism τ_1, and since by the assumptions of the lemma $\hat{S}_i^1$ cannot be mapped into a conjugate of the fundamental group of a subgraph that does not contain Q' in the multi-graded decomposition of the corresponding factor of $MGlim_1(s,z,y,r,p,a)$, Lemma 1.4 implies that there must exist some subsurface of $\hat{S}_i^1$ separated by the collection of curves $\Gamma(S_i^1)$, which we denote $\tilde{S}$, so that $\tilde{S}$ is mapped into a subgroup of finite index of Q', and all the boundary components of $\tilde{S}$ are mapped into conjugates of boundary elements of $S_{Q'}$. Hence, the double of $\tilde{S}$ is mapped into a subgroup of finite index of the double of S'. Therefore, $genus(S') \leq genus(\tilde{S}) \leq genus(S_Q)$, and $|\chi(S')| \leq |\chi(\tilde{S})| \leq |\chi(S_Q)|$. ∎

We continue analyzing the levels of the multi-graded resolution $MGRes(s,z,y,r,p,a)$ in the same way we have analyzed the first level, i.e., the multi-graded limit group $MGlim_1(s,z,y,r,p,a)$. Clearly, Theorem 2.6 and Lemma 2.7 remain valid for the next levels of the multi-graded resolution $MGRes(s,z,y,r,p,a)$. After finitely many steps we are left with either a rigid or solid multi-graded limit group with respect to the subgroups $P, R_1, \ldots, R_\mu$. Like in the minimal rank case, to analyze the multi-graded resolution $MGRes(s,z,y,r,p,a)$ we need to look at **surviving surfaces**.

Definition 2.8: Let Q be a quadratically hanging subgroup in the abelian JSJ decomposition of $Rlim(y,a)$ and let S be its corresponding (punctured) surface. The QH subgroup Q (and the corresponding surface S) is called **surviving** if along the multi-graded resolution $MGRes(s,z,y,r,p,a)$ there exists some quadratically hanging subgroup Q' in the multi-graded abelian JSJ decomposition of a factor of one of the multi-graded limit groups $MGlim_j(s,z,y,r,p,a)$, with corresponding surface S', so that τ_1 maps Q non-trivially into Q', $genus(S') = genus(S)$ and $\chi(S') = \chi(S)$.

By definition, if Q is a non-surviving QH subgroup in the JSJ decomposition of $Rlim(y,a)$, then every QH subgroup Q' in any level of the multi-graded resolution $MGRes(s,z,y,r,p,a)$ into which a subsurface of Q is mapped non-trivially, has either a strictly lower genus or a strictly smaller Euler characteristic than that of the QH subgroup Q. This would eventually "force" the complexity of the multi-graded resolution $MGRes(s,z,y,r,p,a)$ to be smaller than that of the resolution $Res(y,a)$ if one is able to "isolate" the surviving surfaces.

THEOREM 2.9: *Let $Q_1, \ldots, Q_r$ be the surviving QH subgroups in the JSJ decomposition of $Rlim(y,a)$, i.e., those QH vertex groups that are mapped non-trivially into QH subgroups of the same genus and Euler characteristic in some level of the multi-graded resolution $MGRes(s,z,y,r,p,a)$. Then in the taut Makanin–Razborov diagram of $QRlim(s,z,y,a)$ associated with those specializations that are taut with respect to the closure $Cl(Res)(s,z,y,a)$, the multi-graded resolution $MGRes(s,z,y,r,p,a)$ can be replaced by finitely many multi-graded resolutions, each composed from two consecutive parts. The first part is a multi-graded resolution of $QRlim(s,z,y,a)$ with respect to the subgroups $< P, R_2, \ldots, R_\mu, Q_1, \ldots, Q_r >$, which we denote*

$$MGRes(s,z,y,(r,p,Q_1,\ldots,Q_r),a).$$

The second part is a one-step resolution that maps the rigid (solid) terminal multi-graded limit group of $MGRes(s,z,y,(r,p,Q_1,\ldots,Q_r),a)$ to the rigid (solid) terminal multi-graded limit group of the resolution $MGRes(s,z,y,r,p,a)$. The two consecutive parts of the multi-graded resolution have the following properties:

(1) *The multi-graded decomposition corresponding to the second part of the resolution contains a vertex stabilized by the terminal rigid (solid) multi-*

graded limit group of the resolution $MGRes(s,z,y,r,p,a)$ connected to r' surviving QH subgroups $Q_{i_1},\dots,Q_{i_{r'}}$ for some $r' \leq r$ and $1 \leq i_1 < \cdots < i_{r'} \leq r$.

(2) *If the terminal multi-graded limit group of the multi-graded resolution*

$$MGRes(s,z,y,(r,p,Q_1,\dots,Q_r),a)$$

is rigid (solid), so is the terminal multi-graded limit group of the multi-graded resolution $MGRes(s,z,y,r,p,a)$. Furthermore, if they are both solid, their abelian multi-graded decompositions are in correspondence, i.e., the decompositions differ only in the stabilizer of the vertices stabilized by the subgroups P and R_d's. These vertices are stabilized by the subgroups P and R_d's and the subgroups $Q_{i_1},\dots,Q_{i_{r'}}$ in the first part, and by the subgroups P and R_d's in the second.

Proof: Identical to the proof of Theorem 1.7. ∎

Since the resolution $Res(y,a)$ is well-separated, the groups associated with the various connected components of the graph of groups Θ_1^0 that contain non-QH, non-abelian vertex groups $W_1,\dots,W_\mu$ are mapped to different factors of the second restricted limit group along the resolution $Res(y,a)$, $Rlim_1(y,a)$. In accordance with this free decomposition, the completed limit group $Comp(Rlim)_1(z,y,a)$ admits a free decomposition

$$Comp(Rlim)_1 = P * R_2 * \cdots * R_\mu.$$

Furthermore, for every taut maximal rank homomorphism $h\colon Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$ that is taut with respect to the resolution $Res(y,a)$, and its associated homomorphism $\hat{h}\colon Comp(Rlim)(z,y,a) \to F_k * F_{rk(Res(y,a))}$, we have

$$h(Rlim(y,a)) = \hat{h}(Comp(Rlim)(z,y,a)) = \hat{h}(Comp(Rlim)_1(z,y,a)) * \hat{h}(H^1)$$

where H^1 is the free group freely generated by free groups "dropped" in the top level of the resolution $Res(y,a)$ from the circumference of the various QH vertex groups and from abelian factors that appear in this level.

Let $T_1(s,z,y,r,p,a)$ be the terminal rigid or solid multi-graded limit group of the multi-graded resolution $MGRes(s,z,y,r,p,a)$. To continue the analysis of the resolutions associated with the quotient limit group $QRlim(s,z,y,a)$, we need a "linkage" between the free decomposition of the completed limit group $Comp(Rlim)_1(z,y,a)$ and the free decomposition of $T_1(s,z,y,r,p,a)$, a correspondence between (the ranks of) the free groups "dropped" along the top level of the resolution $Res(y,a)$ and along the various levels of the multi-graded resolution $MGRes(s,z,y,r,p,a)$, and a correspondence between the image $\hat{h}(Comp(Rlim)_1(z,y,a))$ and the image $h_1(T_1(s,z,y,r,p,a))$, where $h_1\colon T_1(s,z,y,r,p,a) \to F_k * F_{rk(Res(y,a))}$ is a rigid or solid homomorphism obtained (by shortening along the multi-graded resolution $MGRes$) from a maximal rank taut homomorphism $h\colon Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$ that is taut with respect to both the closure $Cl(Res)(s,z,y,a)$ and for which the corresponding homomorphism $\hat{h}$ is taut with respect to the multi-graded resolution $MGRes(s,z,y,r,p,a)$.

Theorem 2.10: *Let $T_1(s,z,y,r,p,a)$ be the solid or rigid terminal limit group of the multi-graded resolution $MGRes(s,z,y,r,p,a)$, let*

$$h\colon Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$$

be a (general) homomorphism which is taut with respect to the resolution $Res(y,a)$, let $\hat{h}\colon Comp(Rlim)(z,y,a) \to F_k * F_{rk(Res(y,a))}$ be the corresponding homomorphism of the completed limit group, and suppose the homomorphism $\hat{h}$ factors through the closure $Cl(Res)(s,z,y,a)$ and is taut and maximal rank with respect to the multi-graded resolution $MGRes(s,z,y,r,p,a)$. Let

$$h_1\colon T_1(s,z,y,r,p,a) \to F_k * F_{rk(Res(y,a))}$$

be a rigid or solid homomorphism obtained from the homomorphism $\hat{h}$ by the shortening procedure applied along the levels of the multi-graded resolution $MGRes(s,z,y,r,p,a)$. Then the multi-graded resolution $MGRes(s,z,y,r,p,a)$ can be extended in finitely many ways, to give finitely many multi-graded resolutions with respect to the subgroups $P,R_2,\dots,R_\mu$, which we still denote $MGRes(s,z,y,r,p,a)$, so that the terminal rigid or solid multi-graded limit group (which we still denote $T_1(s,z,y,r,p,a)$) has the following properties:

- (i) *The summation of the ranks of the free groups dropped along the various levels of the multi-graded resolution $MGRes(s,z,y,r,p,a)$ is identical to the rank of the free group dropped along the top level of the resolution $Res(y,a)$.*
- (ii) *By our assumptions, the completed limit group $Comp(Rlim)_1(z,y,a)$ admits the free decomposition $Comp(Rlim)_1(z,y,a) = P*R_2*\cdots*R_\mu$. Then the terminal solid or rigid multi-graded limit group $T_1(s,z,y,r,p,a)$ admits a free decomposition $T_1(s,z,y,r,p,a) = N_1*\cdots*N_\mu$, where $P < N_1$ and, for every $d \geq 2$, R_d can be conjugated into N_d.*

(iii) $h_1(T_1(s,z,y,r,p,a)) = \hat{h}(Comp(Rlim)_1(z,y,a))$ *and, with the notation of part (ii),* $h_1(N_1) = \hat{h}(P)$, *and for every* $d \geq 2$, $h_1(N_d) = \hat{h}(R_d)$.

Proof: Since the resolution $Res(y,a)$ is well-separated, and since the homomorphisms $\hat{h}$: $Comp(Rlim)(z,y,a) \to F_k * F_{rk(Res(y,a))}$ are all taut maximal rank homomorphisms with respect to the completed resolution $Comp(Res)(z,y,a)$, the summation of the ranks of the free groups that are dropped along the levels of the multi-graded resolution $MGRes(s,z,y,r,p,a)$ is at most the rank of the free group dropped along the top level of the resolution $Res(y,a)$.

Let F be the free group which is the free product of the free factors dropped along the various levels of the multi-graded resolution $MGRes(s,z,y,r,p,a)$, and let τ_ℓ: $Rlim(y,a) \to T_1(s,z,y,r,p,a) * F$ be the natural map. As we have pointed out, the formulation and proof of Theorem 2.6 which are presented for the top level of the multi-graded resolution $MGRes(s,z,y,r,p,a)$ hold for all its levels, and in particular for its terminal level. By part (iv) of Theorem 2.6 applied to the terminal level of $MGRes$,

$$T_1(s,z,y,r,p,a) * F = M * \hat{H}$$

where all the subgroups $P, R_2, \dots, R_\mu$ can be conjugated into the factor M. Furthermore, let h: $Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$ be a taut maximal rank homomorphism with respect to the resolution $Res(y,a)$, let $\hat{h}$: $Comp(Rlim)(z,y,a) \to F_k * F_{rk(Res(y,a))}$ be the corresponding (completed) homomorphism, and suppose that $\hat{h}$ factors through the closure $Cl(Res)(s,z,y,a)$ and the multi-graded resolution $MGRes(s,z,y,r,p,a)$. Let h_1: $T_1(s,z,y,r,p,a) * F \to F_k * F_{rk(Res(y,a))}$ be the homomorphism obtained from the homomorphism $\hat{h}$ by applying the shortening procedure along the various levels of the multi-graded resolution $MGRes(s,z,y,r,p,a)$. Then by successively applying part (vi) of Theorem 2.6,

$$h_1(T_1(s,z,y,r,p,a) * F) = F_k * F_{rk(Res(y,a))} = h_1(M) * h_1(\hat{H})$$

and h_1 maps the factor $\hat{H}$ isomorphically onto their image. Now, if the rank of the factor F is strictly smaller than the rank of the free group dropped along the top level of the resolution $Res(y,a)$, then for every homomorphism h_1: $T_1(s,z,y,r,p,a) \to F_k * F_{rk(Res(y,a))}$, the image admits a free decomposition of the form $F_{h_1} * < v >$, where $h_1(P) < F_{h_1}$, and the subgroups $h_1(R_2), \dots, h_1(R_\mu)$ can all be conjugated into F_{h_1} in the free group $F_k * F_{rk(Res(y,a))}$, which contradicts our assumption that the homomorphisms h_1: $T_1(s,z,y,r,p,a) \to F_k * F_{rk(Res(y,a))}$ are either rigid or solid multi-graded homomorphisms, and we get part (i) of the theorem.

By composing each taut maximal rank homomorphism h: $Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$ with an automorphism of the free group $F_k * F_{rk(Res(y,a))}$, w.l.o.g. we may assume that the factors $h_1(M), \hat{H}$ are all fixed factors of the free group $F_k * F_{rk(Res(y,a))}$, and similarly that the factors $h_1(P), h_1(R_2), \ldots, h_1(R_\mu)$ are all fixed factors of the free group $F_k * F_{rk(Res(y,a))}$: $B_1, \ldots, B_\mu$. Let T be the Bass–Serre tree corresponding to the free decomposition $B_1 * \cdots * B_\mu$. Naturally, with every homomorphism $h_1: T_1(s,z,y,r,p,a) \to F_k * F_{Res(y,a))}$ we can associate an action $\lambda_{h_1}: T_1(s,z,y,r,p,a) \times T \to T$.

$T_1(s,z,y,r,p,a)$ is constructed from a converging sequence of homomorphisms $\{h_1(n)$: $T_1 \to B_1 * \cdots * B_\mu\}$. This sequence of homomorphisms corresponds to a sequence of actions $\{\lambda_{h_1(n)}\}$ of the group $T_1(s,z,y,r,p,a)$ on the Bass–Serre tree T. Since the sequence of homomorphisms $\{h_1(n)\}$ converges into the subgroup $T_1(s,z,y,r,p,a)$, a converging sequence of actions $\{\lambda_{h_1(n_m)}\}$ converges in the Gromov–Hausdorff topology into a faithful action of the subgroup $T_1(s,z,y,r,p,a)$ on some real tree Y.

Since the resolution $Res(y,a)$ is a well-separated resolution, and the homomorphisms h: $Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$ from which the multi-graded resolution $MGRes(s,z,y,r,p,a)$ are all taut and maximal rank with respect to the resolution $Res(y,a)$, and since by part (i) the summation of the ranks of the free groups dropped along the levels of the multi-graded resolution $MGRes(s,z,y,r,p,a)$ is equivalent to the rank of the free group dropped in the top level of the resolution $Res(y,a)$, $T_1(s,z,y,r,p,a)$ does not admit a free decomposition in which the subgroups $P, R_2, \ldots, R_\mu$ can all be conjugated into one of the factors. Hence, if all the subgroups $P, R_2, \ldots, R_\mu$ fix the same point in the real tree Y, the real tree Y is composed from finitely many orbits of IET and axial components and finitely many orbits of segments with non-trivial (abelian) stabilizers. Therefore, we can successively apply the shortening procedure for the action of $T_1(s,z,y,r,p,a)$ on the real tree Y, and continue the multi-graded resolution $MGRes(s,z,y,r,p,a)$ (canonically) in finitely many ways, until the action of each of the terminal multi-graded limit groups, which we still denote $T_1(s,z,y,r,p,a)$, acts on a discrete tree T', where the stabilizer of each edge in T' is trivial, and the subgroups $P, R_2, \ldots, R_\mu$ fix points in the discrete tree T'. Hence, by standard Bass–Serre theory, the multi-graded limit group $T_1(s,z,y,r,p,a)$ admits a free decomposition of the form $T_1(s,z,y,r,p,a) = N_1 * \cdots * N_\mu$, levels of the multi-graded where $P < N_1$ and, for every $d \geq 2$, $R_d < N_d$, and by the construction of the limit discrete tree T', for every homomorphism $h_1: T_1(s,z,y,r,p,a) \to F_k * F_{rk(Res(y,a))}$ that factors through the obtained multi-graded resolution $MGRes(s,z,y,r,p,a)$ and its corresponding homomorphism, $\hat{h}\colon Comp(Rlim)(z,y,a) \to F_k * F_{rk(Res(y,a))}$: $h_1(T_1(s,z,y,r,p,a)) = \hat{h}(Comp(Rlim)_1(z,y,a))$, $h_1(N_1) = \hat{h}(P)$, and for every $d \geq 2$, $h_1(N_d) = \hat{h}(R_d)$, and we get parts (ii) and (iii) of the theorem. ∎

Theorem 2.10 associates a free decomposition

$$T_1(s,z,y,r,p,a) = N_1 * \cdots * N_\mu * F$$

of the terminal rigid or solid limit group of the multi-graded resolution $MGRes(s,z,y,r,p,a)$ with the free decomposition $Comp(Rlim)_1(z,y,a) = P * R_2 * \cdots * R_\mu$. However, the free decomposition associated with the restricted limit group $Rlim_1(y,a)$ along the resolution $Res(y,a)$, hence the free decomposition of the completed limit group $Comp(Rlim)_1(z,y,a)$ along the completed resolution $Comp(Res)(z,y,a)$, may be finer than the above free decomposition. To continue the analysis of the resolutions of the quotient limit group $QRlim(s,z,y,a)$ and their associated maximal rank taut homomorphisms, we need to show that the terminal multi-graded limit group $T_1(s,z,y,r,p,a)$ admits a free decomposition compatible with the associated finer free decomposition of $Rlim_1(y,a)$.

Proposition 2.11: *Let $Rlim_1(y,a) = L_1 * \cdots * L_s * F^1 * H^1$ be the free decomposition associated with $Rlim_1(y,a)$ along the resolution $Res(y,a)$, where F^1 is a free group, and H^1 is the free group generated by the free groups dropped in the top level of the resolution $Res(y,a)$, and let*

$$Comp(Rlim)_1(z,y,a) = \hat{L}_1 * \cdots * \hat{L}_s * F^1$$

be the corresponding free decomposition of $Comp(Rlim)_1(z,y,a)$. Let $T_1(s,z,y,r,p,a)$ be the solid or rigid terminal limit group of the multi-graded resolution $MGRes(s,z,y,r,p,a)$, let $h\colon Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$ be a (general) maximal rank homomorphism which is taut with respect to the resolution $Res(y,a)$, let $\hat{h}\colon Comp(Rlim)(z,y,a) \to F_k * F_{rk(Res(y,a))}$ be the corresponding homomorphism of the completed limit group, and suppose the homomorphism $\hat{h}$ factors through the closure $Cl(Res)(s,z,y,a)$ and is taut with respect to the multi-graded resolution $MGRes(s,z,y,r,p,a)$. Let $h_1\colon T_1(s,z,y,r,p,a) \to F_k * F_{rk(Res(y,a))}$ be a rigid or solid homomorphism obtained from the homomorphism $\hat{h}$ by the shortening procedure applied along the levels of the multi-graded resolution $MGRes(s,z,y,r,p,a)$.

Then the multi-graded resolution $MGRes(s,z,y,r,p,a)$ can be extended in finitely many ways, to obtain finitely many multi-graded resolutions with respect *to the subgroups* $P, R_2, \dots, R_\mu$, *which we still denote* $MGRes(s,z,y,r,p,a)$, *so that the terminal rigid or solid multi-graded limit group of each of these multi-graded resolutions, which we still denote* $T_1(s,z,y,r,p,a)$, *satisfies*

(1) $T_1(s,z,y,r,p,a) = D_1 * \cdots * D_s * F^1$ *where the natural map*

$$\lambda\colon Comp(Rlim)_1(z,y,a) \to T_1(s,z,y,r,p,a)$$

maps the subgroup F^1 *isomorphically, and each of the factors* $\hat{L}_j$ *into the factor* D_j.

(2) *For every index* j, $1 \le j \le s$, $h_1(D_j) = \hat{h}(\hat{L}_j)$.

Proof: The proof of Theorem 2.11 is basically identical to the proof of parts (ii) and (iii) of Theorem 2.10. By composing each taut maximal rank homomorphism $h\colon Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$ with an automorphism of the free group $F_k * F_{rk(Res(y,a))}$, w.l.o.g. we may assume that the factors $h_1(\hat{L}_1), \dots, h_1(\hat{L}_s)$ are all fixed factors of the free group $F_k * F_{rk(Res(y,a))}$: $B_1, \dots, B_s$ in correspondence, and the generators of the free factor F^1 are mapped to fixed generators of the (target) free group $F_k * F_{rk(Res(y,a))}$. Let $B = B_1 * \cdots * B_s * F^1$, and let T be the Bass–Serre tree corresponding to the free decomposition $B = B_1 * \cdots * B_s * F^1$. Naturally, with every homomorphism $h_1\colon T_1(s,z,y,r,p,a) \to F_k * F_{Res(y,a))}$ we can associate an action $\lambda_{h_1}\colon T_1(s,z,y,r,p,a) \times T \to T$.

The subgroup $T_1(s,z,y,r,p,a)$ is constructed from a converging sequence of homomorphisms $\{h_1(n)\colon T_1(s,z,y,r,p,a) \to B_1 * \cdots * B_s * F^1\}$. This sequence of homomorphisms corresponds to a sequence of actions $\{\lambda_{h_1(n)}\}$ of the subgroup $T_1(s,z,y,r,p,a)$ on the Bass–Serre tree T. Since the sequence of homomorphisms $\{h_1(n)\}$ converges into the subgroup $T_1(s,z,y,r,p,a)$, a converging sequence of actions $\{\lambda_{h_1(n_m)}\}$ converges in the Gromov–Hausdorff topology into a faithful action of the factor $T_1(s,z,y,r,p,a)$ on some real tree Y.

Since the resolution $Res(y,a)$ is a well-separated resolution, and the homomorphisms $h\colon Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$ from which the multi-graded resolution $MGRes(s,z,y,r,p,a)$ was constructed are all taut and maximal rank with respect to the resolution $Res(y,a)$, and since by part (i) of Theorem 2.10 the summation of the ranks of the free groups dropped along the levels of the multi-graded resolution $MGRes(s,z,y,r,p,a)$ is equivalent to the rank of the free group dropped in the top level of the resolution $Res(y,a)$, the subgroup $T_1(s,z,y,r,p,a)$ does not admit a free decomposition in which the subgroups $\hat{L}_1, \dots, \hat{L}_s$ and F^1, which in particular generate the subgroups $P, R_2, \dots, R_\mu$, can all be conjugated into one of the factors. Hence, if all the subgroups $\hat{L}_1, \dots, \hat{L}_s$ and F^1 fix the same point in the real tree Y, the real tree Y is composed from finitely many orbits of IET and axial components and finitely many orbits of segments with non-trivial (abelian) stabilizers. Therefore, we can successively apply the shortening procedure for the action of $T_1(s,z,y,r,p,a)$ on the real tree Y, and continue the multi-graded resolution $MGRes(s,z,y,r,p,a)$ (canonically) in finitely many ways, until the action of each of the terminal multi-graded limit groups, which we still denote $T_1(s,z,y,r,p,a)$, acts on a discrete tree T', where the stabilizer of each edge in T' is trivial, and the subgroups $\lambda(\hat{L}_1),\dots,\lambda(\hat{L}_s)$ and F^1 fix points in the discrete tree T'. Hence, by standard Bass–Serre theory, the multi-graded limit group $T_1(s,z,y,r,p,a)$ admits a free decomposition of the form $T_1(s,z,y,r,p,a) = \hat{D}_1 * \cdots * \hat{D}_s * F^1$, levels of the multi-graded where for every index j, $\lambda(\hat{L}_j) < D_j$, and by the construction of the limit discrete tree T' for every homomorphism $h_1: T_1(s,z,y,r,p,a) \to F_k * F_{rk(Res(y,a))}$ that factors through the obtained multi-graded resolution $MGRes(s,z,y,r,p,a)$ and its corresponding homomorphism $\hat{h}: Comp(Rlim)(z,y,a) \to F_k * F_{rk(Res(y,a))}$, $h_1(D_j) = \hat{h}(\hat{L}_j)$. ∎

In general, the restricted limit group $Rlim_1(y,a)$ is associated with a free decomposition along the resolution $Res(y,a)$, a free decomposition of the form $Rlim_1(y,a) = L_1 * \cdots * L_s * F^1 * H^1$. From this free decomposition, the completed limit group, $Comp(Rlim)_1(z,y,a)$, inherits a free decomposition $Comp(Rlim)_1(z,y,a) = \hat{L}_1 * \cdots * \hat{L}_s * F^1$, where the canonical map $\nu_1: Rlim_1(y,a) \to Comp(Rlim)_1(z,y,a)$ maps each of the factors L_j into the corresponding factor $\hat{L}_j$. Furthermore, given any taut maximal rank homomorphism

$$h: Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$$

that is taut with respect to the resolution $Res(y,a)$, and its corresponding (completed) homomorphism $\hat{h}: Comp(Rlim)(a,y,a) \to F_k * F_{rk(Res(y,a))}$, $\hat{h}(Comp(Rlim)_1(z,y,a)) = \hat{h}(\hat{L}_1) * \cdots * \hat{h}(\hat{L}_s) * F^1$, and we have even assumed (w.l.o.g.) that for each factor $\hat{L}_j$, $\hat{h}(\hat{L}_j)$ is a fixed factor of the free group $F_k * F_{rk(Res(y,a))}$.

Proposition 2.11 shows that the terminal (rigid or solid) multi-graded limit group $T_1(s,z,y,r,p,a)$ of the multi-graded resolution $MGRes(s,z,y,r,p,a)$ admits a free decomposition $T_1(s,z,y,r,p,a) = D_1 * \cdots * D_s * F^1$ which is compatible with the free decomposition of $Comp(Rlim)_1(z,y,a)$, i.e., if

$$\lambda: Comp(Rlim)_1(z,y,a) \to T_1(s,z,y,r,p,a)$$

is the natural map, then for every index j, $1 \le j \le s$, $\lambda(\hat{L}_j) < D_j$. Furthermore, let $h: Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$ be a taut maximal rank homomorphism with respect to the resolution $Res(y,a)$, let

$$\hat{h}\colon Comp(Rlim)_1(z,y,a) \to F_k * F_{rk(Res(y,a))}$$

be the corresponding homomorphism, and suppose that the homomorphism $\hat{h}$ is taut with respect to both the closure $Cl(Res)(s,z,y,a)$ and the multi-graded resolution $MGRes(s,z,y,r,p,a)$, and $h_1\colon T_1(s,z,y,r,p,a) \to F_k * F_{rk(Res(y,a))}$ is the rigid or solid homomorphism obtained from $\hat{h}$ by applying the shortening procedure along the multi-graded resolution $MGRes(s,z,y,r,p,a)$; then $h_1(D_j) = \hat{h}(\hat{L}_j)$.

The compatibility of the free decompositions of

$$Rlim_1(y,a), \quad Comp(Rlim)_1(z,y,a) \quad \text{and} \quad T_1(s,z,y,r,p,a)$$

proved in Proposition 2.11 enables one to analyze the factors D_j of the terminal rigid or solid multi-graded limit group $T_1(s,z,y,r,p,a)$ in parallel, hence it reduces the continuation of the analysis of the resolutions associated with the quotient limit group $QRlim(s,z,y,a)$ to the case in which the free decomposition associated with the restricted limit group $Rlim_1(y,a)$ in the resolution $Res(y,a)$ is the trivial one. This will be our assumption for the rest of this section.

We continue the analysis of the collection of taut maximal rank homomorphisms $h\colon Cl(Res)(s,z,y,a) \to F_k * F$ that factor through the multi-graded resolution $MGRes(s,z,y,r,p,a)$ as we did in the previous section for minimal rank resolutions, i.e., by sequentially reducing the subgroups P and R_d's according to which we construct our multi-graded resolutions.

Let z be a generating set for the entire completion, $Comp(Res)(z,y,a)$, let z_{base} be a subset of the generating set z that generate $Comp(Res)_1(z,y,a)$, and let z_{scbase} be the subset of the generators z that generate $Comp(Res)_2(z,y,a)$, i.e., the subgroup of the completion associated with all its levels except the top two levels.

With the second level of the well-separated resolution $Res(y,a)$ we have associated a graph of groups Θ_1^1. Let $W_1^1,\dots,W_{\mu_1}^1$ be the fundamental groups of those connected components of Θ_1^1 that contain a non-QH vertex group. Since the resolution $Res(y,a)$ is well-separated,

$$\eta_1(Rlim_1(y,a)) = Rlim_2(y,a) = \eta_1(W_1^1) * \cdots * \eta_1(W_{\mu_1}^1) * \hat{H}^2$$

where $\hat{H}^2$ is a free factor of H^2, and each factor $\eta_1(W_d^1)$ is a free product of some (possibly none) of the factors R_j^2, and a (possibly trivial) factor of the free group $F^2_{rk(2)}$. Since the resolution $Res(y,a)$ is well-separated, it is in particular a well-structured resolution, so the free decomposition of $Rlim_2(y,a)$ is inherited by all the restricted limit groups that lie below $Rlim_2(y,a)$ in the resolution $Res(y,a)$ and, hence, by the part of the completed resolution $Comp(Res)(z,y,a)$ that contains all its levels except the two highest ones. Therefore, this (bottom) part of the completed limit group, $Comp(Rlim)_2(z,y,a)$, admits the free decomposition $Comp(Rlim)_2(z,y,a) = U_1 * \cdots * U_{\mu_1}$ where $\eta_1(W^1_d) < U_d$ and $F_k < U_1$. We will denote the z generators that lie in the factor U_d by z^d_{scbase}. Since the homomorphisms in question are taut maximal rank ones, the subgroup $< z_{scbase}, a >< T_1(s,z,y,r,p,a)$ admits the free decomposition $< z_{scbase}, a >=< z^1_{scbase}, a > * < z^2_{scbase} > * \cdots * < z^{\mu_1}_{scbase} >$.

In the sequel we will set the subgroup $P^1 < T_1(s,z,y,r,p,a)$ to be $P^1 = < z^1_{scbase}, a >$ and for each index d, $2 \leq d \leq \mu_1$, we set $R^1_d < T_1(s,z,y,r,p,a)$ to be $R^1_d =< z^d_{scbase} >$. Following the construction of the taut Makanin–Razborov diagram (Proposition 2.5), we construct sequentially a taut multi-graded Makanin–Razborov diagram of the limit group $T_1(s,z,y,r,p,a)$ with respect to the subgroups $P^1, R^1_2, \ldots, R^1_{\mu_1}$, containing all the rigid or solid homomorphisms $h_1\colon T_1(s,z,y,r,p,a) \to F_k * F_{rk(Res(y,a))}$ that are obtained by applying the shortening procedure along the multi-graded resolution $MGRes(s,z,y,r,p,a)$ from homomorphisms

$$\hat{h}\colon Comp(Rlim)(s,z,y,a) \to F_k * F_{rk(Res(y,a))}$$

that correspond to a taut maximal rank homomorphism with respect to the resolution $Res(y,a)$, $h\colon Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$, so that the homomorphism $\hat{h}$ factors through both the closure $Cl(Res)(s,z,y,a)$ and the multi-graded resolution $MGRes(s,z,y,r,p,a)$. We denote each of the obtained multi-graded resolutions in our sequential construction $MGRes(s,z,y,r^1,p^1,a)$.

To analyze the complexity of the multi-graded resolution

$$MGRes(s,z,y,r^1,p^1,a)$$

we start with the top level multi-graded limit group

$$T_1(s,z,y,r,p,a) = T_1(s,z,y,r^1,p^1,a).$$

$T_1(s,z,y,r^1,p^1,a)$ admits a canonical free decomposition $T_1(s,z,y,r^1,p^1,a) = D^1_0 * D^1_1 * \cdots * D^1_{n(1)} * F$, where F is a free group, and the free decomposition is the most refined free decomposition in which $AP =< z^1_{scbase}, a >< D^1_0$ and each of the subgroups $R^1_d =< z^d_{scbase} >$ can be conjugated into one of the factors D^1_i.

Let Λ_i^1 be the multi-graded decomposition associated with the factor D_i^1 with respect to the subgroups $P^1, R_2^1, \ldots, R_{\mu_1}^1$.

Naturally, there exist canonical maps $\tau_1\colon Rlim_1(y,a) \to T_1(s,z,y,r^1,p^1,a)$ and $\lambda_1\colon Comp(Rlim)_1(z,y,a) \to T_1(s,z,y,r^1,p^1,a)$. The image

$$\lambda_1(Comp(Rlim)_1(z,y,a)) < T_1(s,z,y,r^1,p^1,a)$$

admits a canonical free decomposition

$$\lambda_1(Comp(Rlim)_1(z,y,a)) = L_0^1 * L_1^1 * \cdots * L_{s(1)}^1 * F,$$

which is the most refined free decomposition in which $AP =< z_{scbase}^1, a >< L_0^1$ and each of the subgroups $R_d^1 =< z_{scbase}^d >$ can be conjugated into one of the factors L_i^1.

$Rlim_1(y,a)$ was assumed freely indecomposable, so it admits an abelian JSJ decomposition. Let Q be a quadratically hanging subgroup in the JSJ decomposition of $Rlim_1(y,a)$ and let S_Q be the corresponding (punctured) surface. The boundary elements of Q are mapped by τ_1 to non-trivial elements that can be conjugated to vertex groups in the multi-graded decompositions associated with the various factors $\Lambda_0^1, \ldots, \Lambda_{n(1)}^1$.

THEOREM 2.12: *Let $\zeta(S_Q)$ be a maximal (possibly trivial) collection of non-homotopic, non-boundary parallel s.c.c. on the surface S_Q, so that each curve in the collection $\zeta(S_Q)$ is mapped to the trivial element in $T_1(s,z,y,r^1,p^1,a)$ by the map τ_1. The collection $\zeta(S_Q)$ separates S_Q into (punctured) surfaces $S_1^1, \ldots, S_{m(1)}^1$ with fundamental groups $Q_1^1, \ldots, Q_{m(1)}^1$ with the following properties:*

(i) *The fundamental group of each of the subsurfaces S_c^1 that is connected to a non-QH subgroup in the abelian JSJ decomposition of $Rlim_1(y,a)$ is mapped by τ_1 into a non-trivial subgroup of a conjugate of one of the factors D_j^1 of $T_1(s,z,y,r^1,p^1,a)$ that contains either the subgroup AP^1 or a conjugate of one of the subgroups R_d^1. Similarly, the fundamental group of S_c^1 is mapped by λ_1 into a non-trivial subgroup of a conjugate of one of the factors L_i^1 that contains either AP^1 or a conjugate of one of the subgroups R_d^1.*

(ii) *Let S_c^1 be a subsurface that is not connected to any non-QH vertex group in the abelian JSJ decomposition of $Rlim_1(y,a)$, and let $\hat{S}_c^1$ be the closed surface obtained from S_c^1 by adding disks to its boundary components. Given a rigid or solid homomorphism $h_1\colon T_1(s,z,y,r,p,a) \to F_k * F_{rk(Res(y,a))}$ that is obtained from a taut maximal rank homo-*

morphism with respect to the resolution $Res(y,a)$: h: $Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$ *which is also taut with respect to the multi-graded resolution* $MGRes(s,z,y,r,p,a)$ *by sequentially applying the shortening procedure through the various levels of* $MGRes(s,z,y,r,p,a)$, *the subgroup* Q_c^1 *of* Q *inherits an* h_1*-rank*, $rk_{h_1}(Q_c^1)$, *which is the rank of* $h_1(Q_c^1)$ *in the free group* $F_k * F_{rk(Res(y,a))}$. *Then for all the homomorphisms* h *that are maximal rank and taut with respect to* $Res(y,a)$, *and for which the corresponding homomorphism* $\hat{h}$: $Cl(Res)(s,z,y,a) \to F_k$ *factors and is taut with respect to the multi-graded resolution* $MGRes(s,z,y,r,p,a)$, *the ranks* $rk_{\hat{h}}(Q_c^1)$ *are equal and are the rank of a maximal free quotient of* $\pi_1(S_c^1)$.

(iii) *There exists a graded taut Makanin–Razborov diagram of* $T_1(s,z,y,r^1,p^1,a)$ *with respect to the parameter subgroup* $< z_{base}, a >$ *containing the graded resolutions*

$$GRes_1(s,z,y,z_{base},a), \dots, GRes_\ell(s,z,y,z_{base},a)$$

that satisfy the following. Let c, $1 \leq c \leq m(1)$, *be an index for which the punctured surface* S_c^1 *is not connected to any non-*QH *vertex group in the abelian JSJ decomposition of* $Rlim_1(y,a)$. *Then the terminal rigid or solid limit group* $T_j^1(s,z,y,z_{base},a)$ *of the graded resolution* $GRes_j(s,z,y,z_{base},a)$ *has the following properties:*

(1) *For each index* j, $1 \leq j \leq \ell$, *let* λ_j: $Comp(Rlim)_1(z,y,a) \to T_j^1(s,z,y,z_{base},a)$ *be the canonical map. Then* $\lambda_j(Q_c^1)$ *is contained in a factor in a free decomposition of* $T_j^1(s,z,y,z_{base},a)$ *in which all the subgroups* $P^1, R_2^1, \dots, R_{\mu_1}^1$ *and all the subgroups* $\lambda_j(Q_{c'}^1)$, $c' \neq c$, *can be conjugated into other factors.*

(2) *Let* $S_{c_1}^1, \dots, S_{c_v}^1$ *be all the punctured surfaces that are not connected to any non-*QH *vertex groups in the abelian decomposition obtained from the given abelian decomposition of* $Rlim_1(y,a)$ *after cutting the surfaces corresponding to its* QH *subgroups along their associated collections of s.c.c. Then for each index* j, $1 \leq j \leq \ell$, $T_j^1(s,z,y,z_{base},a)$ *admits the free decomposition*

$$T_j^1(s,z,y,z_{base},a) = M_j * N_1 * \cdots * N_v * \hat{H}$$

where $\hat{H}$ *is a free group,* N_t *is the trivial group if* $\lambda_j(Q_{c_t}^1)$ *is the trivial group, all the subgroups* $P^1, R_2^1, \dots, R_{\mu_1}^1$ *can be conjugated into the factor* M_j, *and each of the subgroups* $\lambda_j(Q_{c_t}^1)$ *can be conjugated into an associated factor* N_t.

(iv) *Let* $\lambda_j\colon Comp(Rlim)_1(z,y,a) \to T_j^1(s,z,y,r^1,p^1,a)$, *and let*

$$\lambda_j(Comp(Rlim)_1(z,y,a)) = L_0^1 * \cdots * L_{s(1)}^1 * F$$

be the maximal (most refined) free decomposition in which $AP^1 < L_0^1$ *and the subgroups* R_d^1 *can be conjugated into the various factors* L_j^1. *Then there exists a maximal (most refined) free decomposition of* $T_j^1(s,z,y,r^1,p^1,a)$ *in which the subgroups* AP^1 *and* R_d^1 *can be conjugated into the various factors, in which up to reordering the factors in the free decomposition* $\lambda_j(Comp(Rlim)_1(z,y,a)) = L_0^1 * \cdots * L_{s(1)}^1 * F$, *each of the factors* L_i^1 *can be conjugated into the* i*-th factor in the free decomposition of* $T_j^1(s,z,y,r^1,p^1,a)$.

Proof: The proof of part (i) is identical with the proof of part (ii) in Theorem 2.6, and the proof of part (ii) is identical to the proof of part (iii) in Theorem 2.6. To prove parts (iii)–(iv), we use a similar construction to the one used in proving parts (iv)–(vi) of Theorem 2.6.

Let $S_{c_1}^1,\dots,S_{c_v}^1$ be the punctured surfaces separated by the collection of curves $\zeta(S_{Q_1}),\dots,\zeta(S_{Q_s})$ that do not contain any boundary components of the ambient surfaces $S_{Q_1},\dots,S_{Q_s}$. Let

$$\lambda_1\colon Comp(Rlim)_1(z,y,a) \to T_1(s,z,y,r,p,a)$$

be the natural map, and let the free decomposition $\lambda(Comp(Rlim)_1(z,y,a)) = L_0^1 * \cdots * L_{s(1)}^1 * F$ be the maximal (most refined) free decomposition of $\lambda_1(Comp(Rlim)_1(z,y,a))$ in which the subgroups AP^1 and R_d^1 can be conjugated into the various factors. Let $L_0^1,\dots,L_{q(1)}^1$ be the factors containing conjugates of at least one of the subgroups $AP^1, R_2^1,\dots,R_{\mu_1}^1$. Then for every homomorphism $h_1\colon T_1(s,z,y,r,p,a) \to F_k * F_{rk(Res(y,a))}$ which is obtained from a taut and maximal rank homomorphism with respect to the resolution $Res(y,a)$, $h\colon Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$,

$$\begin{aligned} h_1(T_1(s,z,y,r,p,a)) &= \hat{h}(Comp(Rlim)_1(z,y,a)) \\ &= F * h_1(Q_{c_1}^1) * \cdots * h_1(Q_{c_v}^1) * h_1(\hat{H}) \end{aligned}$$

where $\hat{H}$ is a free group, and F and $h_1(\hat{H})$ are some free factors of the image free group $F_k * F_{rk(Res)}$.

By composing each taut maximal rank homomorphism

$$h\colon Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$$

with an automorphism of the free group $F_k * F_{rk(Res(y,a))}$, w.l.o.g. we may assume that the factors $F, h_1(Q^1_{c_1}), \ldots, h_1(Q^1_{c_v}), h_1(\hat{H})$ are all fixed factors of the free group $F_k * F_{rk(Res(y,a))}$: $F, B_1, \ldots, B_v, \hat{F}$. Let T be the Bass–Serre tree corresponding to the free decomposition $V = C_1 * \cdots * C_{q(1)} * B_1 * \cdots * B_v * \hat{F}$. Naturally, with every homomorphism $h_1: T_1(s,z,y,r,p,a) \to V$ we can associate an action $\alpha_{h_1}: T_1(s,z,y,r,p,a) \times T \to T$.

Now, the multi-graded limit group $T_1(s,z,y,r,p,a)$ is constructed from a converging sequence of homomorphisms $\{(h_1)_n: T_1(s,z,y,r,p,a) \to V\}$. This sequence of homomorphisms corresponds to a sequence of actions $\{\alpha_{(h_1)_n}\}$ of the rigid or solid limit group $T_1(s,z,y,r,p,a)$ on the Bass–Serre tree T. Since the sequence of homomorphisms $\{(h_1)_n\}$ converges into the (rigid or solid) limit group $T_1(s,z,y,r,p,a)$, a converging sequence of actions $\{\alpha_{(h_1)_{n_m}}\}$ converges in the Gromov–Hausdorff topology into a faithful action of the multi-graded limit group $T_1(s,z,y,r,p,a)$ on some real tree Y.

Since we have assumed that the factors $F, h_1(Q^1_{c_1}), \ldots, h_1(Q^1_{c_v}), \hat{F}$ are all fixed factors of the free group $F_k * F_{rk(Res(y,a))}$, and these fixed free factors fix points in the Bass–Serre tree T, every converging sequence of actions $\{\alpha_{(h_1)_{n_m}}\}$ converges into a faithful action of the multi-graded limit group $T_1(s,z,y,r,p,a)$ on a real tree Y, in which the subgroups $Q^1_{c_1}, \ldots, Q^1_{c_v}$ all fix points in Y.

Since the resolution $Res(y,a)$ is a well-separated resolution, and the homomorphisms $h: Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$ from which the multi-graded resolution $MGRes(s,z,y,r,p,a)$ is constructed are all taut and maximal rank with respect to the resolution $Res(y,a)$, and since by part (i) of Theorem 2.10 the summation of the ranks of the free groups dropped along the levels of the multi-graded resolution $MGRes(s,z,y,r,p,a)$ is equivalent to the rank of the free group dropped in the top level of the resolution $Res(y,a)$, $T_1(s,z,y,r,p,a)$ does not admit a free decomposition in which the subgroup $Comp(Rlim)_1(z,y,a)$ is a subgroup of one of the factors. Hence, if the subgroup $Comp(Rlim)_1(z,y,a)$ fixes a point in the real tree Y, the real tree Y is composed from finitely many orbits of IET and axial components and finitely many orbits of segments with non-trivial (abelian) stabilizers. Therefore, we can successively apply the shortening procedure for the action of $T_1(s,z,y,r,p,a)$ on the real tree Y, and continue the multi-graded resolution $MGRes(s,z,y,r,p,a)$ (canonically) in finitely many ways, until the action of each of the terminal multi-graded limit groups, which we still denote $T_1(s,z,y,r,p,a)$, acts on a discrete tree T', where the stabilizer of each edge in T' is trivial, and the subgroups $Q^1_{c_1}, \ldots, Q^1_{c_v}$ fix points in the discrete tree T'. Hence, by standard Bass–Serre theory, each of the terminal multi-graded limit groups of the multi-graded resolutions $MGRes_j(s,z,y,r,p,a)$, that we denote $T_j^1(s,z,y,r,p,a)$, admits a free decomposition of the form

$$T_j^1(s,z,y,z_{base},a) = M_j * N_1 * \cdots * N_v * \hat{H}$$

where N_t is the trivial group if $\lambda_j(Q_{c_t}^1)$ is the trivial group, and all the subgroups $P^1, R_2^1, \ldots, R_{\mu_1}^1$ can be conjugated into the factor M_j; and we finally get parts (iii) and (iv) of the theorem. ∎

We will continue constructing the multi-graded resolutions of $QRlim(s,z,y,a)$ with each of the resolutions $GRes_j(s,z,y,z_{base},a)$ in parallel, so in the sequel we will denote the multi-graded resolution with which we continue by $GRes(s,z,y,z_{base},a) = MGRes(s,z,y,r^1,p^1,a)$, omitting the index. Similarly, we will denote the terminal rigid or solid limit group of $GRes(s,z,y,z_{base},a)$ by $T^1(s,z,y,z_{base},a)$.

Clearly, there exists a canonical map $\tau_1\colon Rlim_1(y,a) \to T^1(s,z,y,z_{base},a)$. If for some QH subgroup Q in the cyclic decomposition of $Rlim_1(y,a)$, the decomposition $\zeta(S_Q)$ can be refined, we apply Theorem 2.12 starting with the multi-graded limit group $T^1(s,z,y,z_{base},a)$ and the further refined decompositions of the QH subgroups $Q_1,\ldots,Q_s$ in the abelian JSJ decomposition of $Rlim_1(y,a)$. Hence, by repeatedly applying Theorem 2.12 we can assume that

$$T^1(s,z,y,z_{base},a) = T^1(s,z,y,r^1,p^1,a) = D_0^1 * \cdots * D_{n(1)}^1 * F$$

is the most refined free decomposition of $T^1(s,z,y,r^1,p^1,a)$ in which $AP^1 < D_0^1$, $R_2^1,\ldots,R_{\mu_1}^1$ can be conjugated into the various factors D_j^1, and the decompositions $\zeta(S_{Q_i})$ corresponding to the QH subgroups $Q_1,\ldots,Q_s$ in the abelian JSJ decomposition of $Rlim_1(y,a)$ and the map $\tau\colon Rlim_1(y,a) \to T^1(s,z,y,r^1,p^1,a)$ cannot be further refined. Each of the factors D_j^1 admits a multi-graded abelian decomposition which we denote Λ_j.

At this point we can apply the relevant part of the argument used in the minimal rank case, for analyzing the complexity of the graded resolution $MGRes(s,z,y,r^1,p^1,a)$. If the image of a QH subgroup Q_i^1 of some QH subgroup Q_t in the abelian JSJ decomposition of $Rlim_1(y,a)$ is in the factor D_j^1, i.e., $\tau^1(Q_i^1) < D_j^1$, then Q_i^1 inherits a (possibly trivial) decomposition from the multi-graded abelian JSJ decomposition Λ_j of D_j^1. This decomposition corresponds to a decomposition of the surface S_i^1, corresponding to the QH subgroup Q_i^1, along a (possibly trivial) collection of disjoint non-homotopic s.c.c. which we denote $\Gamma_i^1(S_i^1)$. Note that by construction every s.c.c. from the defining collection of $\Gamma_i^1(S_i^1)$ is mapped by τ_1 to either a trivial or an elliptic element in the abelian JSJ decomposition of D_j^1.

LEMMA 2.13: *Let Q' be a quadratically hanging subgroup in the multi-graded abelian JSJ decomposition of a factor D_j^1 of $T^1(s,z,y,r^1,p^1,a)$. Suppose that $\tau_1(Q_i^1)$ can be conjugated into a factor D_j^1, but $\tau_1(Q_i^1)$ cannot be conjugated into the fundamental group of a subgraph of the multi-graded decomposition Λ_j that does not contain the vertex stabilized by Q'. Then some subsurface of S_i^1 is mapped into a finite index subgroup of Q', $genus(S') \le genus(S_i^1)$ and $|\chi(S')| \le |\chi(S_i^1)|$.*

Proof: Let $\hat{S}_i^1$ be the surface obtained from the surface S_i^1 by adding disks to those boundary components of S_i^1 which are mapped to the identity by the map τ_1. Let Q_t be the ambient QH subgroup in the abelian JSJ decomposition of $Rlim_1(y,a)$ that contains Q_i^1. Since by our assumptions the collection of curves $\zeta(S_{Q_t})$ is a maximal collection, no s.c.c. on the surface $\hat{S}_i^1$ is mapped to the identity by the homomorphism τ_1, and by the construction of the decomposition $\Gamma(S_i^1)$ all the boundary elements of the surface $\hat{S}_i^1$ are mapped to non-trivial elliptic elements in the decomposition Λ_j. With these properties of the image of the surface $\hat{S}_i^1$ under the map τ_1, the proof of the lemma is identical with the proof of Lemma 2.7. ∎

If Q' is a QH subgroup that appears along the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$ so that no subsurface of a QH subgroup in the abelian JSJ decomposition of $Rlim_1(y,a)$ is mapped non-trivially into Q', then we can bound the genus and Euler characteristic of the QH subgroup Q' in terms of the genus and Euler characteristic of some QH subgroup in the abelian JSJ decomposition of $Rlim(y,a)$. Clearly, there exists a canonical map τ_0: $Rlim(y,a) \to T^1(s,z,y,r^1,p^1,a)$.

LEMMA 2.14: *Let Q' be a quadratically hanging subgroup in the multi-graded abelian JSJ decomposition of a factor D_j^1 of $T^1(s,z,y,r^1,p^1,a)$ with corresponding surface S', and suppose that no subsurface of a QH subgroup Q in the cyclic JSJ decomposition of $Rlim_1(y,a)$ is mapped non-trivially into a finite index subgroup of a conjugate of Q' by the map τ_1.*

Then there exists some QH subgroup Q in the cyclic JSJ decomposition of $Rlim(y,a)$ with a corresponding surface S_Q such that Q is not a surviving surface, and a subsurface S_i^1 of S_Q such that $\tau_0(\pi_1(S_i^1))$ is a finite index subgroup of Q'. In particular, $genus(S') \le genus(S_i^1)$ and $|\chi(S')| \le |\chi(S_i^1)|$. In addition, either $genus(S') < genus(S_Q)$ or $|\chi(S')| < |\chi(S_Q)|$.

Proof: Let Λ_j be the multi-graded abelian decomposition of the factor D_j^1. By the assumptions of the lemma, Q' is a QH subgroup in the graph of groups

Λ_j. By assumption, no subsurface of any QH subgroup Q in the abelian JSJ decomposition of $Rlim_1(y,a)$ is mapped into a finite index subgroup of Q'. Since the resolution $MGRes$ and the abelian decomposition Λ_j are multi-graded, each of the subgroups $P^1, R^1_2, \ldots, R^1_{\mu_1}$ that is mapped by the homomorphism τ_1 into a conjugate of the factor D^1_j, is mapped into a conjugate of a non-QH, non-abelian vertex group in Λ_j. In addition, by part (iv) of Theorem 2.12, if there is a subgroup $P^1, R^1_2, \ldots, R^1_{\mu_1}$ that is mapped by τ_1 into a conjugate of the factor D^1_j, then all the subgroups $P^1, R^1_2, \ldots, R^1_{\mu_1}$ that are mapped into conjugates of the factor D^1_j are mapped into conjugates of the fundamental group of a connected subgraph of Λ_j that does not contain the vertex stabilized by the QH subgroup Q'.

If none of the subgroups $P^1, R^1_2, \ldots, R^1_{\mu_1}$ is mapped into the factor D^1_j, then by part (iv) of Theorem 2.12, there exists some subgroup Q^1_c, which is the fundamental group of a connected subsurface in the decomposition $\zeta(S_{Q_i})$ corresponding to a QH subgroup Q_i in the cyclic JSJ decomposition of $Rlim_1(y,a)$, such that the surface corresponding to Q^1_c in $\zeta(S_{Q_i})$, $S_{Q^1_c}$, contains no boundary component of S_{Q_i}, and $\tau_1(Q^1_c) < D^1_j$. By Lemma 2.13, the subgroup Q^1_c is mapped into a conjugate of the fundamental group of a connected subgraph of Λ_j that does not contain the vertex stabilized by Q'.

In both cases the multi-graded limit group $T^1(s,z,y,r^1,p^1,a)$ admits a graph of groups Δ, obtained from collapsing a connected subgraph of Λ_j and the given free decomposition of $T^1(s,z,y,r^1,p^1,a)$, such that the QH subgroup Q' is a vertex group in Δ, and the homomorphism τ_1 maps the entire limit group $Rlim_1(y,a)$ into a conjugate of a vertex group in Δ, which is not the vertex stabilized by the QH subgroup Q'. Therefore, by Lemma 2.7 there must exist a subgroup of a non-surviving QH subgroup Q in the cyclic JSJ decomposition of $Rlim(y,a)$ that is mapped by the homomorphism τ_0: $Rlim(y,a) \to T^1(s,z,y,r^1,p^1,a)$ into a finite index subgroup in a conjugate of Q'. In particular, if S_Q is the surface corresponding to the QH subgroup Q, then $genus(S') \leq genus(S_Q)$ and $|\chi(S')| \leq |\chi(S_Q)|$.

The homomorphisms h_1: $T_1(s,z,y,r,p,a) \to F_k * F_{rk(Res(y,a))}$ used to construct the limit group $T^1(s,z,y,r^1,p^1,a)$ are assumed rigid (or solid) with respect to the subgroups $P, R_2, \ldots, R_\mu$. Since Q is not a surviving QH subgroup, if $genus(S') = genus(S_Q)$ and $|\chi(S')| = |\chi(S_Q)|$ then the homomorphisms h_1: $T_1(s,z,y,r,p,a) \to F_k * F_{rk(Res(y,a))}$ used to construct the limit group $T^1(s,z,y,r^1,p^1,a)$ are all flexible multi-graded homomorphisms of the rigid or solid limit group $T_1(s,z,y,r,p,a)$, a contradiction. Hence, $genus(S') <$

$genus(S_Q)$ and $|\chi(S')| < |\chi(S_Q)|$, and the lemma follows. ∎

We continue the construction and the analysis of the next levels of the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$ in the same way we have constructed and analyzed the first level. At each step we first construct a "maximal" multi-graded free decomposition of the multi-graded limit group $MGlim_j(s,z,y,r^1,p^1,a)$ in question by repeated applications of Theorem 2.12 and then apply the (multi-graded) shortening procedure for each of the factors. Clearly, Theorem 2.12 and Lemmas 2.13 and 2.14 remain valid for the next levels of the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$. After finitely many steps we are left with either a rigid or solid multi-graded limit group with respect to the subgroups $P^1, R^1_2, \ldots, R^1_{\mu_1}$. Like in the minimal rank case, to analyze the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$ we need to look at **surviving surfaces**.

Definition 2.15: Let Q be a quadratically hanging subgroup in the JSJ decomposition of $Rlim_1(y,a)$ and let S be its corresponding (punctured) surface. The QH subgroup Q (and the corresponding surface S) is called **surviving** if along the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$ there exists some quadratically hanging subgroup Q' in the multi-graded abelian JSJ decomposition of a factor of one of the multi-graded limit groups $MGlim_j(s,z,y,r^1,p^1,a)$, with corresponding surface S', so that τ_j: $Rlim_1(y,a) \to MGlim_j(s,z,y,r^1,p^1,a)$ maps Q non-trivially into Q', $genus(S') = genus(S)$ and $\chi(S') = \chi(S)$.

By definition, if Q is a non-surviving QH subgroup in the JSJ decomposition of $Rlim_1(y,a)$, then every QH subgroup Q' in any level of the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$ into which a subsurface of Q is mapped non-trivially has either a strictly lower genus or a strictly smaller Euler characteristic than that of the QH subgroup Q. This would eventually "force" the complexity of the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$ to be smaller than that of the resolution $Res(y,a)$ if one is able to "isolate" the surviving surfaces.

Theorem 2.16: *Let $Q_1,\ldots,Q_r$ be the surviving QH subgroups in the JSJ decomposition of $Rlim_1(y,a)$, i.e., those QH vertex groups that are mapped non-trivially into QH subgroups of the same genus and Euler characteristic in some level of the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$. Then in the taut Makanin–Razborov diagram of $QRlim(s,z,y,a)$ associated with those specializations that are taut with respect to the closure $Cl(Res)(s,z,y,a)$, the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$ can be replaced by finitely many multi-graded resolutions, each composed from two consecutive parts. The* *first part is a multi-graded resolution of $QRlim(s,z,y,a)$ with respect to the subgroups $< P^1, R^1_2, \dots, R^1_{\mu_1}, Q_1, \dots, Q_r >$, which we denote*

$$MGRes(s,z,y,(r^1,p^1,Q_1,\dots,Q_r),a).$$

The second part is a one-step resolution that maps the rigid (solid) terminal multi-graded limit group of $MGRes(s,z,y,(r^1,p^1,Q_1,\dots,Q_r),a)$ to the rigid (solid) terminal multi-graded limit group of the resolution $MGRes(s,z,y,r^1,p^1,a)$. The two consecutive parts of the multi-graded resolution have the following properties:

(1) *The multi-graded decomposition corresponding to the second part of the resolution contains vertices stabilized by the terminal rigid (solid) multi-graded limit group of the resolution $MGRes(s,z,y,r^1,p^1,a)$ connected to r' surviving QH subgroups $Q_{i_1},\dots,Q_{i_{r'}}$ for some $r' \leq r$ and $1 \leq i_1 < \dots < i_{r'} \leq r$.*

(2) *If the terminal multi-graded limit group of the multi-graded resolution*

$$MGRes(s,z,y,(r^1,p^1,Q_1,\dots,Q_r),a)$$

is rigid (solid), so is the terminal multi-graded limit group of the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$. Furthermore, if they are both solid, their abelian multi-graded decompositions are in correspondence, i.e., the decomposition differs only in the stabilizer of the vertices stabilized by the subgroups P and R^1_d's. These vertices are stabilized by the subgroups P^1 and R^1_d's and the subgroups $Q_{i_1},\dots,Q_{i_{r'}}$ in the first part, and by the subgroups P^1 and R^1_d's in the second.

Proof: Identical to the proof of Theorem 1.7. ∎

Since the resolution $Res(y,a)$ is well-separated, the fundamental groups of the various connected components of the graph of groups Θ^1_1: $W^1_1,\dots,W^1_{\mu_1}$ are mapped to different factors in the given free factorization of the third restricted limit group along the resolution $Res(y,a)$, $Rlim_2(y,a)$. In accordance with this free decomposition, the completed limit group $Comp(Rlim)_2(z,y,a)$ admits a free decomposition $Comp(Rlim)_2 = P^1 * R^1_2 * \dots * R^1_{\mu_1}$. Furthermore, for every taut maximal rank homomorphism h: $Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$, which is taut with respect to the resolution $Res(y,a)$, and its associated homomorphism $\hat{h}$: $Comp(Rlim)(z,y,a) \to F_k * F_{rk(Res(y,a))}$, we have

$$\begin{aligned} h(Rlim(y,a)) = \hat{h}(Comp(Rlim)(z,y,a)) &= \hat{h}(Comp(Rlim)_1(z,y,a)) * \hat{h}(H^1) \\ &= \hat{h}(Comp(Rlim)_2(z,y,a)) * \hat{h}(H^2) \end{aligned}$$

where H^1 is the free group freely generated by free groups "dropped" in the top level and H^2 is the free group freely generated by free groups "dropped" in the first and second levels of the resolution $Res(y,a)$ from the circumference of the various QH vertex groups and from abelian factors that appear in these levels.

Let $T_2(s,z,y,r^1,p^1,a)$ be the terminal rigid or solid multi-graded limit group of the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$. To continue the analysis of the resolutions associated with the quotient limit group $QRlim(s,z,y,a)$, we need a "linkage" between the free decomposition of the completed limit group $Comp(Rlim)_2(z,y,a)$ and the free decomposition of $T_2(s,z,y,r^1,p^1,a)$, a correspondence between (the ranks of) the free groups "dropped" along the second level of the resolution $Res(y,a)$ and along the various levels of the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$, and a correspondence between the image $\hat{h}(Comp(Rlim)_2(z,y,a))$ and the image $h_2(T_2(s,z,y,r^1,p^1,a))$, where h_2: $T_2(s,z,y,r^1,p^1,a) \to F_k * F_{rk(Res(y,a))}$ is a rigid or solid shortening of a maximal rank taut homomorphism h: $Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$ which is taut with respect to both the closure $Cl(Res)(s,z,y,a)$ and the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$.

Theorem 2.17: *Let $T_2(s,z,y,r^1,p^1,a)$ be the solid or rigid terminal limit group of the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$, let h: $Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$ be a homomorphism which is taut with respect to the resolution $Res(y,a)$, let $\hat{h}$: $Comp(Rlim)(z,y,a) \to F_k * F_{rk(Res(y,a))}$ be the corresponding homomorphism of the completed limit group, and suppose the homomorphism $\hat{h}$ factors through the closure $Cl(Res)(s,z,y,a)$ and is taut with respect to the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$. Let h_2: $T_2(s,z,y,r^1,p^1,a) \to F_k * F_{rk(Res(y,a))}$ be a rigid or solid homomorphism obtained from the homomorphism $\hat{h}$ by the shortening procedure applied along the levels of the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$. Then the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$ can be extended in finitely many ways to give finitely many multi-graded resolutions with respect to the subgroups $P^1, R^1_2, \ldots, R^1_{\mu_1}$, which we still denote $MGRes(s,z,y,r^1,p^1,a)$, so that the terminal rigid or solid multi-graded limit group $T_2(s,z,y,r^1,p^1,a)$ of each of these multi-graded resolutions, which we still denote $T_2(s,z,y,r^1,p^1,a)$, has the following properties:*

(i) *The summation of the ranks of the free groups dropped along the various levels of the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$ is identical to the summations of the ranks of the free groups dropped along the top two levels of the resolution $Res(y,a)$.*

(ii) *By our assumptions, the completed limit group*

$$Comp(Rlim)_2(z,y,a)$$

admits the free decomposition $Comp(Rlim)_2(z,y,a) = P^1 * R_2^1 * \cdots * R_{\mu_1}^1$. *Then the terminal solid or rigid multi-graded limit group*

$$T_2(s,z,y,r^1,p^1,a)$$

admits a free decomposition $T_2(s,z,y,r^1,p^1,a) = N_1 * \cdots * N_{\mu_1}$ *where* $P < N_1$ *and, for every* $d \geq 2$, R_d *can be conjugated into* N_d.

(iii) $h_2(T_2(s,z,y,r^1,p^1,a)) = \hat{h}(Comp(Rlim)_2(z,y,a))$ *and with the notation of part (ii),* $h_2(N_1) = \hat{h}(P^1)$, *and for every* $d \geq 2$, $h_2(N_d) = \hat{h}(R_d^1)$.

(iv) *The limit group* $Rlim_1(y,a)$ *admits a natural map into each multi-graded limit group along the multi-graded resolution* $MGRes(s,z,y,r^1,p^1,a)$. *Given the map* τ_1: $Rlim_1(y,a) \to T_2(s,z,y,r^1,p^1,a)$, *the image of every* QH *subgroup* Q *in the second level of the resolution* $Res(y,a)$ *under the map* τ_1 *admits a (possibly trivial) free decomposition from the free decomposition* $T_2(s,z,y,r^1,p^1,a) = N_1 * \cdots * N_{\mu_1}$. *Let* S *be the (punctured) surface corresponding to* Q, *and let* $\Gamma(S)$ *be the decomposition of the surface* S *corresponding to the free decomposition of* $\tau_1(Q)$. *Let* $S_1,\dots,S_m$ *be the punctured subsurfaces of* S *in the decomposition* $\Gamma(S)$ *that contain a boundary component of* S, *let* $\hat{S} = S \setminus (S_1 \cup \cdots \cup S_m)$, *and let* S' *be the closed surface obtained from* $\hat{S}$ *by adding disks to its boundary components. Then the rank* $\hat{h}(\hat{S})$ *is identical for all the homomorphisms* $\hat{h}$ *that factor through the multi-graded resolution* $MGRes(s,z,y,p^1,r^1,a)$.

Proof: Identical to the proof of Theorem 2.10. ∎

We continue the analysis of the limit group $QRlim(s,z,y,a)$ by successively reducing the subgroups that are used for the construction of the multi-graded resolutions sequentially. Recall that in order to obtain the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$, we first used the subgroups $P, R_2, \dots, R_\mu$ for constructing a multi-graded resolution, where

$$Comp(Rlim)_1(z,y,a) = P * R_2 * \cdots * R_\mu$$

and for the connected component W_d of the graph of groups Θ_1^0, $\eta_0(W_1) = P$, and for $d \geq 2$, $\eta_0(W_d) = R_d$. Then we used the subgroups $P^1, R_2^1, \dots, R_{\mu_1}^1$ for constructing a multi-graded resolution, where

$$Comp(Rlim)_2(z,y,a) = P^1 * R_2^1 * \cdots * R_{\mu_1}^1$$

and for the connected component W_d^1 of the graph of groups Θ_1^1, $\eta_1(W_1^1) = P^1$, and for $d \geq 2$, $\eta_1(W_d^1) = R_d^1$. To continue the analysis of the resolutions limit group $QRlim(s,z,y,a)$, we set the subgroups $P^{\ell-1}, R_1^{\ell-1}, \ldots, R_{\mu_{\ell-1}}^{\ell-1}$ to be the factors in the free decomposition

$$Comp(Rlim)_\ell(z,y,a) = P^{\ell-1} * R_1^{\ell-1} * \cdots * R_{\mu_{\ell-1}}^{\ell-1},$$

where for the connected components $W_d^{\ell-1}$ of the graphs of groups $\Theta_j^{\ell-1}$ associated with $Rlim_{\ell-1}(y,a)$, $\eta_{\ell-1}(W_1^{\ell-1}) = P^{\ell-1}$, and for $d \geq 2$, $\eta_{\ell-1}(W_d^{\ell-1}) = R_d^{\ell-1}$. The subgroups $P^{\ell-1}, R_1^{\ell-1}, \ldots, R_{\mu_{\ell-1}}^{\ell-1}$ are used in the ℓ-th part of our sequential construction of multi-graded resolutions.

Let $T_2(s,z,y,r^1,p^1,a)$ be the terminal rigid or solid graded limit group in the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$, which is multi-graded with respect to the subgroups $P^1, R_1^1, \ldots, R_{\mu_1}^1$. We continue the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$ by viewing $T_2(s,z,y,r^1,p^1,a)$ as a multi-graded limit group with respect to the subgroups $P^2, R_1^2, \ldots, R_{\mu_2}^2$ and continue with the procedure used to construct the second part of the multi-graded resolution $MGRes(s,z,y,r^1,p^1,a)$ to obtain a multi-graded resolution $MGRes(s,z,y,r^2,p^2,a)$. We further continue with the rigid or solid terminal limit group $T_3(s,z,y,r^2,p^2,a)$ by viewing it as a multi-graded limit group with respect to the subgroups $P^3, R_1^3, \ldots, R_{\mu_3}^3$, and so on, until we exclude all the z variables appearing in the different levels of the completed resolution $Comp(Res)(z,y,a)$. Note that the final resolution we obtain is an ungraded resolution of the limit group $QRlim(s,z,y,a)$, or of some quotient of it. Also, note that the procedure described produces (canonically) finitely many such resolutions of the quotient limit group $QRlim(s,z,y,a)$. We will denote an (ungraded) resolution obtained by our procedure $QRes(s,z,y,a)$, and call it a **quotient resolution** of the quotient limit group $QRlim(s,z,y,a)$.

PROPOSITION 2.18: *Each of the obtained quotient resolutions, $QRes(s,z,y,a)$, is a well-separated resolution. Furthermore, if h: $Rlim(y,a) \to F_k * F_{rk(Res(y,a))}$ is a taut maximal rank homomorphism with respect to the resolution $Res(y,a)$, and its corresponding (completed) homomorphism, $\hat{h}$: $Comp(Rlim)(z,y,a) \to F_k * F_{rk(Res(y,a))}$, factors through the closure $Cl(Res)(s,z,y,a)$, then $\hat{h}$ is a taut maximal rank homomorphism with respect to at least one of the well-separated (quotient) resolutions $QRes(s,z,y,a)$ constructed by our procedure.*

Proof: By construction, a quotient resolution $QRes(s,z,y,a)$ is a strict resolution. Theorems 2.6, 2.12 and 2.17 guarantee that it is also a well-separated resolution. The quotient resolutions $QRes(s,z,y,a)$ were constructed from the entire collection of (completed) taut maximal rank homomorphisms $\hat{h}$: $Comp(Rlim)(z,y,a) \to F_k * F_{rk(Res(y,a))}$ that factor through the closure $Cl(Res)(s,z,y,a)$ and the quotient limit group $QRlim(s,z,y,a)$, hence every such (completed) taut maximal rank homomorphism $\hat{h}$ is a taut maximal rank homomorphism with respect to at least one of the quotient resolutions $QRes(s,z,y,a)$. ∎

To analyze taut maximal rank homomorphisms, we define the complexity of a resolution in the same way we have defined it in the minimal rank case (Definition 1.14). In order to ensure the termination of the iterative procedure presented in the next sections, we need the complexities of the resolutions obtained in its different steps to decrease. Indeed, the complexity of the (well-separated) resolutions $QRes(s,z,y,a)$ obtained through our procedure for analyzing taut maximal rank homomorphisms is strictly smaller than the complexity of the resolution $Res(y,a)$ we started with.

THEOREM 2.19: *The complexity of a quotient resolution $QRes(s,z,y,a)$ is strictly smaller than the complexity of the (original) resolution $Res(y,a)$, i.e.,*

$$Cmplx(QRes(s,z,y,a)) < Cmplx(Res(y,a)).$$

Proof: Let Q' be a QH subgroup that appears in an abelian decomposition associated with one of the levels of the quotient resolution $QRes(s,z,y,a)$, and let S' be its associated punctured surface. By Lemmas 2.7, 2.13 and 2.14, there exists some QH subgroup Q with associated punctured surface S that appears in an abelian JSJ decomposition associated with one of the levels of the original resolution $Res(y,a)$, so that the fundamental group of a subsurface of S is mapped into a finite index subgroup of S' and, in particular, $genus(S') \leq genus(S)$ and $|\chi(S')| \leq |\chi(S)|$. Furthermore, if $genus(S') = genus(S)$ and $|\chi(S')| = |\chi(S)|$ then Q is by definition a surviving surface, so by Theorems 2.9 and 2.16 the QH subgroup Q is mapped isomorphically onto Q', Q is mapped into a non-abelian, non-QH vertex group in all levels above the one in which Q' appears, and in all the levels below the one containing Q', Q is identified with its image in the completed resolution $Comp(Res)(z,y,a)$.

Hence, for any surviving QH subgroup Q with associated punctured surface S, there corresponds a unique QH subgroup Q' in the resolution $QRes(s,z,y,a)$ with associated punctured surface S', so that S is homeomorphic to S'. If Q' is a QH subgroup in the quotient resolution $QRes(s,z,y,a)$ with an associated punctured surface S', and no surviving surface Q is mapped onto a conjugate of

Q', then for any QH subgroup Q with an associated surface S in $Res(y,a)$, for which the fundamental group of some subsurface of S is mapped into a finite index subgroup of a conjugate of Q', $genus(S') \leq genus(S)$, $|\chi(S')| \leq |\chi(S)|$ and either $genus(S') < genus(S)$ or $|\chi(S')| < |\chi(S)|$. Therefore, if $Q_1, \ldots, Q_m$ are the QH subgroups that appear in the original resolution $Res(y,a)$ and $S_1, \ldots, S_m$ are their associated punctured surfaces, and $Q'_1, \ldots, Q'_{m'}$ are the QH subgroups that appear in the quotient resolution $QRes(s,z,y,a)$ and $S'_1, \ldots, S'_{m'}$ are their associated punctured surfaces, then the tuple

$$((genus(S_1), |\chi(S_1)|), \ldots, (genus(S_m), |\chi(S_m|))$$

is greater than or equal in the lexicographical order to the tuple

$$((genus(S'_1), |\chi(S'_1)|), \ldots, (genus(S'_{m'}), |\chi(S'_{m'}|))$$

and the two tuples are equal if and only if all the QH subgroups $Q_1, \ldots, Q_m$ are surviving QH subgroups.

Since the quotient limit group $QRlim(s,z,y,a)$ is a proper quotient of the closure $Cl(Res)(s,z,y,a)$, if all the QH subgroups $Q_1, \ldots, Q_m$ are surviving, then necessarily $Abrk(QRes(s,z,y,a)) < Abrk(Res(y,a))$, and we finally get that $Cmplx(QRes(s,z,y,a)) < Cmplx(Res(y,a))$. ∎

3. Induced resolutions

In the first section we used an iterative procedure for the validation of a sentence, which assumes that all the limit groups containing the set of the "remaining" y's are of minimal rank. In the second section we modified the procedure to analyze taut maximal rank homomorphisms. In order to construct an iterative procedure for validation of a sentence in the general case, i.e., when the limit groups containing the set of the "remaining" y's are not necessarily of minimal rank, we will need several additional tools and notions. In this section we present the needed tools, and in the next one we use these tools to construct a procedure for validation of a sentence in the general case.

We start with the definitions of a **geometric subresolution** of a resolution $Res(t,y,a)$ and its complexity.

Definition 3.1: Let $Res(t,y,a)$ be a a well-separated resolution and let $Comp(Res)(u,t,y,a)$ be its completion. Let $GSRes(g,y,a)$ be a resolution with the following properties:

(i) The resolution $GSRes(g,y,a)$ is a well-separated resolution.

(ii) The resolution $GSRes(g,y,a)$ is a completed resolution, i.e., the completion of $GSRes(g,y,a)$ is the resolution $GSRes(g,y,a)$ itself.

(iii) There exists a (geometric) embedding

$$\nu \colon GSRes(g,y,a) \to Comp(Res)(u,t,y,a)$$

that maps the subgroup $< y,a >$ of the resolution $GSRes(g,y,a)$ onto the subgroup $< y,a >$ of the completed resolution $Comp(Res)(u,t,y,a)$ elementwise. In addition, the embedding ν has the following properties:

(1) Every QH subgroup in an abelian decomposition associated with one of the various levels of the resolution $GSRes(g,y,a)$ is embedded geometrically into (a finite index subgroup of) a QH subgroup of $Comp(Res)(u,t,y,a)$.

(2) Every abelian vertex group in an abelian decomposition associated with one of the levels of the resolution $GSRes(g,y,a)$ is embedded into an abelian vertex group in one of the abelian decompositions associated with $Comp(Res)(u,t,y,a)$.

(3) Except for the terminal free groups dropped along the various levels of the geometric subresolution, the free and abelian decompositions associated with each level of the resolution $GSRes(g,y,a)$ are the decompositions induced (using Bass–Serre theory) from the embedding ν and the free and abelian decompositions of $Comp(Res)(u,t,y,a)$. Furthermore, the canonical maps between successive levels in the resolution $GSRes(g,y,a)$ are the ones induced from the canonical maps between successive levels in the completion $Comp(Res)(u,t,y,a)$.

We call the resolution $GSRes(g,y,a)$ together with the embedding ν: $GSRes(g,y,a) \to Comp(Res)(u,t,y,a)$ a **geometric subresolution** of the (completion of the) resolution $Res(t,y,a)$. The modular groups associated with a geometric subresolution are set to be the modular groups induced from those of the completed resolution $Comp(Res)(u,t,y,a)$. In particular, the modular groups associated with each of the QH vertex groups in the abelian decompositions of $GSRes(g,y,a)$ are set to be the finite index subgroups of the modular groups of the corresponding QH vertex groups of $Comp(Res)(u,t,y,a)$ into which they are embedded, that preserve the image of the embedded QH subgroup of $GRes(g,y,a)$. Note that the completion, $Comp(Res)(u,t,y,a)$, itself is a geometric subresolution of the (completion of the) resolution $Res(t,y,a)$.

To analyze resolutions in the general case, we need to measure the complexity of geometric subresolutions of their completion. To get such a complexity measure, we modify our previous notion of complexity of a resolution (Definition 1.14) so that the modification takes into account the reduced modular groups associated with the various QH subgroups.

Definition 3.2: Let $Res(t,a)$ be a well-separated completed resolution with (possibly) reduced modular groups associated with each of its various QH subgroups. Let $Q_1,\ldots,Q_m$ be the QH subgroups that appear in the completion $Comp(Res)(t,y,a)$ and let $S_1,\ldots,S_m$ be the (punctured) surfaces associated with the reduced modular group associated with each of the QH vertex group. To each (punctured) surface S_j we may associate an ordered couple $(genus(S_j),|\chi(S_j)|)$. We will assume that the QH subgroups $Q_1,\ldots,Q_m$ are ordered according to the lexicographical (decreasing) order of the ordered couples associated with their corresponding surfaces. Let $rk(Res(t,a))$ be the rank of the resolution $Res(t,a)$ (Definition 2.1), and let $Abrk(Res(t,a))$ be the abelian rank of the resolution (see Definition 1.14).

We set the complexity of the resolution $Res(t,a)$, denoted $Cmplx(Res(t,a))$, to be

$$\begin{aligned} Cmplx(Res(t,a)) = (rk(Res(t,a)),\ &(genus(S_1),|\chi(S_1)|),\ldots \\ &\ldots,(genus(S_m),|\chi(S_m)|),\ Abrk(Res(t,a))). \end{aligned}$$

On the set of completed resolutions with (possibly) reduced modular groups we can define a linear order. Let $Res_1(t_1,a)$ and $Res_2(t_2,a)$ be two completed resolutions with (possibly) reduced modular groups. We say that $Cmplx(Res_1(t_1,a))$ $= Cmplx(Res_2(t_2,a))$ if the tuples defining the two complexities are identical. We say that $Cmplx(Res_1)(t_1,a)) < Cmplx(Res_2(t_2,a))$ if:

(1) The rank $rk(Res_1(t_1,a))$ is smaller than the rank $rk(Res_2(t_2,a))$.
(2) The above ranks are equal and the tuple

$$((genus(S_1^1),|\chi(S_1^1)|),\ldots,(genus(S_{m_1}^1),|\chi(S_{m_1}^1|))$$

is smaller in the lexicographical order than the tuple

$$((genus(S_1^2),|\chi(S_1^2)|),\ldots,(genus(S_{m_2}^2),|\chi(S_{m_2}^2|)).$$

(3) The above ranks and tuples are equal and

$$Abrk(Res_1(t_1,a)) < Abrk(Res_2(t_2,a)).$$

Let $Comp(Res)(t,v,a)$ be a completed well-separated resolution, and let $< v,a >< Rlim(t,v,a)$ be a subgroup of the limit group $Rlim(t,v,a)$. With each level of the completed resolution $Comp(Res)(t,v,a)$ there is associated a (possibly trivial) free decomposition and abelian decompositions of each of the factors. We denote the decompositions associated with the various levels of the completed resolution $Comp(Res)(t,v,a)$: $\Lambda_1,\dots,\Lambda_q$, and the canonical epimorphisms between consecutive levels we denote $\eta_1,\dots,\eta_{q-1}$. Furthermore, each vertex group in these decompositions which is neither quadratically hanging nor abelian is embedded in the next level by the canonical epimorphisms between consecutive levels.

From the completed resolution $Comp(Res)(t,v,a)$ we construct the **induced resolution**, $Ind(Comp(Res))(u,v,a)$, of the limit group $<v,a> \, < Rlim(t,v,a)$, iteratively. We start by describing the first step in the iterative construction.

(i) Using standard Bass–Serre theory, the subgroup $<v,a> < Rlim(t,v,a)$ inherits a decomposition Δ_1 with abelian and trivial edge groups from the decomposition Λ_1. Note that if $<v,a>$ intersects a conjugate of a QH vertex group in the decomposition Λ_1 in a subgroup of finite index, then the intersection appears as a QH vertex group in the inherited decomposition Δ_1. If $<v,a>$ intersects a conjugate of a QH vertex group in the decomposition Λ_1 in a non-trivial, non-boundary subgroup of infinite index, then by Lemma 1.4 the intersection gives rise to a free factorization (and a possible free factor) in the decomposition Δ_1 of the group $<v,a>$.

(ii) Suppose that the free decomposition inherited by the subgroup $<v,a>$ from the decomposition Δ_1 is

$$<v,a> = F_1 * <v_1,a> * <v_2> * \cdots * <v_b>$$

where F_1 is a free group which is the free product of free factors contributed by subgroups of infinite index in QH vertex groups in Λ_1 and Bass–Serre generators of loops with trivial stabilizer in Δ_1. $<v_1,a>$ is the connected component that contains the vertex stabilized by $F_k = <a_1,\dots,a_k>$ itself.

We continue with each of the factors $<v_1,a>, <v_2>, \dots, <v_b>$ separately. We will denote each of these factors by V_i. Each factor V_i inherits an abelian splitting Δ_1^i from the decomposition Λ_1. Each edge e in Δ_1^i that connects two non-QH, non-abelian vertex groups is composed from a couple of edges e_1 and e_2 that are adjacent and are both in the orbit of the same edge e' in the Bass–Serre tree corresponding to the graph of groups Λ_1 of $Comp(Res)(t,v,a)$. Furthermore, e' connects a non-abelian vertex group to an abelian vertex group in the decomposition Λ_1.

Let A be the abelian vertex group that stabilizes the common vertex v of e_1 and e_2 in the Bass–Serre tree corresponding to the decomposition Λ_1. There exists a (unique) element $a \in A$ that conjugates the vertex adjacent to v in e_2 to the vertex adjacent to v in e_1, and for which $\eta_1(a) = 1$. We modify the factor V_i by adding to its generators the element a. Note that the image of the obtained group in the next level of the completed resolution is not changed, i.e., $\eta_1(V_i) = \eta_1(< V_i, a >)$. We act in the same way on the factor V_i, in case an abelian vertex group in Δ_1^i is connected to two non-QH, non-abelian vertex groups, which are necessarily in the same orbit of a non-QH, non-abelian vertex group in the graph of groups Λ_1. Repeating this operation for all the edges connecting two non-QH, non-abelian vertex groups in the decomposition Δ_1^i of the factor V_i, and for all couples of edges connecting an abelian vertex group to two non-QH, non-abelian vertex groups in Δ_1^i, we get a subgroup $\hat{V}_i < Comp(Res)(t, v, a)$, so that $V_i < \hat{V}_i$, and in the decomposition $\hat{\Delta}_1^i$ inherited by $\hat{V}_i$ from the decomposition Λ_1, which is a "folding" of the decomposition Δ_1^i of V_i, a non-QH, non-abelian vertex group is connected only to QH vertex groups and to abelian vertex groups and not to any other non-QH, non-abelian vertex groups. Furthermore, an abelian vertex group in $\hat{\Delta}_1^i$ is connected to at most one non-QH, non-abelian vertex group as it is in the decomposition Λ_1.

(iii) As we did with a general well-separated resolution in Definition 2.2, we associate a decomposition $\hat{\Theta}_1^i$ with the decomposition $\hat{\Delta}_1^i$, by cutting each of the punctured surfaces that correspond to QH vertex groups in $\hat{\Delta}_1^i$ along the collection of disjoint, non-homotopic (non-boundary parallel) s.c.c. which are the pre-images of the s.c.c. along which each of the QH vertex groups in the ambient resolution $Comp(Res)(t, v, a)$ is cut. As we did in completing a general well-structured resolution ([Se2], definition 1.12), we modify the factor $\hat{V}_i$ by adding Bass–Serre elements so that each connected punctured subsurface in the decomposition $\hat{\Theta}_1^i$ that is connected to a non-QH vertex group in $\hat{\Theta}_1^i$ will be connected to a unique non-QH vertex group, and get a subgroup $\tilde{V}_i$ with corresponding graphs of groups $\tilde{\Delta}_1^i$ and $\tilde{\Theta}_1^i$.

As in the construction of the completed resolution of a well-separated resolution, every connected component in the decomposition $\tilde{\Theta}_1^i$ that contains a non-QH vertex group contains a unique non-QH, non-abelian vertex group, and (possibly) few abelian vertex groups all connected (only) to the (unique) non-QH, non-abelian vertex group in their connected component. All the QH vertex groups in a connected component in $\tilde{\Theta}_1^i$ are also connected only to the (unique) non-QH, non-abelian vertex group in that connected component.

Since $Comp(Res)(t,v,a)$ is a completed and well-separated resolution, all the (conjugating) Bass–Serre elements we have added to the factor $\hat{V}_i$ are naturally mapped to elements of the completed resolution $Rlim(t,v,a)$, and the subgroup $\tilde{V}_i$ obtained after adding the Bass–Serre generators is naturally mapped into the completed limit group $Rlim(t,v,a)$ (note that it is not necessarily embedded). Since (the image in $Rlim(t,v,a)$ of) the conjugating (Bass–Serre) elements that we added are mapped to the identity element by the map η_1, the addition of these elements does not change the image of the map η_1, i.e., $\eta_1(\hat{V}_i)=\eta_1(\tilde{V}_i)$.

(iv) With each factor $\tilde{V}_i$ and each connected component in the decomposition $\tilde{\Theta}_1^i$ that contains a non-QH vertex group, we associate a subgroup J_s^i and continue to the second level of the completed resolution $Comp(Res)(t,v,a)$ with each of the subgroups J_s^i separately.

For each index i, and each connected component (indexed by s) in $\tilde{\Theta}_1^i$, we set J_s^i to be the image under the canonical epimorphism η_1 of the fundamental group of the corresponding connected component. Note that the Bass–Serre elements, and the elements from abelian vertex groups that were added to the various factors V_i, do not change its image under the epimorphism η_1.

(v) As we did in part (i), for each of the subgroups J_s^i we use standard Bass–Serre theory to get a decomposition $\Delta_2^{(i,s)}$, inherited by the subgroup J_s^i from the decomposition Λ_2 associated with the second level of the completed resolution $Comp(Res)(t,v,a)$. From the decomposition $\Delta_2^{(i,s)}$, the subgroup J_s^i inherits a free decomposition and an abelian decomposition of each of the factors. We add elements for each edge connecting two non-QH, non-abelian vertex groups in the abelian decompositions of the different factors as we did in part (ii), and conjugating elements corresponding to boundary components of QH vertex groups in one of the abelian decompositions of the different factors using the corresponding decomposition $\Theta_2^{(i,s)}$ associated with $\Delta_2^{(i,s)}$ as we did in part (iii). With each factor of J_s^i we associate finitely many subgroups $M_{(i,s,j)}$ corresponding to the different connected components in the decomposition $\Theta_2^{(i,s)}$ exactly in the same way we associated the subgroups J_s^i with the factors V_i

in part (iv). We continue to the third level of the completed resolution $Comp(Res)(t,v,a)$ with the subgroups $M_{(i,s,j)}$ associated with each of the connected components of the various graphs $\Theta_2^{(i,s)}$ separately, analyze the decompositions inherited by each of the subgroups $M_{(i,s,j)}$ from the the abelian decomposition Λ_3 associated with the third level of the completed resolution $Comp(Res)(t,v,a)$, add additional elements according to parts (ii) and (iii), and subsequently continue to the next levels of the completed resolution $Comp(Res)(t,v,a)$.

The first step in our iterative procedure for constructing the induced resolution constructs a resolution $Res(u,v,a)$ of the subgroup $<v,a>< Rlim(t,v,a)$ by going through the levels of the completed resolution $Comp(Res)(t,v,a)$ and applying steps (i)–(v) above.

To start the second step in our iterative procedure for constructing the induced resolution, we set $IRes_1(u,v,a)$ to be the subgroup of the completed resolution $Comp(Res)(t,v,a)$ generated by the images of the different factors $\tilde{V}_i$ (in the completed resolution $Comp(Res)(u,v,a)$), and their images in the lower levels of the completed resolution $Comp(Res)(t,v,a)$ obtained by steps (i)–(v) above. Having defined the subgroup $IRes_1(u,v,a) < Rlim(t,v,a)$, we start going through the levels of the completed resolution $Comp(Res)(t,v,a)$ starting with the subgroup $IRes_1(u,v,a)$ instead of the subgroup $<v,a>$ we started with in the first step.

$IRes_1(u,v,a)$, being a subgroup of $Rlim(t,v,a)$, inherits a decomposition from the abelian decomposition Λ_1 associated with the first level of the completed resolution $Comp(Res)(t,v,a)$. Let this decomposition be $I\Delta_1$. Since the image of the subgroup $\tilde{V}_i$ in $Comp(Res)(t,v,a)$, and the subgroup $IRes_1(u,v,a)$ differ only in the stabilizers of the unique non-abelian, non-QH vertex group in each connected component of the various decompositions $\tilde{\Theta}_1^i$ associated with the decompositions $\tilde{\Delta}_1^i$ of the factors $\tilde{V}_i$, if the graph of groups $I\Delta_1$ is not combinatorially similar to the graph of groups $\tilde{\Delta}_1$, i.e., if the free decompositions, and the QH and abelian vertex groups that appear in the abelian decomposition inherited by $IRes_1(u,v,a)$ are not identical to those that appear in the abelian decompositions used to construct the subgroups $\tilde{V}_i$, then one of the following occurs:

(1) In the free decomposition $I\Delta_1$, inherited by the subgroup $IRes_1(u,v,a)$ from the graph of groups Λ_1, either the number of factors is dropping, or the rank of the free factor corresponding to Bass–Serre generators of loops with trivial edge stabilizers and free factors contributed by infinite

index subgroups of QH vertex groups in Θ_1 is dropping. In this case each of the factors V_i of $< v, a >$, and the image of the subgroups $\tilde{V}_i$ (in $Comp(Res)(t, v, a)$) associated with it, are subgroups of factors in the free decomposition of $IRes_1(u, v, a)$ inherited from Λ_1.

(2) The number of factors and the rank of the free group corresponding to Bass–Serre generators of loops with trivial edge stabilizers and free factors contributed by infinite index subgroups of QH vertex groups in the free decomposition inherited by $IRes_1(u, v, a)$ from Λ_1 stays identical to their values in the free decomposition inherited by the subgroup $< v, a >$ from the decomposition $\tilde{\Delta}_1$. In this case each of the factors V_i of $< v, a >$ is a subgroup of a unique factor in the free decomposition of $I\Delta_1$ inherited by $IRes_1(u, v, a)$ from Λ_1.

    The combinatorics of the graph of groups $I\Delta_1$ is strictly smaller than the combinatorics of the graphs of groups $\tilde{\Delta}_1^i$ in correspondence, i.e., for at least one of the factors in the decomposition $I\Delta_1$, the combinatorics of its corresponding graph of groups is strictly smaller than the combinatorics of the corresponding graph of groups in $\tilde{\Delta}_1^i$, i.e., either the number of edges and vertices is smaller or the genus or the (absolute value of the) Euler characteristic of some of the QH vertex groups is smaller.

Applying steps (i)–(v) to the subgroup $IRes_1(u, v, a)$, we set $IRes_2(u, v, a)$ to be the subgroup of the completed limit group $Rlim(t, v, a)$ generated by the different factors of the subgroup $IRes_1(u, v, a)$ and their images in the lower levels of the completed resolution $Comp(Res)(t, v, a)$ obtained in the second step of the iterative procedure, i.e., obtained by performing steps (i)–(v) above starting with the subgroup $IRes_1(u, v, a) < Rlim(t, v, a)$. Having defined the subgroup $IRes_2(u, v, a) < Rlim(t, v, a)$, we say that the procedure for the construction of the induced resolution terminates if $IRes_2(u, v, a) = IRes_1(u, v, a)$. Otherwise, we perform the third step of the iterative procedure for constructing the induced resolution by going through the levels of the completed resolution $Comp(Res)(t, v, a)$ starting with the subgroup $IRes_2(u, v, a)$ instead of the subgroups $< v, a >$ and $IRes_1(u, v, a)$ as we did in the first two steps.

In the general step, we say that the iterative procedure for the construction of the induced resolution terminates if $IRes_n(u, v, a) = IRes_{n-1}(u, v, a)$. Otherwise, we continue to the next step by applying steps (i)–(v) starting with the subgroup $IRes_n(u, v, a) < Rlim(t, v, a)$.

Proposition 3.3: *The iterative procedure for the construction of the induced resolution of the subgroup $< v, a >< Comp(Res)(t, v, a)$ terminates.*

Proof: Suppose that for some index n, the subgroup $IRes_n(u,v,a)$ properly contains the subgroup $IRes_{n-1}(u,v,a)$. The subgroup $IRes_{n-1}(u,v,a)$ is obtained by starting with the subgroup $IRes_{n-2}(u,v,a)$ and applying steps (i)–(v), and the subgroup $IRes_n(u,v,a)$ is obtained by starting with the subgroup $IRes_{n-1}(u,v,a)$ and applying steps (i)–(v). Since the subgroup $IRes_{n-1}(u,v,a)$ is properly contained in the subgroup $IRes_n(u,v,a)$, at least one of the decompositions constructed from $IRes_{n-1}(u,v,a)$ through steps (i)–(v) is not similar to the corresponding decomposition constructed from $IRes_{n-2}(u,v,a)$ through steps (i)–(v). By cases (1)–(2) above, if we restrict ourselves to the highest level decompositions constructed from $IRes_{n-1}(u,v,a)$ which are not similar to the corresponding decompositions constructed from $IRes_{n-2}(u,v,a)$, then either the decompositions constructed from $IRes_{n-1}(u,v,a)$ have fewer factors, or the ranks of the free groups generated by Bass–Serre generators of edges with trivial stabilizers and free groups which are infinite index subgroups in QH vertex groups in the corresponding (level) abelian decomposition of the completed resolution $Comp(Res)(t,v,a)$ are smaller in the decompositions constructed from $IRes_{n-1}(u,v,a)$ than they are in the decompositions constructed from $IRes_{n-2}(u,v,a)$, or in case there are equalities in the number of factors and the ranks of the corresponding free groups, then the combinatorics of the graphs of groups constructed from $IRes_{n-1}(u,v,a)$ is smaller than the combinatorics of the graphs of groups constructed from $IRes_{n-2}(u,v,a)$. This decrease in either the number of factors, the ranks of the free factors, or the combinatorics in the highest level decompositions which are not similar forces the iterative procedure for the construction of the induced resolution to terminate. ∎

LEMMA 3.4: *Let $IRes(u,v,a)$ be the terminal resolution obtained by the above procedure. Then $IRes(u,v,a)$ is a geometric subresolution of the completed resolution $Comp(Res)(t,v,a)$.*

Proof: The resolutions $IRes_n(u,v,a)$ constructed in the various steps of the iterative procedure are all well-separated by construction. The terminal resolution $IRes(u,v,a)$ is a completed resolution, since if we start with the subgroup $IRes(u,v,a)$ and apply steps (i)–(v) we get the same subgroup (and resolution) $IRes(u,v,a)$, which implies that $IRes(u,v,a)$ is equivalent to its completion. Since starting with the subgroup $IRes(u,v,a)$ and applying steps (i)–(v) gives us the subgroup $IRes(u,v,a)$ and the resolution used to get it in the first place, the natural embedding of the resolution $IRes(u,v,a)$ is a "geometric" embedding, i.e., it satisfies all the properties listed in part (iii) of Definition 3.1. ∎

Definition 3.5: Let $IRes(u,v,a)$ be the terminal resolution obtained by the above procedure. We say that the resolution obtained from the resolution $IRes(u,v,a)$, by replacing the modular groups associated with each of its QH vertex groups Q with its subgroup corresponding to the modular group of the QH vertex group Q' of the completed resolution $Comp(Res)(t,v,a)$ it is associated with, is the **induced resolution** of the subgroup $<v,a>$ extracted from the completed resolution $Comp(Res)(t,v,a)$. We denote the induced resolution $Ind(Comp(Res))(u,v,a)$.

The terminating iterative procedure presented above associates with a given completed well-separated resolution $Comp(Res)(t,v,a)$, and a subgroup $<v,a>$ of the limit group $Rlim(t,v,a)$, the induced resolution $Ind(Comp(Res))(u,v,a)$. Let $GRes(t,v,p,a)$ be a completed graded well-separated resolution, and let $<v,p_1,a>$ be a subgroup of the graded limit group $GRlim(t,v,p,a)$, where $<p_1>$ is a subgroup of the parameter subgroup $<p>< GRlim(t,v,p,a)$. Applying the procedure used to construct an induced resolution with an (ungraded) completed well-separated resolution and a subgroup of its limit group, we associate a graded induced resolution, $GInd(GRes)(u,v,p_1,a)$, with the graded resolution $GRes(t,v,p,a)$ and the subgroup $<v,p_1,a>$. As in the ungraded case, the procedure for the construction of the graded induced resolution guarantees that the graded induced resolution, $GInd(GRes)(u,v,p_1,a)$, is canonically embedded into the graded resolution $GRes(t,v,p,a)$. In the sequel, we will need the following basic property of the canonical embedding of the graded induced resolution.

LEMMA 3.6: *Let $GRes(t,v,p,a)$ be a completed graded well-separated resolution, and let $<v,p_1,a>$ be a subgroup of the graded limit group $GRlim(t,v,p,a)$, where $<p_1>$ is a subgroup of the parameter subgroup $<p>< GRlim(t,v,p,a)$. Let $GInd(GRes)(u,v,p_1,a)$ be the graded induced resolution associated with $GRes(t,v,p,a)$ and the subgroup $<v,p_1,a>$, let $Term(t,v,p,a)$ be the terminal (rigid or solid) graded limit group of the graded resolution $GRes(t,v,p,a)$, and let $GIndTerm(u,v,p_1,a)$ be the terminal (graded) limit group of the graded induced resolution $GInd(GRes)(u,v,p_1,a)$.*

By construction, the terminal limit group of the graded induced resolution, $GIndTerm(u,v,p_1,a)$, is canonically embedded into the terminal graded limit group of $GRes(t,v,p,a)$, $Term(t,v,p,a)$. Let

$$\nu\colon GIndTerm(u,v,p_1,a) \to Term(t,v,p,a)$$

be that canonical embedding. Then the image of the canonical embedding,

$\nu(GIndTerm(u,v,p_1,a))$, is contained in the subgroup generated by the image of the (original) subgroup $<v,p_1,a>$ in the terminal limit group $Term(t,v,p,a)$.

Proof: Follows immediately from the procedure for the construction of the graded induced resolution, since the elements from the various abelian groups and the Bass–Serre elements that are added along the procedure do not change the bottom level of the (graded) induced resolution. ∎

In our iterative procedure for validation of a sentence, as well as in our iterative procedure for quantifier elimination, we will need to be able to "compare" resolutions constructed at various steps of the iterative procedures. One of the basic tools for making such a "comparison" possible is the existence of a **decorated closure**.

THEOREM 3.7: *Let $Res(t,y,a)$ be a well-structured resolution, let $Comp(Res)(u,t,y,a)$ be its completion, and let $Rlim(y,a)$ be the subgroup generated by $< y,a >$ in the limit group $Rlim(t,y,a)$ associated with the resolution $Res(t,y,a)$. Let $Res_1(y,a),\dots,Res_r(y,a)$ be the resolutions in the taut Makanin–Razborov diagram of the limit group $Rlim(y,a)$.*

There exists a (canonical) covering closure

$$Cl_1(Res)(s,u,t,y,a),\dots,Cl_d(Res)(s,u,t,y,a)$$

of the resolution $Res(t,y,a)$, and for each index i, $1\le i\le d$, there exists an associated resolution $Res_{\delta(i)}(y,a)$, which is one of the resolutions in the taut Makanin–Razborov diagram of the limit group $Rlim(y,a)$, and an associated canonical map

$$\lambda_i\colon Comp(Res_{\delta(i)})(z,y,a)\to Cl_i(Res)(s,u,t,y,a)$$

that satisfy the following.

1. *For every index i, $1\le i\le d$, and every index j, $1\le j\le k$, $\lambda_i(a_j)=a_j$.*
2. *For every index i, $1\le i\le d$, and every index j, $1\le j\le \ell$, $\lambda_i(y_j)=y_j$.*
3. *Let $\{(u_j,t_j,y_j,a)\}_{j=1}^{\infty}$ be an arbitrary test sequence (see Definition 1.20 in [Se2]) that factors through the completed resolution $Comp(Res)(u,t,y,a)$. Then there exists some index i, $1\le i\le d$, and a subsequence of the test sequence (still denoted $\{(u_j,t_j,y_j,a)\}_{j=1}^{\infty}$) for which there exists a corresponding sequence of specializations $\{(z_j,y_j,a)\}_{j=1}^{\infty}$ that factor through and are taut with respect to the completed resolution $Comp(Res_{\delta(i)})(z,y,a)$, and such that the combined sequence $\{(\lambda_i(z_j),u_j,t_j,y_j,a)\}_{j=1}^{\infty}$ factors through the closure $Cl_i(Res)(s,u,t,y,a)$.*

We call each of the closures $Cl_i(Res)(w,s,z,y,a)$ a **decorated closure**, *denoted $DecorCl(s,u,t,y,a)$, and its associated resolution $Res_{\delta(i)}(y,a)$, its* **reference resolution**, *denoted $RefRes(y,a)$.*

Proof: Given a test sequence $\{(u_j,t_j,y_j,a)\}_{j=1}^{\infty}$ that factors through the completion $Comp(Res)(u,t,y,a)$, the universality of the taut Makanin–Razborov diagram implies that it is possible to find a subsequence, (still denoted)

$$\{(u_j,t_j,y_j,a)\}_{j=1}^{\infty},$$

for which there exists a corresponding shortest form sequence of specializations $\{(z_j,y_j,a)\}_{j=1}^{\infty}$ that factor through and are taut with respect to the completed resolution $Comp(Res_{\delta(i)})(z,y,a)$. Hence, the same proof that was used to prove the existence of a formal solution ([Se2], 1.18) gives a proof of Theorem 3.7. ∎

Induced resolutions and graded induced resolutions are the additional tools needed for presenting the iterative procedure for validation of an AE sentence in the general case. During the attempts to find such a procedure, we have constructed several additional tools that eventually were not useful in constructing a (terminating) procedure for validation of a sentence, but these tools seem rather basic and natural to us, and they are likely to be helpful in approaching other problems, even closely related ones. It is our aim to present some of these tools in forthcoming papers, still in the rest of this section we decided to include some notions that seem basic to us, although they are not going to be used in the sequel.

Let $Comp(Res)(t,v,a)$ be a well-separated completed resolution, and let $<v,a> < Rlim(t,v,a)$. The induced resolution $Ind(Comp(Res))(u,v,a)$ is the "basic" geometric subresolution of $Comp(Res)(t,v,a)$ that contains the subgroup $<v,a>$. However, for certain applications, which include the possibility to bound the complexity of resolutions by the complexity of their reference resolutions, the induced resolution needs to be modified. Besides the geometric requirement we will need our (completed) resolutions to be **firm**. Intuitively, we would like the rank of the geometric subresolution we work with to be equal to the rank of the free group corresponding to a generic specialization of the group $<v,a>$ that factors through the ambient resolution $Comp(Res)(t,v,a)$.

Definition 3.8: Let $Comp(Res)(t,v,a)$ be a completed resolution and let $<v,a> < Rlim(t,v,a)$. Let r be the maximal rank of all free groups which are specializations of the subgroup $<v,a>$ that can be extended to a specialization of the limit group $Rlim(t,v,a)$ which factors through the completed resolution $Comp(Res)(t,v,a)$. We call r the **rank** of the subgroup $< v,a >$ with respect to the completed resolution $Comp(Res)(t,v,a)$.

A test sequence $\{t_n,v_n,a\}$ of the completed resolution $Comp(Res)(t,v,a)$ is called a **firm test sequence** for the subgroup $< v,a >$ if, for every index n, the rank of the free group $< v_n,a >$ is equal to the rank of $< v,a >$ with respect to the completed resolution $Comp(Res)(t,v,a)$.

Using firm test sequences we are able to define **firm resolutions**, which are the type of resolutions one needs in order to control the complexity of resolutions by the complexity of their reference resolutions.

Definition 3.9: Let $Comp(Res)(t,v,a)$ be a completed resolution, let $< v,a >$ be a subgroup of the limit group $Rlim(t,v,a)$, and let $FrmGSRes(g,v,a)$ be a geometric subresolution of $Comp(Res)(t,v,a)$. We say that the resolution $FrmGSRes(g,v,a)$ is a **firm subresolution** of the completed resolution $Comp(Res)(t,v,a)$ if it has the following properties:

(i) The rank of the resolution $FrmGSRes(g,v,a)$, $rk(FrmGSRes(g,v,a))$, is equal to the rank of its terminal free group with respect to the completed resolution $Comp(Res)(t,v,a)$.

(ii) There exists a firm test sequence for the terminal free group of the geometric resolution $FrmGSRes(g,v,a)$.

(iii) Let $A_1,\ldots,A_m$ be all the non-cyclic pegged abelian groups that appear along the completed resolution $Comp(Res)(t,v,a)$, let $peg_1,\ldots,peg_m$ be the pegs of the abelian groups $A_1,\ldots,A_m$, and let $\{peg_i,q^i_1,\ldots,q^i_{j_i}\}_{i=1}^m$ be an arbitrary basis for the collection of the subgroups $A_1,\ldots,A_m$. Then for any set of integers $\{(s^i_j,n^i_j)\}$, where $n^i_j \geq 2$ and $0 \leq s^i_j \leq n^i_j$, there exists a firm test sequence $\{t_n,v_n,a\}$ of the subgroup $< v,a >$ such that for every index n, the specialization of each of the pegs peg_i is an element h_i, and the specialization of each of the basis elements q^i_j is $h_i^{r^i_j}$ where $r^i_j = u^i_j \cdot n^i_j + s^i_j$ for some positive integer u^i_j.

Given a completed resolution $Comp(Res)(t,v,a)$ and a subgroup $< v,a >$, the induced resolution, $Ind(Comp(Res))(u,v,a)$, is the basic geometric subresolution of $Comp(Res)(t,v,a)$, which is a resolution of the subgroup $< v,a >$. However, in general the induced resolution is certainly not a firm subresolution of $Comp(Res)(t,v,a)$. Indeed, it may be that $rk(Comp(Res)(t,v,a)) = 0$ whereas the rank of the induced resolution, $Ind(Comp(Res))(u,v,a)$, is arbitrarily large. Hence, for the purpose of controlling the complexity of resolutions we need a different type of resolution. Instead of actually constructing a firm subresolution, we present the notion of an **extremal subresolution**.

Definition 3.10: Let $Comp(Res)(t,v,a)$ be a completed resolution and let $< v,a >$ be a subgroup of the limit group $Rlim(t,v,a)$. A geometric subresolution of $Comp(Res)(t,v,a)$ that contains the subgroup $< v,a >$ is called **extremal**, and denoted $ExtrmRes(u,v,a)$, if for every geometric subresolution of $Comp(t,v,a)$ that contains the subgroup $< v,a >$, $GSRes(g,v,a)$,

$$Cmplx(ExtrmRes(u,v,a)) \leq Cmplx(GSRes(g,v,a)).$$

Before studying some of the basic properties of extremal resolutions we need to show their existence.

Proposition 3.11: *Let $Comp(Res)(t,v,a)$ be a completed resolution and let $< v,a >$ be a subgroup of the limit group $Rlim(t,v,a)$. There exists an extremal subresolution $ExtrmRes(u,v,a)$ of the completed resolution $Comp(Res)(t,v,a)$ that contains the subgroup $< v,a >$.*

Proof: The completed resolution $Comp(Res)(t,v,a)$ itself, and (by Lemma 3.4) the induced resolution, $Ind(Comp(Res))(u,v,a)$, are geometric subresolutions of $Comp(Res)(t,v,a)$ that contain the subgroup $< v,a >$. Hence, the set of geometric subresolutions of $Comp(Res)(t,v,a)$ that contain the subgroup $< v,a >$ is non-empty. By the way, the complexity of a resolution is defined (Definition 3.2); any decreasing sequence of complexities of resolutions terminates in a finite time. Hence, any (strictly) minimizing sequence of geometric subresolutions of $Comp(t,v,a)$ that contain the subgroup $< v,a >$ terminates, so the set of geometric subresolutions of $Comp(Res)(t,v,a)$ that contain the subgroup $< v,a >$ contains extremal subresolutions. ∎

As we have already pointed out, induced resolutions need not be firm. Extremal resolutions must be. Although we do not use extremal resolutions in our iterative procedures for validation of a sentence and for quantifier elimination, we list a few basic properties of these resolutions. The proof of these properties follows from the construction of a core resolution, presented in the next paper in the sequel.

Theorem 3.12: *Let $Comp(Res)(t,v,a)$ be a completed resolution and let $< v,a >$ be a subgroup of the limit group $Rlim(t,v,a)$. Then:*

(1) *An extremal subresolution $ExtrmRes(u,v,a)$ of the completed resolution $Comp(Res)(t,v,a)$ that contains the subgroup $< v,a >$ is a firm subresolution of $Comp(Res)(t,v,a)$.*

(2) *Suppose* $<v,a>=<a>*<v>$, *where* $<v>$ *is isomorphic to a free group* F_s *of rank* s. *Then either an extremal subresolution,* $ExtrmRes(u,v,a)$, *of the completed resolution* $Comp(Res)(t,v,a)$ *that contains the subgroup* $<v,a>$ *is the free group* $<a,v>$ *itself, or the rank of* $ExtrmRes(u,v,a)$ *is strictly smaller than* s.

(3) *Let* $ExtrmRes(u,v,a)$ *be an extremal subresolution of the completed resolution* $Comp(Res)(t,v,a)$ *that contains the subgroup* $<v,a>$. *Let* Q *be a conjugate of a* QH *subgroup in one of the abelian decompositions associated with the various levels of the completed resolution* $Comp(Res)(t,v,a)$. *Then:*

   (i) *If one of the free groups dropped along the various levels of the extremal resolution* $ExtrmRes(u,v,a)$ *intersects* Q *in a subgroup of finite index, then the extremal resolution* $ExtrmRes(u,v,a)$ *contains the entire* QH *subgroup* Q.

   (ii) *If* Q *is a* QH *subgroup in* $Comp(Res)(u,v,a)$ *from which a free factor is dropped in the next level (a* $floating\ QH\ vertex\ group$*), and* $ExtrmRes(u,v,a)$ *intersects* Q *in a subgroup of finite index, then the extremal resolution* $ExtrmRes(u,v,a)$ *contains the entire subgroup* Q.

(4) *Let* $Res(t,y,a)$ *be a well-separated resolution,* $Comp(Res)(u,t,y,a)$ *its completion,* $Rlim(y,a)$ *the subgroup generated by* $<y,a>$ *in* $Rlim(t,y,a)$, *and* $Res_1(y,a),\dots,Res_r(y,a)$ *the resolutions in the taut Makanin–Razborov diagram of the limit group* $Rlim(y,a)$. *Let*

$$DecorCl_1(s,u,t,y,a),\dots,DecorCl_d(s,u,t,y,a)$$

be its canonical set of decorated closures, and let

$$Res_{\delta(1)}(y,a),\dots,Res_{\delta(d)}(y,a)$$

be their associated reference resolutions. For each index i, $1\leq i\leq d$, *let* λ_i: $Comp(Res_{\delta(i)})(z,y,a)\to DecorCl(s,u,t,y,a)$ *be the canonical map from the completion of the* i*-th reference resolution into its associated decorated closure. For each level* m *of the completion of the reference resolution,* $Comp(Res_{\delta(i)})(z,y,a)$, *let* $<z_m,a>$ *be the subgroup associated with the* m*-th level of* $Comp(Res_{\delta(i)})(z,y,a)$. *Let* $<\lambda_i(z_m),a>$ *be the image of the* m*-th level subgroup* $<z_m,a>$ *in the decorated closure* $DecorCl_i(s,u,t,y,a)$. *Let* $ExtrmRes(u_m,\lambda_i(z_m),a)$ *be an extremal resolution of the subgroup* $<\lambda_i(z_m),a>$ *in the decorated closure*

$DecorCl_i(s,u,t,y,a)$.

There exists a test sequence $\{(s_j,u_j,t_j,y_j,a)\}_{j=1}^{\infty}$ that satisfies the following:

(i) *The test sequence factors through all the decorated closures:*

$$DecorCl_1(s,u,t,y,a),\dots,DecorCl_d(s,u,t,y,a).$$

(ii) *The test sequence is a firm test sequence with respect to all the extremal resolutions $ExtrmRes(u_m,\lambda_i(z_m),a)$ of the various subgroups $<\lambda_i(z_m),a>$.*

(iii) *There exists some index i, $1\leq i\leq d$, so that for every index j the specializations $\{(\lambda_i(z)_j,y_j,a)\}$ are shortest form and taut with respect to the completion of the i-th reference resolution $Comp(Res_{\delta(i)})(z,y,a)$.*

(5) *With the notation of (4), there exists some index i, $1\leq i\leq d$, for which*

$$Cmplx(ExtrmRes(u,y,a))\leq Cmplx(Comp(Res_{\delta(i)})(z,y,a))$$

and, in case of equality, there exist extremal subresolutions so that for every level m of the completion $Comp(Res_{\delta(i)})(z,y,a)$, the structure of the extremal resolution $ExtrmRes(u_m,\lambda_i(z_m),a))$ is identical to the structure of the part of the completion $Comp(Res_i)(z,y,a)$ corresponding to the m-th level subgroup $<z_m,a>$. In particular, the complexity of $ExtrmRes(u_m,\lambda_i(z_m),a)$ is identical to the complexity of the corresponding part of $Comp(Res_{\delta(i)})(z,y,a)$.

4. An iterative procedure for validation of a sentence

In Section 1 we used an iterative procedure for validation of a sentence, assuming the restricted limit groups in question are of minimal rank. In Section 2 we modified this procedure to analyze collections of maximal rank homomorphisms which are taut with respect to a given (well-separated) resolution. In Section 3 we studied induced resolutions and decorated closures of a resolution. To construct a terminating iterative procedure for validation of a general AE sentence, which is crucial for our quantifier elimination "trial and error" procedure, we must show that either the Zariski closures, or the complexities of the resolutions associated with the set of specializations for which the validity of the sentence was not yet demonstrated, strictly decrease after finitely many steps of the procedure. To get the desired strict decrease in either the complexity of the associated resolutions or the associated Zariski closures, we combine the tools, ideas and concepts of the (terminating) procedure in the minimal rank case presented in Section 1, with some additional properties of the (canonical) JSJ decomposition and the construction of the induced resolution presented in the previous section. To get this combination we are forced to construct certain "data structure" that keeps track of all the previous steps of the procedure, and not only the last one.

Let

$$\forall y \quad \exists x \quad \Sigma(x,y,a)=1 \wedge \Psi(x,y,a)\neq 1$$

be a true sentence over F_k. Let $F_y =< y_1,\dots,y_\ell >$ be a the free group with a free basis $y_1,\dots,y_\ell$. By Proposition 1.1 of [Se2] there exists a formal solution $x=x(y,a)$, and a finite set of restricted limit groups

$$Rlim_1(y,a),\dots,Rlim_m(y,a)$$

for which:

(i) The words corresponding to the equations in the system $\Sigma(x(y,a),y,a)=1$ represent the trivial word in the free group $F_k * F_y$.

(ii) For every index i, $Rlim_i(y,a)$ is a proper quotient of the free group F_k*F_y.

(iii) Let $B_1(y),\dots,B_m(y)$ be the basic sets corresponding to the restricted limit groups $Rlim_1(y,a),\dots,Rlim_m(y,a)$. If $y\notin B_1(y)\cup\dots\cup B_m(y)$, and $\psi_j(x,y,a)$ is a word corresponding to one of the equations in the system $\Psi(x,y,a)\neq 1$, then $\psi_j(x(y,a),y,a)\neq 1$.

Proposition 1.1 of [Se2] gives a formal solution that proves the validity of the given sentence on a co-basic set $(F_k)^\ell \setminus (B_1(y)\cup\dots\cup B_m(y))$, hence the rest of the procedure needs to construct formal solutions that prove the validity of the sentence on the remaining basic sets $B_1(y),\dots,B_m(y)$. Since our treatment of the restricted limit groups $Rlim_1(y,a),\dots,Rlim_m(y,a)$ which define the basic sets $B_1(y),\dots,B_m(y)$ is independent, we will continue with one of these limit groups, which for brevity we denote $Rlim(y,a)$. Note that $Rlim(y,a)$ is a proper quotient of the free group F_k*F_y, hence the rank of every resolution in the (taut) Makanin–Razborov diagram of $Rlim(y,a)$ is strictly less than ℓ, the rank of the free group F_y.

Let $Rlim(y,a)$ be one of the restricted limit groups

$$Rlim_1(y,a),\dots,Rlim_m(y,a).$$

By Proposition 2.5, with $Rlim(y,a)$ one associates its (canonical) taut Makanin–Razborov diagram. Let $Res_1(y,a),\dots,Res_r(y,a)$ be the resolutions in this taut diagram. Each resolution in the diagram is a well-separated Makanin–Razborov resolution of either the restricted limit group $Rlim(y,a)$ or of a proper quotient of it, and every specialization of the restricted limit group $Rlim(y,a)$ can be extended to a specialization which is taut with respect to at least one of the resolutions $Res_1,(y,a),\dots,Res_r(y,a)$.

In presenting the iterative procedure for validation of a sentence, we won't need to consider all the specializations that factor through the completions of a given well-separated resolution, but only the **shortest form** ones.

Definition 4.1: Let $Rlim(y,a)$ be a restricted limit group, let $Res(y,a)$ be a well-separated resolution of $Rlim(y,a)$, let $Comp(Res)(z,y,a)$ be its completed resolution, and let $Comp(Rlim)(z,y,a)$ be its completed limit group. We fix generating sets for the subgroups of the completion, $Comp(Rlim)(z,y,a)$, associated with the various levels of the resolutions $Res(y,a)$.

We say that a specialization (z,y,a) that factors and is taut with respect to the completed resolution $Comp(Res)(z,y,a)$ is a **shortest form** specialization, if the specialization of the subgroup associated with each level of the resolution $Res(y,a)$ is the shortest possible under the action of the modular group associated with that level.

By the construction of the taut Makanin–Razborov diagram of a limit group (Proposition 2.5), every specialization of a given limit group can be extended to a shortest form specialization of a completion of (at least) one of the resolutions in the taut Makanin–Razborov diagram of the given limit group.

I. The first step. We start the first step of the iterative procedure in the general case, with a (finite) collection of limit groups, the resolutions that appear in their taut Makanin–Razborov diagrams, and the collections of shortest form specializations associated with each of these (well-separated) resolutions.

As in the minimal rank case, the continuation of the iterative procedure in the general case is conducted independently for the different limit groups and each of their resolutions. Hence, for the rest of the procedure, we will denote the restricted limit group in question $Rlim(y,a)$, and its (well-separated) resolution $Res(y,a)$, omitting their indices.

By theorem 1.18 of [Se2], from the validity of our given sentence we get the existence of a covering closure $Cl_1(Res)(s,z,y,a),\dots,Cl_q(Res)(s,z,y,a)$ of the resolution $Res(y,a)$, and for each index $1\le n\le q$ there exists a formal solution $x_n(s,z,y,a)$ for which:

(i) The words corresponding to the equations in the system

$$\Sigma(x_n(s,z,y,a),y,a)=1$$

represent the trivial word in the closure $Cl_n(Res)(s,z,y,a)$.

(ii) There exists some specialization (s_0^n,z_0^n,y_0^n) for which

$$\Psi(x_n(s_0^n,z_0^n,y_0^n,a),y_0^n,a)\neq 1.$$

The first formal solution, defined over the free group $F_y=<y_1,\dots,y_\ell>$, proves the validity of our given sentence on a co-basic set of y's, and we are still required to prove the validity of the sentence on the basic set, which consists of all the specializations of the collection of limit groups

$$Rlim_1(y,a),\dots,Rlim_m(y,a).$$

The formal solutions described above for each of the closures $Cl_n(Res)(s,z,y,a)$ are defined using the sets of variables (s,z,y,a). Hence, they prove the validity of the given sentence on the intersection of Diophantine and co-Diophantine sets, defined by the various sets of variables (s,z,y,a).

To analyze the remaining set of y's, we construct an iterative procedure that produces a sequence of well-separated resolutions and their completions. The iterative procedure for analyzing general resolutions is based on the procedures presented in the first two sections, but the analysis of the set of the remaining y's in each step of the procedure needs to be modified to guarantee (eventual) strict reduction in complexity. Given a resolution $Res(y,a)$, we go over all the indices n, $1\leq n\leq q$, and analyze all the specializations (s_0,z_0,y_0,a) that are shortest form with respect to the closure $Cl_n(Res)(s,z,y,a)$ (hence, in particular, factor and are taut with respect to $Cl_n(Res)(s,z,y,a)$), for which the tuple (s_0,z_0,y_0,a) satisfies the additional equation

$$\psi_j(x^n(s_0,z_0,y_0,a),y_0,a)=1$$

for some index j. Note that the equation $\psi_j(x_n(s_0,z_0,y_0,a),y_0,a)=1$, defined over the set of specializations that factor through the closure $Cl_n(Res)(s,z,y,a)$, is a non-trivial equation by construction. Clearly, if an element $y_0\in F_k$ can be extended to a shortest form specialization with respect to one of the constructed closures and this shortest form specialization does not satisfy any of the additional equations, then the sentence is valid for y_0.

If we fix the index n, then the entire collection of shortest form specializations (s_0,z_0,y_0,a) that factor through the closure $Cl_n(Res)(s,z,y,a)$ and satisfy (at least) one of the additional equations is contained in a finite set of maximal limit groups $QRlim_1^n(s,z,y,a),\dots,QRlim_{u_n}^n(s,z,y,a)$. Since our analysis of these quotient limit groups is conducted in parallel, we will omit the indices from the quotient limit groups and closures, and denote them $Cl(Res)(s,z,y,a)$ and $QRlim(s,z,y,a)$ in correspondence.

The resolution $Res(y,a)$ is well-separated, in particular well-structured, so with each level of the closure, $Cl(Res)(s,z,y,a)$, there is an associated abelian decomposition. By the construction of these decompositions ([Se2], 1.12), a non-QH, non-abelian vertex group is connected (by an edge with non-trivial stabilizer) only to QH and abelian vertex groups, and QH and abelian vertex groups are connected only to non-QH, non-abelian vertex groups. Let $Base_{2,1}^1,\dots,Base_{2,v_1}^1$ be the non-abelian, non-QH vertex groups in the abelian decomposition associated with the top level of the closure, $Cl(Res)(s,z,y,a)$ (the notation $Base_{\ell,j}^s$ indicates the index of the base that will be further explained in the presentation of the general step, s is the step index, and j is the number of the vertex in the abelian decomposition associated with level $\ell-1$ in the completion). In a similar way to what we did in Sections 1 and 2, with the quotient limit group $QRlim(s,z,y,a)$, we associate canonically the multi-graded taut Makanin–Razborov diagram with respect to the subgroups $Base_{2,1}^1,\dots,Base_{2,v_1}^1$. Let

$$MGQRes_1(s,z,y,Base_{2,1}^1,\dots,Base_{2,v_1}^1,a),\dots,$$
$$MGQRes_q(s,z,y,Base_{2,1}^1,\dots,Base_{2,v_1}^1,a)$$

be the quotient multi-graded resolutions in this taut Makanin–Razborov diagram.

We continue with a subset of the resolutions which appear in the multi-graded taut Makanin–Razborov diagram of the quotient limit group $QRlim(s,z,y,a)$, a subset through which all the specializations (y_0,a) of $Rlim(y,a)$ that can be extended to specializations (s_0,z_0,y_0,a) which are taut and shortest form with respect to $Cl(Res(y,a)$, and factor through $QRlim(s,z,y,a)$, do factor, and not with all the quotient multi-graded resolutions in this taut multi-graded diagram. If the subgroup generated by $<y,a>$ of the limit group (generated by) $<s,z,y,a>$ associated with a quotient multi-graded resolution $MGQRes_i$ is a proper quotient of the limit group $Rlim(y,a)$ associated with the resolution $Res(y,a)$ with which we have started this branch of the first step of the procedure, we include the quotient multi-graded resolution $MGQRes_i$ in the subset we continue with. Hence, to define the other quotient multi-graded resolutions in the subset with which we continue, we may assume that the subgroup generated by $< y, a >$ in the limit group associated with a multi-graded resolution $MGQRes_i$ is isomorphic to the limit group $Rlim(y, a)$ associated with the resolution $Res(y, a)$.

For each QH vertex group Q in the abelian decomposition associated with the top level of the closure $Cl(Res)(s, z, y, a)$, the boundary elements of Q can be conjugated (in the limit group associated with the closure $Cl(Res)(s, z, y, a)$) into the subgroups $Base^1_{2,1}, \ldots, Base^1_{2,v_1}$. Since the closure $Cl(Res)(s, z, y, a)$ is canonically mapped onto the quotient limit group $QRlim(s, z, y, a)$, each of the QH subgroups Q in the abelian decomposition associated with the top level of $Cl(Res)(s, z, y, a)$ is naturally mapped into the quotient limit group $QRlim(s, z, y, a)$. Hence, the QH subgroup Q and its corresponding quotients naturally inherit a sequence of abelian decompositions from the multi-graded abelian decompositions associated with the various levels of a multi-graded resolution $MGQRes_i$ abelian decompositions in which the boundary elements of Q are all elliptic (and non-trivial).

The resolution $Res(y, a)$ was assumed to be well-separated, hence on each of the QH vertex groups Q associated with its various levels there exists an additional indication for a collection of s.c.c. on its associated surface S that is mapped to a trivial element in the limit group associated with the next level of the resolution $Res(y, a)$. Given a multi-graded resolution $MGQRes_i$, if for some QH vertex group Q in the abelian decomposition associated with the top level of the closure $Cl(Res)(s, z, y, a)$, the sequence of abelian decompositions Q and its corresponding quotients inherited from the multi-graded resolution $MGQRes_i$ is not compatible with the specific indication of the collection of s.c.c. on the surface S associated with the QH vertex group Q that are mapped to the trivial element in the resolution $Res(y, a)$, we omit the quotient multi-graded resolution $MGQRes_i$ from our list of quotient multi-graded resolutions. Otherwise, we include the quotient multi-graded resolution in the subset of multi-graded resolutions with which we continue.

By construction, for every specialization (y_0, a) of $Rlim(y, a)$ that can be extended to a specialization (s_0, z_0, y_0, a) of $QRlim(s, z, y, a)$, that factors and is taut and shortest form with respect to the closure, $Cl(Res)(s, z, y, a)$, $Cl_n(Res)(s, z, y, a)$, there exists a quotient multi-graded resolution $MGQRes_i$ which is included in the subset of quotient multi-graded resolutions with which we continue, and for which the specialization (s_0, z_0, y_0, a) factors through and is taut with respect to that quotient multi-graded resolution.

Let $Q_1, \ldots, Q_r$ be the QH vertex groups in the abelian decomposition associated with the top level of the closure $Cl(Res)(s,z,y,a)$. By the same arguments used in proving Theorems 1.7 and 2.9, it is possible to modify each multi-graded quotient resolution $MGQRes_i$ to a resolution that terminates in a multi-graded limit group with a multi-graded abelian decomposition containing the entire collection of surviving QH vertex groups $Q_{i_1}, \ldots, Q_{i_{r'}}$ (Definition 2.8), and these surviving QH subgroups are either closed surface subgroups or (in case they correspond to punctured surfaces) they are mapped to their images in the subgroup $< z_2, a >$ in either a rigid or solid multi-graded limit group with respect to the subgroups $Base^1_{2,1}, \ldots, Base^1_{2,v_1}$. Furthermore, the surviving QH vertex groups $Q_{i_1}, \ldots, Q_{i_{r'}}$ are subgroups of the non-abelian, non-QH vertex groups, the ones stabilized by the subgroups $Base^1_{2,1}, \ldots, Base^1_{2,v_1}$, in all levels above the two terminating ones (see Theorem 2.9).

A basic property of a multi-graded quotient resolution, that was proved for the entire resolution (and not just its top part) in Section 1 under the minimal rank assumption, is the following property. It is one of the main properties used in our approach to validation of an AE sentence. Recall that the complexity of a resolution is presented in Definition 1.14 in the minimal rank case, and in Definition 3.2 for general resolutions.

Proposition 4.2: *Suppose that the subgroup generated by $< y, a >$ in the limit group associated with a multi-graded resolution*

$$MGQRes(s,z,y,Base^1_{2,1},\ldots,Base^1_{2,v_1},a),$$

is isomorphic to the limit group $Rlim(y,a)$ associated with the resolution $Res(y,a)$ with which we started the first step. Then the complexity of each of the multi-graded abelian decompositions associated with the various levels of the multi-graded resolution

$$MGQRes(s,z,y,Base^1_{2,1},\ldots,Base^1_{2,v_1},a)$$

is bounded by the complexity of the abelian decomposition associated with the top level of the completion $Comp(Res)(z,y,a)$. In case of equality, the multi-graded resolution $MGQRes$ can be modified to a resolution that has only one level and its structure is identical to the structure of the abelian decomposition associated with the top level of the completion $Comp(Res)(z,y,a)$ (see Theorems 1.7 and 2.9 for a description of the modification).

Proof: Since the sequence of abelian decompositions inherited by each of the QH vertex groups Q in the abelian decomposition associated with the top level of $Res(y,a)$, from the multi-graded resolution $MGQRes$, is compatible with the indicated collection of s.c.c. which are mapped to the trivial element in the resolution $Res(y,a)$, the (multi-graded) rank of the multi-graded resolution $MGQRes$ is bounded by the multi-graded rank of the free factor that is dropped at the top level of the completion $Comp(Res)(z,y,a)$. If the rank of the multi-graded resolution $MGQRes$ is strictly smaller than the rank of the free factor dropped at the top level of the completion $Comp(Res)(z,y,a)$, the proposition follows. If these ranks are equal, the proposition follows from our analysis of taut homomorphisms of maximal rank (Theorem 2.9). ∎

Proposition 4.2 shows in particular, that if the multi-graded resolution $MGQRes$ does not have a single level with the same structure as the abelian decomposition associated with the top level of the completion, $Comp(Res)(z,y,a)$, then the complexity of each of the abelian decompositions associated with the various levels of the multi-graded quotient resolution, $MGQRes$, is strictly smaller than the complexity of the resolution $Res(y,a)$ with which we started. A multi-graded quotient resolution, $MGQRes$, has few other basic properties which are crucial in our approach. Proposition 4.3 is the key for constructing our iterative procedure for validation of an AE sentence.

Proposition 4.3: *Let $MGQRes$ be one of the multi-graded quotient resolutions, for which the subgroup generated by $< y,a >$ in the limit group associated with $MGQRes$ is isomorphic to the limit group $Rlim(y,a)$, associated with the resolution $Res(y,a)$. By construction, the original limit group $Rlim(y,a)$ is naturally mapped into the limit groups associated with each of the levels of the multi-graded quotient resolution $MGQRes$.*

Let $Q_{term}(y,a)$ be the image of $Rlim(y,a)$ in the terminal (rigid or solid) multi-graded limit group of $MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$. If the limit group $Rlim(y,a)$ is not the (coefficient) free group $F_k =< a >$, then $Q_{term}(y,a)$ is a proper quotient of $Rlim(y,a)$.

Proof: If $Rlim(y,a)$ is a free product of free groups, free abelian groups, and (closed) surface groups, then $Q_{term}(y,a)$ is necessarily a proper quotient of $Rlim(y,a)$, unless $Rlim(y,a)$ is the coefficient group F_k. Hence, we may assume that the groups $Base^1_{2,1},\dots,Base^1_{2,v_1}$ are not trivial.

Let $Term(s,z,y,a)$ be the terminal rigid or solid limit group of the multi-graded resolution

$$MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a).$$

Let $\{(s_n, z_n, y_n, a)\}$ be a sequence of rigid, or shortest solid specializations of the terminal limit group $Term(s, z, y, a)$, that converge into the limit group $Term(s, z, y, a)$. By passing to a subsequence, we may assume that the sequence $\{(s_n, z_n, y_n, a)\}$ converges into a non-trivial, stable and faithful action of $Term(s, z, y, a)$ on a real tree Y, with abelian edge stabilizers. Since the sequence of specializations are either rigid or shortest solid, at least one of the subgroups $Base^1_{2,1}, \ldots, Base^1_{2,v_1}$ is not elliptic when acting on Y. Since the specializations used to construct the quotient limit group $Qlim(s, z, y, a)$, and the multi-graded resolution, $MGQRes$, are assumed to be shortest form specializations (Definition 4.1), if at least one of the subgroups $Base^1_{2,1}, \ldots, Base^1_{2,v_1}$ is not elliptic, then at least one of the non-abelian, non-QH vertex groups in the abelian JSJ decomposition of the original limit group, $Rlim(y, a)$, is not elliptic when acting on Y.

The abelian JSJ decomposition is a universal object that encodes all possible stable, abelian, faithful actions of a group on a real tree. In particular, the non-abelian, non-QH vertex groups in the abelian decomposition of $Rlim(y, a)$ are elliptic in every stable, faithful action of $Rlim(y, a)$ on a real tree with abelian edge stabilizers. Hence, the action of $Rlim(y, a)$ on the real tree Y, via the natural map from $Rlim(y, a)$ to $Term(s, z, y, a)$, is not a faithful action. Therefore, the image of $Rlim(y, a)$ in $Term(s, z, y, a)$, $Q_{term}(y, a)$, is a proper quotient of $Rlim(y, a)$. ∎

In Section 1 we have started the procedure for validation of a sentence with a (minimal rank) modular block, and showed that the complexities of the various (quotient) modular blocks associated with it are strictly smaller than the complexity of the original modular block. In the general case, we are not able to get a reduction in the complexity of the obtained modular blocks in each step of our iterative procedure. To "force" the "size" of the set of the remaining specializations y_0 to actually decrease, we need to associate with each modular block information about certain (multi-graded) resolutions and abelian decompositions associated with it together with Zariski closures of some subgroups associated with the modular block. To carry all the information attached to a modular block, we associate a **data structure**, and (canonical) resolutions which we call **developing resolutions**, with each of the quotient resolutions $QRes(s, z, y, a)$ which are associated, or rather derived, from the various multi-graded quotient resolutions

$$MGQRes(s, z, y, Base^1_{2,1}, \ldots, Base^1_{2,v_1}, a).$$

We construct the **data structure** and **developing resolutions** iteratively. We start with the multi-graded quotient resolution

$$MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$$

and continue with Zariski closures and (multi-graded) resolutions of (the image of) various subgroups of the completion $Comp(Res)(z,y,a)$ inherited from the multi-graded quotient resolution $MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$ and (multi-graded quotient) resolutions derived from it. We describe the iterative procedure for constructing the **data structure** and **developing resolutions** starting with the first step, the second step, and then the general step.

(1) Let $Rlim(y,a)$ be the restricted limit group with which we have started. Let $Q(y,a)$ be the limit group generated by $<y,a>$ in the limit group associated with the multi-graded quotient resolution

$$MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a).$$

If $Q(y,a)$ is a proper quotient of the subgroup $Rlim(y,a)$, we continue this branch of the iterative procedure by replacing the limit group $Rlim(y,a)$ by $Q(y,a)$, and continue with the finite collection of resolutions that appear in the taut Makanin–Razborov diagram of the limit group $Q(y,a)$ in the same way we handled the resolutions in the taut Makanin–Razborov diagram of $Rlim(y,a)$. Note that this is always the case if the limit group we started with, $Rlim(y,a)$, is a free product of a free group and some (closed) surface groups.

(2) At this stage we may assume that $Q(y,a)$ is isomorphic to $Rlim(y,a)$. In this part we will also assume that the multi-graded quotient resolution $MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$ is not of maximal complexity, i.e., that the complexities of the abelian decompositions associated with its various levels are strictly smaller than the complexity of the abelian decomposition associated with the top level of the resolution $Res(y,a)$. To continue handling a multi-graded quotient resolutions

$$MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$$

which is not of maximal complexity, we need the following observation.

Proposition 4.4: *Let $MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$ be a multi-graded quotient resolution which is not of maximal complexity. By construction, the original limit group $Rlim(y,a)$ and the subgroup $QRlim(s,z,y,a)$ are* *mapped into the limit group* $MGQlim_2(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$ *associated with the second level of the multi-graded quotient resolution* $MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$. *Let* $Q_2(y,a)$ *and* $Q_2(s,z,y,a)$ *be the images of* $Rlim(y,a)$ *and* $QRlim(s,z,y,a)$ *in the limit group* $MGQlim_2(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$. *Then* $Q_2(y,a)$ *is a quotient of* $Rlim(y,a)$, *and* $Q_2(s,z,y,a)$ *is a proper quotient of the subgroup* $QRlim(s,z,y,a)$.

Proof: The proposition is simply a property of a multi-graded resolution. ∎

By Proposition 4.4, either the image of $Rlim(y,a)$ in the limit group associated with the second level of $MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$, $Q_2(y,a)$, is a proper quotient of $Rlim(y,a)$, or the image of $QRlim(s,z,y,a)$ in the limit group associated with the second level of $MGQRes$, $Q_2(s,z,y,a)$, is a proper quotient of $QRlim(s,z,y,a)$.

Suppose that the subgroup $Q_2(y,a)$ is a proper quotient of $Rlim(y,a)$. Let $QRes_1(y,a),\dots,QRes_q(y,a)$ be the resolutions in the taut Makanin–Razborov diagram of $Q_2(y,a)$. We continue with each of the resolutions $QRes_i(y,a)$ separately and omit its index. With the resolution $QRes(y,a)$ we associate a resolution $CRes(y,a)$, where $CRes(y,a)$ is composed from two parts, the top being the resolution induced (using the construction of the induced resolution presented in Section 3) by the subgroup $<y,a>$ from the completion of the top level of the multi-graded quotient resolution

$$MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$$

and the tail being the resolution $QRes(y,a)$. (Note that by the construction of the induced resolution, the terminal subgroup of the resolution induced by the subgroup $<y,a>$ from the completion of the top level of the multi-graded resolution $MGQRes$, is naturally embedded into a subgroup of $Q_2(y,a)$ (see Lemma 3.6); hence the top and the tail of the resolution $CRes$ can be naturally glued (amalgamated) together.) By theorem 1.18 of [Se2], with the resolution $CRes(y,a)$ we can associate a set of closures

$$Cl_1(CRes)(u,y,a),\dots,Cl_d(CRes)(u,y,a)$$

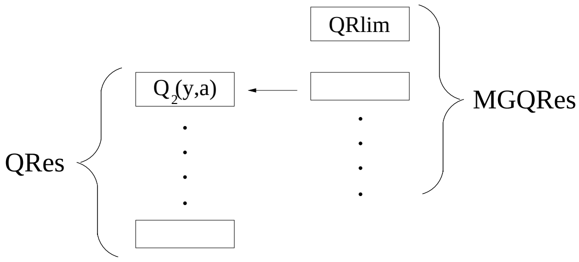
and for each closure a corresponding formal solution defined over it. We continue with each of these closures in parallel.

If the subgroup generated by $<y,a>$ in a closure, $Cl(CRes)(u,y,a)$, is a proper quotient of the limit group $Rlim(y,a)$ with which we started, we replace the particular closure by starting the first step of the procedure with the subgroup generated by $< y, a >$ in it. Hence, for the continuation we can assume that the subgroup generated by $< y, a >$ in the closure is isomorphic to the limit group $Rlim(y, a)$ with which we started.

Fixing a closure, $Cl(CRes)(u, y, a)$, we set the **developing resolution** to be that closure. With the closure, $Cl(CRes)(u, y, a)$, there is an associated formal solution $x(u, y, a)$ defined over it. We further set the **anvils** associated with the developing resolution to be the (canonical) finite set of maximal limit quotients of the group obtained as the amalgamated product of the completion of the developing resolution and the completion of the top level of the multi-graded resolution

$$MGQRes(s, z, y, Base^1_{2,1}, \ldots, Base^1_{2,v_1}, a),$$

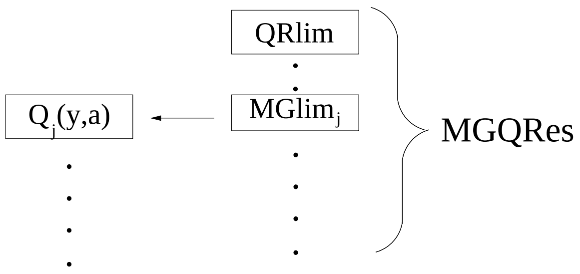
amalgamated along the top part of the developing resolution, which was set to be the resolution induced by the subgroup $< y, a >$ from the top level of $MGQRes$. We denote each of the (finitely many) anvils $Anv(MGQRes)(w, y, a)$. Note that the completion of the developing resolution is canonically mapped into the anvil, hence the formal solution defined over the developing resolution can be naturally defined over the anvil.



Suppose that $Q_2(y, a)$ is isomorphic to $Rlim(y, a)$. In this case we continue to the next levels of the multi-graded quotient resolution

$$MGQRes(s, z, y, Base^1_{2,1}, \ldots, Base^1_{2,v_1}, a).$$

By Theorem 4.3, we are guaranteed that at some level j of $MGQRes$, the image of $Rlim(y, a)$ in the limit group associated with the j-th level of $MGQRes$, $Q_j(y, a)$, is a proper quotient of $Rlim(y, a)$. At this point we continue in a similar way to what we did in case $Q_2(y, a)$ is a proper quotient of $Rlim(y, a)$. We associate with $Q_j(y, a)$ the resolutions that appear in its taut Makanin–Razborov diagram. With each such resolution, we associate a resolution obtained from the resolution induced by the subgroup $< y, a >$ (according to the construction presented in Section 3) from the completion of the resolution obtained from the levels of the multi-graded resolution $MGQRes$ that lie above the j-th level, continued with the given resolution from the taut Makanin–Razborov diagram of $Q_j(y, a)$. By the generalized Merzlyakov's theorem (theorem 1.18 in [Se2]), with each of the obtained resolutions one can associate a covering closure, where on each closure a formal solution is defined. We set the developing resolutions to be the closures in the given covering closures. With each developing resolution we associate a finite set of anvils. We set the anvils to be the maximal limit quotients of the subgroup obtained as the amalgamated product of the completion of the developing resolution and the completion of the multi-graded resolution $MGQRes$ amalgamated over their common part, i.e., over the resolution induced by the subgroup $< y, a >$ from all levels of the multi-graded resolution $MGQRes$, that lie above the j-th level (including the j-th level itself). With the anvil, $Anv(MGQRes)(w, y, a)$, we associate the formal solution $x(w, y, a)$ that was defined over the developing resolution, as we did in case $Q_2(y, a)$ is a proper quotient of $Rlim(y, a)$.



(3) By part (1) we may assume that $Q(y, a)$ is isomorphic to $Rlim(y, a)$. Part (2) treats the case in which the multi-graded quotient resolution

$$MGQRes(s, z, y, Base^1_{2,1}, \ldots, Base^1_{2,v_1}, a)$$

is not of maximal complexity. Hence, the only case left in presenting the first step of our general procedure for validation of a sentence is the case of a multi-graded quotient resolution

$$MGQRes(s, z, y, Base^1_{2,1}, \ldots, Base^1_{2,v_1}, a)$$

of maximal possible complexity, i.e., a multi-graded quotient resolution $MGQRes(s, z, y, Base^1_{2,1}, \ldots, Base^1_{2,v_1}, a)$ consists of a single level, and the abelian and QH vertex groups that appear in the abelian decomposition associated with this level are identical to those that appear in the abelian decomposition associated with the top level of the closure, $Cl(Res)(z,y,a)$, with which we started this step.

We treat this case as we treated it in the minimal rank case. In case the abelian decompositions associated with $MGQRes$ and the top level of $Res(y,a)$ are identical (in the sense indicated above), we use the modular groups associated with the abelian decomposition associated with $MGQRes$ to map the subgroup $Rlim(y,a)$ into the limit group associated with the second level of $Res(y,a)$, $< z_2, a >$, which is naturally mapped into the quotient limit group $QRlim(s,z,y,a)$. We now set the subgroups $Base^1_{3,1},\dots,Base^1_{3,t_1}$ to be the subgroups corresponding to the non-abelian, non-QH vertex groups and edge groups in the multi-graded abelian decomposition associated with the second level of the closure $Cl(Res)(s,z,y,a)$ with which we started the first step.

Since the quotient limit group we started with, $QRlim(s,z,y,a)$, is a proper quotient of the closure, $Cl(Res)(s,z,y,a)$, and since the abelian decomposition associated with the (only) level of

$$MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$$

is identical to the abelian decomposition associated with the top level of $Cl(Res)(s,z,y,a)$, the subgroup $QRlim_2(s,z,y,a)$, which is the subgroup generated by $< s_2, z_2, \dots, s_\ell, z_\ell, a >$ in $QRlim(s,z,y,a)$, is a proper quotient of its corresponding preimage in the closure $Cl(Res)(s,z,y,a)$. Hence, at this point we can analyze the quotient limit group

$$QRlim_2(s,z,y,a)$$

with respect to the subgroups $Base^1_{3,1},\dots,Base^1_{3,t_1}$ exactly as we analyzed the quotient limit group $QRlim(s,z,y,a)$ with respect to the subgroups $Base^1_{2,1},\dots,Base^1_{2,v_1}$, i.e., we associate with $QRlim_2(s,z,y,a)$ all its multi-graded quotient resolutions with respect to the subgroups

$$Base^1_{3,1},\dots,Base^1_{3,t_1}$$

which are its subgroups, and analyze each of the obtained multi-graded quotient resolutions according to parts (1)–(2) and the above portion of part (3). If the multi-graded abelian decomposition associated with a multi-graded quotient resolution of $QRlim_2(s,z,y,a)$ with respect to the subgroups $Base^1_{3,1},\dots,Base^1_{3,t_1}$ is of maximal possible complexity, i.e., if part (3) applies to an obtained quotient multi-graded resolution, we continue by analyzing multi-graded quotient resolutions of the quotient limit group $QRlim_3(s,z,y,a)$ with respect to the subgroups

$$Base^1_{4,1},\dots,Base^1_{4,r_4}.$$

As long as the multi-graded abelian decompositions associated with the constructed graded quotient resolutions are of maximal possible complexity, we continue analyzing multi-graded quotient resolutions associated with various level of the resolution $Res(y,a)$ we started with, until one of the parts (1)–(2) applies to the corresponding multi-graded quotient resolution. At the level that one of the parts (1)–(2) applies, we first construct a resolution composed from the resolution induced by the subgroup $<y,a>$ from the parts of the resolution above that level, followed by the (developing) resolution associated with that level according to one of the parts (1)–(2). By the generalized Merzlyakov's theorem (theorem 1.18 in [Se2]), one is able to associate a covering closure with the obtained resolution, and with each closure one further associates a formal solution. We set each of the closures in the obtained covering closures to be a developing resolution. With each of the developing resolutions we associate a finite collection of anvils, $Anv(MGQRes)(w,y,a)$, that are set to be the maximal limit quotients of the group obtained as the amalgamated product of the given developing resolution and the completion of the multi-graded resolution associated with the relevant level, amalgamated along their common part, i.e., the resolution induced by the subgroup $<y,a>$ from the constructed completion. With the developing resolution and its associated anvil, we naturally associate a formal solution, $x(w,y,a)$, as we did in part (2).

Definition 4.5: Starting with the limit group $QRlim(s,z,y,a)$, the procedure for the construction of the data structure and developing resolutions produces finitely many multi-graded quotient resolutions of $QRlim(s,z,y,a)$ and some of its quotients. With each such multi-graded quotient resolution the procedure associates a developing resolution, and with each developing resolution it associates an anvil. We call the entire collection of the original limit group $Rlim(y,a)$ and the resolution $Res(y,a)$ we started with, the closure $Cl(Res)(s,z,y,a)$, auxiliary multi-graded together with the quotient limit group $QRlim(s,z,y,a)$, its chosen quotient and multi-graded quotient resolution, the associated developing resolution, and the associated anvil, the **data structure**.

We continue to the second step of the iterative procedure for validation of a sentence, with each of the data structures and its associated developing resolution, anvil, and the formal solution defined over the anvil that was constructed using theorem 1.18 of [Se2].

II. The second step. In the first step of the general iterative procedure for validation of a sentence, we used formal solutions defined over the entire set of y's, and then formal solutions defined over closures of the various resolutions of the "remaining" limit groups $Rlim_1(y,a),\dots,Rlim_m(y,a)$. These formal solutions left us with the quotient limit groups $QRlim(s,z,y,a)$ and their various multi-graded quotient resolutions. For presentation purposes, we present the second step of the iterative procedure. Later on we present the general step of the iterative procedure, and finally obtain its termination.

Since each of the remaining specializations of the variables y factors through at least one of the anvils and its associated developing resolution, we start the second step of the iterative procedure with the (canonical) finite set of anvils. Since we treat the anvils in parallel, we present the second step of the procedure with one of the anvils $Anv(MGQRes)(w,y,a)$.

With each anvil we have associated a formal solution $x(w,y,a)$ that satisfies the properties listed in theorem 1.18 of [Se2]. Starting with the anvil $Anv(MGQRes)(w,y,a)$, We look at the set of specializations (w_0,y_0,a) that factor through and are taut and shortest form (see Definition 4.1) with respect to the multi-graded resolution associated with the anvil $Anv(GMQRes)(w,y,a)$, and factor through and are taut and shortest form with respect to the developing resolution, for which for some index j, $\psi_j(x(w_0,y_0,a),y_0,a)=1$. Clearly, for any element $y_0\in F_k$ that cannot be extended to such shortest form specialization, the (finite) set of formal solutions defined in the initial and first steps already prove the validity of our given sentence. Hence, in the next steps of our iterative procedure it is enough to study the collection of these shortest form specializations.

If we fix the index j of the equation $\psi_j(x,y,a)=1$, then the entire collection of shortest form specializations (w_0,y_0,a) that satisfy the corresponding system of equations is contained in a finite set of maximal (second quotient) limit groups $Q^2Rlim_1(w,y,a),\dots,Q^2Rlim_{u_j}(w,y,a)$. Since our analysis of these (second) quotient limit groups is conducted in parallel, we will omit the indices from these (second) quotient limit groups and denote them $Q^2Rlim(w,y,a)$.

We construct the **data structure** and **developing resolutions** associated with the anvil and the (second) quotient limit group $Q^2Rlim(w,y,a)$ iteratively, similarly to the way we have analyzed quotient resolutions in the first step of the procedure, though in a somewhat modified way. The analysis we carry out in the second step depends on the structure of the data structure and the developing resolution constructed in the first step of the procedure, in addition to various possibilities parallel to the ones handled in the first step. As in the first step of the procedure, our aim is to obtain a strict decrease in either the Zariski closure or the complexity of the resolution associated with some level of the data structure we construct.

(1) Let $Q(y,a)$ be the restricted limit group generated by $< y,a >$ in the anvil $Anv(MGQRes)(w,y,a)$. Let $Q^2(y,a)$ be the limit group generated by $< y,a >$ in the second quotient limit group $Q^2Rlim(w,y,a)$. If $Q^2(y,a)$ is a proper quotient of the subgroup $Q(y,a)$, we continue this branch of the iterative procedure by replacing the limit group $Q(y,a)$ by $Q^2(y,a)$, and continue with the finite collection of resolutions that appear in the taut Makanin–Razborov diagram of the limit group $Q^2(y,a)$ in the same way we handled the resolutions in the taut Makanin–Razborov diagram of the original limit group $Rlim(y,a)$. Note that this is always the case if the limit group $Q(y,a)$ was a free group.

(2) At this stage we may assume that $Q^2(y,a)$ is isomorphic to $Q(y,a)$. In this part we assume that the multi-graded quotient resolution

$$MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$$

with which the anvil, $Anv(MGQRes)$, is associated, is of maximal possible complexity. Let

$$MGQ^2Res_1(w,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a),\dots,$$
$$MGQ^2Res_q(w,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$$

be the multi-graded resolutions in the multi-graded taut Makanin–Razborov diagram of $Q^2Rlim(w,y,a)$ with respect to the subgroups $Base^1_{2,1},\dots,Base^1_{2,v_1}$. We will treat the multi-graded quotient resolutions

$$MGQ^2Res_j(w,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$$

in parallel, hence we omit their index.

By Proposition 4.2, the complexities of the abelian decompositions associated with the various levels of each (second) multi-graded quotient resolution $MGQ^2Res(w,y,Base^1_{2,1},\dots,Base^1_{2,v_1}a)$ is bounded by the complexity of the multi-graded quotient abelian decomposition associated with the top level of $MGQRes(s,z,y,Base^1_{2,1},\ldots,Base^1_{2,v_1},a)$, and if the complexity of the abelian decomposition associated with some level of

$$MGQ^2Res(w,y,Base^1_{2,1},\ldots,Base^1_{2,v_1},a)$$

is equal to the complexity of the abelian decomposition associated with the top level of $MGQRes(s,z,y,Base^1_{2,1}\ldots,Base^1_{2,v_1},a)$, then the second quotient resolution

$$MGQ^2Res(w,y,Base^1_{2,1},\ldots,Base^1_{2,v_1},a)$$

has only one level above the terminating solid or rigid limit group, and the structure of the abelian decomposition associated with this level is identical to the structure of the abelian decomposition associated with the top level of $MGQRes(s,z,y,Base^1_{2,1},\ldots,Base^1_{2,v_1},a)$. In this part of the second step of the procedure we will also assume that the complexities of the abelian decompositions associated with the various levels of the second quotient multi-graded resolution $MGQ^2Res(w,y,Base^1_{2,1},\ldots,Base^1_{2,v_1},a)$ are strictly smaller than the complexity of the abelian decomposition associated with the top level of the multi-graded quotient resolution

$$MGQRes(s,z,y,Base^1_{2,1},\ldots,Base^1_{2,v_1},a)$$

associated with the anvil. In this case we treat the (second) multi-graded resolution $MGQ^2Res(w,y,Base^1_{2,1},\ldots,Base^1_{2,v_1},a)$ as in part (2) of the first step of the procedure, and associate with it finitely many anvils, which we denote $Anv(MGQ^2Res)$, and with each anvil we associate a developing resolution precisely as we did in part (2) of the first step of the procedure.

(3) At this stage we may assume that $Q^2(y,a)$ is isomorphic to $Q(y,a)$. In this part we assume that the multi-graded resolution

$$MGQRes(s,z,y,Base^1_{2,1},\ldots,Base^1_{2,v_1},a)$$

is not of maximal complexity. Let $Base^2_{2,1},\ldots,Base^2_{2,v_2}$ be the non-abelian, non-QH vertex groups in the multi-graded abelian decomposition associated with the top level of the anvil, $Anv(MGQRes)(w,y,a)$. Note that the subgroups $Base^1_{2,1},\ldots,Base^1_{2,v_1}$, which are all subgroups of the closure $Cl(Res)(s,z,y,a)$, are naturally mapped into conjugates of the subgroups $Base^2_{2,1},\ldots,Base^2_{2,v_2}$. Let

$$MGQ^2Res_1(w,y,Base^2_{2,1},\ldots,Base^2_{2,v_2},a),\ldots,$$
$$MGQ^2Res_q(w,y,Base^2_{2,1},\ldots,Base^2_{2,v_2},a)$$

be the resolutions in the taut Makanin–Razborov diagram of the limit group $Q^2Rlim(w,y,a)$ with respect to the subgroups

$$Base^2_{2,1},\dots,Base^2_{2,v_2}.$$

As in the first step of the iterative procedure, we do not really need to consider all the multi-graded resolutions in this taut Makanin–Razborov diagram, but only those that are compatible with the indicated collections of s.c.c. that are being mapped to the trivial element on each of the QH vertex groups in the multi-graded resolution

$$MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$$

with which we started this branch of the second step. Since we treat these multi-graded quotient resolutions in parallel, we will omit their indices in the sequel.

Let $MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_2},a)$ be one of these multi-graded quotient resolutions. Let $Q(s,z,y,a)$ be the subgroup generated by $<s,z,y,a>$ in the anvil $Anv(MGQRes)(w,y,a)$, and let $Q^2(s,z,y,a)$ be the subgroup generated by $<s,z,y,a>$ in the limit group corresponding to the multi-graded quotient resolution

$$MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_2},a).$$

By construction, $Q^2(s,z,y,a)$ is a quotient of $Q(s,z,y,a)$. If $Q^2(s,z,y,a)$ is a proper quotient of $Q(s,z,y,a)$, we replace the multi-graded quotient resolution

$$MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_2},a)$$

by starting the first step of the procedure with the subgroup generated by $<s,z,y,a>$ in the completion of

$$MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_2},a),$$

$Q^2(s,z,y,a)$. Since the multi-graded quotient resolution

$$MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$$

we started with is not of maximal possible complexity, in analyzing the limit group $Q^2(s,z,y,a)$ we treat only those multi-graded resolutions $MGQRes_j(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$ which are not of maximal possible complexity, i.e., we do not (need to) treat those multi-graded quotient resolutions of $Q^2(s,z,y,a)$ which have only one level above the terminating rigid or solid limit group, and the structure of the abelian decomposition associated with that level is identical to the structure of the abelian decomposition associated with the completion, $Comp(Res)(z,y,a)$, with which we started.

(4) With the notation of part (3), in this part we assume that the multi-graded quotient resolution $MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$ is not of maximal possible complexity, and that for our given second quotient resolution $MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_2},a)$, the corresponding subgroup $Q^2(s,z,y,a)$ is isomorphic to $Q(s,z,y,a)$. In this part we assume that the (second) graded quotient resolution

$$MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_2},a)$$

is not of maximal possible complexity, i.e., it does not have a single level with a (multi-graded) abelian decomposition that contains similar QH and abelian vertex groups as the ones that appear in the abelian decomposition associated with the top level of the multi-graded quotient resolution $MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$. The case of maximal complexity will be treated in the next part. To treat a (second) multi-graded quotient resolution which is not of maximal possible complexity, we need the following observation, which is similar to Proposition 4.3.

LEMMA 4.6: *Let $MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_2},a)$ be one of the multi-graded quotient resolutions in our list of second multi-graded quotient resolutions, and suppose that MGQ^2Res is not of maximal complexity. Let $MGQ^2lim_{term}(w,y,a)$ be the terminal rigid or solid limit group of the multi-graded quotient resolution MGQ^2Res, and let $Q^2_{term}(s,z,y,a)$ be the image of $Q^2(s,z,y,a)$ in the limit group $MGQ^2lim_{term}(w,y,a)$. Then $Q^2_{term}(s,z,y,a)$ is a proper quotient of $Q^2(s,z,y,a)$.*

Proof: The proof is similar to the proof of Proposition 4.3. If $Q^1(s,z,y,a)$ is a free product of free groups, free abelian groups, and (closed) surface groups (and not the coefficient group F_k, in which case the procedure terminates), $Q^2(s,z,y,a)$ must be a proper quotient of $Q^1(s,z,y,a)$ and part (1) or part (3) of the second step forces us to restart the procedure with $Q^2(s,z,y,a)$ or $Q^2(y,a)$. Hence, we may assume that the groups $Base^2_{2,1},\dots,Base^2_{2,v_2}$ are not trivial.

Clearly, since $Q^2_{term}(y,a) < Q^2_{term}(s,z,y,a)$, to prove the lemma we may assume that $Q^2_{term}(y,a)$ is isomorphic to $Q^2(y,a)$, which is assumed to be isomorphic to $Q^1(y,a)$ by part (1).

Let $Term(w,y,a)$ be the terminal rigid or solid limit group of the multi-graded resolution

$$MGQRes(w,y,Base^2_{2,1},\dots,Base^2_{2,v_2},a).$$

Let $\{(w_n,y_n,a)\}$ be a sequence of rigid, or shortest solid specializations of the terminal limit group $Term(w,y,a)$, that converge into the limit group $Term(w,y,a)$. By passing to a subsequence, we may assume that the sequence $\{(w_n,y_n,a)\}$ converges into a non-trivial, stable and faithful action of $Term(w,y,a)$ on a real tree Y, with abelian edge stabilizers. Since the sequence of specializations are either rigid or shortest solid, at least one of the subgroups $Base^2_{2,1},\dots,Base^2_{2,v_2}$ is not elliptic when acting on Y.

Since $Q^2_{term}(y,a)$ is assumed to be isomorphic to $Q^1(y,a)$, every non-abelian, non-QH vertex group in the abelian JSJ decomposition of $Q^2_{term}(y,a)$ fixes a point in Y, and since we assume that the specializations of the subgroup $Q^2(s,z,y,a)=Q^1(s,z,y,a)$ are shortest form, the subgroups $Base^1_{2,1},\dots,Base^1_{2,v_1}$ are all elliptic when acting on Y.

Since the specializations used to construct the quotient limit group $Q^2(w,y,a)$ and the multi-graded resolution, $MGQRes$, are assumed to be shortest form specializations (Definition 4.1), if all the subgroups $Base^1_{2,1},\dots,Base^1_{2,v_1}$ are elliptic when acting on Y and $Q^2(s,z,y,a)$ is isomorphic to $Q^1(s,z,y,a)$, then all the non-abelian, non-QH vertex groups in the multi-graded abelian JSJ decomposition of $Q^2(s,z,y,a)$ with respect to the subgroups $Base^1_{2,1},\dots,Base^1_{2,v_1}$ are elliptic when acting on Y, and therefore all the subgroups $Base^2_{2,1},\dots,Base^2_{2,v_2}$ are elliptic when acting on Y, a contradiction. ∎

By Lemma 4.6, the image of $Q^2(s,z,y,a)$ in the terminal rigid or solid limit group $MGQ^2lim_{term}(w,y,a)$ of MGQ^2Res is a proper quotient of $Q^2(s,z,y,a)$. To continue our treatment of the (second) multi-graded resolution MGQ^2Res we need the following observation, which is similar to Proposition 4.4.

Lemma 4.7: *Let MGQ^2Res be a (second) multi-graded quotient resolution from our list of (second) quotient multi-graded resolutions, and suppose MGQ^2Res is not of maximal complexity. Let*

$$Q^2(y,a),Q^2(s,z,y,a),Q^2(w,y,a)$$

be the subgroups generated by

$$<y,a>,<s,z,y,a>,<w,y,a>$$

in correspondence, in the limit group associated with the second multi-graded quotient resolution MGQ^2Res. *Let* $Q^2_2(y,a), Q^2_2(s,z,y,a), Q^2_2(w,y,a)$ *be the images in the limit group associated with the second level of* MGQ^2Res,

$$MGQ^2lim_2(w,y,a),$$

of the subgroups $Q^2(y,a), Q^2(s,z,y,a), Q^2(w,y,a)$ *in correspondence. Then* $Q^2_2(y,a)$ *is a quotient of* $Q^2(y,a)$, $Q^2_2(s,z,y,a)$ *is a quotient of the subgroup* $Q^2(s,z,y,a)$, *and* $Q^2_2(w,y,a)$ *is a proper quotient of the subgroup* $Q^2(w,y,a)$.

Proof: The claim is simply one of the basic properties of a multi-graded resolution. ∎

Suppose that the subgroup $Q^2_2(y,a)$ is a proper quotient of $Q^2(y,a)$. In this case we continue as we did in part (2) of the first step. Let

$$QRes_1(y,a),\dots,QRes_q(y,a)$$

be the resolutions in the taut Makanin–Razborov diagram of $Q^2_2(y,a)$. We continue with each of the resolutions $QRes_i(y,a)$ separately, so we omit its index. With the resolution $QRes(y,a)$ we associate a resolution $CRes(y,a)$, where $CRes(y,a)$ is composed of two parts, the top being the resolution induced (according to the construction presented in Section 3) by the subgroup $< y,a >$ from the completion of the top level of the second multi-graded quotient resolution

$$MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_2},a),$$

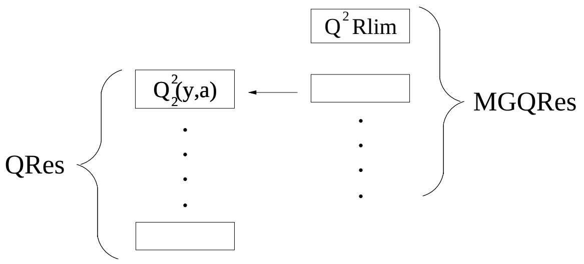
and the tail being the resolution $QRes(y,a)$. By theorem 1.18 of [Se2], with the resolution $CRes(y,a)$ we can associate a set of closures

$$Cl_1(CRes)(u,y,a),\dots,Cl_d(CRes)(u,y,a)$$

and for each closure a corresponding formal solution defined over it. We continue with each of these closures in parallel.

If the subgroup generated by $< y,a >$ in one of these closures is a proper quotient of the limit group $Q^2(y,a)$ with which we started, we replace the particular closure by starting the first step of the procedure with the subgroup generated by $< y,a >$ in it. Hence, for the continuation we can assume that the subgroup generated by $< y,a >$ in a closure is isomorphic to the limit group $Q^2(y,a)$ with which we started.

Fixing a closure $Cl(u,y,a)$, we set the developing resolution to be that closure. With the closure $Cl(u,y,a)$, there is an associated formal solution $x(u,y,a)$ according to theorem 1.18 of [Se2] that we associate with the developing resolution. We further set finitely many anvils associated with a developing resolution to be the maximal limit quotients of the amalgamated product of the completion of the developing resolution and the completion of the top level of the (second) quotient multi-graded resolution $MGQ^2Res(w,y,Base^2_{2,1},\ldots,Base^2_{2,v_2},a)$, amalgamated along the resolution induced by the subgroup $< y,a >$ from the top level of completion of the multi-graded resolution $MGQRes$, and denote them $Anv(MGQ^2Res)(w,y,a)$. Note that the developing resolution is canonically mapped into the anvil, hence the formal solution defined over the developing resolution can be naturally defined over the anvil.



Suppose that $Q^2_2(y,a)$ is isomorphic to $Q^2(y,a)$, and $Q^2_2(s,z,y,a)$ is a proper quotient of $Q^2(s,z,y,a)$.

With the subgroup $Q^2_2(s,z,y,a)$, we associate the multi-graded resolutions that appear in its multi-graded taut Makanin–Razborov diagram with respect to the subgroups $Base^1_{2,1},\ldots,Base^1_{2,v_1}$:

$$MGQRes_1(s,z,y,Base^1_{2,1},\ldots,Base^1_{2,v_1},a),\ldots,\\ MGQRes_t(s,z,y,Base^1_{2,1},\ldots,Base^1_{2,v_1},a).$$

We continue with each of the multi-graded resolutions $MGQRes_j$ in parallel.

Suppose that a multi-graded quotient resolution $MGQRes_j$ is of maximal possible complexity, i.e., it consists of one level with an abelian decomposition which has the same QH and abelian vertex groups as in the abelian decomposition associated with the top level of the resolution $Res(y,a)$. The group $Q^2_2(s,z,y,a)$ is naturally mapped into the limit group $Q^2Rlim(MGQRes_j)(s,z,y,a)$ corresponding to the multi-graded resolution $MGQRes_j$. Since $Q^2_2(s,z,y,a)$ is a proper quotient of $Q^2(s,z,y,a)$ by our assumption, and since the multi-graded resolution $MGQRes_j$ is of maximal possible complexity, and since limit groups are residually finite, so they are necessarily Hopf, the image of the subgroup $Zlim_2$ associated with the second level of the (original) closure, $Cl(Res)(s,z,y,a)$, in the limit group associated with the multi-graded resolution $MGQRes_j$ is necessarily a proper quotient of the image of $Zlim_2$ in $Q^2Rlim(w,y,a)$. Hence, we can replace the resolution $MGQRes_j$ by starting the first step of the procedure with the subgroup generated by $< s,z,y,a >$ in the limit group corresponding to the resolution induced by the subgroup $< s,z,y,a >$ from the top level of the multi-graded quotient resolution $MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_2},a)$ continued with the resolution $MGQRes_j$, which is necessarily a proper quotient of the limit group $QRlim(s,z,y,a)$ with which we started. As we remarked in part (3), we need to treat only those multi-graded resolutions of the limit group $< s,z,y,a >$ that are not of maximal possible complexity.

If the subgroup generated by $< s,z,y,a >$ in the limit group corresponding to the graded resolution $CRes_j(u,y,a)$, where $CRes_j(u,y,a)$ is the multi-graded resolution obtained from the resolution induced by the subgroup $< s,z,y,a >$ from the top level of the completion of the multi-graded resolution $MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_2},a)$, followed by the multi-graded resolution $MGQRes_j(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$, is a proper quotient of $Q^2(s,z,y,a)$, we replace the multi-graded resolution $MGQRes_j$ by starting the first step of the procedure with the subgroup generated by $< s,z,y,a >$ in the limit group corresponding to the resolution $CRes_j(u,y,a)$, and treat only those multi-graded resolutions of this limit group that are not of maximal possible complexity. Hence, we may assume that for the rest of this part, the subgroup generated by $< s,z,y,a >$ in the limit group corresponding to $CRes_j(u,y,a)$ is isomorphic to $Q^2(s,z,y,a)$. In particular, we may assume that each of the multi-graded resolutions $MGQRes_j$ in question is not of maximal possible complexity.

We now treat each of the multi-graded resolutions

$$MGQRes_j(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$$

and their associated resolutions $CRes_j(u,y,a)$ in a similar way to our treatment of multi-graded quotient resolutions in the first step of the procedure. Every QH vertex group Q in the abelian decomposition associated with the top level of the resolution $Res(y,a)$ with which we started this branch of the procedure inherits a sequence of abelian decompositions from the various levels of the resolution $CRes_j(u,y,a)$. Since the resolution $Res(y,a)$ is well-separated, with each such QH vertex group Q, there is an associated collection of s.c.c. that are mapped into the trivial element in the next level of the multi-graded resolution. If for some such QH vertex group Q, its sequence of inherited abelian decompositions is not compatible with its associated collection of s.c.c. that are mapped into the trivial element in the next level of the resolution $Res(y,a)$, we omit the multi-graded resolution $MGQRes_j$ from our list of multi-graded resolutions. Hence, for the rest of this part we assume the resolution $CRes_j(u,y,a)$ is compatible with the collections of s.c.c. associated with each of the QH vertex groups in the abelian decomposition associated with the top level of the resolution $Res(y,a)$.

At this point we apply the procedure presented in part (2) of the first step of our iterative procedure, and associate with the multi-graded resolution $CRes_j$ (obtained from the resolution $MGQRes_j$) a finite collection of anvils and developing resolutions. Given a developing resolution, and an anvil associated with it (constructed according to part (2) of the second step), associated with the resolution $CRes_j$, we set the developing resolution associated with the multi-graded resolution

$$MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_2},a)$$

to be the developing resolution associated with $CRes_j$. With the developing resolution, we associate a finite collection of anvils, denoted $Anv(MGQ^2Res)(w,y,a)$, that are set to be the (finite collection of) maximal limit quotients of the group obtained as the amalgamated product of the completion of the top level of the (second) multi-graded quotient resolution MGQ^2Res, with the anvil associated with $CRes_j$, along the subgroup $Q^2_2(s,z,y,a)$ that is naturally mapped into both. With the developing resolution we associate a formal solution constructed according to theorem 1.18 of [Se2], that is naturally defined over the anvil as well.

We still need to treat the case in which both $Q^2_2(y,a)$ is isomorphic to $Q^2(y,a)$ and $Q^2_2(s,z,y,a)$ is isomorphic to $Q^2(s,z,y,a)$. In this case, we just continue to the next level of the multi-graded quotient resolution $MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_2},a)$, and proceed with the same analysis we applied for the top level of this multi-graded resolution. By Lemma 4.6, there must exist some level j of the second multi-graded resolution MGQ^2Res for which either $Q^2_j(y,a)$ is a proper quotient of $Q^2(y,a)$

or $Q^2_j(s,z,y,a)$ is a proper quotient of $Q^2(s,z,y,a)$. If level j is the highest level of the multi-graded resolution MGQ^2Res for which such proper quotients are obtained, we repeat the procedure described above and associate with the multi-graded resolution MGQ^2Res a finite collection of anvils and developing resolutions.

(5) By part (1) we may assume that $Q^2(y,a)$ is isomorphic to the limit group $Q(y,a)$ associated with the anvil, $Anv(MGQRes)(w,y,a)$, and by part (3) we may assume that $Q^2(s,z,y,a)$ is isomorphic to the subgroup $Q(s,z,y,a)$ associated with the anvil. Part (4) treats the case in which the second multi-graded quotient resolution

$$MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_2},a)$$

is not of maximal complexity. Hence, the only case left in presenting the second step of our general procedure for validation of a sentence, is the case of a second multi-graded quotient resolution

$$MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_2},a)$$

(or $MGQ^2Res(w,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$ in case $MGQRes$ is of maximal possible complexity — see part (2)) of maximal possible complexity, i.e., a multi-graded (second) quotient resolution

$$MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_1},a)$$

consists of a single level, and the QH and abelian vertex groups in the abelian decomposition associated with this level are similar to the ones that appear in the abelian decomposition associated with the top level of the completion of the multi-graded quotient resolution

$$MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$$

with which we started the second step of the procedure.

We treat this case as we treated it in the minimal rank case and in the first step of the general procedure. In case the abelian decompositions associated with $MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_2},a)$ and the top level of (the completion of) $MGQRes(s,z,y,Base^1_{2,1},\dots,Base^1_{2,v_1},a)$ have the same complexity, we use the modular groups associated with the abelian decomposition associated with $MGQ^2Res(w,y,Base^2_{2,1},\dots,Base^2_{2,v_1},a)$ to map the subgroup $<w,y,a>$ into the limit group associated with the (appropriate) next level of the corresponding part of the anvil

$$Anv(MGQRes)(w,y,a).$$

By construction, either the image of the subgroup generated by $< y, a >$ in that limit group is a proper quotient of $Q^2(y, a)$, or the subgroup generated by $< s, z, y, a >$ in that limit group is a proper quotient of $Q^2(s, z, y, a)$. We now analyze the remaining part of the anvil in the same way we have analyzed the second multi-graded quotient resolution, according to parts (1)–(5) above.

As long as the abelian decompositions associated with the constructed multi-graded quotient resolutions are of maximal possible complexity, we continue analyzing multi-graded quotient resolutions associated with various parts of the anvil with which we started, until one of the parts (1)–(4) applies to the corresponding multi-graded quotient resolution. If there exists a level for which one of the parts (1)–(4) applies, we construct a developing resolution and an anvil with the limit group associated with this level according to the part (1)–(4) that applies to it.

We construct a developing resolution associated with the resolution MGQ^2Res we started with to be the resolution induced by the subgroup $< y, a >$ from the parts of the resolution above that level followed by the developing resolution associated with the level for which one of the steps (1)–(4) applies. With it we associate a finite collection of anvils, denoted $Anv(MGQ^2Res)(w, y, a)$, that are set to be the maximal limit quotients of the group obtained as the amalgamated product of the anvil associated with the level for which one of the steps (1)–(4) applies, and the completion of the multi-graded resolutions above that level, amalgamated along the copies of the limit group associated with the level to which one of the steps (1)–(4) was applied. With the developing resolution and its associated anvil, we naturally associate a formal solution according to theorem 1.18 of [Se2].

If all the abelian decompositions associated with the multi-graded resolutions used for the construction of the developing resolution are of maximal complexity, i.e., if none of the parts (1)–(4) applies to any of these multi-graded resolutions, we examine the structure of the developing resolution. The developing resolution is built from a sequence of induced resolutions. Each of the induced resolutions is a resolution induced by the (image of the) subgroup $< y, a >$, and with each level of the induced resolution there is associated an (induced) abelian decomposition (see Section 3 for the construction of the induced resolution).

Proposition 4.8: *Suppose that all the abelian decompositions associated with* *the multi-graded resolutions used for the construction of the developing resolution are of maximal possible complexity. Let* $< v, y, a >$ *be the subgroup generated by the closure of the developing resolution in the anvil,*

$$Anv(MGQRes)(w, y, a).$$

From each of the multi-graded resolutions used to construct the developing resolution, there is a resolution induced by the (image of the) subgroup (generated by) $< v, y, a >$.

Then there exists some level j *such that the structure of the abelian decompositions associated with the resolutions induced by the subgroup* $< v, y, a >$ *above level* j *is identical to the structure of the abelian decompositions associated with the corresponding resolution induced by the subgroup* $< y, a >$*, and in level* j*, either the number of factors in the free decomposition associated with the abelian decomposition associated with the resolution induced by* $< v, y, a >$ *is strictly smaller than the number of factors in the corresponding free decomposition associated with the resolution induced by the subgroup* $< y, a >$*, or in case of equality in the number of factors, the complexity of the abelian decomposition associated with the resolution induced by* $< v, y, a >$ *is strictly smaller than the complexity of the abelian decomposition associated with the resolution induced by* $< y, a >$.

Proof: The subgroup generated by $< y, a >$ is naturally embedded in the subgroup generated by $< v, y, a >$ in the anvil $Anv(MGQRes)(w, y, a)$. If the structure of the developing resolution associated with the anvil, $Anv(MGQRes)$, with which we started the second step, has the same structure as the resolution induced by the subgroup $< v, y, a >$ from the various multi-graded resolutions constructed along the various levels in the second step of the procedure, then the closure of the developing resolution associated with $Anv(MGQRes)(w, y, a)$ is embedded in the quotient limit group $QRlim(w, y, a)$ with which we started the second step of the procedure, a contradiction to the properties of the formal solution associated with the developing resolution that is associated with the anvil, $Anv(MGQRes)(w, y, a)$. Hence, the structure of the resolutions induced by the subgroup $< v, y, a >$ differs from the structure of the developing resolution associated with the anvil, $Anv(MGQRes)$.

As in the proof of the termination of the construction of the induced resolution (Proposition 3.3), if we restrict ourselves to the highest level of the developing resolution, properties (1)–(3) of the construction of the induced resolution (see the construction in Section 3) imply that if we restrict ourselves to the highest level, if the decomposition associated with the subgroup $< v, y, a >$ differs from the decomposition associated with the subgroup $< y, a >$, then either it contains fewer connected components, or in case of equality, the complexity of the decompositions associated with the subgroup $< v, y, a >$ is strictly smaller than the complexity of the decompositions associated with the subgroup generated by $< y, a >$. In case of equality in the complexities of the decompositions associated with the subgroups $< v, y, a >$ and $< y, a >$, we continue to the next level until we get to a level in which either a decrease in the number of connected components or a decrease in the complexity of the associated decompositions occurs. This completes the proof of the proposition. ∎

> Using Proposition 4.8, in case all the multi-graded resolutions used in the construction of the developing resolution are of maximal possible complexity, we replace the resolutions induced by the subgroup $< y, a >$ used to construct the developing resolution, by the resolutions induced by the subgroup $< v, y, a >$. With the obtained (modified) developing resolution we associate a covering closure according to theorem 1.18 of [Se2], with each closure we naturally associate an anvil, denoted $Anv(MGQ^2Res(w, y, a)$, obtained from the anvil $Anv(MGQRes)(w, y, a)$ by adding the appropriate roots corresponding to the specific closure, and with the closure and its associated anvil we associate a formal solution $x(w, y, a)$.

Given the developing resolution, the anvil, and the formal solution defined over them, we construct the data structure according to Definition 4.5. With the anvil constructed in the second step of the iterative procedure we have associated a formal solution $\{x(w, y, a)\}$. We start the third step of the iterative procedure by imposing on the specializations that factor through, and are taut and shortest form with respect to the multi-graded quotient resolutions associated with the anvil, the (finitely many) systems of equations $\psi_j(x(w, y, a), y, a) = 1$.

III. The general step. In the first steps of the iterative procedure for validation of a sentence we used formal solutions defined over the entire set of y's, and then families of formal solutions defined over closures of the various resolutions of the "remaining" limit groups $Rlim_1(y, a), \dots, Rlim_m(y, a)$ and closures of developing resolutions associated with quotient resolutions obtained from these resolutions. At each step, we finally obtained finitely many data structures and their associated developing resolutions and anvils. With each tuple of a developing resolution and anvil, we associated a formal solution defined over the developing resolution, which is embedded into the anvil. By construction, each specialization of the variables y remaining after the first steps can be extended to a (shortest form) specialization that factors through at least one of the modular blocks associated with the various anvils constructed in these steps. After presenting the first steps, we finally present the general step of the procedure for validation of a sentence, and then prove it terminates after finitely many steps.

We define the general step of the procedure inductively. For brevity, we denote the multi-graded resolutions that were obtained in the previous steps of the procedure, $MGQ^mRes(w,y,a)$, where m is the index of the step in which they were constructed. With each such multi-graded quotient resolution there is an associated developing resolution, and an anvil that we denote $Anv(MGQ^mRes)(w,y,a)$. By construction, each of the remaining specializations of the variables y, i.e., the specializations for which none of the formal solutions constructed in previous steps of the iterative procedure proves the validity of the given AE sentence for them, can be extended to a (shortest form) specialization that factors through at least one of the modular blocks associated with the various multi-graded quotient resolutions and their associated anvils. We start the general step of our iterative procedure for validation of a sentence with the (finite) collection of multi-graded quotient resolutions constructed in the previous step, and their associated anvils.

The ultimate goal of the general step of the iterative procedure is to obtain a strict reduction in either the complexity of certain decompositions and resolutions or a strict reduction in the Zariski closures of certain limit groups associated with the anvils constructed in the previous steps of the procedure. The strict reduction in complexity and Zariski closures will finally guarantee the termination of the iterative procedure after finitely many steps.

Since we treat the anvils in parallel, we present the general (n-th) step of the procedure with one of the anvils, $Anv(MGQ^{n-1}Res)(w,y,a)$. With each anvil we have associated a formal solution $x(w,y,a)$ that satisfies the properties listed in theorem 1.18 of [Se2]. Starting with the anvil $Anv(MGQ^{n-1}Res)(w,y,a)$, we use the formal solutions $x(w,y,a)$ associated with the corresponding closure of the developing resolution. We look at the set of specializations (w_0,y_0,a) that factor through and are taut and shortest form with respect to the resolutions associated with the anvil $Anv(MGQ^{n-1}Res)(w,y,a)$ (which includes its associated developing resolution), for which for some index j, $\psi_j(x(w_0,y_0,a),y_0,a)=1$. Clearly, if an element $y_0 \in F_k$ can be extended to a shortest form specialization that factors through one of the anvils, and this specialization does not satisfy any of these (finite) systems of equations, then the sentence is valid for

y_0. Hence, in the next steps of our iterative procedure it is enough to study these specializations of the universal variables y.

If we fix the index j of the equation $\psi_j(x(w,y,a),y,a) = 1$, then the entire collection of shortest form specializations (w_0,y_0,a) that satisfy the corresponding system of equations is contained in a finite set of maximal (n-th) limit groups $Q^nRlim_1(w,y,a),\dots,Q^nRlim_{u_j^n}(w,y,a)$. Since our analysis of these (n-th) quotient limit groups is conducted in parallel, we will omit the indices from these (n-th) quotient limit groups and denote them $Q^nRlim(w,y,a)$.

We construct the data structure and developing resolutions associated with the anvil and the n-th quotient limit group $Q^nRlim(w,y,a)$ iteratively, similarly to the way we have analyzed quotient resolutions in the second step of the procedure. The analysis we carry out in the general step depends on the structure of the data structure, and the developing resolutions constructed in the previous steps of the procedure. As in the first steps of the procedure, our aim is to obtain a strict decrease in either the Zariski closure or the complexity of the resolution associated with some level of the data structure we construct.

(1) Let $Q^{n-1}(y,a)$ be the restricted limit group generated by $< y,a >$ in the anvil $Anv(MGQ^{n-1}Res)(w,y,a)$. Let $Q^n(y,a)$ be the limit group generated by $< y,a >$ in the n-th quotient limit group $Q^nRlim(w,y,a)$. If $Q^n(y,a)$ is a proper quotient of the subgroup $Q^{n-1}(y,a)$, we continue this branch of the iterative procedure by replacing the limit group $Q^{n-1}(y,a)$ by $Q^n(y,a)$, and continue with the finite collection of resolutions that appear in the taut Makanin–Razborov diagram of the limit group $Q^n(y,a)$ in the same way we handled the resolutions in the taut Makanin–Razborov diagram of the original limit group $Rlim(y,a)$. Note that this is always the case if the limit group $Q^{n-1}(y,a)$ was a free group.

(2) At this stage we may assume that $Q^n(y,a)$ is isomorphic to $Q^{n-1}(y,a)$. Along the process used to construct the anvil, $Anv(MGQ^{n-1}Res)(w,y,a)$, we enlarge the parameter subgroups each time the complexity of the abelian decomposition associated with the top level of the corresponding multi-graded quotient resolution is being reduced. At step m, $1 \leq m \leq n-1$, we set the parameter subgroups to be $Base_{2,1}^{s(m)},\dots,Base_{2,v_{s(m)}}^{s(m)}$ and the corresponding multi-graded quotient resolution to be

$$MGQ^mRes(w_m,y,Base_{2,1}^{s(m)},\dots,Base_{2,v_{s(m)}}^{s(m)},a).$$

For each index s, $1 \leq s \leq s(n-1)$, we set $f(s)$ to be the minimal index m, $1 \leq m \leq n-1$, for which $s = s(m)$, and $\ell(s)$ to be the maximal index m for which $s = s(m)$. For each couple of indices m_1, m_2, $1 \leq m_1 \leq m_2 \leq n$, let $Q^{m_2}(w_{m_1},y,a)$ be the subgroup generated by $< w_{m_1},y,a >$ in the limit group $Q^{m_2}Rlim(w_{m_2},y,a)$.

In this part of the general step we assume that the multi-graded quotient resolution $MGQ^{n-1}Res(w,y,Base^{s(n-1)}_{2,1},\dots,Base^{s(n-1)}_{2,v_{s(n-1)}},a)$ is of maximal complexity. Suppose that for some index s, $1 \leq s \leq s(n-1)-1$, $Q^n(w_{\ell(s)},y,a)$ is a proper quotient of $Q^{\ell(s)}(w_{\ell(s)},y,a)$, and let s be the minimal index for which this happens. Then we omit the n-th limit group $Q^nRlim(w,y,a)$ from our list of n-th quotient limit groups, and replace it by going back to the $\ell(s)$-th step of the iterative procedure, and start it with the limit group $Q^n(w_{\ell(s)},y,a)$ instead of the $\ell(s)$-th limit group $Q^{\ell(s)}Rlim(w_{\ell(s)},y,a)$ used in the s-th step of the process that lead to the construction of the anvil, $Anv(MGQ^{n-1}Res)(w,y,a)$. Since, by definition of the index $\ell(s)$, the parameter subgroups were enlarged at step $\ell(s)+1$, in analyzing the quotient limit group $Q^nRlim(w_{\ell(s)},y,a)$ we need to take into account only those multi-graded quotient resolutions which are not of maximal complexity, i.e., only those multi-graded quotient resolutions which do not have a single level with a multi-graded abelian decomposition with the same structure as the abelian decomposition associated with the top level of the multi-graded quotient resolution

$$MGQ^{\ell(s)-1}Res(w_{\ell(s)-1},y,Base^{s(\ell(s)-1)}_{2,1},\dots,Base^{s(\ell(s)-1)}_{2,v_{s(\ell(s)-1)}},a)$$

used in the process of the construction of the anvil

$$Anv(MGQ^{n-1}Res)(w,y,a).$$

Suppose that for $s(n-1)-1$ (hence, for every index s, $1 \leq s \leq s(n-1)-1$), $Q^n(w_{\ell(s)},y,a)$ is isomorphic to $Q^{\ell(s)}(w_{\ell(s)},y,a)$. We set $s(n)=s(n-1)$. Let

$$MGQ^nRes_1(w,y,Base^{s(n)}_{2,1},\dots,Base^{s(n)}_{2,v_{s(n)}},a),\dots,$$
$$MGQ^nRes_q(w,y,Base^{s(n)}_{2,1},\dots,Base^{s(n)}_{2,v_{s(n)}},a)$$

be the multi-graded resolutions in the multi-graded taut Makanin–Razborov diagram of $Q^nRlim(w,y,a)$ with respect to the parameter subgroups

$$Base^{s(n)}_{2,1},\dots,Base^{s(n)}_{2,v_{s(n)}}.$$

We will treat the multi-graded quotient resolutions

$$MGQ^nRes_j(w,y,Base^{s(n)}_{2,1},\dots,Base^{s(n)}_{2,v_{s(n)}},a)$$

in parallel, hence we omit their index.

If for some QH vertex group Q in the abelian decomposition associated with the top level of the multi-graded resolution

$$MGQ^{n-1}Res(w,y,Base_{2,1}^{s(n-1)},\dots,Base_{2,v_{s(n-1)}}^{s(n-1)},a)$$

the sequence of abelian decompositions Q inherits from the multi-graded resolution $MGQ^nRes_j(w,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$ is not compatible with the collection of s.c.c. that are mapped to the trivial element in the next level of the multi-graded resolution

$$MGQ^{n-1}Res(w,y,Base_{2,1}^{s(n-1)},\dots,Base_{2,v_{s(n-1)}}^{s(n-1)},a),$$

we omit the multi-graded resolution

$$MGQ^nRes_j(w,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$$

from our list of multi-graded resolutions of the n-th quotient limit group $Q^n(w,y,a)$.

Otherwise, by Proposition 4.2, the complexities of the abelian decompositions associated with the various levels of each of the n-th multi-graded quotient resolutions $MGQ^nRes(w,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$ are bounded by the complexity of the multi-graded quotient abelian decomposition associated with the top level of

$$MGQ^{n-1}Res(w,y,Base_{2,1}^{s(n-1)},\dots,Base_{2,v_{s(n-1)}}^{s(n-1)},a);$$

and if the complexity of the abelian decomposition associated with some level of $MGQ^nRes(w,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$ is equal to the complexity of the abelian decomposition associated with the top level of

$$MGQ^{n-1}Res(w,y,Base_{2,1}^{s(n-1)},\dots,Base_{2,v_{s(n-1)}}^{s(n-1)},a),$$

then the n-th multi-graded quotient resolution

$$MGQ^nRes(w,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$$

has only one level above the terminating solid or rigid limit group, and the structure of the abelian decomposition associated with this level is identical with the structure of the abelian decomposition associated with the top level of $MGQ^{n-1}Res(w,y,Base_{2,1}^{s(n-1)},\dots,Base_{2,v_{s(n-1)}}^{s(n-1)},a)$. In this part of the n-th step of the procedure we will also assume that the complexities of the abelian decompositions associated with the various levels of the n-th multi-graded quotient resolution

$$MGQ^n Res(w, y, Base^{s(n)}_{2,1}, \ldots, Base^{s(n)}_{2,v_{s(n)}}, a)$$

are strictly smaller than the complexity of the abelian decomposition associated with the top level of the multi-graded quotient resolution

$$MGQ^{n-1} Res(w, y, Base^{s(n-1)}_{2,1}, \ldots, Base^{s(n-1)}_{2,v_{s(n-1)}}, a).$$

In this case we treat the multi-graded resolution

$$MGQ^n Res(w, y, Base^{s(n)}_{2,1}, \ldots, Base^{s(n)}_{2,v_{s(n)}}, a)$$

according to part (4) of step $n-1$ of the procedure, and associate with it a finite collection of developing resolutions and anvils.

(3) At this stage we may assume that $Q^n(y, a)$ is isomorphic to $Q^{n-1}(y, a)$. In this part we assume that the multi-graded quotient resolution

$$MGQ^{n-1} Res(w, y, Base^{s(n-1)}_{2,1}, \ldots, Base^{s(n-1)}_{2,v_{s(n-1)}}, a)$$

is not of maximal possible complexity.

Suppose that for some index s, $1 \leq s \leq s(n-1)$, $Q^n(w_{\ell(s)}, y, a)$ is a proper quotient of $Q^{\ell(s)}(w_{\ell(s)}, y, a)$, and suppose that s is the minimal index for which this happens. Then we omit the n-th limit group $Q^n Rlim(w, y, a)$ from our list of n-th quotient limit groups, and replace it by going back to the $\ell(s)$-th step of the iterative procedure and start it with the limit group $Q^n(w_{\ell(s)}, y, a)$, the subgroup generated by $< w_{\ell(s)}, y, a >$ in the n-th quotient limit group $Q^n Rlim(w, y, a)$, instead of the $\ell(s)$-th limit group $Q^{\ell(s)} Rlim(w_{\ell(s)}, y, a)$ used in the $\ell(s)$-th step of the process that leads to the construction of the anvil, $Anv(MGQ^{n-1}Res)(w, y, a)$. Since, by definition of the index $\ell(s)$, in case $\ell(s) < n-1$ the parameter subgroups were enlarged at step $\ell(s)+1$, and in case $\ell(s) = n-1$ the multi-graded quotient resolution

$$MGQ^{n-1} Res(w, y, Base^{s(n-1)}_{2,1}, \ldots, Base^{s(n-1)}_{2,v_{s(n-1)}}, a)$$

is not of maximal possible complexity, in analyzing the quotient limit group $Q^n Rlim(w_{\ell(s)}, y, a)$ we need to take into account only those multi-graded quotient resolutions which are not of maximal complexity, i.e., only those multi-graded quotient resolutions which do not have a single level with a multi-graded abelian decomposition with the same structure as the abelian decomposition associated with the top level of the multi-graded quotient resolution

$$MGQ^{\ell(s)-1}Res(w_{\ell(s)-1}, y, Base_{2,1}^{s(\ell(s)-1)}, \ldots, Base_{2,v_{s(\ell(s)-1)}}^{s(\ell(s)-1)}, a)$$

used in the process of the construction of the anvil

$$Anv(MGQ^{n-1}Res)(w, y, a).$$

(4) In this part we may assume that the multi-graded quotient resolution $MGQ^{n-1}Res(w, y, Base_{2,1}^{s(n-1)}, \ldots, Base_{2,v_{s(n-1)}}^{s(n-1)}, a)$ is not of maximal complexity, and that $Q^n(w_{n-1}, y, a)$ is isomorphic to $Q^{n-1}(w_{n-1}, y, a)$. We set $s(n) = s(n-1)+1$, and the parameter subgroups $Base_{2,1}^{s(n)}, \ldots, Base_{2,v_{s(n)}}^{s(n)}$ to be the non-abelian, non-QH vertex groups and edge groups in the multi-graded abelian decomposition associated with the top level of the anvil $Anv(MGQ^{n-1}Res)(w, y, a)$. Let

$$MGQ^nRes_1(w, y, Base_{2,1}^{s(n)}, \ldots, Base_{2,v_{s(n)}}^{s(n)}, a), \ldots,$$
$$MGQRes_q(w, y, Base_{2,1}^{s(n)}, \ldots, Base_{2,v_{s(n)}}^{s(n)}, a)$$

be the resolutions in the taut multi-graded Makanin–Razborov diagram of $Q^nRlim(w, y, a)$ with respect to the parameter subgroups

$$Base_{2,1}^{s(n)}, \ldots, Base_{2,v_{s(n)}}^{s(n)}.$$

If a multi-graded resolution $MGQ^nRes_j(w, y, Base_{2,1}^{s(n)}, \ldots, Base_{2,v_{s(n)}}^{s(n)}, a)$ is not compatible with the collections of s.c.c. associated with the various QH vertex groups in the abelian decomposition associated with the top level of the multi-graded resolution

$$MGQ^{n-1}Res(w, y, Base_{2,1}^{s(n-1)}, \ldots, Base_{2,v_{s(n-1)}}^{s(n-1)}, a),$$

and is mapped to a trivial element in the next level of that multi-graded resolution, we omit the multi-graded resolution

$$MGQ^nRes_j(w, y, Base_{2,1}^{s(n)}, \ldots, Base_{2,v_{s(n)}}^{s(n)}, a)$$

from our list of n-th quotient multi-graded resolutions. We analyze the n-th multi-graded quotient resolutions

$$MGQ^nRes_j(w, y, Base_{2,1}^{s(n)}, \ldots, Base_{2,v_{s(n)}}^{s(n)}, a)$$

in parallel, hence we will omit their index.

In this part we will also assume that the n-th multi-graded quotient resolution $MGQ^nRes(w,y,Base_{2,1}^{s(n)},\ldots,Base_{2,v_{s(n)}}^{s(n)},a)$ is not of maximal possible complexity, i.e., it does not have a single level with a (multi-graded) abelian decomposition identical to the abelian decomposition associated with the top level of the multi-graded quotient resolution

$$MGQ^{n-1}Res(w,y,Base_{2,1}^{s(n)},\ldots,Base_{2,v_{s(n)}}^{s(n)},a).$$

The case of maximal complexity will be treated in the next part of the general step. As in the second step of the procedure, to treat an n-th multi-graded quotient resolution which is not of maximal possible complexity we need the following observation, which is similar to Lemma 4.6 and Theorem 4.3.

LEMMA 4.9: *Let $MGQ^nRes(w,y,Base_{2,1}^{s(n)},\ldots,Base_{2,v_{s(n)}}^{s(n)},a)$ be one of the resolutions in our list of n-th multi-graded quotient resolutions and suppose MGQ^nRes is not of maximal complexity. By construction, the limit group $Q^n(w_{n-1},y,a)$ is mapped into the limit group associated with each of the levels of the multi-graded quotient resolution*

$$MGQ^nRes(w,y,Base_{2,1}^{s(n)},\ldots,Base_{2,v_{s(n)}}^{s(n)},a).$$

Let $Q^n_{term}(w_{n-1},y,a)$ be the image of $Q^n(w_{n-1},y,a)$ in the terminal (rigid or solid) limit group of

$$MGQ^nRes(w,y,Base_{2,1}^{s(n)},\ldots,Base_{2,v_{s(n)}}^{s(n)},a).$$

Then $Q^n_{term}(w_{n-1},y,a)$ is a proper quotient of $Q^n(w_{n-1},y,a)$.

Proof: Identical to the proof of Lemma 4.6. ∎

To handle an n-th multi-graded quotient resolution

$$MGQ^nRes(w,y,Base_{2,1}^{s(n)},\ldots,Base_{2,v_{s(n)}}^{s(n)},a)$$

which is not of maximal complexity, we also need the following lemma, which is similar to Lemma 4.7.

LEMMA 4.10: *Let $MGQ^nRes(w,y,Base_{2,1}^{s(n)},\ldots,Base_{2,v_{s(n)}}^{s(n)},a)$ be one of the resolutions in our list of n-th multi-graded quotient resolutions, and suppose that MGQ^nRes is not of maximal complexity. By construction, the limit group*

$Q^n(w,y,a)$ is mapped onto the limit group associated with each of the levels of the multi-graded quotient resolution

$$MGQ^n Res(w,y,Base^{s(n)}_{2,1},\dots,Base^{s(n)}_{2,v_{s(n)}},a).$$

Let $Q^n_2(y,a)$, $Q^n_2(w_{n-1},y,a)$, $Q^n_2(w_n,y,a)$ be the images of the subgroups generated by $Q^n(y,a)$, $Q^n(w_{n-1},y,a)$, $Q^n(w_n,y,a)$ in correspondence, in the limit group associated with the second level of the multi-graded quotient resolution $MGQ^n Res(w,y,Base^{s(n)}_{2,1},\dots,Base^{s(n)}_{2,v_{s(n)}},a)$. Then $Q^n_2(y,a)$ is a quotient of $Q^n(y,a)$, $Q^n_2(w_{n-1},y,a)$ is a quotient of the subgroup $Q^n(w_{n-1},y,a)$, and $Q^n_2(w_n,y,a)$ is a proper quotient of the subgroup $Q^n(w_n,y,a)$.

Proof: The claim is simply one of the basic properties of a multi-graded resolution. ∎

Suppose that the subgroup $Q^n_2(y,a)$ is a proper quotient of $Q^n(y,a)$. Let $QRes_1(y,a),\dots,QRes_q(y,a)$ be the resolutions in the taut Makanin–Razborov diagram of $Q^n_2(y,a)$. We continue with each of the resolutions $QRes_i(y,a)$ separately, hence we omit their index. With the resolution $QRes(y,a)$ we associate a resolution $CRes(y,a)$, where $CRes(y,a)$ is composed from two parts, the top being the resolution induced by the subgroup $< y,a >$ from the completion of the top level of the multi-graded quotient resolution $MGQ^n Res(w,y,Base^{s(n)}_{2,1},\dots,Base^{s(n)}_{2,v_{s(n)}},a)$, and the tail being the resolution $QRes(y,a)$. By theorem 1.18 of [Se2], with the resolution $CRes(y,a)$ we can associate a set of closures

$$Cl_1(CRes)(u,y,a),\dots,Cl_d(CRes)(u,y,a)$$

and for each closure a corresponding formal solution defined over it. We continue with each of these closures in parallel.

If the subgroup generated by $< y,a >$ in the developing resolution is a proper quotient of the limit group $Q^2(y,a)$ we started with, we replace the particular closure by starting the first step of the procedure with the subgroup generated by $< y,a >$ in it, hence for the continuation we can assume that the subgroup generated by $< y,a >$ in the closure is isomorphic to the limit group $Q^n(y,a)$ with which we started.

Fixing a closure, $Cl(u,y,a)$, we set the developing resolution to be that closure. With each closure, $Cl(u,y,a)$, there is an associated formal solution $x(u,y,a)$ (constructed according to theorem 1.18 of [Se2]). With the developing resolution, we further associate a finite collection of anvils that are set to be the maximal limit quotients of the group obtained as the amalgamated product of the completion of the developing resolution and the completion of the top level of the (second) multi-graded resolution $MGQ^nRes(w,y,Base^{s(n)}_{2,1},\dots,Base^{s(n)}_{2,v_{s(n)}},a)$, amalgamated along the top part of the developing resolution, and denote it $Anv(MGQ^nRes)(w,y,a)$. Note that the developing resolution is canonically mapped into the anvil, hence the family of formal solutions defined over the developing resolution can be naturally defined over the anvil.

Suppose that $Q^n_2(y,a)$ is isomorphic to $Q^n(y,a)$, and $Q^n_2(w_{n-1},y,a)$ is a proper quotient of $Q^n(w_{n-1},y,a)$. In this case we set s, $1\leq s\leq s(n-1)$, to be the minimal index for which $Q^n_2(w_{\ell(s)},y,a)$ is a proper quotient of $Q^n(w_{\ell(s)},y,a)=Q^{\ell(s)}(w_{\ell(s)},y,a)$. Let

$$MGQRes_1(w_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a),\dots,\\ MGQRes_d(w_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a)$$

be the resolutions in the taut multi-graded diagram of $Q^n_2(w_{\ell(s)},y,a)$ with respect to the parameter subgroups $Base^s_{2,1},\dots,Base^s_{2,v_s}$.

Suppose that a multi-graded quotient resolution

$$MGQRes(w_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a)$$

is of maximal possible complexity, i.e., that it has a single level with an abelian decomposition of the same structure as the abelian decomposition associated with the top level of the multi-graded quotient resolution $MGQ^{\ell(s)-1}Res(w_{\ell(s)-1},y,Base^{s(\ell(s)-1)}_{2,1},\dots,Base^{s(\ell(s)-1)}_{2,v_{s(\ell(s)-1)}},a)$. Let $Dw_{\ell(s)}$ be the subgroup generated by the stabilizers of the distinguished vertices in the anvil

$$Anv(MGQ^{\ell(s)-1}Res)(w_{\ell(s)},y,a).$$

Since the subgroup $Q^n_2(w_{\ell(s)},y,a)$ is a proper quotient of $Q^n(w_{\ell(s)},y,a)$, and since limit groups are residually finite and hence satisfy the Hopf property, the image of the subgroup $Dw_{\ell(s)}$ in the (closure of the) multi-graded quotient resolution $MGQRes(w_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a)$ is a proper quotient of the subgroup $Dw_{\ell(s)}$. In this case of

$$MGQRes(w_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a)$$

being a resolution of maximal possible complexity, we do the following. We set the multi-graded quotient resolution

$$CRes(w_{\ell(s)},y,Base^s_{2,1},\dots,Base^s_{2,v_s},a)$$

to be the multi-graded resolution obtained from the resolution induced by the subgroup $< w_{\ell(s)}, y, a >$ from the top level of the completion of the multi-graded quotient resolution

$$MGQ^n Res(w, y, Base_{2,1}^{s(n)}, \dots, Base_{2,v_{s(n)}}^{s(n)}, a)$$

followed by the multi-graded resolution

$$MGQRes(w_{\ell(s)}, y, Base_{2,1}^{s}, \dots, Base_{2,v_s}^{s}, a).$$

We set q, $1 \leq q \leq s$, to be the minimal index for which the subgroup generated by $< w_{\ell(q)}, y, a >$ in (the closure of)

$$CRes(w_{\ell(s)}, y, Base_{2,1}^{s}, \dots, Base_{2,v_s}^{s}, a)$$

is a proper quotient of $Q^n(w_{\ell(q)}, y, a) = Q^{\ell(q)}(w_{\ell(q)}, y, a)$. Let t be the minimal index for which $s(q) = t$. We now replace the multi-graded quotient resolution $MGQRes(w_{\ell(s)}, y, Base_{2,1}^{s}, \dots, Base_{2,v_s}^{s}, a)$ of the limit group $Q_2^n(w_{\ell(s)}, y, a)$, by starting the $\ell(q)$-th step of our process with the limit group generated by $< w_{\ell(q)}, y, a >$ in (the closure of)

$$CRes(w_{\ell(s)}, y, Base_{2,1}^{s}, \dots, Base_{2,v_s}^{s}, a)$$

instead of the limit group

$$Q^{\ell(q)} Rlim(w_{\ell(q)}, y, a) = Q^n Rlim(w_{\ell(q)}, y, a)$$

used in the $\ell(q)$-th step of the procedure.

By the above argument, for the rest of part (4) of the general step of the procedure we may consider only those multi-graded quotient resolutions $MGQRes(w_{\ell(s)}, y, Base_{2,1}^{s}, \dots, Base_{2,v_s}^{s}, a)$ which are not of maximal possible complexity. In this case we analyze each of the multi-graded quotient resolutions $MGQRes(w_{\ell(s)}, y, Base_{2,1}^{s}, \dots, Base_{2,v_s}^{s}, a)$ as we did in step $\ell(s)$ of our iterative procedure. First, we associate with the multi-graded resolution $MGQRes(w_{\ell(s)}, y, Base_{2,1}^{s}, \dots, Base_{2,v_s}^{s}, a)$ a multi-graded resolution $CRes(w_{\ell(s)}, y, Base_{2,1}^{s}, \dots, Base_{2,v_s}^{s}, a)$, obtained from the resolution induced by the subgroup $< w_{\ell(s)}, y, a >$ from the top level of the completion of the multi-graded resolution

$$MGQ^n Res(w, y, Base_{2,1}^{s(n)}, \dots, Base_{2,v_{s(n)}}^{s(n)}, a)$$

followed by the multi-graded resolution

$$MGQRes(w_{\ell(s)}, y, Base_{2,1}^{s}, \dots, Base_{2,v_s}^{s}, a).$$

If the multi-graded resolution $CRes(w_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a)$ is not compatible with the collections of s.c.c. associated with the various QH vertex groups in the multi-graded abelian decomposition associated with the top level of the multi-graded resolution

$$MGQ^{\ell(s-1)}Res(w_{\ell(s-1)}, y, Base^{s-1}_{2,1}, \ldots, Base^{s-1}_{2,v_{s-1}}, a),$$

we omit the multi-graded resolution

$$MGQRes(w_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a)$$

from our list of multi-graded resolutions of $Q^n_{term}(w_{\ell(s)}, y, a)$. Otherwise, we continue as in step $\ell(s)$ of the iterative procedure, and associate with the multi-graded resolution $MGQRes(w_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a)$ a canonical collection of developing resolutions and their associated anvils and formal solutions. As we did in part (4) of the second step of our iterative procedure, given an anvil associated with the graded resolution $MGQRes(w_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a)$, we set a resolution $CRes(u, y, a)$ obtained from the resolution induced by the subgroup $< y, a >$ from the top level of the n-th multi-graded quotient resolution

$$MGQ^nRes(w, y, Base^{s(r)}_{2,1}, \ldots, Base^{s(r)}_{2,v_{s(r)}}, a)$$

followed by the developing resolution in the anvil associated with the multi-graded resolution $MGQRes(w_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a)$. We set the developing resolution to be the resolution $CRes(u, y, a)$. We set the anvil, $Anv(MGQ^nRes)(w, y, a)$, to be the group generated by the corresponding closure of the top level of the multi-graded resolution

$$MGQ^nRes(w, y, Base^{s(n)}_{2,1}, \ldots, Base^{s(n)}_{2,v_{s(n)}}, a)$$

and the anvil associated with

$$MGQRes(w_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a)$$

amalgamated along the corresponding images of the subgroup $Q^n_2(w_{\ell(s)}, y, a)$ in the second level of the completion of MGQ^nRes and in the anvil associated with the multi-graded resolution $MGQRes(w_{\ell(s)}, y, Base^s_{2,1}, \ldots, Base^s_{2,v_s}, a)$. With the developing resolution we associate the formal solutions $x(w, y, a)$ (constructed according to theorem 1.18 of [Se2]), and since the developing resolution is mapped into the anvil, $Anv(MGQ^n Res)(w,y,a)$, the family of formal solutions is naturally defined over the anvil as well.

We still need to treat the case in which both $Q_2^n(y,a)$ is isomorphic to $Q^n(y,a)$ and $Q_2^n(w_{n-1},y,a)$ is isomorphic to $Q^n(w_{n-1},y,a)$. In this case, we just continue to the next level of the multi-graded quotient resolution $MGQ^n Res(w,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$, and proceed with the same analysis we applied for the top level of this multi-graded resolution. By Lemma 4.6, there must exist some level j for which either $Q_j^n(y,a)$ is a proper quotient of $Q^n(y,a)$ or $Q_j^n(w_{n-1},y,a)$ is a proper quotient of $Q^n(w_{n-1},y,a)$. If j is the highest such level in the multi-graded resolution $MGQ^n Res(w,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$, we associate with it a finite collection of developing resolutions and anvils according to the various cases described above.

(5) By part (1) we may assume that $Q^n(y,a)$ is isomorphic to the limit group $Q(y,a)$ associated with the anvil, $Anv(MGQ^{n-1}Res)(w,y,a)$, and by parts (2)–(3) we may assume that $Q^n(w_{s(n)-1},y,a)$ is isomorphic to the subgroup $Q^{n-1}(w_{s(n)-1},y,a)$ associated with the anvil. Part (4) treats the case in which the n-th multi-graded quotient resolution

$$MGQ^n Res(w,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$$

is not of maximal complexity. Hence, the only case left in presenting the general step of our procedure for validation of a sentence is the case of an n-th multi-graded quotient resolution

$$MGQ^n Res(w,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$$

of maximal possible complexity, i.e., a multi-graded quotient resolution $MGQ^n Res(w,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$ consists of a single level, and the QH and abelian vertex groups in the abelian decomposition associated with this level are similar to the ones that appear in the abelian decomposition associated with the top level of the multi-graded quotient resolution $MGQ^{n-1}Res(w,y,Base_{2,1}^{s(n-1)},\dots,Base_{2,v_{s(n-1)}}^{s(n-1)},a)$ with which we started the n-th step of the procedure.

We treat this case as we treated it in the minimal rank case and in the first steps of the general procedure. In case the abelian decomposition associated with $MGQ^n Res(w,y,Base_{2,1}^{s(n)},\dots,Base_{2,v_{s(n)}}^{s(n)},a)$ is of maximal complexity, we use the modular groups associated with the abelian decomposition associated with $MGQ^n Res$ to map the subgroup $Q^n(w,y,a)$

into the limit group associated with the next level of the corresponding part of the anvil, $Anv(MGQ^{n-1}Res)(w,y,a)$, i.e., into the subgroup of the anvil associated with the next level of the developing resolution associated with that anvil. By the construction of the developing resolution associated with the anvil, $Anv(MGQ^{n-1}Res)(w,y,a)$, this image of the appropriate subgroup of $Q^n(w,y,a)$ is a proper quotient of itself, and the relevant part of the developing resolution associated with the anvil, $Anv(MGQ^{n-1}Res)(w,y,a)$, is composed of a smaller number of levels. We now analyze the remaining part of the anvil in the same way we analyzed the n-th multi-graded quotient resolution, according to parts (1)–(5).

As long as the abelian decompositions associated with the constructed multi-graded quotient resolutions are of maximal possible complexity, we continue analyzing multi-graded quotient resolutions associated with various parts of the anvil we started with until one of the parts (1)–(4) applies to the corresponding multi-graded quotient resolution. If there exists a level for which one of the parts (1)–(4) applies, we set a developing resolution and an anvil with the limit group associated with this level according to the part (1)–(4) that applies to it.

Given such an anvil, we set the developing resolution to be the resolution induced by the subgroup $< y,a >$ from the parts of the resolution above that level followed by the developing resolution associated with that level. We set the anvils, $Anv(MGQ^nRes)(w,y,a)$, to be the maximal limit quotients of the group generated by the anvil associated with that level and the completion of the resolutions above it, amalgamated along the copies of the limit group associated with the level to which one of the steps (1)–(4) was applied. With the developing resolution and its associated anvil, we naturally associate a formal solution, $x(w,y,a)$, according to theorem 1.18 of [Se2].

If all the abelian decompositions associated with the multi-graded resolutions used for the construction of the developing resolution are of maximal complexity, i.e., if none of the parts (1)–(4) applies to any of these multi-graded resolutions, we examine the structure of the developing resolution. The developing resolution is built from a sequence of induced resolutions. Each of the induced resolutions is a resolution induced by the (image of the) subgroup $< y,a >$, and with each level of the induced resolution there is associated an (induced) abelian decomposition (see Section 3 for the construction of the induced resolution).

PROPOSITION 4.11: *Suppose that all the abelian decompositions associated with the multi-graded resolutions used for the construction of the developing resolution are of maximal possible complexity. Let* $< v, y, a >$ *be the subgroup generated by the closure of the developing resolution in the anvil* $Anv(MGQ^{n-1}Res)(w, y, a)$. *From each of the multi-graded resolutions used to construct the developing resolution, there is a resolution induced by the (image of the) subgroup* $< v, y, a >$.

Then there exists some level j *such that the structure of the abelian decompositions associated with the resolutions induced by the subgroup* $< v, y, a >$ *above level* j *are identical to the structure of the abelian decompositions associated with the corresponding resolution induced by the subgroup* $< y, a >$, *and in level* j, *either the number of factors in the free decomposition associated with the abelian decomposition associated with the resolution induced by* $< v, y, a >$ *is strictly smaller than the number of factors in the corresponding free decomposition associated with the resolution induced by the subgroup* $< y, a >$, *or in case of equality in the number of factors, the complexity of the abelian decomposition associated with the resolution induced by* $< v, y, a >$ *is strictly smaller than the complexity of the abelian decomposition associated with the resolution induced by* $< y, a >$.

Proof: Identical to the proof of Proposition 4.8. ∎

Using Proposition 4.11, in case all the multi-graded resolutions used in the construction of the developing resolution are of maximal possible complexity, we replace the resolutions induced by the subgroup $< y, a >$ used to construct the developing resolution, by the resolutions induced by the subgroup $< v, y, a >$. With the obtained (modified) developing resolution we associate a covering closure according to theorem 1.18 of [Se2], with each closure we naturally associate an anvil, denoted $Anv(MGQ^n Res(w, y, a)$, obtained from the anvil $Anv(MGQ^{n-1}Res)(w, y, a)$ by adding the appropriate roots corresponding to the specific closure, and with the closure and its associated anvil we associate a formal solution $x(w, y, a)$ according to Theorem 1.18 of [Se2].

Given the developing resolution, the anvil, and the family of formal solutions defined over them, we construct the data structure according to Definition 4.5. With the anvil constructed in the general step of the iterative procedure, we have associated a formal solution $\{x(w, y, a)\}$. We start step $n + 1$ of the iterative procedure by imposing on the specializations that factor through and are taut and shortest form with respect to the resolutions associated with the anvil, the (finitely many) systems of equations $\psi_j(x(w,y,a),y,a)=1$.

IV. Termination of the iterative procedure. Having defined the first, second and general steps of our iterative procedure for validation of a sentence, we are still required to prove its termination. To prove termination of our iterative procedure in the minimal rank case (Section 1), we used the strict decrease in the complexity of the resolutions associated with successive steps of the procedure, a strict decrease that forces termination. Unlike our procedure in the minimal rank case, in the general procedure we do not obtain a strict decrease in the complexity of the resolutions associated with successive steps of the procedure. To obtain termination in the general case, we need to look at limit groups (or alternatively Zariski closures) and complexities of various resolutions and decompositions associated with the data structures and anvils constructed along the steps of the procedure. Our ultimate goal in proving the termination of the procedure is to show that after finitely many steps of it, the iterative procedure is applied (effectively) not to specializations of the limit group $Rlim(y,a)$ but rather to specializations of a proper quotient of it. Clearly, once we show that, the descending chain condition for limit groups ([Se1], 5.1) guarantees the termination of our iterative procedure for validation of a sentence.

Theorem 4.12: *The iterative procedure for validation of an AE sentence terminates after finitely many steps.*

Proof: Suppose that the iterative procedure for validation of an AE sentence does not terminate after finitely many steps, which implies that the procedure must contain an infinite path. Each time part (1) of the general step of the procedure is applied along the given infinite path of the procedure, the limit group $Q^n(y,a)$ is replaced by its proper quotient. Hence, by the descending chain condition for limit groups, for the rest of the argument we may assume that part (1) of the general step is not applied along our given path of the procedure. If part (3) is applied to an anvil along the infinite path, then the limit group $Q^{\ell(s)}(w_{\ell(s)},y,a)$ is replaced by its proper quotient, $Q^n(w_{\ell(s)},y,a)$ for some index s, $1\leq s\leq s(n-1)$. Hence, by the descending chain condition for limit groups, for any fixed s, part (3) can be applied to the limit group $Q^n(w_{\ell(s)},y,a)$ only finitely many times along the infinite path.

Each time part (4) of the general step of the sieve procedure is applied to a multi-graded resolution along the infinite path, the index of the based subgroups,

$s(n)$, is increased by 1, and the complexity of the (multi-graded) abelian decomposition associated with the top level of the constructed multi-graded resolution strictly decreases (see Definition 3.2 for the complexity of a well-structured resolution that restricts to the complexity of an abelian decomposition). Since a strictly decreasing sequence of complexities of (multi-graded) abelian decompositions terminates after finitely many steps, parts (1), (3) and (4) can be applied to the multi-graded resolution associated with the top level of the developing resolution (or anvil) along our given infinite path of the iterative procedure only finitely many times.

Hence, there exists some step n_0 along the given path of the iterative procedure such that for every step $n > n_0$:

(i) $s(n) = s(n_0)$;

(ii) for every s, $1 \leq s \leq s(n_0) - 1$,

$$Q^{\ell(s)}(w_{\ell(s)}, y, a)$$

is isomorphic to $Q^n(w_{\ell(s)}, y, a)$;

(iii) the multi-graded resolution associated with the top level of the developing resolution constructed in step n of the iterative procedure, MGQ^nRes, is of maximal complexity.

Since for every $n > n_0$, the multi-graded resolutions MGQ^nRes associated with the top level of the developing resolution are of maximal complexity, for every $n > n_0$, the procedure is effectively applied to the limit group associated with the terminal (second) level of the maximal complexity (one level) multi-graded resolutions, MGQ^nRes. By construction, in the limit group associated with the terminal level of the maximal complexity resolutions MGQ^nRes, the image of the limit group $Q^n(w_{\ell(s(n_0)-1)}, y, a)$, $Q^n_{term}(w_{\ell(s(n_0)-1)}, y, a)$, is a proper quotient of $Q^{\ell(s(n_0)-1)}(w_{\ell(s(n_0)-1)}, y, a)$. If at some step $n > n_0$, the image of $Q^n(y, a)$ in the terminal level of MGQ^nRes, $Q^n_{term}(y, a)$, is a proper quotient of $Q^1(y, a)$, then for some n_1, for $n > n_1$, the infinite path of the iterative procedure is effectively applied to a proper quotient of the limit group $Q^1(y, a)$ with which we started. Otherwise, let $s_1 \leq s(n_0) - 1$ be the minimal index s for which $Q^n_{term}(w_{\ell(s)}, y, a)$ is a proper quotient of $Q^{\ell(s)}(w_{\ell(s)}, y, a)$ for some step $n > n_0$. Repeating the arguments used for analyzing the multi-graded resolutions associated with the top level of the developing resolution along our infinite path, parts (1), (3) and (4) can be applied only finitely many times to the multi-graded resolutions associated with the second level of the developing resolution (anvil) along our given infinite path of the iterative procedure, hence there must exist some index n_2 such that for every $n > n_2$, the multi-graded resolutions associated with the top two levels of the developing resolution (anvil) along the given infinite path of the iterative procedure are of maximal complexity.

By the descending chain condition for limit groups ([Se1], 5.1), if we repeat this argument inductively, we obtain either:

(i) a level d, and an index n_d, for which the multi-graded resolutions associated with the top d levels of the associated developing resolutions are of maximal complexity at all steps $n > n_d$, and the image of $Q^n(y, a)$ in the limit group associated with the $d + 1$ level of the developing resolution is a proper quotient of $Q^1(y, a)$ for all $n > n_d$; or

(ii) an infinite sequence of limit groups $Q^n(w_{\ell(s(n))}, y, a)$, with a corresponding sequence of (multi-graded) abelian decompositions, such that the sequence of complexities of these abelian decompositions is strictly decreasing.

Since a sequence of strictly decreasing complexities of abelian decompositions terminates after finitely many steps, part (ii) cannot exist. Hence part (i) holds, so there exists some index n_q for which for every step $n > n_q$ along the infinite path of the iterative procedure, the procedure is effectively applied to a proper quotient of the limit group $Rlim(y, a)$ with which we started the procedure. Therefore, the descending chain condition for limit groups contradicts the existence of an infinite path, which implies that the iterative procedure for validation of a sentence terminates after finitely many steps. ∎

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