THE STACK OF LOCAL SYSTEMS WITH RESTRICTED VARIATION AND GEOMETRIC LANGLANDS THEORY WITH NILPOTENT SINGULAR SUPPORT

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Introduction

0.1. **Starting point.** Classically, Langlands conjectured a bijection between irreducible automorphic representations for a reductive group G and spectral data involving the dual group $\check{G}$.

P. Deligne (for GL_1), V. Drinfeld (for GL_2) and G. Laumon (for GL_n) realized Langlands-style phenomena in algebraic geometry. In their setting, the fundamental objects of interest are *Hecke eigensheaves*. This theory works over an arbitrary ground field k, and takes as an additional input a *sheaf theory* for varieties over that field. Specializing to $k = \overline{\mathbb{F}}_q$ and étale sheaves, one recovers special cases of Langlands's conjectures by taking the trace of Frobenius.

Inspired by these works, Beilinson and Drinfeld proposed the *categorical Geometric Langlands Conjecture*

$$\text{D-mod}(\text{Bun}_G) \simeq \text{IndCoh}_{\text{Nilp}}(\text{LocSys}_{\check{G}}(X)) \tag{0.1}$$

for X a smooth projective curve over a field k of characteristic zero. Here the left-hand side $\text{D-mod}(\text{Bun}_G)$ is a sheaf-theoretic analogue of the space of unramified automorphic functions, and the right hand side is defined in [AG].

There are (related) discrepancies between this categorical conjecture and more classical conjectures.

(i) Hecke eigensheaves make sense in any sheaf theory, while the Beilinson-Drinfeld conjecture applies only in the setting of D-modules.

(ii) Hecke eigensheaves categorify the arithmetic Langlands correspondence through the trace of Frobenius construction, while the Beilinson-Drinfeld conjecture bears no direct relation to automorphic functions.

(iii) Langlands's conjecture parametrizes *irreducible* automorphic representations, while the Beilinson-Drinfeld conjecture provides a spectral decomposition of (a sheaf-theoretic analogue of) the whole space of (unramified) automorphic functions.

These differences provoke natural questions:

–Is there a categorical geometric Langlands conjecture that applies in any sheaf-theoretic context, in particular, in the étale setting over finite fields?

–The trace construction attaches automorphic functions to particular étale sheaves on Bun_G; is there a direct relationship between the *category* of étale sheaves on Bun_G and the *space* of automorphic functions?

–Is it possible to give a spectral description of the space of classical automorphic functions, not merely its irreducible constituents?

0.2. **Summary.**

0.2.1. In this paper, we provide positive answers to the three questions raised above.

Our Conjecture 20.2.7 provides an analogue of the categorical Geometric Langlands Conjecture that is suited to any ground field and any sheaf theory.

Our Conjecture 21.3.7 proposes a closer relationship between sheaves on Bun_G and unramified automorphic functions than was previously considered. As such, it allows one to extract new, concrete conjectures on automorphic functions from our categorical Geometric Langlands Conjecture, see right below.

Our Conjecture 23.8.6 describes the space of unramified automorphic functions over a function field in spectral terms, refining Langlands's conjectures in this setting.

In sum, the main purpose of this work is to propose a variant of the categorical Beilinson-Drinfeld conjecture that makes sense over finite fields, and in that setting, to connect it with the arithmetic Langlands program.

0.2.2. This paper contains two main ideas. The first of them is the introduction of a space

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$$

of G-*local systems with restricted variation* on X. In our Conjecture 20.2.7, $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ replaces $\mathrm{LocSys}_{\check{G}}(X)$ from the original conjecture of Beilinson and Drinfeld.

We discuss $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ in detail later in the introduction. For now, let us admit it into the discussion as a black box.

Then our Conjecture 20.2.7 asserts

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)), \tag{0.2}$$

where the left-hand side is the category of ind-constructible sheaves on Bun_G with nilpotent singular support; we study this category in detail in Sect. 13.

0.2.3. In addition, we make some progress toward Conjecture 20.2.7.

Our Theorem 13.3.2 provides an action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ compatible with Hecke functors. We regard this result as a spectral decomposition of the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ over $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$. This theorem is a counterpart of [Ga5, Corollary 4.5.5], which applied in the D-module setting and whose proof used completely different methods.

Using these methods, we settle long-standing conjectures on the structure of Hecke eigensheaves. Our Corollary 13.4.10 shows that Hecke eigensheaves have nilpotent singular support, as predicted by G. Laumon in [Lau, Conjecture 6.3.1]. In addition, our Corollary 15.5.7 shows that in the D-module setting, any Hecke eigensheaf has regular singularities, as predicted by Beilinson-Drinfeld in [BD1, Sect. 5.2.7].

0.2.4. The second main idea of this paper is that of categorical trace. It appears in our Conjecture 21.3.7, which we title the *Trace Conjecture*. This conjecture predicts a stronger link between geometric and arithmetic Langlands than was previously considered:

Suppose $k = \overline{\mathbb{F}}_q$ and that X and G are defined over $\mathbb{F}_q$, and therefore carry Frobenius endomorphisms. The Trace Conjecture asserts that the categorical trace of the functor $(\mathrm{Frob}_{\mathrm{Bun}_G})_*$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ maps isomorphically to the space of (compactly supported) unramified automorphic functions

$$\mathrm{Autom} := \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)).$$

More evocatively: we conjecture that a trace of Frobenius construction recovers the *space* of automorphic forms from the *category* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, much as one classically extracts a automorphic functions from an automorphic sheaf by a trace of Frobenius construction.

Combined with our Theorem 13.3.2, the Trace Conjecture gives rise to the spectral decomposition of Autom along the set of isomorphism classes of semi-simple Langlands parameters, recovering the (unramified case of) V. Lafforgue's result.

Moreover, if we combine the Trace Conjecture with our version of the categorical Geometric Langlands Conjecture (i.e., Conjecture 20.2.7), we obtain a full description of the space of (unramified) automorphic functions in terms of Langlands parameters (and not just the spectral decomposition):

$$\mathrm{Autom} \simeq \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \omega_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}),$$

where $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ is the algebraic stack of Frobenius-fixed points, i.e.,

$$\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X) := (\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))^{\mathrm{Frob}},$$

where Frob is the automorphism of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ induced by the geometric Frobenius on X. This is our Conjecture 23.8.6, as referenced in Sect. 0.2.1.

0.3. **Some antecedents.** Before discussing the contents of this paper in more detail, we highlight two points that are *not* original to our work.

0.3.1. *Work of Ben-Zvi and Nadler.* Observe that in Conjecture 20.2.7 we consider the subcategory $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$ of (ind-constructible) sheaves with nilpotent singular support, a hypothesis with no counterpart in the Beilinson-Drinfeld setting of D-modules.

The idea of considering this subcategory, which is so crucial to our work, is due to D. Ben-Zvi and D. Nadler, who did so in their setting of *Betti* Geometric Langlands, see [BN].

0.3.2. Let us take a moment to clarify the relationship between our work and [BN].

For $k = \mathbb{C}$, Ben-Zvi and Nadler consider the larger category $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)$ of *all* (possibly not ind-constructible) sheaves on $\mathrm{Bun}_G(\mathbb{C})$, considered as a complex stack via its analytic topology. Let $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)$ be the full subcategory consisting of objects with nilpotent singular support. Ben-Zvi and Nadler conjectured an equivalence

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}(X)), \tag{0.3}$$

where in the right-hand side $\mathrm{LocSys}_{\check{G}}(X)$ is the Betti version of the stack of $\check{G}$-local systems on X.

Let us compare this conjectural equivalence with the Beilinson-Drinfeld version (0.1). The latter is particular to D-modules, while (0.3) is particular to topological sheaves. Our (0.2) sits in the middle between the two: when $k = \mathbb{C}$ our $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ can be thought of as a full subcategory of both $\mathrm{D\text{-}mod}(\mathrm{Bun}_G)$ and $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Similarly, our $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ is an algebro-geometric object that is embedded into both the de Rham and Betti versions of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$. Now, the point is that $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ can be defined abstractly, so that it makes sense in any sheaf-theoretic context, along with the conjectural equivalence (0.2).

Remark 0.3.3. We should point out another source of initial evidence towards the relationship between $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$:

It was a discovered by D. Nadler and Z. Yun in [NY1] that when we apply Hecke functors to objects from $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, we obtain objects in $\mathrm{Shv}(\mathrm{Bun}_G \times X)$ that *behave like local systems along* X; see Theorem 13.2.4 for a precise assertion.

Remark 0.3.4. We should also emphasize that what enabled us to even talk about $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ in the context of ℓ-adic sheaves was the work of A. Beilinson [Be2] and T. Saito [Sai], where the singular support of étale sheaves over any ground field was defined and studied.

0.3.5. *Work of V. Lafforgue.* Our Trace Conjecture is inspired by the work [VLaf1] of V. Lafforgue on the arithmetic Langlands correspondence for function fields.

A distinctive feature of Geometric Langlands is that Hecke functors are defined not merely at points $x \in X$ of a curve, but extend over all of X, and moreover, extend over X^I for any finite set I. These considerations lead to the distinguished role played by the *factorization algebras* of [BD2] and *Ran space* in geometric Langlands theory.

In his work, V. Lafforgue showed that the existence of Hecke *functors* over powers of a curve has implications for automorphic *functions*. Specifically, he used the existence of these functors to construct *excursion operators*, and used these excursion operators to define the spectral decomposition of automorphic functions (over function fields) as predicted by the Langlands conjectures.

0.3.6. In [GKRV], a subset of the authors of this paper attempted to reinterpret V. Lafforgue's constructions using categorical traces. It provided a toy model for the spectral decomposition in [VLaf1] in the following sense:

In *loc.cit.* one starts with an abstract category $\mathcal{C}$ equipped with an action of Hecke functors *in the Betti setting* and an endofunctor $\Phi : \mathcal{C} \to \mathcal{C}$ (to be thought of as a prototype of Frobenius), and obtains a spectral decomposition of the vector space $\mathrm{Tr}(\Phi, \mathcal{C})$ along a certain space, which could be thought of as a Betti analog of the coarse moduli space of arithmetic Langlands parameters.

Now, the present work allows to carry the construction of [GKRV] in the actual setting of applicable to the study of automorphic functions: we take our $\mathcal{C}$ to be the ℓ-adic version of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (for a curve X over $\overline{\mathbb{F}}_q$).

In Sect. 23, we revisit V. Lafforgue's work, and show how our Trace Conjecture recovers and (following ideas of V. Drinfeld) refines the main results of [VLaf1] in the unramified case.

0.4. **Contents.** This paper consists of four parts.

In Part I we define and study the properties of the stack $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$.

In Part II we establish a general spectral decomposition result that produces an action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ on a category $\mathbf{C}$, equipped with what one can call a *lisse Hecke action*.

In Part III we study the properties of the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. We should say right away that in this Part we prove two old-standing conjectures: that Hecke eigensheaves have nilpotent singular support, and that (in the case of D-modules) all sheaves with nilpotent singular support have regular singularities.

In Part IV we study the applications of the theory developed hereto to the Langlands theory.

Below we will review the main results of each of the Parts.

0.5. **Overview: the stack $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.** Let G be an arbitrary affine algebraic group over a field of coefficients e of characteristic 0.

0.5.1. Let us start by recalling the definition of the (usual) algebraic stack $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$ of G-local systems on X in the context of sheaves in the classical topology (to be referred to as the *Betti* context).

On the first pass, let us take $\mathsf{G} = GL_n$.

Choose a base point $x \in X$. For an affine test scheme $S = \mathrm{Spec}(A)$ over e, an S-point of $\mathrm{LocSys}_{GL_n}(X)$ is an A-module E_S, locally free of rank n, equipped with an action of $\pi_1(X, x)$.

For an arbitrary G, the definition is obtained from the one for GL_n via Tannakian formalism.

0.5.2. We now give the definition of $\mathrm{LocSys}^{\mathrm{restr}}_{GL_n}(X)$, still in the Betti context. Namely $\mathrm{LocSys}^{\mathrm{restr}}_{GL_n}(X)$ is a subfunctor of $\mathrm{LocSys}_{GL_n}(X)$ that corresponds to the following condition:

We require that the action of $\pi_1(X, x)$ on E_S be e-locally finite, i.e., each element of E_S is contained in a finite-dimensional e-vector subspace, preserves by the action of $\pi_1(X, x)$.

For an arbitrary G, one imposes this condition for each finite-dimensional representation $\mathsf{G} \to GL_n$ (or, equivalently, for one faithful representation).

When A is Artinian, the above condition is automatic, so the formal completions of $\mathrm{LocSys}_{\mathsf{G}}(X)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ at any point are the same. The difference appears for A that have positive Krull dimension.

With that we should emphasize that $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is *not* entirely formal, i.e., it is *not* true that any S-point of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ factors though an S'-point with S' Artinian. For example, for $\mathsf{G} = \mathbb{G}_a$, the map

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X)$$

is an isomorphism.

0.5.3. Let us explain the terminology "restricted variation", again in the example of $\mathsf{G} = GL_n$,

The claim is that when we move along $S = \mathrm{Spec}(A)$, the corresponding representation of $\pi_1(X, x)$ does not change too much, in the sense that the isomorphism class of its semi-simplification is constant (as long as S is connected).

Indeed, let us show that for every $\gamma \in \pi_1(X, x)$ and every $\lambda \in \mathsf{e}$, the generalized λ-eigenspace of γ on $E_s := E_S \underset{A,s}{\otimes} \mathsf{e}$ has a constant dimension as s moves along S.

Indeed, due to the locally finiteness condition, we can decompose E_S into a direct sum of generalized eigenspaces for γ

$$E_S = \underset{\lambda}{\oplus} E_S^{(\lambda)},$$

where each $E_S^{(\lambda)}$ is an A-submodule, and being a direct summand of a locally free A-module, it is itself locally free.

The same phenomenon will happen for any G: an S-point of $\mathrm{LocSys}_{\mathsf{G}}(X)$ factors through $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ if and only for all e-points of S, the resulting G-local systems on X all have the same semi-simplification.

0.5.4. We are now ready to give the general definition of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$.

Within the given sheaf theory, we consider the full subcategory

$$\mathrm{Lisse}(X) \subset \mathrm{Shv}(X)^{\mathrm{constr}}$$

of local systems (of finite rank).

Consider its ind-completion, denoted $\mathrm{IndLisse}(X)$. Finally, let $\mathrm{QLisse}(X)$ be the left completion of $\mathrm{IndLisse}(X)$ in the natural t-structure[1]. Now, for an affine test scheme $S = \mathrm{Spec}(A)$, an S-point of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ is a symmetric monoidal functor

$$\mathrm{Rep}(\mathsf{G}) \to A\text{-mod} \otimes \mathrm{QLisse}(X),$$

required to be right t-exact with respect to the natural t-structures.

By definition, e-points of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ are just G-local systems on X.

Two remarks are in order:

(i) In the definition of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ one can (and should!) allow S to be a *derived* affine scheme over e (i.e., we allow A to be a connective commutative e-algebra). Thus, $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ is inherently an object of derived algebraic geometry[2].

(ii) The definition of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ uses the *large* category $A\text{-mod} \otimes \mathrm{QLisse}(X)$. When we evaluate our functor on *truncated* affine schemes, we can replace $\mathrm{QLisse}(X)$ by $\mathrm{IndLisse}(X) = \mathrm{Ind}(\mathrm{Lisse}(X))$ (see Proposition 2.1.7), and so we can express the definition in terms of small categories. But for an arbitrary S, it is essential to work with the entire $\mathrm{QLisse}(X)$, to ensure *convergence* (see Sect. 2.1).

0.5.5. As defined above, $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ is just a functor on (derived) affine schemes, so is just a prestack. But what kind of prestack is it? I.e., can we say something about its geometric properties?

The majority of Part I is devoted to investigating this question.

While the geometric properties we find are exotic, this study plays a key role in Part III, where the geometry of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ (for $\mathsf{G} = \check{G}$, the Langlands dual of G) has concrete consequences for the category the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

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[1]The last step of left completion is unnecessary if X is a *categorical* $K(\pi, 1)$, see Sect. E.2.1, which is the case of curves of genus > 0. However, left completion is *non-trivial* for $X = \mathbb{P}^1$, i.e., $\mathrm{IndLisse}(\mathbb{P}^1) \neq \mathrm{QLisse}(\mathbb{P}^1)$, see Sect. E.2.6

[2]In fact, our definition of the usual $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X)$ in the Betti context was a bit of a euphemism: for the correct definition in the context of derived algebraic geometry, one has to use the entire fundamental groupoid of X, and not just π_1; the difference does not matter, however, when we evaluate on classical test affine schemes, while the distinction between $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X)$ and $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ happens at the classical level.

0.5.6. First, let us illustrate the shape that $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ has in the Betti context. Recall that in this case we have the usual moduli stack $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$, which is a quotient of the affine scheme $\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\mathsf{G}}(X)$ (that classifies local systems with a trivialization at x) by G.

Assume that G is reductive, and let

$$\mathrm{LocSys}^{\mathrm{Betti,coarse}}_{\mathsf{G}}(X) := \mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\mathsf{G}}(X)/\!/\mathsf{G} := \mathrm{Spec}(\Gamma(\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X), \mathcal{O}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)}))$$

be the corresponding coarse moduli space. We have the tautological map

$$\mathbf{r} : \mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X) \to \mathrm{LocSys}^{\mathrm{coarse}}_{\mathsf{G}}(X), \tag{0.4}$$

and recall that two e-points of $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$ lie in the same fiber of this map if and only if they have isomorphic semi-simplifications.

We can describe $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ as the disjoint union of formal completions of the fibers of $\mathbf{r}$ over e-points of $\mathrm{LocSys}^{\mathrm{Betti,coarse}}_{\mathsf{G}}(X)$ (see Theorem 4.8.4).

In particular, we note one thing that $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is *not*: it is *not* an algebraic stack (or union of such), because it has all these formal directions.

Remark 0.5.7. The above explicit description of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ in the Betti case may suggest that it is in general a "silly" object. Indeed, why would we want a moduli space in which all irreducible local systems belong to different connected components?

However, as the results in Parts III and IV of this paper show, $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is actually a natural object to consider, in that it is perfectly adapted to the study of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, and thereby to applications to the arithmetic theory.

For example, formula (0.9) below is the reflection on the automorphic side of the above decomposition of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ as a disjoint union. See also (0.11) for a version of the Geometric Langlands Conjecture with nilpotent singular support. Finally, see formula (0.19) for an expression for the space of automorphic functions in terms of Frobenius-fixed locus on $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

Looked at from a different angle, in the Betti and de Rham contexts, there are the "honest" moduli spaces of local systems, denoted $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$ and $\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$, respectively. However, in the étale context, $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is the best algebro-geometric approximation to the moduli of local systems that we can imagine.

0.5.8. For a general sheaf theory, we prove the following theorem concerning the structure of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$. Let $\mathrm{LocSys}^{\mathrm{restr,rigid}_x}_{\mathsf{G}}(X)$ be the fiber product

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \underset{\mathrm{pt}/\mathsf{G}}{\times} \mathrm{pt},$$

where

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{pt}/\mathsf{G}$$

is the map corresponding to taking the fiber at a chosen base point $x \in X$. So

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \simeq \mathrm{LocSys}^{\mathrm{restr,rigid}_x}_{\mathsf{G}}(X)/\mathsf{G}.$$

We prove (in Theorem 1.4.5) that $\mathrm{LocSys}^{\mathrm{restr,rigid}_x}_{\mathsf{G}}(X)$ is a disjoint union of ind-affine ind-schemes $\mathcal{Y}$ (locally almost of finite type), each of which is a *formal affine scheme*.

We recall that a prestack $\mathcal{Y}$ is a formal affine scheme if it an be written as a formal completion

$$\mathrm{Spec}(R)^\wedge_Y,$$

where R is a connective e-algebra (but not necessarily almost of finite type over e) and $Y \simeq \mathrm{Spec}(R')$ is a Zariski closed subset in $\mathrm{Spec}(R)$, where R' is a (classical, reduced) e-algebra of finite type.

This all may sound technical, but the upshot is that the $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ fails to be an algebraic stack precisely to the same extent as in the Betti case, and the extent of this failure is such that we can control it very well.

To illustrate the latter point, in Sect. 7 we study the category $\mathrm{QCoh}(\mathcal{Y})$ on formal affine schemes (or quotients of these by groups) and show that its behavior is very close to that of $\mathrm{QCoh}(-)$ on affine schemes (which is *not at all* the case of $\mathrm{QCoh}(-)$ on arbitrary ind-schemes).

0.5.9. As we have seen in Sect. 0.5.6, in the Betti context, the prestack $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ splits into a disjoint union of prestacks $\mathcal{Z}_\sigma$ parameterized by isomorphism classes of semi-simple G-local systems[3] σ on X. Moreover, the underlying reduced prestack of each $\mathcal{Z}_\sigma$ is an algebraic stack.

In Sect. 3 we prove that the same is true in any sheaf theory. Furthermore, for each σ, we construct a *uniformization map*

$$\underset{\mathsf{P}}{\sqcup} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \to \mathcal{Z}_\sigma,$$

which is proper and surjective on geometric points, where:

- The disjoint union runs over the set over parabolic subgroups P, such that σ can be factored via an *irreducible* local system σ_{M} for some/any Levi splitting $\mathsf{P} \hookleftarrow \mathsf{M}$ (here M is the Levi quotient of P);
- $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X)$ is the *algebraic stack*

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P}}(X) \underset{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{M}}(X)}{\times} \mathrm{pt}\,/\,\mathrm{Aut}(\sigma_{\mathsf{M}}).$$

0.5.10. Let G be again reductive. For a general sheaf theory, we do not have the picture involving (0.4) that we had in the Betti case. However, we do have a formal part of it.

Namely, let $\mathcal{Z}$ be a connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$. This is an ind-algebraic stack, which can be written as

$$\underset{i}{\mathrm{colim}}\, \mathcal{Z}_i,$$

where each $\mathcal{Z}_i$ is an algebraic stack isomorphic to the quotient of a (derived) affine scheme by G.

We can consider the ind-affine ind-scheme

$$\mathcal{Z}^{\mathrm{coarse}} := \underset{i}{\mathrm{colim}}\, \mathrm{Spec}(\Gamma(\mathcal{Z}_i, \mathcal{O}_{\mathcal{Z}_i})),$$

and the map

$$\mathbf{r} : \mathcal{Z} \to \mathcal{Z}^{\mathrm{coarse}}. \tag{0.5}$$

In Theorem 5.4.2 we prove that:

(i) $\mathcal{Z}^{\mathrm{coarse}}$ is a formal affine scheme (see Sect. 0.5.8 for what this means) whose underlying reduced scheme is pt;

(ii) The map (0.5) makes $\mathcal{Z}$ into a *relative algebraic stack* over $\mathcal{Z}^{\mathrm{coarse}}$.

0.6. **Overview: spectral decomposition.** Part II contains one of the two the main results of this paper, Theorem 8.1.4.

0.6.1. We again start with a motivation in the Betti context.

Let $\mathcal{X}$ be a connected space, and let $\mathbf{C}$ be a DG category.

In this case, we have the notion of action of $\mathrm{Rep}(\mathsf{G})^{\otimes \mathcal{X}}$ on $\mathbf{C}$, see [GKRV, Sect. 1.7]. It consists of a compatible family of functors

$$\mathrm{Rep}(\mathsf{G})^{\otimes I} \to \mathrm{End}(\mathbf{C}) \otimes (\mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}})^{\otimes I}, \quad I \in \mathrm{fSet},$$

where $\mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}$ is the DG category $\mathrm{Funct}(\mathcal{X}, \mathrm{Vect}_{\mathsf{e}})$ (it can be thought of as the category of local systems of vector spaces on $\mathcal{X}$, *not necessarily* of finite rank), and fSet is the category of finite sets.

Now, we have the stack of Betti local systems $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(\mathcal{X})$ and we can consider actions of the symmetric monoidal category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(\mathcal{X}))$ on $\mathbf{C}$.

[3]When G is not reductive, the parameterization is by the same set for the maximal reductive quotient of G.

The tautological defined symmetric monoidal functor

$$\mathrm{Rep}(\mathsf{G}) \otimes \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X})) \to \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}$$

gives rise to a map (of ∞-groupoids)

$$\{\text{Actions of } \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X})) \text{ on } \mathbf{C}\} \to \{\text{Actions of } \mathrm{Rep}(\mathsf{G})^{\otimes \mathcal{X}} \text{ on } \mathbf{C}\}. \tag{0.6}$$

A relatively easy result (see [GKRV, Theorem 1.5.5]) says that the map (0.6) is an equivalence (of ∞-groupoids).

0.6.2. We now transport ourselves to the context of algebraic geometry. Let X be a connected scheme over k and $\mathbf{C}$ be a e-linear DG category. By an action of $\mathrm{Rep}(\mathsf{G})^{\otimes X\text{-lisse}}$ on $\mathbf{C}$ we shall mean a compatible collection of functors

$$\mathrm{Rep}(\mathsf{G})^{\otimes I} \to \mathrm{End}(\mathbf{C}) \otimes \mathrm{QLisse}(X)^{\otimes I}, \quad I \in \mathrm{fSet}\,.$$

As before, we have the tautological symmetric monoidal functor

$$\mathrm{Rep}(\mathsf{G}) \otimes \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)) \to \mathrm{QLisse}(X),$$

and we obtain a map

$$\{\text{Actions of } \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(\mathcal{X})) \text{ on } \mathbf{C}\} \to \{\text{Actions of } \mathrm{Rep}(\mathsf{G})^{\otimes X\text{-lisse}} \text{ on } \mathbf{C}\}. \tag{0.7}$$

One can can ask whether the map (0.7) is an isomorphism as well, and our Spectral Decomposition theorem, namely, Theorem 8.3.7 in the main body of the paper, says that it is.

0.6.3. Unfortunately, our proof of Theorem 8.3.7 is not aesthetically very satisfactory. In fact, we conjecture that a more general statement along the same lines holds (see Conjecture 8.3.6), when we replace the category $\mathrm{QLisse}(X)$ by what we call a *gentle* Tannakian category $\mathbf{H}$.

Our proof of Theorem 8.3.7 is very specific to $\mathbf{H}$ being $\mathrm{QLisse}(X)$, where X is a smooth proper curve.

Namely, we use the fact that (0.6) is an equivalence to prove that the assertion of Theorem 8.3.7 holds in the Betti context (i.e., when $\mathrm{QLisse}(X)$ is the left completion of the ind-completion of the category of *finite-dimensional* Betti local systems on X).

Using Riemann-Hilbert, this formally implies the assertion of Theorem 8.3.7 holds in the de Rham context (i.e., when $\mathrm{QLisse}(X)$ is the left completion of the ind-completion of the category of de Rham local systems on X).

Finally, we show that in the étale context, the assertion of Theorem 8.3.7 follows formally from its validity in the Betti context, essentially because the étale $\mathrm{QLisse}(X)$ (over any algebraically closed ground field) can be realized as a direct factor of the Betti version of $\mathrm{QLisse}(X')$ for some complex curve X'.

Remark 0.6.4. The particularly troublesome aspect of our proof of Theorem 8.3.7 is that it is not applicable to the case when X is a non-complete curve, while this case is of interest if we have an eye on extending our theory to the ramified case.

0.7. **Overview: the category** $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$**.** In Part III of the paper, we take G to be a reductive group and we will study the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ of sheaves on Bun_G (within any of our contexts) with singular support in the nilpotent cone $\mathrm{Nilp} \subset T^*(\mathrm{Bun}_G)$.

0.7.1. The stack Bun_G is non quasi-compact, and what allows us to work efficiently with the category $\mathrm{Shv}(\mathrm{Bun}_G)$ is the fact that we can simultaneously think of it as a *limit*, taken over poset of quasi-compact open substacks $\mathcal{U} \subset \mathrm{Bun}_G$,

$$\lim_{\mathcal{U}} \mathrm{Shv}(\mathcal{U}),$$

with transition functors given by restriction, and *also as a colimit*

$$\operatorname{colim}_{\mathcal{U}} \mathrm{Shv}(\mathcal{U}),$$

with transition functors given by !-extension.

We now take $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. More or less by definition, we still have

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) := \lim_{\mathcal{U}} \mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U}),$$

but we run into trouble with the colimit presentation:

In order for such presentation to exist, we should be able to find a cofinal family of quasi-compact opens, such that for every pair $\mathcal{U}_1 \overset{j}{\hookrightarrow} \mathcal{U}_2$ from this family, the functor $j_!$ sends

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U}_1) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U}_2).$$

Fortunately, we can find such a family; its existence is guaranteed by Theorem 13.1.5.

0.7.2. Thus, we can access the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ via the corresponding categories on quasi-compact open substacks. But our technical troubles are not over:

We do not know whether the categories $\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U})$ are compactly generated. Such questions (for an arbitrary algebraic stack or even scheme $\mathcal{Y}$, with a fixed $\mathcal{N} \subset T^*(\mathcal{Y})$) may be non-trivial. For example, it is *not* true in general that $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ is generated by objects that are compact in $\mathrm{Shv}(\mathcal{Y})$. We refer the reader to Sect. F where some general facts pertaining to these issues are summarized.

Although we conjecture that $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is generated by objects that are compact in the ambient category $\mathrm{Shv}(\mathrm{Bun}_G)$, we were not able to prove this in full generality. We do, however, prove this in the de Rham and Betti contexts.

That said, we were able to prove that $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is compactly generated as a DG category, and hence is dualizable. The latter is important for Part IV of the paper, in order for the trace of the Frobenius endofunctor on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ to be well-defined.

0.7.3. We now proceed to formulating the other results in Part III.

We consider the Hecke action on $\mathrm{Shv}(\mathrm{Bun}_G)$. Now, the subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$$

has the following key feature with respect to this action:

According to [NY1], combined with [GKRV, Theorem A.3.8], the Hecke functors

$$\mathrm{H}(-,-) : \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times X^I), \quad I \in \mathrm{fSet}, \tag{0.8}$$

send the subcategory

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(\mathrm{Bun}_G)$$

to

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X)^{\otimes I} \subset \mathrm{Shv}(\mathrm{Bun}_G \times X^I).$$

This means that $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ carries an action of $\mathrm{Rep}(\check{G})^{\otimes X\text{-lisse}}$, i.e., we find ourselves in the setting of the Spectral Decomposition theorem.

Thus, combined with Theorem 8.3.7 described above, we obtain the following assertion (it appears as Theorem 13.3.2 in the main body of the paper):

Theorem 0.7.4. *The category* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ *has a natural structure of module category over* $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

Theorem 0.7.4 has an obvious ideological significance. For example, it immediately implies that the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ splits as a direct sum

$$\underset{\sigma}{\oplus}\, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_\sigma, \tag{0.9}$$

indexed by isomorphism classes of semi-simple $\check{G}$-local systems.

However, in addition, we use Theorem 0.7.4 extensively to prove a number of structural results about $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. For example, we use it to prove: (i) the compact generation of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (this is Theorem 15.1.1); (ii) the fact that in the de Rham context, objects from $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ have regular singularities (this is Corollary 15.5.6); (iii) the tensor product property of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (Theorem 15.3.3, see below).

0.7.5. We now come to the second main result of this paper (it appears as Theorem 13.4.4 in the main body of the paper), which is in some sense a converse to the assertion of [NY1] mentioned above:

Theorem 0.7.6. *Let $\mathcal{F}$ be an object of $\mathrm{Shv}(\mathrm{Bun}_G)$, such that the Hecke functors (0.8) send it to*

$$\mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X),$$

then $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

A particular case of this assertion was conjectured by G. Laumon. Namely, [Lau, Conjecture 6.3.1] says that Hecke eigensheaves have nilpotent singular support.

0.7.7. The combination of Theorems 0.7.4 and 0.7.6 allows us to establish a whole array of results about $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, in conjunction with another tool: Beilinson's spectral projector, whose definition we will now recall.

Let us first start with a single $\check{G}$-local system σ. We can consider the category

$$\mathrm{Hecke}_\sigma(\mathrm{Shv}(\mathrm{Bun}_G))$$

of Hecke eigensheaves on Bun_G with respect to σ.

We have a tautological forgetful functor

$$\mathbf{oblv}_{\mathrm{Hecke}_\sigma} : \mathrm{Hecke}_\sigma(\mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{Shv}(\mathrm{Bun}_G), \tag{0.10}$$

and Beilinson's spectral projector is a functor

$$\mathsf{P}^{\mathrm{enh}}_\sigma : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Hecke}_\sigma(\mathrm{Shv}(\mathrm{Bun}_G)),$$

left adjoint to (0.10).

A feature of the functor $\mathsf{P}^{\mathrm{enh}}_\sigma$ is that the composition

$$\mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathsf{P}^{\mathrm{enh}}_\sigma}{\longrightarrow} \mathrm{Hecke}_\sigma(\mathrm{Shv}(\mathrm{Bun}_G)) \overset{\mathbf{oblv}_{\mathrm{Hecke}_\sigma}}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G)$$

is given by an explicit *integral Hecke functor*[4].

However, now that we have $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$, we can consider a version of the functor $\mathsf{P}^{\mathrm{enh}}_\sigma$ is families:

[4]I.e., a colimit of functors (0.8) for explicit objects of $\mathrm{Rep}(\check{G})^{\otimes I}$, as I ranges over the category of finite sets.

0.7.8. Let $\mathcal{Z}$ be a prestack over the field of coefficients e, equipped with a map

$$f : \mathcal{Z} \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X).$$

Then it again makes sense to consider the category of *Hecke eigensheaves* parametrized by S:

$$\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)).$$

It is endowed with a forgetful functor

$$\mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}} : \mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(\mathrm{Bun}_G)$$

(i.e., forget the eigenproperty).

We have a version of Beilinson's spectral projector, which is now a functor, denoted in this paper by

$$\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}} : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)),$$

left adjoint to the composition

$$\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G).$$

Let us note that the definition of functor $\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}}$ only uses the existence of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$. We do not need to use Theorems 0.7.4 and 0.7.6 to prove its existence or to establish its properties.

0.7.9. However, let us now use the functor $\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}}$ in conjunction with Theorems 0.7.4 and 0.7.6.

First, Theorem 0.7.6 implies that the inclusion

$$\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \subset \mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G))$$

is an equality.

And Theorem 0.7.4 implies that the category $\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))$ identifies with

$$\mathrm{QCoh}(\mathcal{Z}) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

Thus, we obtain that the functor $\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}}$ provides a left adjoint to the functor

$$\mathrm{QCoh}(\mathcal{Z}) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{f_*\otimes \mathrm{Id}}{\longrightarrow} \mathrm{QCoh}(\mathcal{Z}) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq$$
$$\simeq \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G),$$

provided that $\mathcal{O}_{\mathcal{Z}}$ is compact.

This construction has a number of consequences:

(i) It allows us to prove the compact generation of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (left adjoints can be used to construct compact generators); this is Theorem 15.1.1.

(ii) We construct explicit generators of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ by applying the functor $\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}}$ (for some particularly chosen $f : \mathcal{Z} \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$) to δ-function objects in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. This leads to the theorem that all objects in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ have regular singularities (in the de Rham context); this is Main Corollary 15.5.6. Combined with Corollary 13.4.10, we obtain that all Hecke eigensheaves have regular singularities; this is Main Corollary 15.5.7.

(iii) We use the above generators of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ to prove the (unexpected, but important for future applications) property that the tensor product functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}\times\mathrm{Nilp}}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$$

is an equivalence; this is Theorem 15.3.3 (see the discussion in Sect. 15.3.1 regarding why such an equivalence is not something we should expect on general grounds).

Remark 0.7.10. The properties of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ mentioned above indicate that this category exhibits behavior similar to that of $\mathrm{Shv}(\mathcal{Y})$, where $\mathcal{Y}$ is an algebraic stack (equal to the union of open substacks) with a finite number of isomorphism classes of k-points (e.g., $N\backslash G/B$ or its affine counterparts), or to the category of character sheaves on G.

The analogy is in fact not too far-fetched, as for $X = \mathbb{P}^1$, our $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is all of $\mathrm{Shv}(\mathrm{Bun}_G)$, and Bun_G is indeed an affine parabolic version of $N\backslash G/B$.

0.8. **Overview: Langlands theory.** Let X be a curve over a ground field k, and we will work with any of the sheaf-theoretic contexts from our list.

0.8.1. Having set up the theories of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ and $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, we are now in the position to state a version of the (categorical) Geometric Langlands Conjecture, with nilpotent singular support: this is Conjecture 20.2.7. It says that we have an equivalence

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)), \tag{0.11}$$

as categories equipped with an action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

Here in the right-hand side, $\mathrm{IndCoh}_?(-)$ stands for the category of ind-coherent sheaves with prescribed *coherent* singular support, a theory developed in [AG]. (In *loc.cit.*, this theory was developed for quasi-smooth schemes/algebraic stacks, but in Sect. 20.1 we show that it equally applicable to objects such as our $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}$.) In our case $? = \mathrm{Nilp}$, the *global nilpotent cone* in $\mathrm{Sing}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$, see Sect. 20.2.5[5].

0.8.2. Note that Conjecture 20.2.7 may be the first instance when a categorical statement is suggested for automorphic sheaves in the context of ℓ-adic sheaves.

That said, both the de Rham and Betti contexts have their own forms of the (categorical) Geometric Langlands Conjecture. In the de Rham context, this is an equivalence

$$\text{D-mod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X)), \tag{0.12}$$

as categories equipped with an action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))$.

In the Betti context, this is an equivalence

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)), \tag{0.13}$$

as categories equipped with an action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))$, where $\mathrm{Shv}^{\mathrm{all}}_?(-)$ stands for the category of all sheaves (i.e., not necessarily ind-constructible ones) with a prescribed singular support.

We show that in each of these contexts, our Conjecture 20.2.7 is a formal consequence of (0.12) (resp., (0.13)), respectively. In fact, we show that the two sides in Conjecture 20.2.7 are obtained from the two sides in (0.12) (resp., (0.13)) by

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{?}_{\check{G}}(X))}{\otimes} -$$

for ?= dR or Betti.

That said, we show (assuming Hypothesis 20.4.2) that the restricted version of GLC (i.e., (0.11)) actually implies the full de Rham version, i.e., (0.12). Probably, a similar argument can show that (0.11) implies the full Betti version (i.e., (0.13)) as well.

[5]It should not be confused with $\mathrm{Nilp} \subset T^*(\mathrm{Bun}_G)$: the two uses of Nilp have different meanings, and occur on different sides of Langlands duality.

0.8.3. For the rest of this subsection we will work over the ground field $k = \overline{\mathbb{F}}_q$, but assume that our geometric objects (i.e., X and G) are defined over $\mathbb{F}_q$, so that they carry the geometric Frobenius endomorphism.

We now come to the second main theme of this paper, the Trace Conjecture.

For any (quasi-compact) algebraic stack $\mathcal{Y}$ over $\overline{\mathbb{F}}_q$, but defined over $\mathbb{F}_q$. we can consider the endomorphism (in fact, automorphism) of $\mathrm{Shv}(\mathcal{Y})$ given by Frobenius pushforward, $(\mathrm{Frob}_{\mathcal{Y}})_*$. Since $\mathrm{Shv}(\mathcal{Y})$ is a compactly generated (and, hence, dualizable) category, we can consider the categorical trace of $(\mathrm{Frob}_{\mathcal{Y}})_*$ on $\mathrm{Shv}(\mathcal{Y})$:

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \in \mathrm{Vect}_{\mathsf{e}}\,.$$

The Grothendieck passage from Weil sheaves on $\mathcal{Y}$ to functions on $\mathcal{Y}(\mathbb{F}_q)$ can be upgraded to a map

$$\mathrm{LT} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)),$$

compatible with *-pullbacks and !-pushforwards, see Theorem 21.1.9.

However, the map LT is *not at all* an isomorphism (unless $\mathcal{Y}$ has finitely many isomorphism classes of $\overline{\mathbb{F}}_q$-points).

0.8.4. We apply the above discussion to $\mathcal{Y} = \mathrm{Bun}_G$. Since Bun_G is not quasi-compact, the local term map is in this case a map

$$\mathrm{LT} : \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)), \tag{0.14}$$

where $\mathrm{Funct}_c(-)$ stands for functions with finite support.

In what follows we will denote

$$\mathrm{Autom} := \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)).$$

This is the space of compactly supported unramified automorphic functions.

As we just mentioned, the map (0.14) is *not* an isomorphism (unless X is of genus 0).

0.8.5. We now consider the full category

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G).$$

It is stable under the action of the Frobenius, and is dualizable as a DG category. Hence, it makes sense to consider the object

$$\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \in \mathrm{Vect}_{\mathsf{e}}\,.$$

Our Trace Conjecture (Conjecture 21.3.7) says that there exists a canonical isomorphism

$$\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \simeq \mathrm{Autom}\,. \tag{0.15}$$

Remark 0.8.6. The sheaves-functions correspondence has been part of the geometric Langlands program since its inception by V. Drinfeld: in his 1983 paper [Dri], he constructed a Hecke eigenfunction corresponding to a 2-dimensional local system on X by first constructing the corresponding sheaf and then taking the associated functions.

Constructions of this sort allow to produce particular elements in Autom that satisfy some desired properties.

Our Trace Conjecture is an improvement in that it, in principle, allows to deduce statements about the *space* Autom from statements of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ as a *category*.

0.8.7. In fact, the Trace Conjecture is a particular case of a more general statement, Conjecture 21.5.7, which we refer to as the Shtuka Conjecture.

Namely, for a finite set I and $V \in \mathrm{Rep}(\check{G})^{\otimes I}$ consider the Hecke functor

$$\mathrm{H}(V,-) : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X^I).$$

Generalizing the categorical trace construction, we can consider the trace of this functor, precomposed with $(\mathrm{Frob}_{\mathrm{Bun}_G})_*$. The result will be an object that we denote

$$\widetilde{\mathrm{Sht}}_{I,V} \in \mathrm{QLisse}(X^I) \subset \mathrm{Shv}(X^I).$$

Our Shtuka Conjecture says that we have a canonical isomorphism

$$\widetilde{\mathrm{Sht}}_{I,V} \simeq \mathrm{Sht}_{I,V}, \tag{0.16}$$

where $\mathrm{Sht}_{I,V} \in \mathrm{Shv}(X^I)$ is the shtuka cohomology, see Sect. 21.5.1 where we recall the definition.

Note that the validity of (0.16) implies that the objects $\mathrm{Sht}_{I,V}$ belong to $\mathrm{QLisse}(X^I) \subset \mathrm{Shv}(X^I)$.

The latter fact has been unconditionally established by C. Xue in [Xue2], which provides a reality check for our Shtuka Conjecture.

0.8.8. We will now explain how the Trace Conjecture recovers V. Lafforgue's spectral decomposition of Autom along the arithmetic Langlands parameters.

The ind-stack $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ (which is an algebro-geometric object over $\mathsf{e} = \overline{\mathbb{Q}}_\ell$) carries an action of Frobenius, by transport of structure; we denote it by Frob. Denote

$$\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}} := \left(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)\right)^{\mathrm{Frob}}.$$

A priori, $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ is also an *ind*-algebraic stack, but we prove (see Theorem 23.1.4) that $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ is an actual algebraic stack (locally almost of finite type). We also prove that it is quasi-compact (i.e., even though $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ had infinitely many connected components, only finitely many of them are Frobenius-invariant).

The algebra

$$\mathcal{E}xc := \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathcal{O}_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)})$$

receives a map from V. Lafforgue's algebra of excursion operators; this map is surjective at the level oh H^0, see Sect. 23.2.2.

0.8.9. The categorical meaning of $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ is that the category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$ identifies with the category of Hochschild chains of Frob^* acting on $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

We will now apply the relative version of the trace construction from [GKRV, Sect. 3.8], and attach to the pair

$$(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G), (\mathrm{Frob}_{\mathrm{Bun}_G})_*),$$

viewed as acted on by the pair

$$(\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)), \mathrm{Frob}^*),$$

its class

$$\mathrm{cl}(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G), (\mathrm{Frob}_{\mathrm{Bun}_G})_*) \in \mathrm{HH}_\bullet(\mathrm{Frob}^*, \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)).$$

We denote the resulting object of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$ by

$$\mathrm{Drinf} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)).$$

Applying a version of [GKRV, Theorem 3.8.5], we have

$$\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathrm{Drinf}) \simeq \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)).$$

Combining with the Trace Conjecture (see (0.15)) we thus obtain an isomorphism

$$\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathrm{Drinf}) \simeq \mathrm{Autom}. \tag{0.17}$$

In particular, the tautological action of $\mathcal{E}xc$ on $\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathrm{Drinf})$ gives rise to an action of $\mathcal{E}xc$ on Automorphisms. This recovers V. Lafforgue's spectral decomposition.

0.8.10. The ideological significance of the isomorphism (0.17) is that it provides a *localization* picture for Autom.

Namely, it says that behind the vector space Autom stands a finer object, namely, a quasi-coherent sheaf (this is our Drinf) on the moduli *stack* of Langlands parameters (this is our $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$), such that Autom is recovered as its global sections.

In other words, Autom is something that lives over the *coarse moduli space*

$$\mathrm{LocSys}^{\mathrm{arithm,coarse}}_{\check{G}}(X) := \mathrm{Spec}(\mathcal{E}xc),$$

and it is obtained as direct image along the tautological map

$$\mathbf{r} : \mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X) \to \mathrm{LocSys}^{\mathrm{arithm,coarse}}_{\check{G}}(X)$$

from a finer object, namely Drinf, on the *moduli stack*.

Remark 0.8.11. The notation Drinf has the following origin: upon learning of V. Lafforgue's work [VLaf1], V. Drinfeld suggested that the objects

$$\widetilde{\mathrm{Sht}}_{I,V}$$

mentioned above should organize themselves into an object of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$. (However, at the time there was not yet a definition of $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$.)

Now, with our definition of $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$, the Shtuka Conjecture, i.e., (0.16), is precisely the statement that the object Drinf constructed above realizes Drinfeld's vision.

0.8.12. A particular incarnation of the localization phenomenon of Autom is the following.

Fix an e-point of $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ corresponding to an *irreducible* Weil $\check{G}$-local system σ. In Theorem 23.1.6 we show that such a point corresponds to a connected component of $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$, isomorphic to $\mathrm{pt} / \mathrm{Aut}(\sigma)$.

The restriction of Drinf to this connected component is then a representation of the (finite) group $\mathrm{Aut}(\sigma)$. The corresponding direct summand on Autom is obtained by taking $\mathrm{Aut}(\sigma)$-invariants in this representation.

0.8.13. Finally, let us juxtapose the Trace Conjecture with the Geometric Langlands Conjecture (0.11). We obtain an isomorphism

$$\mathrm{Autom} \simeq \mathrm{Tr}(\mathrm{Frob}^!, \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))).$$

Now, a (plausible, and much more elementary) Conjecture 23.6.9 says that the inclusion

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \hookrightarrow \mathrm{IndCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$$

induces an isomorphism

$$\mathrm{Tr}(\mathrm{Frob}^!, \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))) \simeq \mathrm{Tr}(\mathrm{Frob}^!, \mathrm{IndCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))). \tag{0.18}$$

Now, for any quasi-smooth stack $\mathcal{Y}$ with an endomorphism ϕ, we have

$$\mathrm{Tr}(\phi^!, \mathrm{IndCoh}(\mathcal{Y})) \simeq \Gamma(\mathcal{Y}^\phi, \omega_{\mathcal{Y}^\phi}).$$

Hence, the right-hand side in (0.18) identifies with

$$\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \omega_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}).$$

Summarizing, we obtain that the combination of the above three conjectures yields an isomorphism

$$\mathrm{Autom} \simeq \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \omega_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}). \tag{0.19}$$

This gives a conjectural expression for the space of (unramified) automorphic functions purely in terms of the stack of arithmetic Langlands parameters.

0.9. **Notations and conventions.** The notations in this paper will largely follow those adopted in [GKRV].

0.9.1. *Algebraic geometry.* There will be two algebraic geometries present in this paper.

On the one hand, we fix a ground field k (assumed algebraically closed, but of arbitrary characteristic) and we will consider algebro-geometric objects over k. This algebraic geometry will occur on the *geometric/automorphic* side of Langlands correspondence.

Thus, X will be a scheme over k (in Parts III and IV of the paper, X will be a complete curve), G will be a reductive group over k, Bun_G will be the stack of G-bundles on X, etc.

The algebro-geometric objects over k will be *classical*, i.e., *non-derived*; this is because we will study sheaf theories on them that are insensitive to the derived structure (such as ℓ-adic sheaves, or D-modules).

All algebro-geometric objects over k will be *locally of finite type* (see [GR1, Chapter 2, Sect. 1.6.1] for what this means). We will denote by $\mathrm{Sch}_{\mathrm{ft}/k}$ the category of schemes of finite type over k.

On the other hand, we will have a field of coefficients e (assumed algebraically closed *and of characteristic zero*), and we will consider *derived* algebro-geometric objects over e, see Sect. 0.9.6 below.

The above two kinds of algebro-geometric objects do not generally mix unless we work with D-modules, in which case $k = \mathsf{e}$ is a field of characteristic zero.

0.9.2. *Higher categories.* This paper will substantially use the language of ∞-categories[6], as developed in [Lu1].

We let Spc denote the ∞-category of spaces.

Given an ∞-category $\mathbf{C}$, and a pair of objects $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}$, we let $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) \in \mathrm{Spc}$ the mapping space between them.

Recall that given an ∞-category $\mathbf{C}$ that contains filtered colimits, an object $\mathbf{c} \in \mathbf{C}$ is said to be compact if the Yoneda functor $\mathrm{Maps}_{\mathbf{C}}(\mathbf{c}, -) : \mathbf{C} \to \mathrm{Spc}$ preserves filtered colimits. We let $\mathbf{C}^c \subset \mathbf{C}$ denote the full subcategory spanned by compact objects.

Given a functor $F : \mathbf{C}_1 \to \mathbf{C}_2$ between ∞-categories, we will denote by F^R (resp., F^L) its right (resp., left) adjoint, provided that it exists.

0.9.3. *Higher algebra.* Throughout this paper we will be concerned with *higher algebra* over a field of coefficients, denoted e (as was mentioned above, throughout the paper, e will be assumed algebraically closed and of characteristic zero).

We will denote by $\mathrm{Vect}_{\mathsf{e}}$ the stable ∞-category of chain complexes of e-modules, see, e.g., [GaLu, Example 2.1.4.8].

We will regard $\mathrm{Vect}_{\mathsf{e}}$ as equipped with a symmetric monoidal structure (in the sense on ∞-categories), see, e.g., [GaLu, Sect. 3.1.4]. Thus, we can talk about commutative/associative algebra objects in $\mathrm{Vect}_{\mathsf{e}}$, see, e.g., [GaLu, Sect. 3.1.3].

0.9.4. *DG categories.* We will denote by DGCat the ∞-category of presentable stable ∞-categories, *equipped with a module structure over* $\mathrm{Vect}_{\mathsf{e}}$ *with respect to the symmetric monoidal structure on the* ∞*-category of presentable stable* ∞*-categories given by the Lurie tensor product*, see [Lu2, Sect. 4.8.1]. We will refer to objects of DGCat as "DG categories". We emphasize that 1-morphisms in DGCat are in particular colimit-preserving.

For a given DG category $\mathbf{C}$, and a pair of objects $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}$, we have a well-defined "inner Hom" object $\mathcal{H}om_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) \in \mathrm{Vect}_{\mathsf{e}}$, characterized by the requirement that

$$\mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(V, \mathcal{H}om_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)) \simeq \mathrm{Maps}_{\mathbf{C}}(V \otimes \mathbf{c}_1, \mathbf{c}_2), \quad V \in \mathrm{Vect}_{\mathsf{e}}.$$

[6]We will often omit the adjective "infinity" and refer to ∞-categories simply as "categories".

The category DGCat itself carries a symmetric monoidal structure, given by Lurie tensor product over Vect:

$$\mathbf{C}_1, \mathbf{C}_2 \rightsquigarrow \mathbf{C}_1 \otimes \mathbf{C}_2.$$

In particular, we can talk about the ∞-category of associative/commutative algebras in DGCat, which we denote by $\mathrm{DGCat}^{\mathrm{Mon}}$ (resp., $\mathrm{DGCat}^{\mathrm{SymMon}}$), and refer to as monoidal (resp., symmetric monoidal) DG categories.

Unless specified otherwise, all monoidal/symmetric monoidal DG categories will be assumed unital. Given a monoidal/symmetric monoidal DG category $\mathcal{A}$, we will denote by $\mathbf{1}_{\mathcal{A}}$ its unit object.

0.9.5. *t-structures.* Given a DG category $\mathbf{C}$, we can talk about a t-structures on it. For example, the category $\mathrm{Vect}_{\mathsf{e}}$ carries a natural t-structure.

Given a t-structure on $\mathbf{C}$, we will denote by

$$\mathbf{C}^{\leq n},\ \mathbf{C}^{\geq n},\ \mathbf{C}^{\heartsuit}$$

the corresponding subcategories (according to *cohomological* indexing conventions), and also

$$\mathbf{C}^{<\infty} = \underset{n}{\cup} \mathbf{C}^{\leq n},\ \mathbf{C}^{>-\infty} = \underset{n}{\cup} \mathbf{C}^{\geq -n}.$$

We will refer to the objects of $\mathbf{C}^{\leq 0}$ (resp., $\mathbf{C}^{\geq 0}$) as *connective* (resp., *coconnective*) with respect to the given t-structure.

0.9.6. *Derived algebraic geometry over* e. Most of the work in the present paper involves algebraic geometry on the spectral side of the Langlands correspondence. This is somewhat atypical to most work in geometric Langlands.

As was mentioned above, algebraic geometry on the spectral side occurs over the field e and is derived. The starting point of derived algebraic geometry over e is the category $\mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ of *derived affine schemes* over e, which is by definition the opposite category of the category of connective commutative algebras in $\mathrm{Vect}_{\mathsf{e}}$.

All other algebro-geometric objects over e will be *prestacks*, i.e., accessible functors

$$(\mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}})^{\mathrm{op}} \to \mathrm{Spc}\,.$$

Inside the category $\mathrm{PreStk}_{/\mathsf{e}}$ of all prestacks, one singles out various subcategories. One such subcategory is $\mathrm{PreStk}_{\mathrm{laft}\,/\mathsf{e}}$ that consists of prestacks *locally almost of finite type* (see [GR1, Chapter 2, Sect. 1.7]). We set

$$\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}\,/\mathsf{e}} := \mathrm{PreStk}_{\mathrm{laft}\,/\mathsf{e}} \cap \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}\,.$$

We refer the reader to [GR1, Chapter 3] for the assighment

$$\mathcal{Y} \in \mathrm{PreStk}_{/\mathsf{e}} \rightsquigarrow \mathrm{QCoh}(\mathcal{Y}) \in \mathrm{DGCat}\,.$$

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Part I: the (pre)stack $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ and its properties

Let us briefly describe the contents of this Part.

In Sect. 1 we define the prestack $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ and state Theorem 1.4.5, pertaining to its geometric properties. We study $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ in the following general context: we consider prestacks of the form $\mathbf{Maps}(\mathsf{G}), \mathbf{H})$, where $\mathbf{H}$ is a *gentle Tannakian category*. We recover $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ by taking $\mathbf{H}$ to be the category $\mathrm{QLisse}(X)$ of lisse sheaves on X.

In Sect. 2 we establish the deformation theory properties of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ (and of its variant $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)$), leading to the conclusion that $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)$ is an ind-affine ind-scheme.

In Sect. 3 we finish the proof of Theorem 1.4.5 by combining the following two results. One is Theorem 3.1.2, which says that the underlying *reduced* prestack of $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)$ is a disjoint union of affine schemes. The other is a general result due to J. Lurie (we quote it as Theorem 3.1.4), which gives a deformation theory criterion for an ind-scheme to be a formal scheme (for completeness, we supply a proof in Sect. A). We prove Theorem 3.1.2 by constructing a uniformization of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ using parabolic subgroups of G and *irreducible* local systems for their Levi subgroups. In the process, we show that the set of connected components of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is in bijection with the set of isomorphism classes of *semi-simple* G-local systems on X.

In Sect. 4 we compare $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ with the *usual* $\mathrm{LocSys}_{\mathsf{G}}(X)$ in the two contexts in which the latter is defined, i.e., de Rham and Betti. We show that in both cases, the resulting map $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X)$ identifies $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ with the disjoint union of formal completions of closed substacks, each corresponding to G-local systems with a fixed semi-simplification. Additionally, in the Betti context, we show that these closed substacks are exactly the fibers of the map

$$\mathbf{r} : \mathrm{LocSys}_{\mathsf{G}}(X) \to \mathrm{LocSys}^{\mathrm{coarse}}_{\mathsf{G}}(X),$$

where $\mathrm{LocSys}^{\mathrm{coarse}}_{\mathsf{G}}(X)$ is the corresponding coarse moduli space.

In Sect. 5, we assume that G is reductive. First, we establish two more geometric properties of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$: namely, that it is *mock-affine* and *mock-proper*. We then state another structural result, Theorem 5.4.2, which says that $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ admits a coarse moduli space whose connected components are *formal affine schemes*.

In Sect. 6 we prove Theorem 5.4.2.

In Sect. 7 we show that the category $\mathrm{QCoh}(-)$ of a formal affine scheme has properties largely analogous to that of $\mathrm{QCoh}(-)$ of an affine scheme. The material from this section will be applied when we will study the action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

1. The restricted version of the stack of local systems

Let X be a connected scheme over the ground field k.

We will begin this section by discussing several versions of the category of lisse sheaves on X.

We will then proceed to the definition of our main object of study–the prestack $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, formulate a structural result about it, Theorem 1.4.5, and consider a few examples.

We will also introduce a rigidified version $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)$ of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, which gets rid of the "stackiness".

1.1. **Sheaves.** In this subsection we will define what we mean by the category of sheaves.

1.1.1. We will work in one of the following sheaf-theoretic contexts:

$$X \mapsto \mathrm{Shv}(X)^{\mathrm{constr}}, \quad X \in \mathrm{Sch}_{\mathrm{ft}\,/k}\,.$$

- Constructible sheaves of e-vector spaces on the topological space underlying X, when $k = \mathbb{C}$ (to be referred to as the *Betti context*);
- Holonomic D-modules on X, when $\mathrm{char}(k) = 0$ (to be referred to as the *de Rham context*);
- Constructible $\overline{\mathbb{Q}}_\ell$-adic étale sheaves on X, when $\mathrm{char}(k) \neq \ell$ (to be referred to as the *étale context*).

In the Betti context, we will sometimes consider also the category of all sheaves of vector spaces on X; we will denote it by $\mathrm{Shv}^{\mathrm{all}}(X)$, see Sect. G.1.1.

In the de Rham context we will sometimes consider also the category of all D-modules on X, denoted $\mathrm{D\text{-}mod}(X)$.

1.1.2. We will denote by e the field of coefficients of our sheaves, which will always be algebraically closed and of characteristic 0. In the three contexts above, this is e (an arbitrary algebraically closed field of characteristic 0), k and $\overline{\mathbb{Q}}_\ell$, respectively.

1.1.3. The category $\mathrm{Shv}(X)^{\mathrm{constr}}$ carries two symmetric monoidal structures. One is given by the "usual" tensor product, denoted $\overset{*}{\otimes}$, for which the unit is the constant sheaf, denoted $\underline{\mathsf{e}}_X$. The other is given by the $\overset{!}{\otimes}$ tensor product, and its unit is the dualizing sheaf, denoted ω_X.

By contrast, in the Betti context, the category $\mathrm{Shv}^{\mathrm{all}}(X)$ only carries the $\overset{*}{\otimes}$ symmetric monoidal structure, and in the de Rham context, the category $\mathrm{D\text{-}mod}(X)$ only carries the $\overset{!}{\otimes}$ symmetric monoidal structure.

1.1.4. In any of the above three contexts, we set

$$\mathrm{Shv}(X) := \mathrm{Ind}(\mathrm{Shv}(X)^{\mathrm{constr}}).$$

The two symmetric monoidal structures on $\mathrm{Shv}(X)^{\mathrm{constr}}$ define by ind-extension the corresponding two symmetric monoidal structures on $\mathrm{Shv}(X)$.

Unless explicitly stated otherwise, when talking about a symmetric monoidal structure on $\mathrm{Shv}(X)$, we will be referring to the $\overset{!}{\otimes}$ one.

1.1.5. The perverse t-structure on $\mathrm{Shv}(X)^{\mathrm{constr}}$ uniquely extends to a t-structure on $\mathrm{Shv}(X)$ compatible with filtered colimits. Its heart is the ind-completion $\mathrm{Ind}(\mathrm{Perv}(X))$ of the category $\mathrm{Perv}(X)$ of perverse sheaves on X.

Since the above t-structure on $\mathrm{Shv}(X)$ is compactly generated (see Sect. E.7.4 for what this means), it is automatically right-complete.

We record the following result, proved in Sect. E.1:

Theorem 1.1.6. *The category* $\mathrm{Shv}(X)$ *is left-complete in its t-structure.*

1.1.7. In addition to the perverse t-structure on $\mathrm{Shv}(X)^{\mathrm{constr}}$, one can consider the *usual* t-structure.

It is a characterized by the requirement that the functors of *-fiber at closed points of X are t-exact. By ind-extension, the usual t-structure on $\mathrm{Shv}(X)^{\mathrm{constr}}$ defines a t-structure on $\mathrm{Shv}(X)$, which we will refer to as the "usual" t-structure.

We note that the analog of Theorem 1.1.6 remains valid for the usual t-structure, due to the fact that the two t-structures are a finite distance apart (bounded by $\dim(X)$).

That said, unless explicitly stated otherwise, we will work with the perverse t-structure.

1.2. **Lisse sheaves.** In this subsection we will introduce one of our main actors–the category of lisse sheaves on X.

1.2.1. We define the full (abelian) subcategory

$$\mathrm{Lisse}(X)^{\heartsuit} \subset \mathrm{Shv}(X)^{\mathrm{constr}}$$

to consist of objects in the heart of the *usual* t-structure that are dualizable in the $\overset{*}{\otimes}$ symmetric monoidal structure.

We define the full DG subcategory

$$\mathrm{Lisse}(X) \subset \mathrm{Shv}(X)^{\mathrm{constr}} \tag{1.1}$$

to consist of objects whose cohomologies with respect to the *usual* t-structure belong to $\mathrm{Lisse}(X)^{\heartsuit}$.

Remark 1.2.2. Note that one can also characterize the subcategory

$$\mathrm{Lisse}(X) \subset \mathrm{Shv}(X)$$

as the subcategory of objects dualizable with respect to the $\overset{*}{\otimes}$ symmetric monoidal structure on $\mathrm{Shv}(X)$.

1.2.3. *Examples.* In the sheaf-theoretic contexts of Sect. 1.1.1, the subcategory (1.1) can be characterized as follows:

- In the Betti context, $\mathrm{Lisse}(X)^{\heartsuit}$ is the abelian category of topological local systems on X of finite rank;
- In the étale context, $\mathrm{Lisse}(X)^{\heartsuit}$ consists of ℓ-adic étale local systems on X;
- In the de Rham context (if X is smooth of dimension n), $\mathrm{Lisse}(X)^{\heartsuit}[n]$ consists of $\mathcal{O}$-coherent (right) D-modules;

1.2.4. Suppose for a moment that X is smooth of dimension n. Set

$$\mathrm{Lisse}(X)^{\heartsuit}[n] =: \mathrm{Perv}_{\mathrm{lisse}}(X) \subset \mathrm{Perv}(X).$$

be the full subcategory of *lisse* objects.

In other words, in each of these sheaf-theoretic contexts of Sect. 1.1.1, the condition that an object $\mathcal{F} \in \mathrm{Perv}(X)$ belong to $\mathrm{Perv}_{\mathrm{lisse}}(X)$ means that $\mathrm{SingSupp}(\mathcal{F}) = \{0\} \subset T^*(X)$, see Sect. E.5.1 for the notations involving singular support.

It is easy to see that the subcategory (1.1) can be alternatively characterized as consisting of objects whose cohomologies with respect to the *perverse* t-structure belong to $\mathrm{Perv}_{\mathrm{lisse}}(X)$.

Thus, in the notations of Sect. E.5.4,

$$\mathrm{Lisse}(X) = \mathrm{Shv}_{\{0\}}(X)^{\mathrm{constr}}.$$

1.2.5. We define the full (abelian) subcategory

$$\mathrm{QLisse}(X)^{\heartsuit} \subset \mathrm{Shv}(X)$$

to consist of objects in the heart of the *usual* t-structure that could be written as filtered colimits of objects from $\mathrm{Lisse}(X)^{\heartsuit}$.

The following definition is central for this paper:

Definition 1.2.6. *Let*

$$\mathrm{QLisse}(X) \subset \mathrm{Shv}(X) \tag{1.2}$$

be the full DG subcategory consisting of objects whose cohomologies with respect to the usual *t-structure belong to* $\mathrm{QLisse}(X)^{\heartsuit}$.

1.2.7. Each of the above categories:

$$\mathrm{Lisse}(X)^{\heartsuit},\ \mathrm{Lisse}(X),\ \mathrm{QLisse}(X)^{\heartsuit},\ \mathrm{QLisse}(X)$$

acquires a symmetric monoidal structures, induced by the $\overset{*}{\otimes}$ symmetric monoidal structure on $\mathrm{Shv}(X)$.

The categories $\mathrm{Lisse}(X)$ and $\mathrm{QLisse}(X)$ carry t-structures, inherited from the *usual* t-structure on $\mathrm{Shv}(X)$, and their hearts identify with $\mathrm{Lisse}(X)^{\heartsuit}$ and $\mathrm{QLisse}(X)^{\heartsuit}$, respectively.

1.2.8. Given a point $x \in X$, consider the (symmetric monoidal) functor

$$\text{QLisse}(X) \overset{\text{ev}_x}{\longrightarrow} \text{Vect}_{\mathsf{e}}, \tag{1.3}$$

given by taking the *-fiber at x.

Note that the functor ev_x is t-exact and conservative. We will regard (1.3) as a fiber functor on $\text{QLisse}(X)$, viewed as a symmetric monoidal category.

1.2.9. Assume for a moment again that X is smooth. Let $\text{Ind}(\text{Perv}_{\text{lisse}}(X))$ be the ind-completion of $\text{Perv}_{\text{lisse}}(X)$, viewed as a full abelian subcategory in $\text{Ind}(\text{Perv}(X))$.

It is easy to see that the subcategory (1.2) can be alternatively characterized as consisting of objects whose cohomologies with respect to the *perverse* t-structure belong to

$$\text{Ind}(\text{Perv}_{\text{lisse}}(X)) \subset \text{Ind}(\text{Perv}(X)).$$

In other words,

$$\text{QLisse}(X) = \text{Shv}_{\{0\}}(X)$$

in the notations of Sect. E.5.3.

1.2.10. We can define a *different* embedding

$$\text{QLisse}(X) \hookrightarrow \text{Shv}(X), \quad E \mapsto E \overset{*}{\otimes} \omega_X. \tag{1.4}$$

This embedding is a symmetric monoidal functor, when we regard $\text{Shv}(X)$ as a symmetric monoidal category via the $\overset{!}{\otimes}$ operation.

That said, when X is smooth, the above two embeddings $\text{QLisse}(X) \rightrightarrows \text{Shv}(X)$ differ by a cohomological shift (by $\dim(X)$), and in particular, they have the same essential image.

1.2.11. By Theorem 1.1.6, the category $\text{QLisse}(X)$ is left-complete in its t-structure.

Unfortunately, for a general X we will be able to say very little about the general categorical properties of $\text{QLisse}(X)$. For example, we do not know whether it is compactly generated or even dualizable.

That said, our main application is when X is a smooth algebraic curve, in which case we do know that $\text{QLisse}(X)$ is compactly generated, see Sects. E.2.6-E.2.7.

1.3. **Another version of lisse sheaves.** In addition to $\text{QLisse}(X)$ we can consider its variant, denoted $\text{IndLisse}(X)$, introduced below. The main advantage of $\text{IndLisse}(X)$ is that it is compactly generated, by definition.

1.3.1. Set

$$\text{IndLisse}(X) := \text{Ind}(\text{Lisse}(X)).$$

In other words, $\text{IndLisse}(X)$ is the full subcategory in $\text{Shv}(X) := \text{Ind}(\text{Shv}(X)^{\text{constr}})$ generated by $\text{Lisse}(X)$.

The t-structure on $\text{Lisse}(X)$ uniquely extends to a t-structure on $\text{IndLisse}(X)$ compatible with filtered colimits.

1.3.2. We have a tautologically defined fully faithful functor

$$\text{IndLisse}(X) \to \text{QLisse}(X). \tag{1.5}$$

The functor (1.5) sends compact generators of IndLisse(X) to compact objects of QLisse(X). This implies that (1.5) is fully faithful.

The functor (1.5) is t-exact since the t-structure on QLisse(X) is also compatible with filtered colimits. Moreover, it is easy to see that the functor (1.5) induces an equivalence on the hearts. Hence, it induces an equivalence

$$(\text{IndLisse}(X))^{\geq -n} \to (\text{QLisse}(X))^{\geq -n}$$

for any n. From here it follows that the functor (1.5) identifies QLisse(X) with the left completion of IndLisse(X).

1.3.3. Note, however, the functor (1.5) is *not* always an equivalence. For example, it fails to be such for $X = \mathbb{P}^1$, see Sect. E.2.7.

Equivalently, the category IndLisse(X) is *not* necessarily left-complete in its t-structure.

That said, as we will see in Sect. E.2.6, the functor (1.5) is an equivalence for all smooth connected curves X (projective or affine) different from $\mathbb{P}^1$.

Remark 1.3.4. The procedure by which we obtained IndLisse(X) from QLisse(X) is similar to the procedure by which one produces IndCoh(S) from QCoh(S) (where S is a scheme almost of finite type).

In that situation we also have a functor

$$\text{IndCoh}(S) \to \text{QCoh}(S). \tag{1.6}$$

However, unlike (1.5), the functor (1.6) is only fully faithful when S is smooth, in which case it is an equivalence.

If S is not smooth but eventually coconnective, the functor (1.6) is a co-localization (i.e., admits a fully faithful left adjoint). So, in a sense, the functor (1.6) exhibits a behavior opposite to that of (1.5).

1.3.5. One should consider IndLisse(X) as a "really nice" symmetric monoidal category, in that it is compactly generated and rigid (see [GR1, Chapter 1, Sect. 9.2] for what this means).

Moreover, one can pick compact generators that belong to the heart of IndLisse(X), and they will have cohomological dimension bounded by dim(X).

One thing that IndLisse(X) is *not* is that it is *not* the derived category of its heart, see Sect. E.2.1.

1.4. **Definition of** $\text{LocSys}_{\mathsf{G}}^{\text{restr}}(X)$ **as a functor.** For the duration of Part I, we let G be a connected algebraic group.

1.4.1. Recall that if $\mathbf{C}$ and $\mathbf{C}'$ are DG categories, each equipped with a t-structure[7], then the tensor product

$$\mathbf{C} \otimes \mathbf{C}'$$

carries a naturally defined t-structure, characterized by the property that its connective part $(\mathbf{C} \otimes \mathbf{C}')^{\leq 0}$ is generated by objects

$$\mathbf{c} \otimes \mathbf{c}', \quad \mathbf{c} \in \mathbf{C}^{\leq 0}, \ \mathbf{c}' \in (\mathbf{C}')^{\leq 0}.$$

Suppose for a moment that $\mathbf{C}'$ is of the form R-mod, where $R \in \text{AssocAlg}(\text{Vect}_{\mathsf{e}}^{\leq 0})$. Then the above t-structure is characterized by the property that the forgetful functor

$$\mathbf{C} \otimes R\text{-mod} \simeq R\text{-mod}(\mathbf{C}) \to \mathbf{C}$$

is t-exact.

[7]Recall that according to our conventions, we require that t-structures be compatible with filtered colimits.

1.4.2. Let X be a connected scheme of finite type over k. We define the prestack

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$$

over the field e of coefficients, by sending $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ to the space of *right t-exact symmetric monoidal functors*

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X). \tag{1.7}$$

Note that e-points of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ are what we usually call G-local systems on X (within our sheaf theory).

1.4.3. *Example.* Let $X = \mathrm{pt}$. Then $\mathrm{QLisse}(X) = \mathrm{Vect}_{\mathsf{e}}$ and we have, by Tannaka duality (see e.g. [Lu3, Corollary 9.4.4.7]), that $\mathrm{LocSys}^{\mathrm{restr}}(X) = \mathrm{pt}/\mathsf{G}$, the classifying stack of G.

1.4.4. The main result of this Part is the following:

Main Theorem 1.4.5. *The prestack* $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ *is an étale stack, equal to a disjoint union of étale stacks each of which can be written as an étale-sheafified quotient by* G *of an étale stack* $\mathcal{Y}$ *with the following properties:*

(a) $\mathcal{Y}$ *is locally almost of finite type*[8]*;*

(b) ${}^{\mathrm{red}}\mathcal{Y}$ *is an affine (classical, reduced) scheme;*

(c) $\mathcal{Y}$ *is an ind-scheme;*

(d) $\mathcal{Y}$ *can be written as*

$$\underset{n\geq 0}{\mathrm{colim}}\ \mathrm{Spec}(R_n) \tag{1.8}$$

where R_n *are connective commutative* e*-algebras of the following form: there exists a connective commutative* e*-algebra* R *and elements*[9] $f_1, ..., f_m \in R$ *so that*

$$R_n = R \underset{\mathsf{e}[t_1,...,t_m]}{\otimes} \mathsf{e}[t_1, ..., t_m]/(t_1^n, ..., t_m^n), \quad t_i \mapsto f_i \in R,\ i = 1, ..., m.$$

Remark 1.4.6. Points (a,b,c) of Theorem 1.4.5 can be combined to the following statement: $\mathcal{Y}$ can be written as *filtered* colimit

$$\mathcal{Y} \simeq \underset{i}{\mathrm{colim}}\, Y_n, \tag{1.9}$$

where all Y_n are affine schemes almost of finite type[10], and the maps $Y_{n_1} \to Y_{n_2}$ are closed nil-isomorphisms (i.e., closed embeddings that induce isomorphisms of the underlying reduced prestacks), see [GR2, Chapter 2, Corollary 1.8.6(a)].

Remark 1.4.7. Note, however, that points (a,b,c) of Theorem 1.4.5 do *not* include the assertion contained in (d). For example, if we take

$$\mathcal{Y} := \underset{n}{\mathrm{colim}}\, (\mathbb{A}^n)^{\wedge}_0,$$

then this $\mathcal{Y}$ admits a presentation as in (1.9) (with the specified properties), but it does *not* admit a presentation (1.8) (the reason is that prestacks of the latter form admit *cotangent spaces*, while the former only *pro-cotangent spaces*, see Sect. 2.2).

The property of admitting a presentation as in (1.8) insures, among other things, that the category $\mathrm{QCoh}(\mathcal{Y})$ is particularly well-behaved (has many properties similar to those of $\mathrm{QCoh}(-)$ of an affine scheme, see Sect. 7).

Prestacks $\mathcal{Y}$ satisfying (d) are called *formal affine schemes.*

Finally, we emphasize that the commutative algebra R that appears in (d) is *not* necessarily almost of finite type over e.

---

[8]See [GR1, Chapter 2, Sect. 1.7.2] for what this means.

[9]By an element of R we mean a point in the space corresponding to R via Dold-Kan correspondence.

[10]See [GR1, Chapter 2, Sect. 1.7.1] for what this means.

Remark 1.4.8. For the validity of Theorem 1.4.5 in the Betti context, we can work more generally: instead of starting with an algebraic variety X over $\mathbb{C}$, we can let X be a topological space homotopy equivalent to a (retract of a) finite CW complex.

1.5. **Some examples.**

1.5.1. Let $\mathsf{G} = \mathbb{G}_m$. As we shall see in Corollary 3.4.3, in this case the *underlying reduced prestack* of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is a disjoint union, over the set of isomorphism classes of one-dimensional local systems on X, of copies of $\mathrm{pt}/\mathbb{G}_m$.

In Sect. 2.2 we will see that for each 1-dimensional local system (i.e., an e-point of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_m}(X)$), the tangent space to $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ at this point identifies with

$$\mathrm{C}^{\cdot}(X, \mathsf{e}_X)[1] \in \mathrm{Vect}_{\mathsf{e}}, \tag{1.10}$$

i.e., it looks like the tangent space of the "usual would-be" $\mathrm{LocSys}_{\mathbb{G}_m}(X)$. (Tangent spaces are defined for prestacks that admit deformation theory and are locally almost of finite type, see [GR2, Chapter 1, Sect. 4.4].)

1.5.2. Let $\mathsf{G} = \mathbb{G}_a$. We claim that in this case $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is the algebraic stack associated with the object (1.10), i.e.,

$$\mathrm{Maps}(\mathrm{Spec}(R), \mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_a}(X)) = \tau^{\leq 0}(R \otimes \mathrm{C}^{\cdot}(X, \mathsf{e}_X)[1]), \tag{1.11}$$

where we view an object of $(\mathrm{Vect}_{\mathsf{e}})^{\leq 0}$ as a space by the Dold-Kan functor (see [GR1, Chapter 1, Sect. 10.2.3]).

Indeed,

$$\mathrm{Rep}(\mathbb{G}_a) \simeq \mathsf{e}[\xi]\text{-mod}, \quad \deg(\xi) = 1,$$

so the space of symmetric monoidal functors from $\mathrm{Rep}(\mathbb{G}_a)$ to any symmetric monoidal category $\mathbf{A}$ identifies with

$$\tau^{\leq 0}(\mathcal{E}nd_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}})[1]).$$

In our case $\mathbf{A} = R\text{-mod} \otimes \mathrm{QLisse}(X)$ so $\mathbf{1}_{\mathbf{A}} = R \otimes \mathsf{e}_X$, whence (1.11).

1.5.3. *Notation.* In what follows, for $V \in \mathrm{Vect}_{\mathsf{e}}$ we will use the notation $\mathrm{Tot}(V)$ for the corresponding prestack, i.e.,

$$\mathrm{Hom}(S, \mathrm{Tot}(V)) = \tau^{\leq 0}(V \otimes \Gamma(S, \mathcal{O}_S)), \quad S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}. \tag{1.12}$$

For example, when $V \in \mathrm{Vect}^{\heartsuit}_{\mathsf{e}} \cap \mathrm{Vect}^{c}_{\mathsf{e}}$, we have

$$\mathrm{Tot}(V) = \mathrm{Spec}(\mathrm{Sym}(V^{\vee})).$$

Thus, (1.11) is saying that

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_a}(X) \simeq \mathrm{Tot}(\mathrm{C}^{\cdot}(X, \mathsf{e}_X)[1]).$$

1.5.4. This is a preview of the material in Sect. 4:

Let our sheaf-theoretic context be either Betti or de Rham, so in both cases we have the usual algebraic stack $\mathrm{LocSys}_{\mathsf{G}}(X)$. In this case we will show that there exists a forgetful map

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}_{\mathsf{G}}(X), \tag{1.13}$$

which identifies $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ with the disjoint union of formal completions of a collection of pairwise non-intersecting Zariski-closed reduced substacks of $\mathrm{LocSys}_{\mathsf{G}}(X)$, such that every e-point of $\mathrm{LocSys}_{\mathsf{G}}(X)$ belongs to (exactly) one of these substacks. Furthermore, we will be able to describe the corresponding reduced substacks explicitly.

1.6. **Rigidification.** Let us choose a base point $x \in X$. We will introduce a cousin of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, denoted $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)$, that has to do with choosing a trivialization of our local systems at x.

1.6.1. Given a base point $x \in X$, consider the functor

$$\mathrm{QLisse}(X) \xrightarrow{\mathrm{ev}_x} \mathrm{Vect}_{\mathsf{e}}, \tag{1.14}$$

of (1.3).

Consider the prestack $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ that sends $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ to the space of symmetric monoidal functors (1.7), equipped with an isomorphism between the composition

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X) \overset{\mathrm{Id}\otimes\mathrm{ev}_x}{\longrightarrow} \mathrm{QCoh}(S)$$

and

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}} \overset{\mathrm{unit}}{\longrightarrow} \mathrm{QCoh}(S) \tag{1.15}$$

(as symmetric monoidal functors). Note that the latter identification implies that the functor (1.7) is right t-exact: this is due to the fact that ev_x is t-exact and conservative.

In other words,

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X) \simeq \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) \underset{\mathrm{pt}/\mathsf{G}}{\times} \mathrm{pt}, \tag{1.16}$$

where the map

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) \to \mathrm{pt}/\mathsf{G}$$

is given by (1.3) (see Example 1.4.3).

1.6.2. From the above description of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ as a fiber product, we obtain that there is a natural action of G on $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$, and we will show in Corollary 2.2.5 that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ identifies with the étale sheafification of the quotient of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ by this action.

Hence, in order to prove Theorem 1.4.5, it will suffice to show the following:

Theorem 1.6.3. *The prestack* $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr},\mathrm{rigid}_x}(X)$ *is an étale stack equal to a disjoint union of étale stacks* $\mathcal{Y}$ *with properties* (a)-(d) listed in Theorem 1.4.5.

1.7. **Gentle Tannakian categories.** For the proof of Theorem 1.4.5 it will be convenient to replace QLisse(X) by an abstract symmetric monoidal category, to be denoted $\mathbf{H}$, that possesses certain properties.

In this subsection we will introduce the notion of *gentle Tannakian category*. This will be the class of symmetric monoidal categories for which will state and prove an appropriate generalization of Theorem 1.4.5.

1.7.1. Let $\mathbf{H}$ be a symmetric monoidal category, equipped with a t-structure and a conservative t-exact symmetric monoidal functor[11] $\mathbf{oblv}_{\mathbf{H}}$ to $\mathrm{Vect}_{\mathsf{e}}$.

Note that the assumptions on $\mathbf{oblv}_{\mathbf{H}}$ imply that the t-structure on $\mathbf{H}$ is compatible with filtered colimits and right-complete.

We will additionally assume that $\mathbf{H}$ is left-complete in its t-structure.

[11]Recall that all our functors are by default assumed to commute with all colimits.

1.7.2. Let $\mathbf{H}$ be as above. We will make the following assumptions, which can be summarized in saying that $\mathbf{H}$ is a particularly well-behaved Tannakian category:

- The following classes of objects in $\mathbf{H}^\heartsuit$ coincide:
  - (i) Objects contained in $\mathbf{H}^c \cap \mathbf{H}^\heartsuit$;
  - (ii) Objects that are sent to compact objects in $\mathrm{Vect}_{\mathsf{e}}$ by $\mathbf{oblv}_{\mathbf{H}}$;
  - (iii) Dualizable objects.
- The object $\mathbf{1}_{\mathbf{H}}$ has the following properties:
  - (i) The functor $\mathcal{H}om_{\mathbf{H}}(\mathbf{1}_{\mathbf{H}}, -)$ has a finite cohomological amplitude;
  - (ii) For any $\mathbf{h} \in \mathbf{H}^c \cap \mathbf{H}^\heartsuit$, the cohomologies of $\mathcal{H}om_{\mathbf{H}}(\mathbf{1}_{\mathbf{H}}, \mathbf{h}) \in \mathrm{Vect}_{\mathsf{e}}$ are finite-dimensional.
- The category $\mathbf{H}^\heartsuit$ is generated under colimits by $\mathbf{H}^c \cap \mathbf{H}^\heartsuit$.

We will call a symmetric mononidal category with the above properties a *gentle Tannakian category.*

1.7.3. Note that, given the first bullet point, the second one can be reformulated as saying that for any pair of objects $\mathbf{h}_1, \mathbf{h}_2 \in \mathbf{H}^c \cap \mathbf{H}^\heartsuit$, we have

$$\mathcal{H}om_{\mathbf{H}}(\mathbf{h}_1, \mathbf{h}_2) \in \mathrm{Vect}_{\mathsf{e}}^c.$$

1.7.4. *Example.* The main is example of interest for us is of course

$$\mathbf{H} = \mathrm{QLisse}(X).$$

In this case, we take the fiber functor $\mathbf{oblv}_{\mathbf{H}}$ to be ev_x, see Sect. 1.2.8.

Another example to keep in mind is $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$, where H is an algebraic group of finite type over e, and $\mathbf{oblv}_{\mathbf{H}}$ is the tautological forgetful functor (to be henceforth denoted $\mathbf{oblv}_{\mathsf{H}}$).

1.7.5. Let $\mathbf{H}^{\mathrm{access}}$ be the full subcategory of $\mathbf{H}$ generated by $\mathbf{H}^c \cap \mathbf{H}^\heartsuit$. In other words, $\mathbf{H}^{\mathrm{access}}$ is the ind-completion of the small DG subcategory $\mathbf{H}^{\mathrm{access},c} \subset \mathbf{H}$ consisting of *cohomologically bounded* objects all of whose cohomologies belong to $\mathbf{H}^c \cap \mathbf{H}^\heartsuit$.

Since $\mathbf{H}^c \cap \mathbf{H}^\heartsuit$ is closed under the monoidal operation, the category $\mathbf{H}^{\mathrm{access}}$ inherits a symmetric monoidal structure.

By construction, $\mathbf{H}^{\mathrm{access}}$ is *rigid* as a symmetric monoidal category (see [GR1, Chapter 1, Sect. 9.2] for what this means)[12].

1.7.6. Consider the tautological embedding

$$\mathbf{H}^{\mathrm{access}} \to \mathbf{H}. \tag{1.17}$$

The t-structure on $\mathbf{H}$ restricts to a t-structure on $\mathbf{H}^{\mathrm{access},c}$, and the latter gives rise to a t-structure $\mathbf{H}^{\mathrm{access}}$. The functor (1.17) is t-exact and induces an equivalence

$$(\mathbf{H}^{\mathrm{access}})^{\geq -n} \to \mathbf{H}^{\geq -n}. \tag{1.18}$$

It follows that the functor (1.17) realizes $\mathbf{H}$ as the left-completion of $\mathbf{H}^{\mathrm{access}}$.

Since the functor (1.17) sends the compact generators of $\mathbf{H}^{\mathrm{access}}$ to compact objects of $\mathbf{H}$, we obtain that (1.17) is fully faithful.

Remark 1.7.7. The mechanism by which (1.17) fails to be an equivalence is that the subcategory $\mathbf{H}^c \cap \mathbf{H}^\heartsuit$ does not necessarily generate $\mathbf{H}$ under colimits.

This may happen even if $\mathbf{H}$ itself compactly generated: its compact generators may be unbounded below.

[12]Note that $\mathbf{H}$ was not necessarily rigid.

1.7.8. *Example.* When $\mathbf{H} = \mathrm{QLisse}(X)$, the category $\mathbf{H}^{\mathrm{access}}$ is the category $\mathrm{IndLisse}(X)$ introduced in Sect. 1.3.1.

In the example $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$, where H is an algebraic group of finite type, the functor (1.17) is an equivalence.

1.7.9. The conditions on $\mathbf{H}$ imply that the unit object $\mathbf{1}_{\mathbf{H}} \in \mathbf{H}$ is compact. Let $\mathbf{inv}_{\mathbf{H}}$ denote the functor

$$\mathcal{H}om_{\mathbf{H}}(\mathbf{1}_{\mathbf{H}}, -) : \mathbf{H} \to \mathrm{Vect},$$

i.e., the right adjoint of the unit functor $\mathrm{Vect}_{\mathsf{e}} \overset{\mathsf{e} \mapsto \mathbf{1}_{\mathbf{H}}}{\longrightarrow} \mathbf{H}$.

For the sequel, we record the following:

Lemma 1.7.10. *The unit functor* $\mathrm{Vect}_{\mathsf{e}} \to \mathbf{H}$ *admits a left adjoint (to be denoted* $\mathbf{coinv}_{\mathbf{H}}$*).*

Proof. First, the functor $\mathbf{coinv}_{\mathbf{H}}$ is defined on objects from $\mathbf{H}^c \cap \mathbf{H}^{\heartsuit}$. Indeed,

$$\mathbf{coinv}_{\mathbf{H}}(\mathbf{h}) = \left(\mathcal{H}om_{\mathbf{H}}(\mathbf{h}, \mathbf{1}_{\mathbf{H}})\right)^{\vee}.$$

Since $\mathbf{H}^{\geq -n}$ is generated under filtered colimits by $\mathbf{H}^c \cap \mathbf{H}^{\heartsuit}$, we obtain that $\mathbf{coinv}_{\mathbf{H}}$ is defined on $\mathbf{H}^{\geq -n}$ for any n.

We now claim that for an arbitrary $\mathbf{h} \in \mathbf{H}$, the value of $\mathbf{coinv}_{\mathbf{H}}$ on it is given by

$$\lim_{n} \mathbf{coinv}_{\mathbf{H}}(\tau^{\geq -n}(\mathbf{h})).$$

Indeed, for every m, the m-th cohomology of the system

$$n \mapsto \mathbf{coinv}_{\mathbf{H}}(\tau^{\geq -n}(\mathbf{h}))$$

stabilizes (since $\mathbf{coinv}_{\mathbf{H}}$ is right t-exact), and the above object has the required adjunction property by the left-completeness of $\mathbf{H}$.

□

1.7.11. *Example.* For $\mathbf{H} = \mathrm{QLisse}(X)$, the functor $\mathbf{coinv}_{\mathbf{H}}$ identifies with the functor of "cochains with compact supports"

$$E \mapsto \mathrm{C}^{\cdot}_c(X, E \overset{*}{\otimes} \omega_X). \tag{1.19}$$

For $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$, the functors $\mathbf{inv}_{\mathbf{H}}$ and $\mathbf{coinv}_{\mathbf{H}}$ are the usual functors H-invariants and coinvariants, respectively, to be henceforth denoted $\mathbf{inv}_{\mathsf{H}}$ and $\mathbf{coinv}_{\mathsf{H}}$.

1.8. **The prestack** $\mathrm{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ **and the abstract version of Theorem 1.4.5.** In this subsection we will introduce an abstract version of the prestacks $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

1.8.1. Let $\mathbf{H}$ be as in Sect. 1.7.1, and let G be a connected algebraic group.

We define the prestack $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ by sending an affine scheme S to the space of right t-exact symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathbf{H}. \tag{1.20}$$

1.8.2. We are now ready to state an abstract version of Theorem 1.4.5:

Theorem 1.8.3. *Assume that* $\mathbf{H}$ *is a gentle Tannakian category. Then the prestack* $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ *has the properties listed in Theorem 1.4.5.*

1.8.4. *A rigidified version.* Along with $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$, we can consider its rigidified version. We define the prestack $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ by sending an affine scheme S to the space of symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathbf{H}, \tag{1.21}$$

equipped with an identification of the composition

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathbf{H} \overset{\mathrm{Id}_{\mathrm{QCoh}(S)} \otimes \mathbf{oblv}_{\mathbf{H}}}{\longrightarrow} \mathrm{QCoh}(S)$$

with the forgetful functor

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\to} \mathrm{Vect}_{\mathsf{e}} \overset{\mathcal{O}_S}{\longrightarrow} \mathrm{QCoh}(S),$$

as symmetric monoidal functors. In other words,

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}} \simeq \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \underset{\mathrm{pt}/\mathsf{G}}{\times} \mathrm{pt}\,. \tag{1.22}$$

The remarks pertaining to the replacement

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \rightsquigarrow \mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X)$$

apply verbatim to the replacement

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}.$$

In particular, Theorem 1.8.3 will follow once we prove its rigidified version:

Theorem 1.8.5. *Let* $\mathbf{H}$ *be as in Theorem 1.8.3. Then the prestack* $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ *is an étale stack equal to a disjoint union of étale stacks* $\mathcal{Y}$ *with properties* (a)-(d) listed in Theorem 1.4.5.

2. Ind-representability and the beginning of proof of Theorem 1.8.3

In this section we will study deformation theory (=infinitesimal) properties of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, or more generally, $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$. Most of these properties follow easily from the definitions, apart from some issues of convergence.

We will conclude that $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ is an ind-scheme. We will show this by reducing to the assertion that for a pair of algebraic groups H and G, the prestack of maps $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$ is an ind-affine ind-scheme.

2.1. **Convergence.** In this subsection we begin the investigation of infinitesimal properties of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ (resp., $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$).

We start with the most basic one–the property of being convergent[13], which is one of the ingredients in the condition of being almost of finite type, stated in Theorem 1.8.3.

2.1.1. By the definition of convergence, we need to show that for a (derived) affine test-scheme S, the map

$$\mathrm{Maps}(S, \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \to \lim_{n} \mathrm{Maps}({}^{\leq n}S, \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \tag{2.1}$$

is an isomorphism, where $S \mapsto {}^{\leq n}S$ denotes the n-th coconnective truncation, i.e., the operation

$$R \mapsto \tau^{\geq -n}(R)$$

at the level of rings.

[13]See [GR1, Chapter 2, Sect. 1.4] for what this means.

2.1.2. In what follows we will use the following assertion:

Lemma 2.1.3. *Let* $\mathbf{C}$ *be a category equipped with a t-structure. Then for* S *as above we have:*

(a) *If* $\mathbf{C}$ *is left-complete, then* $\mathbf{C} \otimes \mathrm{QCoh}(S)$ *is also left-complete.*

(a') *More generally, if* $\mathbf{C}^\wedge$ *is the left completion of* $\mathbf{C}$*, then the functor*

$$\mathrm{QCoh}(S) \otimes \mathbf{C} \to \mathrm{QCoh}(S) \otimes \mathbf{C}^\wedge$$

identifies $\mathrm{QCoh}(S) \otimes \mathbf{C}^\wedge$ *with the left completion of* $\mathrm{QCoh}(S) \otimes \mathbf{C}$.

(b) *If* $\mathbf{C}$ *is left-complete, then the functor*

$$(\mathbf{C} \otimes \mathrm{QCoh}(S))^{\leq 0} \to \lim_n \, (\mathbf{C} \otimes \mathrm{QCoh}({}^{\leq n}S))^{\leq 0}$$

is an equivalence.

Proof. Points (a) and (a') follow from the fact that the functor

$$\mathbf{C} \otimes \mathrm{QCoh}(S) \overset{\mathrm{Id} \otimes \Gamma(S,-)}{\longrightarrow} \mathbf{C}$$

is t-exact, conservative and commutes with limits.

Point (b) follows from point (a) and the fact that for any n, the functor

$$(\mathbf{C} \otimes \mathrm{QCoh}(S))^{\leq 0, \geq -n} \to (\mathbf{C} \otimes \mathrm{QCoh}({}^{\leq m}S))^{\leq 0, \geq -n}$$

is an equivalence, whenever $m \geq n$.

□

2.1.4. We are now ready to prove that (2.1) is an equivalence.

Proof. Since $\mathrm{Rep}(\mathsf{G})$ is the derived category of its heart and the monoidal operation is t-exact, the space of right t-exact symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathbf{H}$$

is isomorphic to the space of symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G})^\heartsuit \to (\mathrm{QCoh}(S) \otimes \mathbf{H})^{\leq 0},$$

and similarly for every ${}^{\leq n}S$.

The assertion now follows from the assumption that $\mathbf{H}$ is left-complete in its t-structure and Lemma 2.1.3(b).

□

Remark 2.1.5. Note that the above argument used the fact that $\mathbf{H}$ is left-complete in its t-structure (in order to be able to apply Lemma 2.1.3).

This is the reason that we have to work with $\mathrm{QLisse}(X)$ (resp., $\mathbf{H}$) rather than with the more manageable category $\mathrm{IndLisse}(X)$ (resp., $\mathbf{H}^{\mathrm{access}}$).

2.1.6. Despite the previous remark, we will now show that one can work $\mathbf{H}^{\mathrm{access}}$ instead of $\mathbf{H}$ as long as we evaluate our prestack on *eventually coconnective* affine schemes.

Recall that an affine scheme S is said to be eventually coconnective if it equals ${}^{\leq n}S$ for some n (i.e., if its structure ring has cohomologies in finitely many degrees).

Proposition 2.1.7. *Suppose that* S *is eventually coconnective. Then the functor* (1.17) *defines an isomorphism from the space of (right t-exact) symmetric monoidal functors*

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathbf{H}^{\mathrm{access}}$$

to the space of (right t-exact) symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathbf{H}.$$

Proof. The space of (continuous) symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathbf{H}^{\mathrm{access}}$$

maps isomorphically to the space of symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G})^c \to \mathrm{QCoh}(S) \otimes \mathbf{H}^{\mathrm{access}},$$

and similarly for $\mathbf{H}^{\mathrm{access}}$ replaced by $\mathbf{H}$.

Since every object of $\mathrm{Rep}(\mathsf{G})^c$ is dualizable, it suffices to show that the embedding

$$\mathrm{QCoh}(S) \otimes \mathbf{H}^{\mathrm{access}} \hookrightarrow \mathrm{QCoh}(S) \otimes \mathbf{H} \tag{2.2}$$

induces an equivalence on the subcategories of dualizable objects. The functor (2.2) is a priori fully faithful because the functor (1.17) is such, while $\mathrm{QCoh}(S)$ is dualizable.

Note that by Lemma 2.1.3(a'), the functor (2.2) identifies $\mathrm{QCoh}(S) \otimes \mathbf{H}$ with the left completion of $\mathrm{QCoh}(S) \otimes \mathbf{H}^{\mathrm{access}}$.

Hence, it suffices to show that (for S eventually coconnective), any dualizable object in the category $\mathrm{QCoh}(S) \otimes \mathbf{H}$ is bounded below (in the sense of the t-structure).

Since the functor $\mathbf{oblv_H}$ is conservative and t-exact, the functor

$$\mathrm{QCoh}(S) \otimes \mathbf{H} \overset{\mathrm{Id} \otimes \mathbf{oblv_H}}{\longrightarrow} \mathrm{QCoh}(S)$$

is also t-exact and conservative.

Hence, it is enough to show that $\mathrm{Id} \otimes \mathbf{oblv_H}$ sends dualizable objects to objects bounded below. However, $\mathrm{Id} \otimes \mathbf{oblv_H}$ is symmetric monoidal, the assertion follows from the fact that dualizable objects in $\mathrm{QCoh}(S)$ (for S eventually coconnective) are bounded below.

□

2.2. **Deformation theory: statements.** In this subsection we will formulate the deformation theory properties of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$, along with its version $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$.

This will be an ingredient in the proof of that fact that $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ is an ind-scheme, stated in Theorem 1.8.5.

2.2.1. We will prove:

Proposition 2.2.2.
(a) *The prestack* $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ *admits a* (-1)*-connective corepresentable deformation theory.*
(b) *For* $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ *and an* S*-point*

$$\mathsf{F} : \mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathbf{H}$$

of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$*, the cotangent space* $T^*_{\mathsf{F}}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \in \mathrm{QCoh}(S)^{\leq 1}$ *identifies with*

$$(\mathrm{Id} \otimes \mathbf{coinv_H})(\mathsf{F}(\mathsf{g}^\vee))[-1],$$

where g *is the Lie algebra of* G*.*

Remark 2.2.3. We refer the reader to [GR2, Chapter 1, Definition 7.1.5(a)], where it is explained what it means to admit a $(-n)$-connective corepresentable deformation theory. In fact, there are three conditions:

(i) The first one is that the prestack admits deformation theory (i.e. admits pro-cotangent spaces that are functorial in the test-scheme, and is infinitesimally cohesive);

(ii) The adjective "corepresentable" refers to the fact that the pro-cotangent spaces are actually objects (of $\mathrm{QCoh}(S)^{<\infty}$, where S is the test-scheme), and not only pro-objects.

(iii) The adjective "$(-n)$-connective" refers to the fact that cotangent spaces actually belong to $\mathrm{QCoh}(S)^{\leq n}$.

2.2.4. As a consequence, we deduce:

Corollary 2.2.5. *The prestack* $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ *satisfies étale descent. In particular, it identifies with the étale quotient* $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}/\mathsf{G}$.

Proof. By Proposition 2.2.2(a) and [GR2, Chapter 1, Proposition 8.2.2], it suffices to show that the underlying classical prestack satisfies étale descent. Thus, by Proposition 2.1.7, it suffices to show that the functor

$$S \mapsto \{\text{right } t\text{-exact symmetric monoidal functors } \mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathbf{H}^{\mathrm{access}}\}$$

satisfies étale descent, for S a classical affine scheme.

Since $\mathbf{H}^{\mathrm{access}}$ is compactly generated (and in particular dualizable), the functor $-\otimes \mathbf{H}^{\mathrm{access}}$ preserves limits. The result now follows from the fact that $\mathrm{QCoh}(S)$ satisfies étale descent.

Now, the assertion that

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \simeq \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}/\mathsf{G}$$

follows from the fact that $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ satisfies étale descent and the identification (1.22). □

Using the presentation of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ as (1.22), from Proposition 2.2.2, we obtain:

Corollary 2.2.6.
(a) *The prestack* $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ *admits a connective corepresentable deformation theory.*
(b) *For* $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ *and an* S*-point of* $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$*, we have a canonical identification*

$$T^*_{\mathsf{F}}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}) \simeq \mathrm{Fib}\left((\mathcal{O}_S \otimes \mathsf{g}^\vee) \to (\mathrm{Id} \otimes \mathbf{coinv}_{\mathbf{H}})(\mathsf{F}(\mathsf{g}^\vee))\right).$$

(b') *The object* $T^*_{\mathsf{F}}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}) \in \mathrm{QCoh}(S)^{\leq 0}$ *belongs to* $\mathrm{QCoh}(S)^{\leq 0} \cap \mathrm{QCoh}(S)^c$.

Proof. The fact that $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ admits corepresentable deformation theory follows formally from Proposition 2.2.2(a) and (1.22).

The functoriality of the identification in Proposition 2.2.2(b) with respect to $\mathbf{H}$ implies that the codifferential of the map

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \to \mathrm{pt}/\mathsf{G}$$

at a point $\mathsf{F} \in \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ is given by

$$(\mathrm{Id} \otimes \mathbf{oblv}_{\mathbf{H}})(\mathsf{F}(\mathsf{g}^\vee))[-1] \to (\mathrm{Id} \otimes \mathbf{coinv}_{\mathbf{H}})(\mathsf{F}(\mathsf{g}^\vee))[-1]. \tag{2.3}$$

This implies the assertion of point (b). Since the map (2.3) induces a *surjection*

$$H^0\left((\mathrm{Id} \otimes \mathbf{oblv}_{\mathbf{H}})(\mathsf{F}(\mathsf{g}^\vee))\right) \to H^0\left((\mathrm{Id} \otimes \mathbf{coinv}_{\mathbf{H}})(\mathsf{F}(\mathsf{g}^\vee))\right),$$

this implies that the cotangent spaces of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ are connective.

For point (b') we will show that for any dualizable $V \in \mathrm{Rep}(\mathsf{G})$, the object

$$(\mathrm{Id} \otimes \mathbf{coinv}_{\mathbf{H}})(\mathsf{F}(V)) \in \mathrm{QCoh}(S)^{\leq 0}$$

belongs to $\mathrm{QCoh}(S)^{\leq 0} \cap \mathrm{QCoh}(S)^c$.

It suffices to show that this happens after restriction to any truncation of S. Hence, we can assume that S is eventually coconnective. In this case, by Proposition 2.1.7, we can regard F as a functor

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathbf{H}^{\mathrm{access}}.$$

The object $\mathsf{F}(V) \in \mathrm{QCoh}(S) \otimes \mathbf{H}^{\mathrm{access}}$ is dualizable. Since both $\mathrm{QCoh}(S)$ and $\mathbf{H}^{\mathrm{access}}$ are rigid, the category $\mathrm{QCoh}(S) \otimes \mathbf{H}^{\mathrm{access}}$ is also rigid. Hence $\mathsf{F}(V)$ is compact as an object of $\mathrm{QCoh}(S) \otimes \mathbf{H}^{\mathrm{access}}$.

The composite functor

$$\mathbf{H}^{\mathrm{access}} \overset{(1.17)}{\to} \mathbf{H} \overset{\mathbf{coinv}_{\mathbf{H}}}{\to} \mathrm{Vect}_{\mathsf{e}}$$

is the left adjoint of the unit functor. Hence, it preserves compactness. Hence, so does the functor

$$\mathrm{Id} \otimes \mathbf{coinv}_{\mathbf{H}} : \mathbf{H}^{\mathrm{access}} \otimes \mathrm{QCoh}(S) \to \mathrm{QCoh}(S).$$

□

2.3. **Establishing deformation theory.** This subsection is devoted to the proof of the fact that $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ admits deformation theory, which is part of the assertion of Proposition 2.2.2(a).

2.3.1. By [GR2, Chapter 1, Proposition 7.2.5], we need to show that for a push-out diagram of affine schemes

$$\begin{array}{ccc} S_1 & \longrightarrow & S_2 \\ \downarrow & & \downarrow \\ S_1' & \longrightarrow & S_2', \end{array} \tag{2.4}$$

where $S_1 \to S_1'$ is a nilpotent embedding, the diagram

$$\begin{array}{ccc} \mathrm{Maps}(S_2', \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) & \longrightarrow & \mathrm{Maps}(S_1', \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \\ \downarrow & & \downarrow \\ \mathrm{Maps}(S_2, \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) & \longrightarrow & \mathrm{Maps}(S_1, \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \end{array}$$

is a pullback square of spaces.

Since $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ is convergent, we can assume that affine schemes in (2.4) are eventually coconnective.

2.3.2. Using Proposition 2.1.7, for S eventually coconnective, we interpret $\mathrm{Maps}(S, \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))$ as the space of right t-exact symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G})^c \to (\mathrm{QCoh}(S) \otimes \mathbf{H}^{\mathrm{access}})^{\mathrm{dualizable}}.$$

Hence, it suffices to show that in the situation of (2.4), the diagram

$$\begin{array}{ccc} (\mathrm{QCoh}(S_2') \otimes \mathbf{H}^{\mathrm{access}})^{\mathrm{dualizable}} & \longrightarrow & (\mathrm{QCoh}(S_1') \otimes \mathbf{H}^{\mathrm{access}})^{\mathrm{dualizable}} \\ \downarrow & & \downarrow \\ (\mathrm{QCoh}(S_2) \otimes \mathbf{H}^{\mathrm{access}})^{\mathrm{dualizable}} & \longrightarrow & (\mathrm{QCoh}(S_1) \otimes \mathbf{H}^{\mathrm{access}})^{\mathrm{dualizable}} \end{array} \tag{2.5}$$

is a pullback square of (small, symmetric monoidal) categories.

2.3.3. Note that by [GR2, Chapter 1, Proposition 1.4.2], the functor

$$\mathrm{QCoh}(S_2') \to \mathrm{QCoh}(S_2) \underset{\mathrm{QCoh}(S_1)}{\times} \mathrm{QCoh}(S_1')$$

is fully faithful (but *not* necessarily an equivalence, see [GR2, Chapter 1, Remark 1.4.3]).

Since $\mathbf{H}^{\mathrm{access}}$ is dualizable as a DG category, the functor

$$\mathrm{QCoh}(S_2') \otimes \mathbf{H}^{\mathrm{access}} \to (\mathrm{QCoh}(S_2) \otimes \mathbf{H}^{\mathrm{access}}) \underset{\mathrm{QCoh}(S_1) \otimes \mathbf{H}^{\mathrm{access}}}{\times} (\mathrm{QCoh}(S_1') \otimes \mathbf{H}^{\mathrm{access}}) \tag{2.6}$$

is also fully faithful.

Hence, the functor

$$(\mathrm{QCoh}(S_2') \otimes \mathbf{H}^{\mathrm{access}})^{\mathrm{dualizable}} \to \\ \to (\mathrm{QCoh}(S_2) \otimes \mathbf{H}^{\mathrm{access}})^{\mathrm{dualizable}} \underset{(\mathrm{QCoh}(S_1) \otimes \mathbf{H}^{\mathrm{access}})^{\mathrm{dualizable}}}{\times} (\mathrm{QCoh}(S_1') \otimes \mathbf{H}^{\mathrm{access}})^{\mathrm{dualizable}} \tag{2.7}$$

is fully faithful.

It remains to prove that (2.7) is essentially surjective. The argument that follows is applicable to $\mathbf{H}^{\mathrm{access}}$ replaced by any proper compactly generated rigid symmetric monoidal category $\mathbf{A}$.

2.3.4. For an affine scheme S, the monoidal category $\mathrm{QCoh}(S)$ is rigid (see [GR1, Chapter 1, Sect. 9] for what this means). Since $\mathbf{A}$ was assumed rigid as well, we obtain that so are the categories of the form

$$\mathrm{QCoh}(S) \otimes \mathbf{A}.$$

In particular,

$$(\mathrm{QCoh}(S) \otimes \mathbf{A})^{\mathrm{dualizable}} = (\mathrm{QCoh}(S) \otimes \mathbf{A})^c$$

as subcategories of $\mathrm{QCoh}(S) \otimes \mathbf{A}$.

Since $\mathbf{A}$ is rigid, it is in particular self-dual (see [GR1, Chapter 1, Sect. 9.2]); i.e. we have a canonical equivalence

$$\mathbf{A} \simeq \mathbf{A}^\vee.$$

Now, since $\mathbf{A}$ is *proper* (i.e., $\mathcal{H}om$'s between compact objects lie in $\mathrm{Vect}_{\mathsf{e}}^c$), the equivalence

$$\mathrm{QCoh}(S) \otimes \mathbf{A}^\vee \simeq \mathrm{Funct}(\mathbf{A}, \mathrm{QCoh}(S))$$

restricts to a fully faithful embedding

$$(\mathrm{QCoh}(S) \otimes \mathbf{A}^\vee)^c \hookrightarrow \mathrm{Funct}(\mathbf{A}^c, \mathrm{QCoh}(S)^c).$$

Thus, we have a fully faithful embedding

$$(\mathrm{QCoh}(S) \otimes \mathbf{A})^c \simeq (\mathrm{QCoh}(S) \otimes \mathbf{A}^\vee)^c \hookrightarrow \mathrm{Funct}(\mathbf{A}^c, \mathrm{QCoh}(S)^c) = \mathrm{Funct}(\mathbf{A}^c, \mathrm{QCoh}(S)^{\mathrm{dualizable}}).$$

We now apply [GR2, Chapter 8, Proposition 3.3.2], which implies that for the diagram (2.4), the diagram of categories

$$\begin{array}{ccc} \mathrm{QCoh}(S_2')^{\mathrm{dualizable}} & \longrightarrow & \mathrm{QCoh}(S_1')^{\mathrm{dualizable}} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(S_2)^{\mathrm{dualizable}} & \longrightarrow & \mathrm{QCoh}(S_1)^{\mathrm{dualizable}} \end{array}$$

is a pull-back square. Hence, so is the diagram

$$\begin{array}{ccc} \mathrm{Funct}(\mathbf{A}^c, \mathrm{QCoh}(S_2')^{\mathrm{dualizable}}) & \longrightarrow & \mathrm{Funct}(\mathbf{A}^c, \mathrm{QCoh}(S_1')^{\mathrm{dualizable}}) \\ \downarrow & & \downarrow \\ \mathrm{Funct}(\mathbf{A}^c, \mathrm{QCoh}(S_2)^{\mathrm{dualizable}}) & \longrightarrow & \mathrm{Funct}(\mathbf{A}^c, \mathrm{QCoh}(S_1)^{\mathrm{dualizable}}). \end{array}$$

2.3.5. Hence, given an object M in the right-hand side of

$$(2.8) \quad (\mathrm{QCoh}(S_2') \otimes \mathbf{A})^{\mathrm{dualizable}} \to \\ \to (\mathrm{QCoh}(S_2) \otimes \mathbf{A})^{\mathrm{dualizable}} \underset{(\mathrm{QCoh}(S_1) \otimes \mathbf{A})^{\mathrm{dualizable}}}{\times} (\mathrm{QCoh}(S_1') \otimes \mathbf{A})^{\mathrm{dualizable}},$$

we can create an object M' in $\mathrm{Funct}(\mathbf{A}^c, \mathrm{QCoh}(S_2')^{\mathrm{dualizable}})$, so that M and M' have the same image in

$$\mathrm{Funct}(\mathbf{A}^c, \mathrm{QCoh}(S_2)^{\mathrm{dualizable}}) \underset{\mathrm{Funct}(\mathbf{A}^c, \mathrm{QCoh}(S_1)^{\mathrm{dualizable}})}{\times} \mathrm{Funct}(\mathbf{A}^c, \mathrm{QCoh}(S_1')^{\mathrm{dualizable}}).$$

Using the fully faithful embedding

$$\mathrm{Funct}(\mathbf{A}^c, \mathrm{QCoh}(S)^{\mathrm{dualizable}}) \hookrightarrow \mathrm{QCoh}(S) \otimes \mathbf{A},$$

we obtain that there exists an object M'' in the left-hand side of

$$(2.9) \qquad \mathrm{QCoh}(S_2') \otimes \mathbf{A} \to (\mathrm{QCoh}(S_2) \otimes \mathbf{A}) \underset{\mathrm{QCoh}(S_1) \otimes \mathbf{A}}{\times} (\mathrm{QCoh}(S_1') \otimes \mathbf{A}),$$

whose image in the right-hand side is isomorphic to that of M.

Thus, it remains to show that M'' is compact.

2.3.6. Since the functor (2.9) commutes with colimits and is fully faithful, an object in the left-hand side of (2.9) is compact if its image in the right-hand side of (2.9) is compact.

This implies that M'' is compact, since M, viewed as an object of the right-hand side of (2.9), is compact.

□

2.4. **Calculating the (co)tangent spaces.** In this subsection we will prove the remaining assertions of Proposition 2.2.2. To do so, it suffices to perform the calculation of point (b).

2.4.1. Let $\mathcal{M}$ be an object of $\mathrm{QCoh}(S)^{\leq 0}$, and let $S_{\mathcal{M}} \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathbf{e}}$ be the corresponding split square-zero extension of S. Unwinding the definitions, we obtain that we need to construct an isomorphism

$$\mathrm{Maps}(S_{\mathcal{M}}, \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \underset{\mathrm{Maps}(S, \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))}{\times} \{*\} \simeq$$
$$\simeq \tau^{\leq 0}\left(\mathcal{H}om_{\mathrm{QCoh}(S)\otimes \mathbf{H}}\left(\mathcal{O}_S \otimes \mathbf{1}_{\mathbf{H}}, (\mathcal{M}\otimes \mathbf{1}_{\mathbf{H}}) \otimes \mathsf{F}(\mathfrak{g})\right)[1]\right).$$

2.4.2. Let $\mathbf{A}$ be a symmetric monoidal DG category and let $\mathbf{a} \in \mathbf{A}$ be an object. We regard $\mathbf{1}_{\mathbf{A}} \oplus \mathbf{a}$ as an object of $\mathrm{ComAlg}(\mathbf{A})$, the square-zero extension of $\mathbf{1}_{\mathbf{A}}$ by means of $\mathbf{a}$. Consider the category

$$(\mathbf{1}_{\mathbf{A}} \oplus \mathbf{a})\text{-mod}(\mathbf{A})$$

as a symmetric monoidal category, equipped with a symmetric monoidal functor back to $\mathbf{A}$, given by

$$- \underset{\mathbf{1}_{\mathbf{A}}\oplus \mathbf{a}}{\otimes} \mathbf{1}_{\mathbf{A}}.$$

We have the following general assertion:

Lemma 2.4.3. *Given a symmetric monoidal functor*

$$\mathsf{F} : \mathrm{Rep}(\mathsf{G}) \to \mathbf{A},$$

the space of its lifts to a functor

$$\mathrm{Rep}(\mathsf{G}) \to (\mathbf{1}_{\mathbf{A}} \oplus \mathbf{a})\text{-mod}(\mathbf{A})$$

identifies canonically with

$$\mathrm{Maps}_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}, \mathbf{a} \otimes \mathsf{F}(\mathfrak{g})[1]).$$

Applying this to

$$\mathbf{A} := \mathrm{QCoh}(S) \otimes \mathbf{H}, \quad \mathbf{a} = \mathcal{M} \otimes \mathbf{1}_{\mathbf{H}},$$

we obtain the result stated in Sect. 2.4.1.

2.5. **Proof of ind-representability.** In this subsection we will begin the proof that the prestack $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ is an ind-affine ind-scheme locally almost of finite type.

We will do so by reducing to the case when instead of the category $\mathbf{H}$ we are dealing with the category $\mathrm{Rep}(\mathsf{H})$ of representations of an algebraic group H.

2.5.1. Recall (see [GR2, Chapter 2, Definition 1.1.2]) that prestack $\mathcal{Y}$ is an *ind-scheme* if it can be written as a *filtered* colimit

$$\mathcal{Y} \simeq \underset{i}{\mathrm{colim}}\, Y_i, \tag{2.10}$$

where Y_i are (quasi-compact) schemes and the transition maps $Y_i \to Y_j$ are closed embeddings.

An ind-scheme is *ind-affine* if all Y_i can be chosen to be affine.

An ind-scheme is locally almost of finite type as a prestack if all Y_i can be chosen to be almost of finite type see ([GR2, Chapter 2, Corollary 1.7.5(a)]).

2.5.2. By Corollary 2.2.6(a) combined with [GR2, Chapter 2, Corollary 1.3.13], in order to show that $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ is an ind-affine ind-scheme, it suffices to show that its classical truncation ${}^{\mathrm{cl}}\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ is a classical ind-affine ind-scheme.

Similarly, by Corollary 2.2.6(b') combined with [GR2, Chapter 1, Theorem 9.1.2], in order to show that the prestack $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ is locally almost of finite type, it suffices to show that ${}^{\mathrm{cl}}\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ is locally of finite type as a classical prestack.

2.5.3. Let H be a pro-algebraic group. Consider the prestack

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$$

that sends an affine scheme S to the space of homomorphisms of group-schemes over S

$$\phi : S \times \mathsf{H} \to S \times \mathsf{G}. \tag{2.11}$$

We have a naturally defined map

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G}) \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathrm{Rep}(\mathsf{H}))^{\mathrm{rigid}},$$

and it follows from Tannaka duality that it is actually an isomorphism.

We will prove:

Proposition 2.5.4. *The prestack* $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$ *is an ind-affine ind-scheme locally almost of finite type.*

The proof is given in Sect. 2.6 below.

Remark 2.5.5. Let us note that for a general pro-algebraic group H (as opposed to an algebraic group of finite type, the category $\mathrm{Rep}(\mathsf{H})$ is generally *not* a gentle Tannakian category (for example, $\mathrm{Ext}^1_{\mathrm{Rep}(\mathsf{H})}(\mathsf{e}, \mathsf{e})$ can be infinite-dimensional).

As a result, the connected components of the ind-scheme $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$ are *not*, in general, formal affine schemes.

2.5.6. Consider the *abelian* symmetric monoidal category $\mathbf{H}^{\heartsuit}$, equipped with the fiber functor $\mathbf{oblv}_{\mathbf{H}}$.

By Tannaka duality, there exists a pro-algebraic group, to be denoted H, such that $\mathbf{H}^{\heartsuit}$ identifies with the abelian category of algebraic representations of H and $\mathbf{oblv}_{\mathbf{H}}$ corresponds to the tautological forgetful functor.

Remark 2.5.7. For $\mathbf{H} = \mathrm{QLisse}(X)$, the resulting group H is $\mathrm{Gal}(X, x)_{\text{Pro-alg}}$, the pro-algebraic completion of the (unramified) Galois group $\mathrm{Gal}(X, x)$ of X with base point x.

2.5.8. We claim:

Proposition 2.5.9. *There exists a canonical isomorphism of* classical *prestacks*

$${}^{\mathrm{cl}}\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}} \simeq {}^{\mathrm{cl}}\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G}).$$

Note that this proposition, combined with Proposition 2.5.4, implies that ${}^{\mathrm{cl}}\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ is representable by a classical ind-affine ind-scheme locally of finite type. By Sect. 2.5.2, this implies that $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ is an ind-affine ind-scheme locally almost of finite type.

Proof of Proposition 2.5.9. Let $S = \mathrm{Spec}(R)$ be a classical affine scheme. As in Sect. 2.1.4, the value of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ on S is the category of symmetric monoidal functors

$$\mathsf{F} : \mathrm{Rep}(\mathsf{G})^{\heartsuit} \to (R\text{-mod} \otimes \mathbf{H})^{\leq 0}, \tag{2.12}$$

equipped with an identification of the composition

$$\mathrm{Rep}(\mathsf{G})^{\heartsuit} \to (R\text{-mod} \otimes \mathbf{H})^{\leq 0} \overset{\mathrm{Id} \otimes \mathbf{oblv}_{\mathbf{H}}}{\longrightarrow} R\text{-mod}$$

with

$$\mathrm{Rep}(\mathsf{G})^{\heartsuit} \overset{\mathbf{oblv}_{\mathsf{G}}}{\to} \mathrm{Vect}_{\mathsf{e}}^{\heartsuit} \overset{\mathrm{unit}}{\to} R\text{-mod}.$$

Such functors F necessarily take values in the abelian monoidal category

$$(R\text{-mod} \otimes \mathbf{H})^{\heartsuit} \simeq R\text{-mod}(\mathbf{H}^{\heartsuit}).$$

Similarly, S-values of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$ are symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G})^{\heartsuit} \to R\text{-mod}(\mathrm{Rep}(\mathsf{H})^{\heartsuit}),$$

equipped with an identification of the composition

$$\mathrm{Rep}(\mathsf{G})^{\heartsuit} \to R\text{-mod}(\mathrm{Rep}(\mathsf{H})^{\heartsuit}) \overset{\mathbf{oblv}_{\mathsf{H}}}{\to} R\text{-mod}$$

with

$$\mathrm{Rep}(\mathsf{G})^{\heartsuit} \overset{\mathbf{oblv}_{\mathsf{G}}}{\to} \mathrm{Vect}_{\mathsf{e}}^{\heartsuit} \overset{\mathrm{unit}}{\to} R\text{-mod}.$$

Thus, the two sets of data are manifestly isomorphic.

□

2.6. **Proof in the case of algebraic groups.** In this subsection we will prove Proposition 2.5.4.

2.6.1. We claim that in order to prove Proposition 2.5.4, it suffices to consider the case when H is an algebraic group of finite type. Indeed, write H as an cofiltered limit

$$\mathsf{H} := \lim_{\alpha} \mathsf{H}^{\alpha},$$

where H^{α} are algebraic groups of finite type and the transition maps are surjective.

Then,

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G}) \simeq \operatorname*{colim}_{\alpha} \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}^{\alpha}, \mathsf{G}).$$

and the transition maps are closed embeddings (see Remark 2.6.7 for the latter statement).

Hence, it suffices to show the following:

Proposition 2.6.2. *For a pair of algebraic groups H and G of finite type, the prestack $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$ is an ind-affine ind-scheme locally almost of finite type.*

The rest of this subsection is devoted to the proof of Proposition 2.6.2.

2.6.3. We will show that the *classical prestack underlying* $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$ is an ind-affine ind-scheme locally of finite type. This will imply that $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$ is an ind-affine ind-scheme locally almost of finite type by the argument in Sect. 2.5.2 applied to $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$.

Remark 2.6.4. The description of the cotangent space in Corollary 2.2.6(b) can be translated as follows:

For an affine test scheme S, and an S-point ϕ of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$, the cotangent space

$$T^*_{\phi}(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})) \in \mathrm{QCoh}(S)$$

identifies with

$$\mathrm{Fib}\left(\mathsf{g}^{\vee} \otimes \mathcal{O}_S \to \mathrm{coinv}_{\mathsf{H}}(\mathsf{g}^{\vee} \otimes \mathcal{O}_S)\right),$$

where $\mathrm{coinv}_{\mathsf{H}}$ stands for H-coinvariants, and $\mathsf{g}^{\vee} \otimes \mathcal{O}_S$ acquires a structure of H-module via ϕ.

2.6.5. From now on until the end of this section, we will consider the underlying classical prestacks and omit the superscript "cl" from the notation.

For a pair of affine schemes of finite type Y_1, Y_2, consider the prestack

$$\mathbf{Maps}_{\mathrm{Sch}}(Y_1, Y_2), \quad S \mapsto \mathrm{Hom}(S \times Y_1, Y_2).$$

We claim that $\mathbf{Maps}_{\mathrm{Sch}}(Y_1, Y_2)$ is representable by an ind-affine ind-scheme locally of finite type.

Indeed, the formation of $\mathbf{Maps}_{\mathrm{Sch}}(Y_1, Y_2)$ commutes with limits in Y_2, and every affine scheme of finite type can be written as a (finite) limit of copies of $\mathbb{A}^1$. This reduces the assertion to the case when $Y_2 = \mathbb{A}^1$.

However, for any prestack $\mathcal{Y}$

$$\mathbf{Maps}_{\mathrm{Sch}}(\mathcal{Y}, \mathbb{A}^1) \simeq \mathrm{Tot}(W), \quad W := \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}),$$

while for any $W \in (\mathrm{Vect}_{\mathsf{e}})^{\geq 0}$, the prestack $\mathrm{Tot}(W)$ is indeed a classical ind-affine ind-scheme locally of finite type: write

$$W \simeq \underset{i}{\mathrm{colim}}\, W_i,$$

with W_i finite dimensional, and we have

$$\mathrm{Tot}(W) \simeq \underset{i}{\mathrm{colim}}\ \mathrm{Tot}(W_i),$$

while

$$\mathrm{Tot}(W_i) \simeq \mathrm{Spec}(\mathrm{Sym}(W_i^\vee)).$$

2.6.6. Setting $Y_1 = \mathsf{H}$, $Y_2 = \mathsf{G}$, we obtain that

$$\mathbf{Maps}_{\mathrm{Sch}}(\mathsf{H}, \mathsf{G})$$

is an ind-affine ind-scheme locally of finite type.

Now, $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$ can be expressed as a fiber product of copies of $\mathbf{Maps}_{\mathrm{Sch}}(\mathsf{H}, \mathsf{G})$ and $\mathbf{Maps}_{\mathrm{Sch}}(\mathsf{H}^2, \mathsf{G})$. This implies the assertion of Proposition 2.6.2.

□[Proposition 2.6.2]

Remark 2.6.7. For future reference, we note that if $Y_2 \hookrightarrow Y_2'$ is a closed embedding of affine schemes, then the corresponding map

$$\mathbf{Maps}_{\mathrm{Sch}}(Y_1, Y_2) \to \mathbf{Maps}_{\mathrm{Sch}}(Y_1, Y_2')$$

is a closed embedding of functors.

In particular, if $\mathsf{G}' \hookrightarrow \mathsf{G}$ is a closed subgroup, then the map

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G}') \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$$

is a closed embedding.

Similarly, for a surjection $\mathsf{H}' \twoheadrightarrow \mathsf{H}$, the map

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G}) \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}', \mathsf{G})$$

is a closed embedding, since

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G}) \simeq \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}', \mathsf{G}) \underset{\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}'', \mathsf{G})}{\times} \mathrm{pt},$$

where $\mathsf{H}'' := \ker(\mathsf{H}' \to \mathsf{H})$.

3. Uniformization and the end of proof of Theorem 1.8.3

In this section we will finish the proof of Theorem 1.8.3, while introducing a tool of independent interest: a uniformization of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ by *algebraic stacks* associated to parabolic subgroups in G and *irreducible* local systems for their Levi quotients.

3.1. **What is there left to prove?**

3.1.1. We claim that in order to finish the proof of Theorem 1.8.5, it remains to show the following:

Theorem 3.1.2. *The underlying reduced prestack of* $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}$ *is a disjoint union of affine schemes, sheafified in the Zariski/étale topology.*

Let us show how Theorem 3.1.2 implies Theorem 1.8.5.

3.1.3. Indeed, we have already shown that $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}$ is an ind-affine ind-scheme locally almost of finite type. Combined with Theorem 3.1.2, this implies points (a,b,c) of Theorem 1.4.5.

To prove point (d), we quote the following result, which is a particular case of [Lu3, Theorem 18.2.3.2] (combined with [GR3, Proposition. 6.7.4]):

Theorem 3.1.4. *Let* $\mathcal{Y}$ *be an ind-scheme locally almost of finite type with the following properties:*

(i) ${}^{\mathrm{red}}\mathcal{Y}$ *is an affine scheme;*

(ii) *For any* $(S,y)\in\mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}$*, the cotangent space* $T^*_y(\mathcal{Y})\in\mathrm{Pro}(\mathrm{QCoh}(S)^{\leq 0})$ *actually belongs to* $\mathrm{QCoh}(S)^{\leq 0}$.

Then $\mathcal{Y}$ *can be written in the form* (1.8).

For the sake of completeness, we will outline the proof of Theorem 3.1.4 in Sect. A.

3.2. **Uniformization.** In this subsection we will begin the proof of Theorem 3.1.2. The method is based on constructing an algebraic stack that maps dominantly onto ${}^{\mathrm{red}}\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})$. This construction will also shed some light on "what $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})$ looks like".

3.2.1. Having proved that $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}$ is an ind-affine ind-scheme locally almost of finite type, we know that each connected component ${}^{\mathrm{red}}\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}$ is a filtered colimit of reduced affine schemes along closed embeddings. Hence, in order to prove Theorem 3.1.2, it suffices to show that these colimits stabilize.

We will achieve this by the following construction. We will find an *algebraic stack* locally almost of finite type $\widetilde{\mathbf{Maps}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})$, equipped with a map

$$\pi:\widetilde{\mathbf{Maps}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})\to\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})$$

with the following properties:

(1) Each connected component of $\widetilde{\mathbf{Maps}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})$ is quasi-compact and irreducible;
(2) The map π is schematic and proper on every connected component of $\widetilde{\mathbf{Maps}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})$;
(3) The map π is surjective on geometric points;
(4) The set of connected components of $\widetilde{\mathbf{Maps}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})$ splits as a union of finite clusters, and elements from different clusters have non-intersecting images in $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})$.

It is clear that an existence of such a pair $(\widetilde{\mathbf{Maps}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}),\pi)$ would imply the required properties of ${}^{\mathrm{red}}\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}$.

Properties (1) and (2) will be established in Sect. 3.3; Property (3) in Sect. 3.4, and Property (4) in Sect. 3.7.

We will now proceed to the construction of $\widetilde{\mathbf{Maps}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})$.

3.2.2. Let $\mathrm{Par}(\mathsf{G})$ be the (po)set of standard parabolics in G. For every $\mathsf{P}\in\mathrm{Par}(\mathsf{G})$, let M denote its Levi quotient. Note that by convention, for $\mathsf{P}=\mathsf{G}$, the corresponding Levi quotient is $\mathsf{G}_{\mathrm{red}}$, the quotient of G by its unipotent radical.

The maps

$$\mathsf{G}\leftarrow\mathsf{P}\to\mathsf{M}$$

induce the maps

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})\overset{\mathsf{p}_{\mathsf{P}}}{\leftarrow}\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}),\mathbf{H})\overset{\mathsf{q}_{\mathsf{P}}}{\to}\mathbf{Maps}(\mathrm{Rep}(\mathsf{M}),\mathbf{H}).$$

3.2.3. Let e' be an algebraically closed field containing e. Let us call an e'-point of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ *irreducible* if it does not factor through the above map q_{P} for any *proper* parabolic $\mathsf{P} \subsetneq \mathsf{G}$.

3.2.4. Let σ_{M} be an *irreducible* e-point of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{M}), \mathbf{H})$. Choose its lift to an e-point of the (ind)-scheme $\mathbf{Maps}(\mathrm{Rep}(\mathsf{M}), \mathbf{H})^{\mathrm{rigid}}$. Let $\mathrm{Stab}_{\mathsf{M}}(\sigma_{\mathsf{M}}) \subset \mathsf{M}$ be its stabilizer with respect to the M-action on $\mathbf{Maps}(\mathrm{Rep}(\mathsf{M}), \mathbf{H})^{\mathrm{rigid}}$. (Note that the subgroup $\mathrm{Stab}_{\mathsf{M}}(\sigma_{\mathsf{M}})$ depends on the choice of a lift, and a change of the choice by $m \in \mathsf{M}(\mathsf{e})$ results in conjugating $\mathrm{Stab}_{\mathsf{M}}(\sigma_{\mathsf{M}})$ by m.)

We obtain a locally closed embedding

$$\mathrm{pt} / \mathrm{Stab}_{\mathsf{M}}(\sigma_{\mathsf{M}}) \hookrightarrow \mathbf{Maps}(\mathrm{Rep}(\mathsf{M}), \mathbf{H}). \tag{3.1}$$

We claim, however:

Proposition 3.2.5. *The map* (3.1) *is a* closed *embedding.*

The proof will be given in Sect. 3.5.5.

Remark 3.2.6. As we will see in Corollary 3.7.5, the map (3.1) is actually the embedding of a connected component *at the reduced level*.

3.2.7. Denote

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})_{\sigma_{\mathsf{M}}} := \mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H}) \underset{\mathbf{Maps}(\mathrm{Rep}(\mathsf{M}), \mathbf{H})}{\times} \mathrm{pt} / \mathrm{Stab}_{\mathsf{M}}(\sigma_{\mathsf{M}}).$$

Finally, we set

$$\widetilde{\mathbf{Maps}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) := \underset{\mathsf{P} \in \mathrm{Par}(\mathsf{G})}{\sqcup} \underset{\sigma_{\mathsf{M}}}{\sqcup} \mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})_{\sigma_{\mathsf{M}}}.$$

The maps p_{P} define the sought-for map

$$\pi : \widetilde{\mathbf{Maps}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}).$$

Remark 3.2.8. The prestacks $\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})$ and $\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})_{\sigma_{\mathsf{M}}}$ have a very transparent meaning in the main example of $\mathbf{H} = \mathrm{QLisse}(X)$.

In this case,

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H}) =: \mathrm{LocSys}_{\mathsf{P}}^{\mathrm{restr}}(X)$$

is the prestack classifying local systems with a reduction to P, and

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})_{\sigma_{\mathsf{M}}} =: \mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}^{\mathrm{restr}}(X)$$

is the prestack of local systems with a reduction to P, whose induced M-local system is isomorphic to σ_{M}.

So the properties of the resulting map

$$\pi : \widetilde{\mathrm{LocSys}}_{\mathsf{G}}^{\mathrm{restr}}(X) \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$$

say that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ is uniformized by the disjoint union of the prestacks $\mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}^{\mathrm{restr}}(X)$.

3.3. **Properness of the uniformization morphism.** In this subsection we will show that the prestack $\widetilde{\mathbf{Maps}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ is an algebraic stack, each of whose connected components is quasi-compact and irreducible, and that the map π is proper on each connected component. This will establish Properties (1) and (2) from Sect. 3.2.1.

3.3.1. We will first show that each $\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})_{\sigma_\mathsf{M}}$ is an algebraic stack, which is quasi-compact and irreducible.

For this, it is sufficient to show that the map

$$\mathsf{q}_\mathsf{P} : \mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H}) \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{M}), \mathbf{H})$$

is a relative algebraic stack (i.e., its base change by a derived affine scheme yields an algebraic stack) with fibers that are quasi-compact and irreducible.

The property of a map between prestacks to be a relative algebraic stack with fibers that are quasi-compact and irreducible is stable under compositions. Filtering the unipotent radical of P we reduce the assertion to the following:

Proposition 3.3.2. *Let*

$$1 \to \mathrm{Tot}(V) \to \mathsf{G}_1 \to \mathsf{G}_2 \to 1 \tag{3.2}$$

be a short exact sequence of algebraic groups, where $\mathrm{Tot}(V)$ *is the vector group associated with a finite-dimensional* G_2*-representation* V*. Then the resulting map*

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}_1), \mathbf{H}) \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}_2), \mathbf{H})$$

is a relative algebraic stack whose fibers are quasi-compact and irreducible.

3.3.3. Before we prove Proposition 3.3.2, we make the following observation.

First, the datum of (3.2) is equivalent to that of an object

$$\mathrm{cl}_{\mathsf{G}_1} \in \mathrm{Maps}_{\mathrm{Rep}(\mathsf{G}_2)}(\mathrm{triv}, V[2]).$$

Let $\mathbf{A}$ be a symmetric monoidal category, and let us be given a symmetric monoidal functor

$$\mathsf{F} : \mathrm{Rep}(\mathsf{G}_2) \to \mathbf{A}.$$

Consider the object $\mathsf{F}(V) \in \mathbf{A}$ and

$$\mathsf{F}(\mathrm{cl}_{\mathsf{G}_1}) \in \mathrm{Maps}(\mathbf{1}_\mathbf{A}, \mathsf{F}(V)[2]).$$

Lemma 3.3.4. *Under the above circumstances, the space of lifts of* F *to a functor*

$$\mathrm{Rep}(\mathsf{G}_1) \to \mathbf{A}$$

identifies with the space of null-homotopies of $\mathsf{F}(\mathrm{cl}_{\mathsf{G}_1})$*.*

Proof. To simplify the notation, we will assume that V is the trivial 1-dimensional representation of G_2. Then the datum of G_1 is equivalent to that of a map of prestacks

$$s : B(\mathsf{G}_2) \to B^2(\mathbb{G}_a),$$

so that

$$B(\mathsf{G}_1) \simeq B(\mathsf{G}_2) \underset{B^2(\mathbb{G}_a)}{\times} \mathrm{pt}\,.$$

It follows that

$$\mathrm{QCoh}(B(\mathsf{G}_1)) \simeq \mathrm{QCoh}(B(\mathsf{G}_2)) \underset{\mathrm{QCoh}(B^2(\mathbb{G}_a))}{\otimes} \mathrm{Vect}_\mathsf{e}\,. \tag{3.3}$$

Note that

$$\mathrm{QCoh}(B^2(\mathbb{G}_a)) \simeq \mathsf{e}[\eta]\text{-mod}, \quad \deg(\eta) = 2.$$

The pullback of η, viewed as a point in

$$\mathrm{Maps}_{\mathrm{QCoh}(B^2(\mathbb{G}_a))}(\mathcal{O}_{B^2(\mathbb{G}_a)}, \mathcal{O}_{B^2(\mathbb{G}_a)}[2])$$

by means of s is our

$$\mathrm{cl}_{\mathsf{G}_1} \in \mathrm{Maps}_{\mathrm{QCoh}(B(\mathsf{G}_2))}(\mathcal{O}_{B(\mathsf{G}_2)}, \mathcal{O}_{B(\mathsf{G}_2)}[2]).$$

Note that for a symmetric monoidal category $\mathbf{A}'$, and $A \in \mathrm{ComAlg}(\mathrm{Vect}_\mathsf{e})$, the space of symmetric monoidal functors

$$A\text{-mod} \to \mathbf{A}'$$

is isomorphic to the space of maps in $\mathrm{ComAlg}(\mathrm{Vect}_{\mathsf{e}})$

$$A \to \mathrm{Maps}_{\mathbf{A}'}(\mathbf{1}_{\mathbf{A}'}, \mathbf{1}_{\mathbf{A}'}).$$

For $A = \mathsf{e}[\eta]$, this space is further isomorphic to

$$\mathrm{Maps}_{\mathbf{A}'}(\mathbf{1}_{\mathbf{A}'}, \mathbf{1}_{\mathbf{A}'}[2]).$$

Combining this with (3.3), we obtain that

$$\mathrm{Rep}(\mathsf{G}_1) \simeq \mathrm{Rep}(\mathsf{G}_2) \underset{\mathsf{e}[\eta]\text{-mod}}{\otimes} \mathrm{Vect}_{\mathsf{e}},$$

where $\mathsf{e}[\eta]\text{-mod} \to \mathrm{Rep}(\mathsf{G}_2)$ is given by

$$\eta \mapsto \mathrm{cl}_{\mathsf{G}_1} \in \mathrm{Maps}_{\mathrm{Rep}(\mathsf{G}_2)}(\mathbf{1}_{\mathrm{Rep}(\mathsf{G}_2)}, \mathbf{1}_{\mathrm{Rep}(\mathsf{G}_2)}[2]).$$

I.e., symmetric monoidal functors $\mathrm{Rep}(\mathsf{G}_1) \to \mathbf{A}$ are the same as symmetric monoidal functors $\mathsf{F} : \mathrm{Rep}(\mathsf{G}_2) \to \mathbf{A}$, equipped with the homotopy of the induced symmetric functor

$$\mathsf{e}[\eta]\text{-mod} \to \mathbf{A}, \quad \eta \mapsto \mathsf{F}(\mathrm{cl}_{\mathsf{G}_1}) \in \mathrm{Maps}_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}, \mathbf{1}_{\mathbf{A}}[2])$$

with

$$\mathsf{e}[\eta]\text{-mod} \to \mathrm{Vect}_{\mathsf{e}} \overset{\mathsf{e} \mapsto \mathbf{1}_{\mathbf{A}}}{\longrightarrow} \mathbf{A},$$

as required.

□

3.3.5. *Proof of Proposition 3.3.2.* Let us be given an affine scheme S and an S-point

$$\mathsf{F} : \mathrm{Rep}(\mathsf{G}_2) \to \mathrm{QCoh}(S) \otimes \mathbf{H}$$

of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}_2), \mathbf{H})$. Consider the object

$$\mathsf{F}(V) \in \mathrm{QCoh}(S) \otimes \mathbf{H},$$

and the object

$$\mathcal{E} := (\mathrm{Id}_{\mathrm{QCoh}(S)} \otimes \mathcal{H}om_{\mathbf{H}}(\mathbf{1}_{\mathbf{H}}, -))(\mathsf{F}(V)) \in \mathrm{QCoh}(S).$$

According to Lemma 3.3.4, we have a point

$$\mathsf{F}(\mathrm{cl}_{\mathsf{G}_1}) \in \Gamma(S, \mathcal{E}[2]),$$

and the fiber product

$$S \underset{\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}_2), \mathbf{H})}{\times} \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}_1), \mathbf{H})$$

is the functor that sends $S' \to S$ to the space of null-homotopies of $\mathsf{F}(\mathrm{cl}_{\mathsf{G}_1})|_{S'}$.

Hence, it remains to show that the above functor of null-homotopies is indeed an algebraic stack over S with fibers that are quasi-compact and irreducible. For that it suffices to show that, locally on S, the object $\mathcal{E}$ is perfect of amplitude $[0, d]$ for some d, i.e., can be represented by a finite complex

$$\mathcal{E}_0 \to \mathcal{E}_1 \to ... \to \mathcal{E}_d,$$

where each $\mathcal{E}_i$ is locally free of finite rank.

3.3.6. Note that

$$\mathrm{QCoh}(S)^{\mathrm{perf}} \simeq \lim_n \mathrm{QCoh}({}^{\leq n}S)^{\mathrm{perf}}.$$

Hence, we can assume that S is eventually coconnective. By Proposition 2.1.7, we can view F as a functor with values in $\mathrm{QCoh}(S) \otimes \mathbf{H}^{\mathrm{access}}$. Moreover, $\mathsf{F}(V)$ is compact in $\mathrm{QCoh}(S) \otimes \mathbf{H}^{\mathrm{access}}$ (see the proof of Corollary 2.2.6(b')).

The functor $\mathcal{H}om_{\mathbf{H}^{\mathrm{access}}}(\mathbf{1}_{\mathbf{H}}, -)$ preserves compactness, hence, the object $\mathcal{E}$ belongs to

$$\mathrm{QCoh}(S)^c = \mathrm{QCoh}(S)^{\mathrm{perf}},$$

and we only need to estimate its cohomological amplitude.

It is easy to see that if $\mathcal{E} \in \mathrm{QCoh}(S)^{\mathrm{perf}}$ is such that $\mathcal{E}|_{{}^{\mathrm{cl}}S}$ has amplitude $[d_1, d_2]$, then so does $\mathcal{E}$ itself. Hence, we can assume that S is classical. Furthermore, since the prestacks involved are locally (almost) of finite type, we can assume that S is of finite type.

In this case, if $\mathcal{E} \in \mathrm{QCoh}(S)^{\mathrm{perf}}$ is such that its *-fiber at any e-point of S lives in degrees $[d_1, d_2]$, then $\mathcal{E}$ has amplitude $[d_1, d_2]$. Hence, we have reduced the assertion to the case when $S = \mathrm{pt} = \mathrm{Spec}(\mathsf{e})$.

Now, the required property follows from the fact that for

$$\mathsf{F}(V) =: \mathbf{h} \in \mathbf{H}^{\heartsuit},$$

we have

$$\mathcal{H}om_{\mathbf{H}}(\mathbf{1}_{\mathbf{H}}, \mathbf{h}) \in (\mathrm{Vect}_{\mathsf{e}})^{\geq 0, \leq d}$$

for some d (i.e., d is the cohomological amplitude of the functor $\mathcal{H}om_{\mathbf{H}}(\mathbf{1}_{\mathbf{H}}, -)$, which is finite by the assumption on $\mathbf{H}$).

□[Proposition 3.3.2]

3.3.7. We will now show that π is schematic and proper when restricted to every connected component of $\widetilde{\mathbf{Maps}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$. Given Proposition 3.2.5 (which will be proved independently), it is sufficient to show that at the level of the underlying reduced prestacks, the map

$$\mathsf{p}_{\mathsf{P}} : \mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H}) \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$$

is schematic, quasi-compact and proper.

However, using the fact that G/P is proper, the assertion follows from the next observation:

Proposition 3.3.8. *Let G' be a subgroup of G. Then the map*

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}'), \mathbf{H}) \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \underset{\mathrm{pt}/\mathsf{G}}{\times} \mathrm{pt}/\mathsf{G}',$$

given by $\mathbf{oblv}_{\mathbf{H}}$, is a closed embedding.

Proof. The statement is equivalent to the assertion that

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}'), \mathbf{H})^{\mathrm{rigid}} \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$$

is a closed embedding.

By Proposition 2.5.9, it suffices to show that for an algebraic group H, the map

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G}') \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$$

is a closed embedding.

However, this is the content of Remark 2.6.7.

□

3.4. **Surjectivity of the uniformization morphism.** In this subsection we will prove that π is surjective, i.e., Property (3) from Sect. 3.2.1.

3.4.1. We need to show that for an algebraically closed field e' containing e, any e'-point σ_{G} of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ equals the image of an e'-point of $\widetilde{\mathbf{Maps}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$.

We will argue by induction on the semi-simple rank of G. If σ_{G} equals the image of a e'-point σ_{P} of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})$ for a *proper* parabolic $\mathsf{P} \subsetneq \mathsf{G}$, we are done by the induction hypothesis.

Hence, we can assume that σ_{G} is irreducible, and we need to show the following:

Proposition 3.4.2. *Let G be reductive. Then for an algebraically closed field extension $\mathsf{e}' \supseteq \mathsf{e}$, any irreducible e'-point of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ factors through an e-point.*

Note that as a particular case, we obtain:

Corollary 3.4.3. *For $\mathsf{G} = \mathsf{T}$ being a torus, the prestack ${}^{\mathrm{red}}\mathbf{Maps}(\mathrm{Rep}(\mathsf{T}), \mathbf{H})$ is the disjoint union of copies of pt/T over the set of isomorphism classes of e-points of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{T}), \mathbf{H})$.*

3.4.4. We proceed with the proof of Proposition 3.4.2. By Proposition 2.5.9, the statement of Proposition 3.4.2 is equivalent to the following:

Proposition 3.4.5. *Let G, H be algebraic groups with G reductive. Let $\mathsf{e}' \supseteq \mathsf{e}$ be an algebraically closed field extension, and let $\phi : \mathsf{H} \to \mathsf{G}$ be a homomorphism defined over e'. Assume that the image of ϕ is not contained in any proper parabolic of G defined over e'. Then ϕ is G-conjugate to a homomorphism defined over e.*

3.5. **Irreducible homomorphisms of reductive groups.** This subsection is devoted to the proof of Proposition 3.4.5.

3.5.1. Consider the Levi decomposition of H

$$1 \to \mathsf{H}_u \to \mathsf{H} \to \mathsf{H}_{\mathrm{red}} \to 1.$$

We claim that ϕ factors via a homomorphism

$$\mathsf{H}_{\mathrm{red}} \to \mathsf{G}.$$

Let $\mathsf{H}' = \mathsf{H}/\ker(\phi)$. Thus, we have an injective homomorphism

$$\mathsf{H}' \to \mathsf{G}.$$

We need to show that H' is reductive. We now recall the following assertion from [Se, Proposition 4.2]:

Theorem 3.5.2. *For a connected reductive group G and a subgroup $\mathsf{H} \subset \mathsf{G}$ the following conditions are equivalent:*

(i) *H is reductive;*

(ii) *Whenever there exists a parabolic $\mathsf{P} \subset \mathsf{G}$ that contains H, there also exists a Levi splitting $\mathsf{P} \leftrightarrows \mathsf{M}$ such that $\mathsf{H} \subset \mathsf{M}$.*

By the irreducibility assumption, our subgroup H' satisfies (ii) in Theorem 3.5.2. Hence, it is reductive as claimed.

3.5.3. Thus, in order to prove Proposition 3.4.5, it suffices to establish the following:

Proposition 3.5.4. *Let H and G be a pair of algebraic groups with H reductive. Then the ind-scheme $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$ is the disjoint union over isomorphism classes of homomorphisms*

$$\phi : \mathsf{H} \to \mathsf{G}$$

of the (classical) schemes $\mathsf{G}/\mathrm{Stab}_{\mathsf{G}}(\phi)$, where the stabilizer is taken with respect to the action of G on $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})$ by conjugation.

This proposition is well-known. We will supply a proof for completeness.

Proof. It is enough to show that for any e-point of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})$, the resulting locally closed embedding

$$\mathsf{G}/\operatorname{Stab}_{\mathsf{G}}(\phi)\to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})$$

induces an isomorphism at the level of tangent spaces.

Let our e-point correspond to a homomorphism $\phi:\mathsf{H}\to\mathsf{G}$. Thus, we have to show that the map

$$\operatorname{coFib}(H^0(\mathbf{inv}_{\mathsf{H}}(\mathfrak{g}))\to\mathfrak{g})\to T_\phi(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})) \tag{3.4}$$

is an isomorphism, where $\mathbf{inv}_{\mathsf{H}}$ stands for H-invariants, and $\mathfrak{g}$ is viewed as a H-representation via ϕ and the adjoint action.

Comparing with the formula for $T^*_\phi(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G}))$ in Remark 2.6.4, we obtain that we need to show that

$$\mathbf{inv}_{\mathsf{H}}(\mathfrak{g})$$

is concentrated in cohomological degree 0.

However, this follows from the assumption that H is reductive (and hence the category $\operatorname{Rep}(\mathsf{H})$ is semi-simple).

NB: note that validity of Proposition 3.5.4 depends on the assumption that we work over a field of characteristic 0 (in our case this is the field of coefficients e).

□

3.5.5. *Proof of Proposition 3.2.5.* It suffices to show that if G is reductive and $\phi:\mathsf{H}\to\mathsf{G}$ is a homomorphism of algebraic groups that does not factor through a parabolic, then the $\operatorname{Ad}(\mathsf{G})$-orbit of ϕ is closed in $\mathbf{Maps}(\mathsf{H},\mathsf{G})$.

Set $\mathsf{H}':=\mathsf{H}/\ker(\phi)$, so that ϕ factors through a map $\phi':\mathsf{H}'\to\mathsf{G}$. It is enough to check that the $\operatorname{Ad}(\mathsf{G})$-orbit of ϕ' is closed in $\mathbf{Maps}(\mathsf{H}',\mathsf{G})$. We will show that it is in fact a connected component.

Indeed, by Theorem 3.5.2, H' is reductive, and the assertion follows from Proposition 3.5.4.
□[Proposition 3.2.5]

3.6. **Associated pairs and semi-simple local systems.** In this subsection we will make a digression and discuss the classification of *semi-simple* points of $\mathbf{Maps}(\operatorname{Rep}(\mathsf{G}),\mathbf{H})$ in terms of irreducible ones of $\mathbf{Maps}(\operatorname{Rep}(\mathsf{M}),\mathbf{H})$, where M is a Levi subgroups of G.

3.6.1. *Terminology.* In what follows, for an algebraic group G', it will be convenient to refer to e-points of $\mathbf{Maps}(\operatorname{Rep}(\mathsf{G}'),\mathbf{H})$ as "G'-local systems". They are literally such in the key example $\mathbf{H}=\operatorname{QLisse}(X)$, in which case

$$\mathbf{Maps}(\operatorname{Rep}(\mathsf{G}'),\mathbf{H})=\operatorname{LocSys}^{\mathrm{restr}}_{\mathsf{G}'}(X).$$

For a homomorphism $\phi:\mathsf{G}'_1\to\mathsf{G}'_2$, and a G'_1-local system

$$\sigma_{\mathsf{G}'_1}\in\mathbf{Maps}(\operatorname{Rep}(\mathsf{G}'_1),\mathbf{H})(\mathsf{e}),$$

we will refer to its image

$$\sigma_{\mathsf{G}'_2}\in\mathbf{Maps}(\operatorname{Rep}(\mathsf{G}'_2),\mathbf{H})(\mathsf{e})$$

as "the G'_2-local system induced by $\sigma_{\mathsf{G}'_1}$ by means of ϕ".

Vice versa, given $\sigma_{\mathsf{G}'_2}$, we will refer to $\sigma_{\mathsf{G}'_1}$ as a "reduction of $\sigma_{\mathsf{G}'_2}$ to G'_1".

3.6.2. Let P be a standard parabolic in G. We will view the partial flag variety G/P as the space of parabolics conjugate to P.

Let P_1 and P_2 be a pair of standard parabolics in G, each equipped with an irreducible local system, σ_{M_i} with respect to the corresponding Levi quotient M_i.

We shall say that the pairs $(\mathsf{P}_1, \sigma_{\mathsf{M}_1})$ and $(\mathsf{P}_2, \sigma_{\mathsf{M}_2})$ are *associated* if there exists a G-orbit O in $\mathsf{G}/\mathsf{P}_1 \times \mathsf{G}/\mathsf{P}_2$, such that for some/any pair of points $(\mathsf{P}'_1, \mathsf{P}'_2) \in \mathsf{O}$ the following holds:

- The maps
$$\mathsf{M}_1 \leftarrow \mathsf{P}'_1 \leftarrow \mathsf{P}'_1 \cap \mathsf{P}'_2 \to \mathsf{P}'_2 \to \mathsf{M}_2$$
identify M_i, $i = 1, 2$, with the Levi quotient of $\mathsf{P}'_1 \cap \mathsf{P}'_2$;
- Under the resulting isomorphism $\mathsf{M}_1 \simeq \mathsf{M}_2$, the local systems σ_{M_1} and σ_{M_2} are isomorphic.

3.6.3. We claim:

Lemma 3.6.4. *Two pairs $(\mathsf{P}_1, \sigma_{\mathsf{M}_1})$ and $(\mathsf{P}_2, \sigma_{\mathsf{M}_2})$ are associated if and only if there exists a G-local system σ_{G}, equipped with reductions to both P_1 and P_2, such that the induced M_i-local systems are σ_{M_i}, respectively.*

Proof. One direction is clear: if $(\mathsf{P}_1, \sigma_{\mathsf{M}_1})$ and $(\mathsf{P}_2, \sigma_{\mathsf{M}_2})$ are associated, choose a pair $(\mathsf{P}'_1, \mathsf{P}'_2)$ on the corresponding orbit, and choose a Levi splitting of $\mathsf{P}'_1 \cap \mathsf{P}'_2$. Then the resulting local system with respect to $\mathsf{P}'_1 \cap \mathsf{P}'_2$ projects to σ_{M_1} and σ_{M_2}, respectively.

For the other implication, the two reductions of σ_{G} correspond to a G-orbit O on $\mathsf{G}/\mathsf{P}_1 \times \mathsf{G}/\mathsf{P}_2$. We will show that this orbit satisfies the two conditions of Sect. 3.6.2.

By assumption, we can choose parabolics P'_1 and P'_2, conjugate to P_1 and P_2, respectively, and lying on O, so that σ_{G} admits a reduction to $\mathsf{P}'_1 \cap \mathsf{P}'_2$; denote this reduction by $\sigma_{1,2}$. Furthermore, σ_{M_i}, $i = 1, 2$, is induced from $\sigma_{1,2}$ along the map

$$\mathsf{P}'_1 \cap \mathsf{P}'_2 \hookrightarrow \mathsf{P}'_i \twoheadrightarrow \mathsf{M}_i. \tag{3.5}$$

We note that (for any pair of parabolics) the image of (3.5) is a parabolic subgroup in M_i, $i = 1, 2$.

Hence, by the assumption on σ_{M_i}, the maps (3.5) are surjective, and hence identify M_i as a Levi quotient of $\mathsf{P}'_1 \cap \mathsf{P}'_2$.

□

3.6.5. We will say that a G-local system σ_{G} is *semi-simple* if whenever it admits a reduction to a local system σ_{P} with $\mathsf{P} \subset \mathsf{G}$ a parabolic, then σ_{P} admits a further reduction to a local system σ_{M} for M for *some* Levi splitting

$$\mathsf{P} \leftrightarrows \mathsf{M}.$$

3.6.6. By Theorem 3.5.2, when we think of σ_{G} as a conjugacy class of homomorphisms

$$\phi : \mathsf{H} \to \mathsf{G}$$

(for H as in Sect. 2.5.6), semi-simplicity is equivalent to the condition that the image of ϕ be reductive.

The latter interpretation has the following consequence:

Lemma 3.6.7. *Let $\mathsf{G}' \hookrightarrow \mathsf{G}$ be an injection of algebraic groups. Then a G'-local system is semi-simple if and only if the induced G-local system is semi-simple.*

3.6.8. Let $\mathsf{P} \subset \mathsf{G}$ be a standard parabolic and choose a Levi splitting $\mathsf{P} \leftrightarrows \mathsf{M}$. Let σ_{M} be an irreducible M-local system, and let σ_{G} denote the induced G-local system via $\mathsf{M} \to \mathsf{P} \to \mathsf{G}$. From Lemma 3.6.7 we obtain that σ_{G} is semi-simple.

From Lemma 3.6.4, we obtain:

Corollary 3.6.9. *For two pairs $(\mathsf{P}_1, \sigma_{\mathsf{M}_1})$ and $(\mathsf{P}_2, \sigma_{\mathsf{M}_2})$, the G-local systems $\sigma_{\mathsf{G},1}$ and $\sigma_{\mathsf{G},2}$ are isomorphic if and only if $(\mathsf{P}_1, \sigma_{\mathsf{M}_1})$ and $(\mathsf{P}_2, \sigma_{\mathsf{M}_2})$ are associated.*

And further:

Corollary 3.6.10.

(a) *Association is an equivalence relation on the set of isomorphism classes of pairs $(\mathsf{P}, \sigma_{\mathsf{M}})$, where P is a parabolic, M is its Levi quotient, and σ_{M} is an irreducible local system with respect to M.*

(b) *The assignment $(\mathsf{P}, \sigma_{\mathsf{M}}) \mapsto \sigma_{\mathsf{G}}$ establishes a bijection between classes of association of pairs $(\mathsf{P}, \sigma_{\mathsf{M}})$ and isomorphism classes of semi-simple G-local systems.*

3.6.11. For future use we notice:

Lemma 3.6.12. *Each equivalence class of associated pairs $(\mathsf{P}, \sigma_{\mathsf{M}})$ contains only finitely many elements.*

Proof. Follows from the fact that for every pair of standard parabolics P_1 and P_2, there are finitely many G-orbits on $\mathsf{G}/\mathsf{P}_1 \times \mathsf{G}/\mathsf{P}_2$. □

3.6.13. Given two G-local systems σ_1 and σ_2, we shall say that σ_2 is a *semi-simplification* of σ_1 if

- σ_2 is semi-simple;
- there exists a parabolic P and reductions $\sigma_{1,\mathsf{P}}$ and $\sigma_{2,\mathsf{P}}$ of σ_1 and σ_2, respectively, such that the induced local systems with respect to the Levi quotient of P are isomorphic.

It is clear that every local system σ_{G} admits a semi-simplification: take the minimal standard parabolic P to which σ_{G} can be reduced, and let σ'_{G} be the G-local system induced from the reduction of σ_{G} to P via the homomorphism

$$\mathsf{P} \to \mathsf{M} \to \mathsf{P} \to \mathsf{G}$$

for some Levi splitting of M.

From Lemma 3.6.4 and Corollary 3.6.10(b) we obtain:

Corollary 3.6.14. *For a given local system, its semi-simplification is well-defined up to isomorphism.*

3.7. **Analysis of connected/irreducible components.** In this subsection, we will establish Property (4) of the map

$$\pi : \widetilde{\mathbf{Maps}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}).$$

from Sect. 3.2.1, namely, that the set of connected components of $\widetilde{\mathbf{Maps}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ is a union of finite clusters, and elements from different clusters have non-intersecting images in $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ along π.

In addition, we will describe explicitly the set of connected components of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$.

3.7.1. We will prove:

Proposition 3.7.2. *There exists a bijection between the set of connected components of the prestack $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ and the set of isomorphisms classes of semi-simple G-local systems, characterized by either of the following two properties:*

(a) *Two e-points of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ belong to the same connected component if and only if they have isomorphic semi-simplifications.*

(b) *For a standard parabolic P and an irreducible M-local system σ_{M}, the map*

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})_{\sigma_{\mathsf{M}}} \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$$

lands in the connected component corresponding via the bijection of Corollary 3.6.10(b) to the association class of $(\mathsf{P}, \sigma_{\mathsf{M}})$.

Note that the assertion of Proposition 3.7.2 combined with that of Lemma 3.6.12 implies Property (4) from Sect. 3.2.1, which was the last one remaining to establish.

3.7.3. We note that point (a) of Proposition 3.7.2 contains the following statement:

Corollary 3.7.4. *Each connected component of* $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ *contains a unique isomorphism class of* e*-points corresponding to a semi-simple* G*-local system.*

Note also the following consequence of Proposition 3.7.2:

Corollary 3.7.5. *Let* G *be reductive. Then a connected component of* $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ *containing an irreducible local system has a unique isomorphism class of* e*-valued points.*

3.7.6. *Proof of Proposition 3.7.2.* For a given class of association $\mathfrak{P}$ of pairs $(\mathsf{P}, \sigma_{\mathsf{M}})$, let

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})_{\mathfrak{P}}$$

be the union of images of the maps

$$\pi : \mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})_{\sigma_{\mathsf{M}}} \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$$

from $\mathfrak{P}$. This union is finite by Lemma 3.6.12. Since the maps π are proper, this is a closed substack of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$.

We will show that the substacks $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})_{\mathfrak{P}}$ are:

(i) Pairwise disjoint;

(ii) Connected;

(iii) The semi-simplification of every e-point of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})_{\mathfrak{P}}$ corresponds under the bijection of Corollary 3.6.10(b) to $\mathfrak{P}$.

Point (i) follows readily from Lemma 3.6.4.

Let σ_{G} be the semi-simple G-local system corresponding to $\mathfrak{P}$, and consider the corresponding map

$$\pi : \mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})_{\sigma_{\mathsf{M}}} \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$$

for $(\mathsf{P}, \sigma_{\mathsf{M}}) \in \mathfrak{P}$.

By definition, all e-points in the image of this map have σ_{G} as their semi-simplification. This proves point (iii).

Further, σ_{G} itself is contained in the image of the above map π. This proves point (ii), since each $\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})_{\sigma_{\mathsf{M}}}$ is irreducible (and hence connected), and their images for $(\mathsf{P}, \sigma_{\mathsf{M}}) \in \mathfrak{P}$ all intersect at σ_{G}.

□[Proposition 3.7.2]

Remark 3.7.7. It is easy to see from the above argument that for given a local system σ, the map

$$\mathrm{pt} / \mathrm{Stab}_{\mathsf{G}}(\sigma) \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$$

is a closed embedding if σ is semi-simple (cf. Propositions 4.3.5 and 4.7.12). Furthermore, if G is reductive, then the above assertion is "if and only if".

Indeed, for every $(\mathsf{P}, \sigma_{\mathsf{M}})$, it is clear that the P-local system σ_P^0 induced from σ_{M} via a Levi splitting $\mathsf{M} \to \mathsf{P}$ is a closed point in

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H}) \underset{\mathbf{Maps}(\mathrm{Rep}(\mathsf{M}), \mathbf{H})}{\times} \mathrm{pt},$$

and the assertion follows from the fact that π is proper.

Further, if G is reductive, the action of the center M contracts any e-point of the above fiber product to σ_P^0.

Remark 3.7.8. It is clear that the image of each $\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})_{\sigma_{\mathsf{M}}}$ along π is irreducible. However, it is *not* true that the images of different of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}_1), \mathbf{H})_{\sigma_{\mathsf{M}_1}}$ and $\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}_2), \mathbf{H})_{\sigma_{\mathsf{M}_2}}$ in $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ will always produce different irreducible components:

For example, take $G = GL_2$, $\mathsf{P}_1 = \mathsf{P}_2 = \mathsf{B}$, so $\mathsf{M}_1 = \mathsf{M}_2 = \mathbb{G}_m \times \mathbb{G}_m$. Take $\mathbf{H} = \mathrm{QLisse}(X)$ and let σ_{M_1} and σ_{M_2} be given by

$$(E_1, E_2) \text{ and } (E_2, E_1),$$

where E_1 and E_2 are non-isomorphic one-dimensional local systems.

Then if X is a curve of genus ≥ 2, the images of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}_1}}(X)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}_2}}(X)$ are two distinct irreducible components of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

By contrast, if X is a curve of genus 1, both $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}_1}}(X)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}_2}}(X)$ are set-theoretically isomorphic to $\mathrm{pt}/(\mathbb{G}_m \times \mathbb{G}_m)$, and they map onto the same closed subset of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

4. Comparison with the Betti and de Rham versions of $\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$

In this section we study the relationship between $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ with $\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$ in the two contexts when the latter is defined: de Rham and Betti.

We will show that in both cases, the map

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$$

is a *formal isomorphism* with an explicit image at the reduced level.

4.1. **Relation to the Rham version.** In this subsection we will take our ground field k to be of characteristic 0. We will take $\mathsf{e} = k$ and let $\mathrm{Shv}(-)$ to be the sheaf theory of ind-holonomic D-modules.

We will study the relationship between $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ and the "usual" stack $\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$ classifying de Rham local systems.

4.1.1. Let X be a scheme of finite type over k. Recall (see, e.g., [AG, Sects. 10.1-2]), that the prestack of de Rham local systems on X, denoted $\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$, is defined by sending $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ to the space of right t-exact symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{D\text{-}mod}(X).$$

It is shown in [AG, Proposition 10.3] that $\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$ is laft (=locally almost of finite type) and admits (-1)-connective corepresentable deformation theory.

Remark 4.1.2. Note that the prestack $\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$ is of the form $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$, where $\mathbf{H}$ is the symmetric monoidal category $\mathrm{D\text{-}mod}(X)$.

4.1.3. We have a tautologically defined symmetric monoidal functor

$$\mathrm{QLisse}(X) \hookrightarrow \mathrm{Shv}(X) \to \mathrm{D\text{-}mod}(X), \tag{4.1}$$

which gives rise to a map of prestacks

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X). \tag{4.2}$$

We observe:

Lemma 4.1.4. *The map* (4.2) *is a monomorphism (i.e., is a monomorphism of spaces when evaluated on any affine scheme).*

Proof. Note that objects of $\mathrm{Shv}(X)^{\mathrm{constr}}$ are compact as objects of $\mathrm{D\text{-}mod}(X)$. Hence, the functor

$$\mathrm{Shv}(X) \to \mathrm{D\text{-}mod}(X),$$

obtained by ind-extending the tautological embedding is fully faithful.

Therefore, so is the composite functor (4.1). Since $\mathrm{QCoh}(S)$ is dualizable, the functor

$$\mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X) \to \mathrm{QCoh}(S) \otimes \mathrm{D\text{-}mod}(X) \tag{4.3}$$

is also fully faithful. This implies the assertion of the lemma.

□

4.1.5. *Example.* Let us explain how the difference between $\mathrm{LocSys}^{\mathrm{dR}}_{\mathbb{G}_m}(X)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_m}(X)$ plays out in the simplest cases when $\mathsf{G} = \mathbb{G}_m$ and $\mathsf{G} = \mathbb{G}_a$.

Take S to be classical. Then S-points of $\mathrm{LocSys}^{\mathrm{dR}}_{\mathbb{G}_m}(X)$ are line bundles over $S \times X$, equipped with a connection along X. Trivializing this line bundle locally, the connection corresponds to a section of

$$\mathcal{O}_S \boxtimes \Omega^{1,\mathrm{cl}}_X,$$

i.e., a function on S with values in closed 1-forms on X.

By contrast, if our S-point lands in $\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_m}(X)$, and if we further assume that S is integral, by Example 1.5.1, our line bundle with connection is necessarily pulled back from X.

Let us now take $\mathsf{G} = \mathbb{G}_a$. Then it follows from Sect. 1.5.2 that the map

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_a}(X) \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathbb{G}_a}(X)$$

is an isomorphism.

4.1.6. Recall that a map of prestacks $\mathcal{Y}_1 \to \mathcal{Y}_2$ is said to be a *formal isomorphism* if $\mathcal{F}_1$ identifies with its own formal completion inside $\mathcal{Y}_2$, i.e., if the map

$$\mathcal{Y}_1 \to (\mathcal{Y}_1)_{\mathrm{dR}} \underset{(\mathcal{Y}_2)_{\mathrm{dR}}}{\times} \mathcal{Y}_2$$

is an isomorphism.

4.1.7. We claim:

Proposition 4.1.8. *The map* (4.2) *is a formal isomorphism, i.e., identifies* $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ *with its formal completion inside* $\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$.

Proof. We need to show that for $S \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ and a map

$$S \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X), \tag{4.4}$$

such that the composite map

$${}^{\mathrm{red}}S \to S \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$$

factors through $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, the initial map (4.4) factors though $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ as well.

Since both $\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ are prestacks locally almost of finite type, we can assume that S is eventually coconnective and almost of finite type.

Thus, we need to show that given a functor

$$\mathsf{F} : \mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{D\text{-}mod}(X), \tag{4.5}$$

such that the composite functor

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{D\text{-}mod}(X) \to \mathrm{QCoh}({}^{\mathrm{red}}S) \otimes \mathrm{D\text{-}mod}(X),$$

lands in

$$\mathrm{QCoh}({}^{\mathrm{red}}S) \otimes \mathrm{QLisse}(X) \subset \mathrm{QCoh}({}^{\mathrm{red}}S) \otimes \mathrm{D\text{-}mod}(X), \tag{4.6}$$

the functor (4.5) also lands in

$$\mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X) \subset \mathrm{QCoh}(S) \otimes \mathrm{D\text{-}mod}(X). \tag{4.7}$$

Since S was assumed eventually coconnective, by Proposition 2.1.7, in (4.6) and (4.7), we can replace $\mathrm{QLisse}(X)$ by $\mathrm{IndLisse}(X)$.

Let ι denote the embedding $\mathrm{IndLisse}(X) \hookrightarrow \mathrm{D\text{-}mod}(X)$. It sends compacts to compacts, hence admits a continuous right adjoint, to be denoted ι^R.

We need to show that the natural transformation

$$(\mathrm{Id} \otimes \iota) \circ (\mathrm{Id} \otimes \iota^R) \circ \mathsf{F} \to \mathsf{F}$$

is an isomorphism.

Let f denote the embedding ${}^{\mathrm{red}}S \to S$. We know that

$$(f^* \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \iota) \circ (\mathrm{Id} \otimes \iota^R) \circ \mathsf{F} \simeq (\mathrm{Id} \otimes \iota) \circ (\mathrm{Id} \otimes \iota^R) \circ (f^* \otimes \mathrm{Id}) \circ \mathsf{F} \to (f^* \otimes \mathrm{Id}) \circ \mathsf{F}$$

is an isomorphism.

This implies the assertion since for $S \in {}^{<\infty}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}\,/\mathsf{e}}$, the functor

$$f^* \otimes \mathrm{Id} : \mathrm{QCoh}(S) \otimes \mathbf{C} \to \mathrm{QCoh}({}^{\mathrm{red}}S) \otimes \mathbf{C}$$

is conservative for any DG category $\mathbf{C}$ (indeed, $\mathrm{QCoh}(S)$ is generated under finite limits by the essential image of f_*).

□

4.1.9. From now on, until the end of this subsection we will assume that X is proper. In this case by [AG, Sects. 10.3.8 and 10.4.3], we know that $\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$ is an algebraic stack locally almost of finite type.

We claim:

Theorem 4.1.10. *The map*

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$$

is a closed embedding at the reduced level *for* each connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

This theorem will be proved in Sect. 4.3. In the course of the proof we will also describe the closed substacks of ${}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$ that arise as images of connected components of ${}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

Combined with Proposition 4.1.8, we obtain:

Corollary 4.1.11. *The subfunctor*

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \subset \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$$

is the disjoint union[14] *of formal completions of a collection of pairwise non-intersecting closed substacks of* ${}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$.

Remark 4.1.12. The closed substacks of ${}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$ appearing in Corollary 4.1.11 will be explicitly described in Remark 4.3.6.

Remark 4.1.13. Let $\mathsf{G}' \to \mathsf{G}$ be a closed embedding. It is not difficult to show that the diagram

$$\begin{array}{ccc} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}'}(X) & \longrightarrow & \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}'}(X) \\ \downarrow & & \downarrow \\ \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) & \longrightarrow & \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X) \end{array}$$

is a fiber square.

4.2. **A digression: ind-closed embeddings.**

[14] Sheafified in the Zariski/étale topology.

4.2.1. Let us recall the notion of *ind-closed embedding* of prestacks (see [GR3, Sect. 2.7.2]).

First, if S is an affine scheme and $\mathcal{Y}$ is a prestack mapping to it, we shall say that this map is an *ind-closed* embedding if $\mathcal{Y}$ is an ind-scheme and for some/any presentation of $\mathcal{Y}$ as (2.10), the resulting maps

$$Y_i \to S$$

are closed embeddings.

We shall say that a map of prestacks $\mathcal{Y}_1 \to \mathcal{Y}_2$ is an ind-closed embedding if its base change by an affine scheme yields an ind-closed embedding.

Remark 4.2.2. Let us emphasize the difference between "ind-closed embedding" and "closed embedding". For example, the inclusion of the disjoint union of infinitely many copies of pt onto $\mathbb{A}^1$ is an ind-closed embedding but not a closed embedding. Similarly, the map

$$\mathrm{Spf}(\mathsf{e}[\![t]\!]) \to \mathbb{A}^1$$

is an ind-closed embedding but not a closed embedding.

4.2.3. From Corollary 4.1.11 we obtain:

Corollary 4.2.4. *The map* $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$ *is an ind-closed embedding.*

Remark 4.2.5. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map, where $\mathcal{Y}_2$ is an algebraic stack, locally almost of finite type. Assume that f is a *formal isomorphism*. It is not difficult to see that the following conditions on f are equivalent:

(i) It is an ind-closed embedding;

(ii) ${}^{\mathrm{red}}\mathcal{Y}_1$ is a union of closed subfunctors of ${}^{\mathrm{red}}\mathcal{Y}_2$;

(iii) $\mathcal{Y}_1$ is obtained as the completion of $\mathcal{Y}_2$ along a subfunctor of $\mathcal{Z} \subset {}^{\mathrm{red}}\mathcal{Y}_2$ equal to a union of closed subfunctors.

4.3. **Uniformization and the proof of Theorem 4.1.10.**

4.3.1. For a standard parabolic P consider the diagram

$$\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X) \overset{\mathsf{p_P}}{\leftarrow} \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{P}}(X) \overset{\mathsf{q_P}}{\to} \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{M}}(X).$$

Fix an irreducible local system σ_{M} for M and denote

$$\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) := \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{P}}(X) \underset{\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{M}}(X)}{\times} \mathrm{pt} / \mathrm{Stab}_{\mathsf{M}}(\sigma_{\mathsf{M}}).$$

4.3.2. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) & \longrightarrow & \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \\ \downarrow & & \downarrow \\ \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) & \longrightarrow & \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X). \end{array}$$

Consider the composite morphism

$$\begin{array}{ccc} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) & \longrightarrow & \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \\ & & \downarrow \\ & & \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X) \end{array} \tag{4.8}$$

Given that the map $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$ is a monomorphism, an easy diagram chase, using properties (2) and (3) of the uniformization morphism in Sect. 3.2.1, shows that in order to prove Theorem 4.1.10, it suffices to show that the composite morphism (4.8) is schematic and proper.

This follows from the combination of the next three assertions:

Proposition 4.3.3. *The map*

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X)$$

is an isomorphism.

Proposition 4.3.4. *The map*

$$\mathsf{p} : \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{P}}(X) \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$$

is schematic and proper.

Proposition 4.3.5. *For a reductive group* G *and an irreducible local system* σ*, the resulting map*

$$\mathrm{pt}\,/\,\mathrm{Stab}_{\mathsf{G}}(\sigma) \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$$

is a closed embedding.

Remark 4.3.6. Note that the combination of the above three propositions describes the ind-closed substack

$$^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \subset {}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X).$$

Namely, it equals the disjoint union over classes of association of $(\mathsf{P}, \sigma_{\mathsf{M}})$ of the unions of the images of the maps

$$^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \to {}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$$

within a given class.

4.3.7. We now prove the above three propositions.

The assertion of Proposition 4.3.3 follows by tracing the proof of Proposition 3.3.2: namely, in the situation of *loc.cit.*, for an S-point of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}_2}$, the map

$$S \underset{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}_2}}{\times} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}_1} \to S \underset{\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}_2}(X)}{\times} \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}_1}(X)$$

is an isomorphism. Indeed, in both cases, this fiber product classifies null-homotopies for the same class.

Proposition 4.3.4 is well-known: it follows from the fact that the map

$$\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{P}}(X) \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X) \underset{\mathrm{pt}\,/\mathsf{G}}{\times} \mathrm{pt}\,/\mathsf{P}$$

is a closed embedding, where $\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X) \to \mathrm{pt}\,/\mathsf{G}$ is obtained by taking the fiber at some point $x \in X$.

It remains to prove Proposition 4.3.5.

Analytic proof. We can assume that $k = \mathbb{C}$. Clearly,

$$\mathrm{pt}\,/\,\mathrm{Stab}_{\mathsf{G}}(\sigma) \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$$

is a locally closed embedding. To prove that it is a closed embedding, it is enough to show that its image is closed *in the analytic topology.*

Using Riemann-Hilbert, we identify the *analytic stack* underlying $^{\mathrm{cl}}\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$ with its Betti version $^{\mathrm{cl}}\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$ (see Sect. 4.5.8). Hence, it is enough to show that the map

$$\mathrm{pt}\,/\,\mathrm{Stab}_{\mathsf{G}}(\sigma) \to \mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$$

is a closed embedding.

However, in this case the assertion follows from Proposition 4.7.12 below.

□

4.4. **Algebraic proof of Proposition 4.3.5.**

4.4.1. The map $\mathrm{pt}\,/\,\mathrm{Stab}_{\mathsf{G}}(\sigma) \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$ is a priori a locally closed embedding. Hence, in order to prove that it is actually a closed embedding, it is enough to show that it is proper. We will do so by applying the valuative criterion.

Thus, is enough to show that for a smooth affine curve C over e and a point $c \in C$, given a map

$$C \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X), \tag{4.9}$$

such that the composite map

$$(C - c) \to C \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$$

factors as

$$(C - c) \to \mathrm{pt}\,/\,\mathrm{Stab}_{\mathsf{G}}(\sigma) \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X),$$

then the initial map (4.9) also factors as

$$C \to \mathrm{pt}\,/\,\mathrm{Stab}_{\mathsf{G}}(\sigma) \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X). \tag{4.10}$$

Furthermore, it is enough to show that there exists a covering

$$\widetilde{C} \to C,$$

allowed to be branched at c, such that the composition

$$\widetilde{C} \to C \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X) \tag{4.11}$$

factors as

$$\widetilde{C} \to \mathrm{pt}\,/\,\mathrm{Stab}_{\mathsf{G}}(\sigma) \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X).$$

4.4.2. The assertion is easy if G is a torus. Hence, we obtain that for the induced bundles with respect to $\mathsf{G}/[\mathsf{G},\mathsf{G}]$, the given isomorphism indeed extends over over all $C \times X$. Modifying by means of a local system with respect to Z_{G}, we can thus assume that the induced local systems for $\mathsf{G}/[\mathsf{G},\mathsf{G}]$ are trivial. Hence, we can replace G by $[\mathsf{G},\mathsf{G}]$, i.e., we can assume that G is semi-simple.

4.4.3. Since σ was assumed irreducible and G semi-simple, the group $\mathrm{Stab}_{\mathsf{G}}(\sigma)$ is finite. The given map $(C-c) \to \mathrm{pt}\,/\,\mathrm{Stab}_{\mathsf{G}}(\sigma)$ corresponds to an étale covering of $C-c$. Let $\widetilde{C}$ denote its normalization over C; let $\widetilde{c}$ be the preimage of c in $\widetilde{C}$.

By construction, the map

$$(\widetilde{C} - \widetilde{c}) \to (C - c) \to \mathrm{pt}\,/\,\mathrm{Stab}_{\mathsf{G}}(\sigma) \to \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$$

factors as

$$(\widetilde{C} - \widetilde{c}) \to \mathrm{pt} \overset{\sigma}{\to} \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X).$$

We will show that the map (4.11) also factors

$$\widetilde{C} \to \mathrm{pt} \overset{\sigma}{\to} \mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X), \tag{4.12}$$

in a way compatible with the restriction to $\widetilde{C} - \widetilde{c}$.

4.4.4. The maps (4.11) and (4.12) correspond to G-bundles $\mathcal{F}^1_{\mathsf{G}}$ and $\mathcal{F}^2_{\mathsf{G}}$ on $\widetilde{C} \times X$, each equipped with a connection, and we are given an isomorphism of these data over $(\widetilde{C} - \widetilde{c}) \times X$. We wish to show that this isomorphism extends over all $\widetilde{C} \times X$.

Let η_X denote the generic point of X. Then the relative position of $\mathcal{F}^1_{\mathsf{G}}$ and $\mathcal{F}^2_{\mathsf{G}}$ at $\widetilde{c} \times \eta_X$ is a cell of the affine Grassmannian of G, and hence corresponds to a coweight λ of G, which is 0 if and only if the isomorphism between $\mathcal{F}^1_{\mathsf{G}}$ and $\mathcal{F}^2_{\mathsf{G}}$ extends over all $\widetilde{C} \times X$.

Furthermore, the restrictions of both $\mathcal{F}^1_{\mathsf{G}}$ and $\mathcal{F}^2_{\mathsf{G}}$ to $\widetilde{c} \times \eta_X$ acquire a reduction to the corresponding standard parabolic P (it corresponds to those vertices i of the Dynkin diagram, for which $\langle \check{\alpha}_i, \lambda \rangle = 0$). These reductions to P are horizontal with respect to the connection along η_X.

4.4.5. Note that $\mathcal{F}^2_{\mathsf{G}}$ is isomorphic to the constant family corresponding to σ, so $\mathcal{F}^2_{\mathsf{G}}|_{\widetilde{c}\times X}$ is also given by σ. By the valuative criterion for G/P, its reduction to P over $\widetilde{c}\times\eta_X$ extends to all of $\widetilde{c}\times X$. However, since σ was assumed irreducible, we have $\mathsf{P}=\mathsf{G}$. Hence, $\lambda=0$, as required.

□[Proposition 4.3.5]

4.5. **The Betti version of** $\mathrm{LocSys}_{\mathsf{G}}(X)$**.** From this point until the end of this section we let e be an arbitrary algebraically closed field of characteristic 0.

4.5.1. Let $\mathcal{X}$ be a connected object of Spc. We define the prestack $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(\mathcal{X})$ to be

$$(\mathrm{pt}/\mathsf{G})^{\mathcal{X}}=\mathbf{Maps}(\mathcal{X},\mathrm{pt}/\mathsf{G}).$$

I.e., for $S\in\mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$,

$$\mathrm{Maps}_{\mathrm{PreStk}}(S,\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(\mathcal{X}))=\mathrm{Maps}_{\mathrm{Spc}}(\mathcal{X},\mathrm{Maps}_{\mathrm{PreStk}}(S,\mathrm{pt}/\mathsf{G})).$$

The fact that pt/G admits (-1)-connective corepresentable deformation theory formally implies that the same is true for $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(\mathcal{X})$.

4.5.2. Assume for a moment that $\mathcal{X}$ is compact, i.e., is a retract of a space that can be obtained by a finite operation of taking push-outs from $\{*\}\in\mathrm{Spc}$.

In this case, it is clear from the definitions that $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(\mathcal{X})$ is locally almost of finite type.

4.5.3. We claim:

Proposition 4.5.4. *The prestack* $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(\mathcal{X})$ *is a derived algebraic stack. It can be realized as a quotient of an affine scheme (to be denoted* $\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\mathsf{G}}(\mathcal{X})$*) by an action of* G.

Proof. Choose a base point $x\in\mathcal{X}$. Denote

$$\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\mathsf{G}}(\mathcal{X}):=\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(\mathcal{X})\underset{\mathrm{pt}/\mathsf{G}}{\times}\mathrm{pt},$$

where the map $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(\mathcal{X})\to\mathrm{pt}/\mathsf{G}$ is given by restriction to x.

We have a natural action of G on $\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\mathsf{G}}(\mathcal{X})$ so that

$$\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(\mathcal{X})\simeq\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\mathsf{G}}(\mathcal{X})/\mathsf{G}.$$

We will show that $\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\mathsf{G}}(\mathcal{X})$ is an affine scheme. The fact that $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(\mathcal{X})$ admits (-1)-connective corepresentable deformation theory implies that $\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\mathsf{G}}(\mathcal{X})$ admits connective corepresentable deformation theory (we argue as in Corollary 2.2.6(b) and use the assumption that $\mathcal{X}$ is connected).

Hence, by [Lu3, Theorem 18.1.0.1], in order to show that $\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\mathsf{G}}(\mathcal{X})$ is an affine scheme, it suffices to show that

$${}^{\mathrm{cl}}\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\mathsf{G}}(\mathcal{X})$$

is a classical affine scheme.

Denote

$$\Gamma:=\pi_1(\mathcal{X},x).$$

It follows from the definitions that for $S\in{}^{\mathrm{cl}}\mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$, the space $\mathrm{Maps}(S,\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\mathsf{G}}(\mathcal{X}))$ is a set of homomorphisms $\Gamma\to\mathsf{G}$, parameterized by S.

I.e., $\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\mathsf{G}}(\mathcal{X})$ is a subfunctor of

$$S\mapsto\mathrm{Maps}(S,\mathsf{G})^{\Gamma}\simeq\mathrm{Maps}(S,\mathsf{G}^{\Gamma}),$$

consisting of elements that obey the group law, i.e.,

$$\mathsf{G}^{\Gamma}\underset{\mathsf{G}^{\Gamma\times\Gamma}}{\times}\mathrm{pt}.$$

Since G^{Γ} and $\mathsf{G}^{\Gamma\times\Gamma}$ are affine schemes, we obtain that so is $\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\mathsf{G}}(\mathcal{X})$.

□

Remark 4.5.5. It follows from Sect. 4.5.2 that if $\mathcal{X}$ compact, then $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti,rigid}_x}(\mathcal{X})$ is almost of finite type.

4.5.6. Let us now rewrite the definition of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X})$ slightly differently. Consider the DG category

$$\mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}} \simeq \mathrm{Funct}(\mathcal{X}, \mathrm{Vect}_{\mathsf{e}}),$$

see [GKRV, Sects. 1.4.1-2].

For any DG category $\mathbf{C}$, we have a tautological functor

$$\mathbf{C} \otimes \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}} \to \mathbf{C}^{\mathcal{X}}, \tag{4.13}$$

which is an equivalence if $\mathbf{C}$ is dualizable (or if $\mathcal{X}$ is compact).

Furthermore $\mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}$ has a natural symmetric monoidal structure, and if $\mathbf{C}$ is also symmetric monoidal, the functor (4.13) is symmetric monoidal.

Assume for a moment that $\mathbf{C}$ has a t-structure. Then $\mathbf{C}^{\mathcal{X}}$ also acquires a t-structure (an object is connective/coconnetive if its value for any $x \in X$ is connective/coconnective). In particular, $\mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}$ has a t-structure. With respect to these t-structures, the functor (4.13) is t-exact.

4.5.7. By the definitions of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X})$ and of pt/G, the value of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X})$ on an affine scheme S is the space of functors

$$\mathcal{X} \times \mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S)$$

that are symmetric monoidal and right t-exact in the second variable. By the above, this is the same as the space of right t-exact symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}.$$

Thus, the prestack $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X})$ is also of the form $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$, for $\mathbf{H} := \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}$.

4.5.8. Let now X be CW complex. Let $\mathrm{Shv}_{\mathrm{loc.const.}}^{\mathrm{all}}(X)$ be the category of sheaves of e-vector spaces *with locally constant cohomologies.*

We define the prestack $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X)$ as follows. It sends an affine scheme S to the space of right t-exact symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathrm{Shv}_{\mathrm{loc.const.}}^{\mathrm{all}}(X).$$

In other words,

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X) := \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$$

for $\mathbf{H} := \mathrm{Shv}_{\mathrm{loc.const.}}^{\mathrm{all}}(X)$.

4.5.9. Let us write X as a geometric realization of an object $\mathcal{X}$ of Spc.

In this case we have a canonical t-exact equivalence of symmetric monoidal categories

$$\mathrm{Shv}_{\mathrm{loc.const.}}^{\mathrm{all}}(X) \simeq \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}.$$

Hence, we obtain that in this case we have a canonical isomorphism of prestacks

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X) \simeq \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X}).$$

Thus, the results pertaining to $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X})$ that we have reviewed above carry over to $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X)$ as well.

4.6. **The coarse moduli space of Betti local systems.** In this subsection we will make a digression and discuss the coarse moduli space of Betti local systems (a.k.a. character variety).

In this subsection we will assume that G is reductive.

4.6.1. Let $\mathcal{X}$ be as in Sect. 4.5. Consider the object of ComAlg(Vect$_{\mathsf{e}}$) given by

$$\Gamma(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X}), \mathcal{O}_{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X})}). \tag{4.14}$$

Note that it is connective: this follows from the presentation of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X})$ as

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti},\mathrm{rigid}_x}(\mathcal{X})/\mathsf{G},$$

so that

$$\Gamma(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X}), \mathcal{O}_{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X})}) = \mathbf{inv}_{\mathsf{G}}\left(\Gamma(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{rigid}_x}(\mathcal{X}), \mathcal{O}_{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{rigid}_x}(\mathcal{X})})\right),$$

and using the fact that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti},\mathrm{rigid}_x}(\mathcal{X})$ is an affine scheme and G is reductive, so the functor $\mathbf{inv}_{\mathsf{G}}$ is t-exact.

Note that if $\mathcal{X}$ is compact, so that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X})$ is almost of finite type, the algebra (4.14) is also almost of finite type.

4.6.2. Set

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti},\mathrm{coarse}}(\mathcal{X}) := \mathrm{Spec}\left(\Gamma(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X}), \mathcal{O}_{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X})})\right).$$

We have a tautologically defined map

$$\mathbf{r} : \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X}) \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti},\mathrm{coarse}}(\mathcal{X}). \tag{4.15}$$

4.6.3. Let us describe the classical affine scheme underlying $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti},\mathrm{coarse}}(\mathcal{X})$. Recall the notation

$$\Gamma := \pi_1(X, x),$$

and consider the affine scheme

$$\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G}),$$

which is acted on by G by conjugation.

Set

$$\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\!/\mathrm{Ad}(\mathsf{G}) = \mathrm{Spec}\left(\mathbf{inv}_{\mathsf{G}}\left(\Gamma(\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G}), \mathcal{O}_{\mathbf{Maps}_{\mathrm{Grp}}(\Gamma,\mathsf{G})})\right)\right).$$

As we have seen in the course of the proof of Proposition 4.5.4, we have the isomorphisms

$${}^{\mathrm{cl}}\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{rigid}_x}(\mathcal{X}) \simeq {}^{\mathrm{cl}}\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})$$

and

$${}^{\mathrm{cl}}\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X}) \simeq {}^{\mathrm{cl}}\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\mathrm{Ad}(\mathsf{G}).$$

Hence, since G reductive, we have

$${}^{\mathrm{cl}}\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti},\mathrm{coarse}}(\mathcal{X}) \simeq {}^{\mathrm{cl}}\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\!/\mathrm{Ad}(\mathsf{G}).$$

4.6.4. For future use we now quote the following fundamental result of [Ri]:

Theorem 4.6.5. *Let Γ be an abstract group. Then two e-points of the stack $\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\mathrm{Ad}(\mathsf{G})$ map to the same point in the affine scheme $\mathbf{Maps}_{\mathrm{Grp}}(\Gamma, \mathsf{G})/\!/\mathrm{Ad}(\mathsf{G})$ if and only if they have isomorphic semi-simplifications.*

By the above, we immediately obtain:

Corollary 4.6.6. *Two e-points of the stack $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(\mathcal{X})$ get sent by $\mathbf{r}$ to the same point in the affine scheme $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti},\mathrm{coarse}}(\mathcal{X})$ if and only if they have isomorphic semi-simplifications.*

4.7. **Relationship of the restricted and Betti versions.** In this subsection we let X be a smooth connected algebraic variety[15] over $\mathbb{C}$.

---

[15] The material of this and the next subsection is equally applicable, when instead of X we take a connected finite CW complex. In this case we let $\mathrm{QLisse}(X)$ be the full subcategory of $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(X)$ consisting of objects such that each of their cohomologies (with respect to the usual t-structure) is locally finite as a representation of $\pi_1(X, x)$.

4.7.1. Consider the functor

$$\mathrm{QLisse}(X) \to \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(X) \tag{4.16}$$

We claim:

Proposition 4.7.2. *The functor* (4.16) *is fully faithful.*

Remark 4.7.3. Note that, unlike the de Rham case, in the Betti setting, the fully faithulness of (4.16) is not a priori evident (because objects from $\mathrm{Shv}(X)^{\mathrm{constr}}$ are *not* compact as objects in the category of *all* sheaves of e-vector spaces on X).

Proof. Since both categories are left-complete and (4.16) is t-exact, it is sufficient to show that it induces fully faithful functors

$$(\mathrm{QLisse}(X))^{\geq -n} \to (\mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(X))^{\geq -n}. \tag{4.17}$$

Now,

$$(\mathrm{IndLisse}(X))^{\geq -n} \to (\mathrm{QLisse}(X))^{\geq -n}$$

is an equivalence, and hence the functor (4.17) sends compacts to compacts.

Since $(\mathrm{IndLisse}(X))^{\geq -n}$ is compactly generated (by $(\mathrm{Lisse}(X))^{\geq -n}$) and

$$\mathrm{Lisse}(X) \to \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(X)$$

is fully faithful, we obtain that (4.17) is fully faithful.

□

4.7.4. The functor (4.16) defines a map

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X). \tag{4.18}$$

As in the de Rham case, from Proposition 4.7.2 we obtain that the map (4.18) is a monomorphism.

Remark 4.7.5. This remark is parallel to Remark 4.1.5. Let us explain how the difference between $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ plays out in the simplest cases when $\mathsf{G} = \mathbb{G}_m$ and $\mathsf{G} = \mathbb{G}_a$. Take $S = \mathrm{Spec}(R)$ to be classical.

In this case, an S-point of $\mathrm{LocSys}_{\mathbb{G}_m}(X)$ is a homomorphism

$$\pi_1(X) \to R^\times.$$

By contrast, if we further assume S to be reduced, then an S-point of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_m}(X)$ is a homomorphism

$$\pi_1(X) \to \mathsf{e}^\times.$$

Take now $\mathsf{G} = \mathbb{G}_a$. In this case, by Example Sect. 1.5.2, the map

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_a}(X) \to \mathrm{LocSys}_{\mathbb{G}_a}(X)$$

is an isomorphism.

Remark 4.7.6. A remark parallel to Remark 4.1.13 holds in the Betti context as well, i.e., for a closed embedding $\mathsf{G}' \to \mathsf{G}$, the diagram

$$\begin{array}{ccc} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}'}(X) & \longrightarrow & \mathrm{LocSys}_{\mathsf{G}'}(X) \\ \downarrow & & \downarrow \\ \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) & \longrightarrow & \mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X) \end{array}$$

is a fiber square.

4.7.7. We have also the following statements that are completely parallel with the de Rham situation (with the same proofs):

Proposition 4.7.8. *The map* (4.18) *is a formal isomorphism, i.e., identifies* $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ *with its formal completion inside* $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$.

Theorem 4.7.9. *The map*

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$$

is a closed embedding at the reduced level *for* each connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

Corollary 4.7.10. *The subfunctor*

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \subset \mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$$

is the disjoint union of formal completions of a collection of pairwise non-intersecting closed substacks of ${}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$.

Corollary 4.7.11. *The map* $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$ *is an ind-closed embedding.*

Note, however, that we still have to supply a proof of the Betti version of Proposition 4.3.5:

Proposition 4.7.12. *For a reductive group* G *and an irreducible local system* σ*, the resulting map*

$$\mathrm{pt}\,/\,\mathrm{Stab}_{\mathsf{G}}(\sigma) \to \mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$$

is a closed embedding.

The proof is given in Sect. 4.8.2 below.

Remark 4.7.13. Note that as in Remark 4.3.6, we obtain that the image of

$$ {}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to {}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X) \tag{4.19}$$

is the ind-closed substack that equals the disjoint union over classes of association of $(\mathsf{P}, \sigma_{\mathsf{M}})$ of the unions of the images of the maps

$${}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \to {}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$$

within a given class.

In Sect. 4.8 below we will give an alternative description of the image of (4.19), which is specific to the Betti situation.

4.8. **Comparison of** $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ **vs** $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$ **via the coarse moduli space.** Let X be as in Sect. 4.7. We will give a more explicit description of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ as a subfunctor of $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$.

4.8.1. Let $\mathsf{G}_{\mathrm{red}}$ denote the reductive quotient of G. We have a fiber square

$$\begin{array}{ccc} \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) & \longrightarrow & \mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X) \\ \downarrow & & \downarrow \\ \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}_{\mathrm{red}}}(X) & \longrightarrow & \mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}_{\mathrm{red}}}(X). \end{array} \tag{4.20}$$

Hence, in order to describe $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ as a subfunctor of $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$, it is enough to do so for G replaced by $\mathsf{G}_{\mathrm{red}}$. So, from now until the end of this subsection we will assume that G is reductive.

4.8.2. First, we are going to deduce Proposition 4.7.12 from Corollary 4.6.6:

Proof. Let σ be irreducible, and consider the closed substack

$$\mathrm{pt} \underset{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti,coarse}}(X)}{\times} \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X) \subset \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X),$$

where

$$\mathrm{pt} \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti,coarse}}(X)$$

is given by $\mathbf{r}(\sigma)$.

By Corollary 4.6.6 and the irreducibility assumption on σ, the above stack contains a unique isomorphism class of e-points. Hence, the map

$$\mathrm{pt} / \mathrm{Stab}_{\mathsf{G}}(\sigma) \to \mathrm{pt} \underset{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti,coarse}}(X)}{\times} \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X)$$

is an isomorphism of the underlying reduced substacks. In particular, it is a closed embedding.

□

4.8.3. We now claim:

Theorem 4.8.4. *The subfunctor*

$${}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) \subset {}^{\mathrm{red}}\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X)$$

is the disjoint union of the fibers of the map $\mathbf{r}$ *of* (4.15).

Combining with Corollary 4.7.10, we obtain:

Corollary 4.8.5. *The subfunctor* $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) \subset \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X)$ *is the disjoint union of formal completions of the fibers of the map*

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X) \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti,coarse}}(X).$$

The rest of this subsection is devoted to the proof of Theorem 4.8.4.

4.8.6. We will prove the following slightly more precise statement (which would imply Theorem 4.8.4 in view of Remark 4.7.13):

Fix a class of association of pairs $(\mathsf{P}, \sigma_{\mathsf{M}})$. For each element in this class pick a Levi splitting

$$\mathsf{P} \leftrightarrows \mathsf{M},$$

and consider the induced G-local system. Note, however, that by Corollary 3.6.10, these G-local systems are all isomorphic (for different elements $(\mathsf{P}, \sigma_{\mathsf{M}})$ in the given class); denote the resulting local system by σ_{G}.

We will show that the reduced substack underlying

$$\mathrm{pt} \underset{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti,coarse}}(X)}{\times} \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X) \tag{4.21}$$

(where $\mathrm{pt} \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti,coarse}}(X)$ is given by $\mathbf{r}(\sigma_{\mathsf{G}})$), equals the union of the images of the maps

$$\mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}^{\mathrm{Betti}}(X) \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X), \tag{4.22}$$

where the union is taken over the pairs $(\mathsf{P}, \sigma_{\mathsf{M}})$ in our chosen class of association.

4.8.7. We claim:

Proposition 4.8.8. *Let* $\mathsf{P} \leftrightarrows \mathsf{M}$ *be a parabolic with a Levi splitting. Let* σ_{M} *be an irreducible* M*-local system, and let* σ_{G} *be the induced* G*-local system. Then the composite*

$$^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \to {}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X) \overset{\mathbf{r}}{\to} {}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{Betti,coarse}}_{\mathsf{G}}(X)$$

factors as

$$^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X) \to \mathrm{pt} \overset{\mathbf{r}(\sigma_{\mathsf{G}})}{\longrightarrow} {}^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{Betti,coarse}}_{\mathsf{G}}(X).$$

Proof. Note that all e-points of $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$ obtained from e-points of $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{P},\sigma_{\mathsf{M}}}(X)$ have σ_{G} as their semi-simplification.

Hence, the assertion of the proposition follows from Theorem 4.6.5.

□

We will now deduce from Proposition 4.8.8 the description of (4.21) as the union of the images of the maps (4.22).

4.8.9. Indeed, on the one hand, Proposition 4.8.8 implies that the images of the maps (4.22) (at the reduced level) indeed lie in the fiber (4.21).

On the other hand, take an e-point σ'_{G} in the fiber (4.21), and let $(\mathsf{P}', \sigma_{\mathsf{M}'})$ be a pair such σ'_{G} lies in the image of

$$\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{P}',\sigma_{\mathsf{M}'}}(X) \to \mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X).$$

We need to show that $(\mathsf{P}', \sigma_{\mathsf{M}'})$ lies in our class of association. However, by Proposition 4.8.8, the G-local system, induced from $\sigma_{\mathsf{M}'}$, is isomorphic to σ_{G}. This implies the result by Corollary 3.6.10.

□[Theorem 4.8.4]

5. Geometric properties of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$

In this section we will assume that G is reductive. The goal of this section is to establish a version, adapted to $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, of the picture

$$\mathbf{r} : \mathrm{LocSys}_{\mathsf{G}}(X) \to \mathrm{LocSys}^{\mathrm{coarse}}_{\mathsf{G}}(X)$$

that we have in the Betti case (see Sect. 4.6.2). This will be stated as Theorem 5.4.2, which constructs the desired picture

$$\mathbf{r} : \mathcal{Z} \to \mathcal{Z}^{\mathrm{restr}}$$

for each connected component $\mathcal{Z}$ of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

Prior to doing so, we show that $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ has the following two geometric properties: it is *mock-affine* and *mock-proper*.

5.1. "Mock-properness" of $^{\mathrm{red}}\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

5.1.1. Let $\mathcal{Z}$ be a quasi-compact algebraic stack locally almost of finite type over e. Let

$$\mathrm{Coh}(\mathcal{Z}) \subset \mathrm{QCoh}(\mathcal{Z})$$

be the full subcategory consisting of objects whose pullback under a smooth cover (equivalently, any map)

$$S \to \mathcal{Z}, \quad S \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{aft}\,/\mathsf{e}}$$

belongs to $\mathrm{Coh}(S) \subset \mathrm{QCoh}(S)$.

We shall say that $\mathcal{Z}$ is *mock-proper* if the functor

$$\Gamma(\mathcal{Z}, -) : \mathrm{QCoh}(\mathcal{Z}) \to \mathrm{Vect}_{\mathsf{e}}$$

sends $\mathrm{Coh}(\mathcal{Z})$ to $\mathrm{Vect}^c_{\mathsf{e}}$.

Remark 5.1.2. This definition is equivalent to one in [Ga3, Sect. 6.5]. Indeed, the subcategory

$$\text{D-mod}(\mathcal{Z})^c \subset \text{D-mod}(\mathcal{Z})$$

is generated under finite colimits by the image of $\text{Coh}(\mathcal{Z})$ along induction functor

$$\mathbf{ind}_{\text{D-mod}} : \text{QCoh}(\mathcal{Z}) \to \text{D-mod}(\mathcal{Z}).$$

5.1.3. *Examples.*

(i) If $\mathcal{Z}$ is a scheme, then it is mock-proper as a stack if and only it is proper as a scheme.

(ii) The stack pt/H is mock-proper for any algebraic group H.

(iii) For a (finite-dimensional) vector space V, the stack $\text{Tot}(V)/\mathbb{G}_m$ is mock-proper. (This is just the fact that for a finitely generated graded $\text{Sym}(V^\vee)$-module, its degree 0 component is finite-dimensional as a vector space.)

5.1.4. Let $\mathbf{H}$ be a gentle Tannakian category, and let $\mathcal{Z}$ be a connected component of $\mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathbf{H})$. Recall that according to Theorem 1.8.3, its underlying reduced prestack ${}^{\text{red}}\mathcal{Z}$ is actually a quasi-compact algebraic stack.

We will prove:

Theorem 5.1.5. *The algebraic stack* ${}^{\text{red}}\mathcal{Z}$ *is mock-proper.*

Of course, our main application is when $\mathbf{H} = \text{QLisse}(X)$, so that $\mathcal{Z}$ is a connected component of $\text{LocSys}_{\mathsf{G}}^{\text{restr}}(X)$.

The rest of the subsection is devoted to the proof of Theorem 5.1.5.

5.1.6. Recall (see Sect. 3.6) that to $\mathcal{Z}$ there corresponds a class of association of pairs $(\mathsf{P}, \sigma_{\mathsf{M}})$, where P is a parabolic in G and σ_{M} is an irreducible local system with respect to the Levi quotient M of P.

Moreover, the resulting morphism

$$\pi : \underset{(\mathsf{P},\sigma_{\mathsf{M}})}{\sqcup} {}^{\text{red}}\mathbf{Maps}(\text{Rep}(\mathsf{P}), \mathbf{H})_{\sigma_{\mathsf{M}}} \to {}^{\text{red}}\mathcal{Z}$$

(the union is taken over the given class of association) is proper and surjective at the level of geometric points.

We claim that the category $\text{Coh}({}^{\text{red}}\mathcal{Z})$ is generated under finite colimits and retracts by the essential image of

$$\text{Coh}\left(\underset{(\mathsf{P},\sigma_{\mathsf{M}})}{\sqcup} {}^{\text{red}}\mathbf{Maps}(\text{Rep}(\mathsf{P}), \mathbf{H})_{\sigma_{\mathsf{M}}}\right)$$

along π_*.

Indeed, this follows from the next general assertion:

Lemma 5.1.7. *Let* $\pi : \mathcal{Z}' \to \mathcal{Z}$ *be a proper map between algebraic stacks, surjective at the level of geometric points. Then* $\text{Coh}(\mathcal{Z})$ *is generated under finite colimits and retracts by the essential image of* $\text{Coh}(\mathcal{Z}')$ *along* π_*.

Proof. First, since π is proper, the functor π_* does indeed send $\text{Coh}(\mathcal{Z}')$ to $\text{Coh}(\mathcal{Z})$. Since $\text{IndCoh}(\mathcal{Z}')$ is generated by $\text{Coh}(\mathcal{Z}')$ (see [DrGa1, Proposition 3.5.1]), the assertion of the lemma is equivalent to the fact that the essential image of $\text{IndCoh}(\mathcal{Z}')$ along

$$\pi_*^{\text{IndCoh}} : \text{IndCoh}(\mathcal{Z}') \to \text{IndCoh}(\mathcal{Z})$$

generates $\text{IndCoh}(\mathcal{Z})$. This is equivalent to the fact that the right adjoint

$$\pi^! : \text{IndCoh}(\mathcal{Z}) \to \text{IndCoh}(\mathcal{Z}')$$

is conservative. However, the latter is [Ga4, Proposition 8.1.2]. □

5.1.8. Thus, we obtain that it suffices to show that for a parabolic P with Levi quotient M and a M-local system σ_{M}, the algebraic stack

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})_{\sigma_{\mathsf{M}}}$$

is mock-proper.

We will consider separately two cases: when $\mathsf{P} = \mathsf{G}$ and when P is a proper parabolic. If $\mathsf{P} = \mathsf{G}$,

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})_{\sigma_{\mathsf{M}}} = \mathrm{pt} / \mathrm{Aut}(\sigma_{\mathsf{M}})$$

and the assertion obvious. Hence, from now on we will assume that P is a proper parabolic.

5.1.9. Let $\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})^{\mathrm{rigid}}_{\sigma_{\mathsf{M}}}$ be the following (algebraic) stack: it classifies the data of

$$(\sigma_{\mathsf{P}}, \alpha, \epsilon),$$

where:

- σ_{P} is a point of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})$,
- α is an identification $\mathsf{M} \overset{\mathsf{P}}{\times} \sigma_{\mathsf{P}} \simeq \sigma_{\mathsf{M}}$, so that the pair $(\sigma_{\mathsf{P}}, \alpha)$ is a point of
$$\mathrm{pt} \underset{\mathbf{Maps}(\mathrm{Rep}(\mathsf{M}), \mathbf{H})}{\times} \mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H});$$
- ϵ is an identification
$$\mathbf{oblv_H}(\sigma_{\mathsf{P}}) \simeq \mathsf{P} \overset{\mathsf{M}}{\times} \mathbf{oblv_H}(\sigma_{\mathsf{M}}),$$
as points of pt / P, compatible with the datum of α.

The stack $\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})^{\mathrm{rigid}}_{\sigma_{\mathsf{M}}}$ carries an action of $\mathrm{Aut}(\sigma_{\mathsf{M}})$ (by changing the datum of α); in particular, it is acted on by $Z(\mathsf{M})^0$, the connected component of the center of M. In addition, it carries an action of the (unipotent) group

$$(\mathsf{N_P})_{\mathbf{oblv_H}(\sigma_{\mathsf{M}})}$$

(by changing the datum of ϵ), where:

- $\mathsf{N_P}$ is the unipotent radical of P;
- $(\mathsf{N_P})_{\mathbf{oblv_H}(\sigma_{\mathsf{M}})}$ is the twist of $\mathsf{N_P}$ by the M-torsor $\mathbf{oblv_H}(\sigma_{\mathsf{M}})$, using the adjoint action of M on $\mathsf{N_P}$.

Combining, we obtain an action on $\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})^{\mathrm{rigid}}_{\sigma_{\mathsf{M}}}$ of the semi-direct product

$$\mathrm{Aut}(\sigma_{\mathsf{M}}) \ltimes (\mathsf{N_P})_{\mathbf{oblv_H}(\sigma_{\mathsf{M}})}.$$

We have:

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})^{\mathrm{rigid}}_{\sigma_{\mathsf{M}}} / \mathrm{Aut}(\sigma_{\mathsf{M}}) \ltimes (\mathsf{N_P})_{\mathbf{oblv_H}(\sigma_{\mathsf{M}})} \simeq \mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})_{\sigma_{\mathsf{M}}}.$$

5.1.10. Choose a coweight $\mathbb{G}_m \to Z(\mathsf{M})^0$, dominant and regular with respect to P (i.e., one such that the adjoint action of $\mathbb{G}_m$ on $\mathfrak{n}_{\mathsf{P}}$ has positive eigenvalues). Such a coweight exists by the assumption that P is a proper parabolic.

We claim that it suffices to show that the algebraic stack

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})^{\mathrm{rigid}}_{\sigma_{\mathsf{M}}} / \mathbb{G}_m \ltimes (\mathsf{N_P})_{\mathbf{oblv_H}(\sigma_{\mathsf{M}})} \tag{5.1}$$

is mock-proper. Indeed, the space global sections of an object in $\mathcal{F} \in \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})_{\sigma_{\mathsf{M}}})$ can be expressed as invariants with respect to the quotient group $\mathrm{Aut}(\sigma_{\mathsf{M}})/\mathbb{G}_m$ on the space of global sections of the pullback of $\mathcal{F}$ to (5.1).

Furthermore, we claim that it suffices to show that the algebraic stack

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})^{\mathrm{rigid}}_{\sigma_{\mathsf{M}}} / \mathbb{G}_m$$

is mock proper. Indeed, let $\mathcal{F}'$ be a quasi-coherent sheaf on (5.1), and let $\mathcal{F}''$ denote its pullback to $\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}), \mathbf{H})^{\mathrm{rigid}}_{\sigma_{\mathsf{M}}} / \mathbf{G}_m$. Since the group $(\mathsf{N_P})_{\mathbf{oblv_H}(\sigma_{\mathsf{M}})}$ is unipotent, using the Chevalley complex that computes Lie algebra cohomology, we obtain that the space of global sections of $\mathcal{F}'$ admits a finite filtration with subquotients of the form

$$\Gamma\left(\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}),\mathbf{H})^{\mathrm{rigid}}_{\sigma_{\mathsf{M}}}/\mathbb{G}_m, \mathcal{F}''\otimes\Lambda^{\cdot}((\mathfrak{n}_{\mathsf{P}})_{\mathbf{oblv_H}(\sigma_{\mathsf{M}})})\right),$$

where $(\mathfrak{n}_{\mathsf{P}})_{\mathbf{oblv_H}(\sigma_{\mathsf{M}})}$ is the Lie algebra of $(\mathsf{N}_{\mathsf{P}})_{\mathbf{oblv_H}(\sigma_{\mathsf{M}})}$.

5.1.11. Note that the proof in Sects. 3.3.1-3.3.5 of the fact that the morphism

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}),\mathbf{H})\to\mathbf{Maps}(\mathrm{Rep}(\mathsf{M}),\mathbf{H})$$

is a relative algebraic stack implies that $\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}),\mathbf{H})^{\mathrm{rigid}}_{\sigma_{\mathsf{M}}}$ is actually an affine (derived) scheme.

Furthermore, the fact that $\mathbb{G}_m$ acts on $\mathfrak{n}_{\mathsf{P}}$ with positive eigenvalues implies that the action of $\mathbb{G}_m$ on $\mathbf{Maps}(\mathrm{Rep}(\mathsf{P}),\mathbf{H})^{\mathrm{rigid}}_{\sigma_{\mathsf{M}}}$ is *contracting*:

Recall (see [DrGa3, Sect. 1.4.4]) that an action of $\mathbb{G}_m$ on an affine scheme Z is said to be contracting if it can be extended to an action of the monoid $\mathbb{A}^1$, so that the action of $0\in\mathbb{A}^1$ factors as

$$Z\to\mathrm{pt}\to Z.$$

The required result follows now from the next general assertion, which generalizes Example (iii) in Sect. 5.1.3:

Lemma 5.1.12. *Let Z be an affine scheme almost of finite type, equipped with a contracting action of $\mathbb{G}_m$. Then the algebraic stack $Z/\mathbb{G}_m$ is mock-proper.*

Proof. Write $Z=\mathrm{Spec}(A)$. The $\mathbb{G}_m$-action on Z equips A with a grading. The fact that the $\mathbb{G}_m$-action is contracting is equivalent to the fact that the grading on A is non-negative and that the map $\mathsf{e}\to A^0$ is an isomorphism.

The category $\mathrm{QCoh}(Z/\mathbb{G}_m)$ consists of complexes M of graded A-modules. The subcategory $\mathrm{Coh}(Z/\mathbb{G}_m)\subset\mathrm{QCoh}(Z/\mathbb{G}_m)$ corresponds to the condition that M is cohomologically bounded and all $H^i(M)$ are finitely generated over $H^0(A)$.

The functor

$$\Gamma(Z/\mathbb{G}_m,-):\mathrm{Coh}(Z/\mathbb{G}_m)\to\mathrm{Vect}_{\mathsf{e}}$$

takes M to its degree 0 component M^0, which is finite-dimensional.

This implies the assertion of the lemma.

□

5.2. **A digression: ind-algebraic stacks.**

5.2.1. Let $\mathcal{Z}$ be a prestack.

We shall say that $\mathcal{Z}$ is an *ind-algebraic stack* if it is *convergent* and for every n, the nth coconnective truncation ${}^{\leq n}\mathcal{Z}$, can be written as

$$^{\leq n}\mathcal{Z}\simeq\underset{i\in I}{\mathrm{colim}}\,\mathcal{Z}_{i,n}, \tag{5.2}$$

where:

- Each $\mathcal{Z}_{i,n}$ is a quasi-compact n-coconnective algebraic stack locally of finite type;
- The category I of indices is filtered;
- The transition maps $\mathcal{Z}_{i,n}\to\mathcal{Z}_{j,n}$ are closed embedding.

We claim:

Lemma 5.2.2. *Let $\mathcal{Z}$ be an n-coconnective ind-algebraic stack. Then:*

(a) *The maps $\mathcal{Z}_{i,n} \to \mathcal{Z}$ are closed embeddings.*

(b) *The family*

$$i \mapsto (\mathcal{Z}_{i,n} \to \mathcal{Z})$$

is cofinal in the category of n-coconnective algebraic quasi-compact stacks equipped with a closed embedding into $\mathcal{Z}$.

The proof is parallel to [GR3, Lemma 1.3.6][16].

5.2.3. We now claim:

Lemma 5.2.4. *Let a prestack $\mathcal{Z}$ be equal to the quotient $\mathcal{Y}/\mathsf{G}$, where $\mathcal{Y}$ is an ind-scheme locally almost of finite type, and G is an algebraic group. Then $\mathcal{Z}$ is an ind-algebraic stack.*

Proof. The convergence condition easily follows from the fact that both $\mathcal{Y}$ and pt/G are convergent. Thus, we may assume that $\mathcal{Y}$ is n-coconnective. We need to show that we can write $\mathcal{Y}$ as a filtered colimit

$$\mathcal{Y} \simeq \underset{i \in I}{\mathrm{colim}}\, Y_i,$$

where:

- Each Y_i is a quasi-compact scheme almost of finite type, stable under the G-action;
- The transition maps $Y_i \to Y_j$ are closed embeddings, compatible with the action of G.

We will first show that such a presentation exists but without the condition that Y_i be almost of finite type.

Recall (see [GR3, Sect. 3.1.6]) that for a map of schemes $f : Y \to Z$, it makes sense to consider the closure of the image of Y inside Z, to be denoted $\overline{\mathrm{Im}(f)}$. This is the universal closed subscheme of Z for which there exists a factorization of f as

$$Y \to \overline{\mathrm{Im}(f)} \to Z.$$

Furthermore, if $i : Z \to Z'$ is a closed embedding and $f' := i \circ f$, then the natural map

$$\overline{\mathrm{Im}(f)} \to \overline{\mathrm{Im}(f')}$$

is an isomorphism. In particular, it makes sense to talk about the closure of the image in the target that is an ind-scheme.

Write $\mathcal{Y}$ as a filtered colimit of closed (but not necessarily G-invariant) subschemes

$$\mathcal{Y} \simeq \underset{i \in I}{\mathrm{colim}}\, Y'_i.$$

Now, let Y_i be the closure of the image of the map

$$\mathsf{G} \times Y'_i \to \mathsf{G} \times \mathcal{Y} \to \mathcal{Y},$$

where the last arrow is the action map. By construction, the closed subschemes Y_i are G-invariant, and the resulting map

$$\underset{i \in I}{\mathrm{colim}}\, Y_i \to \mathcal{Y}$$

is an isomorphism.

Now, starting from the family of subschemes constructed above, we apply a G-equivariant version of [GR3, Proposition 1.7.7] (proved in *loc.cit.* Sect. 3.5.2) to produce a family that consists of G-equivariant schemes almost of finite type.

□

5.2.5. As corollary, combining with Theorem 1.8.3, we obtain:

Corollary 5.2.6. *Every connected component of* $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ *is an ind-algebraic stack.*

[16] The n-coconnectivity condition is important here: we use it when we say that an n-coconnective quasi-compact algebraic stack can be written as a *finite* colimit of affine schemes, sheafified in the étale/fppf topology.

5.3. **Mock-affineness and coarse moduli spaces.**

5.3.1. Let $\mathcal{Z}$ be an algebraic stack. We shall say that $\mathcal{Z}$ is *mock-affine* if the functor of global sections

$$\Gamma(\mathcal{Z}, -) : \mathrm{QCoh}(\mathcal{Z}) \to \mathrm{Vect}_{\mathsf{e}}$$

is t-exact.

Clearly, $\mathcal{Z}$ is mock-affine if and only if its underlying classical stack ${}^{cl}\mathcal{Z}$ is mock-affine.

5.3.2. *Example.* Let $\mathcal{Z}$ be of the form Y/G, where Y is affine scheme and G is a *reductive* algebraic group. Then (assuming that e has characteristic zero) the stack $\mathcal{Z}$ is mock-affine.

5.3.3. Let $\mathcal{Z}$ be an ind-algebraic stack. We shall say that $\mathcal{Z}$ is mock-affine if ${}^{cl}\mathcal{Z}$ admits a presentation (5.2) whose terms are mock-affine.

By Lemma 5.2.2, this is equivalent to requiring that for every algebraic stack $\mathcal{Z}'$ equipped with a closed embedding $\mathcal{Z}' \to \mathcal{Z}$, the stack $\mathcal{Z}'$ is mock-affine.

5.3.4. From Theorem 1.8.3, combined with Lemma 5.2.4 and Example 5.3.2, we obtain that each connected component of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ is mock-affine.

5.3.5. Let $\mathcal{Z}$ be a mock-affine algebraic stack. In particular, the e-algebra

$$\Gamma(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$$

is connective.

Further, for every n,

$$\tau^{\geq -n}(\Gamma(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})) \simeq \Gamma({}^{\leq n}\mathcal{Z}, \mathcal{O}_{\leq n\mathcal{Z}}).$$

We define the coarse moduli space $\mathcal{Z}^{\mathrm{coarse}}$ of $\mathcal{Z}$ to be the affine scheme

$$\mathrm{Spec}(\Gamma(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})).$$

By construction, we have a canonical projection

$$\mathbf{r} : \mathcal{Z} \to \mathcal{Z}^{\mathrm{coarse}}.$$

5.3.6. Let $\mathcal{Z}$ be a mock-affine ind-algebraic stack. For every n consider the n-coconnective ind-affine ind-scheme

$${}^{\leq n}\mathcal{Z}^{\mathrm{coarse}} := \underset{i}{\mathrm{colim}}\ \mathrm{Spec}(\Gamma(\mathcal{Z}_{i,n}, \mathcal{O}_{\mathcal{Z}_{i,n}}))$$

for ${}^{\leq n}\mathcal{Z}$ written as in Sect. 5.2 (by Lemma 5.2.2, this definition is independent of the presentation).

We define the ind-affine ind-scheme $\mathcal{Z}^{\mathrm{coarse}}$ to be the convergent completion[17] of

$$\underset{n}{\mathrm{colim}}\ {}^{\leq n}\mathcal{Z}^{\mathrm{coarse}}. \tag{5.3}$$

I.e., this is a convergent prestack whose value on eventually coconnective affine schemes is given by the colimit (5.3).

We have a canonical projection

$$\mathbf{r} : \mathcal{Z} \to \mathcal{Z}^{\mathrm{coarse}}.$$

[17]See [GR1, Chapter 2, Sect. 1.4.8] for what this means.

5.3.7. We claim:

Lemma 5.3.8. *Let $\mathcal{Z}$ be a mock-affine ind-algebraic stack satisfying:*

- *$\mathcal{Z}$ is locally almost of finite type;*
- *${}^{\mathrm{red}}\mathcal{Z}$ is a mock-proper algebraic stack;*
- *${}^{\mathrm{red}}\mathcal{Z}$ is connected.*

Then $\mathcal{Z}^{\mathrm{coarse}}$ has the following properties:

- *It is locally almost of finite type;*
- *${}^{\mathrm{red}}(\mathcal{Z}^{\mathrm{coarse}}) \simeq \mathrm{pt}$.*

Proof. To prove that $\mathcal{Z}^{\mathrm{coarse}}$ is locally almost of finite type, it suffices to show that for every n, and a presentation of ${}^{\leq n}\mathcal{Z}$ as in Sect. 5.2, the rings $\Gamma(\mathcal{Z}_{i,n}, \mathcal{O}_{\mathcal{Z}_{i,n}})$ are finite-dimensional over e. However, this follows from the mock-properness assumption.

This also implies that ${}^{\mathrm{red}}(\mathcal{Z}^{\mathrm{coarse}})$ is Artinian, i.e., is the union of finite many copies of pt. The connectedness assumption on ${}^{\mathrm{red}}\mathcal{Z}$ implies that there is only one copy.

□

5.4. Coarse moduli spaces for connected components of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

5.4.1. Let $\mathbf{H}$ be again a gentle Tannakian category. We will apply the discussion from Sect. 5.3 to $\mathcal{Z}$ being a connected component of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$.

Note that $\mathcal{Z}$ satisfies the conditions of Lemma 5.3.8 by the combination of Theorems 1.8.3 and 5.1.5. In particular, we obtain that $\mathcal{Z}^{\mathrm{coarse}}$ is an ind-affine ind-scheme locally almost of finite type, and ${}^{\mathrm{red}}(\mathcal{Z}^{\mathrm{coarse}}) \simeq \mathrm{pt}$.

We are now ready to state the main result of this subsection:

Main Theorem 5.4.2. *Let $\mathcal{Z}$ being a connected component of* $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$*, and consider the corresponding map*

$$\mathbf{r} : \mathcal{Z} \to \mathcal{Z}^{\mathrm{coarse}}.$$

We have:

(a) *The map* $\mathbf{r}$ *makes $\mathcal{Z}$ into a relative algebraic stack over $\mathcal{Z}^{\mathrm{coarse}}$, i.e., the base change of* $\mathbf{r}$ *by an affine scheme yields an algebraic stack.*

(b) *The ind-scheme $\mathcal{Z}^{\mathrm{coarse}}$ is a formal affine scheme (see Remark 1.4.7 for what this means).*

The proof of Theorem 5.4.2 (for a general gentle Tannakian category $\mathbf{H}$) will given in Sect. 6. In the particular case when $\mathbf{H} = \mathrm{QLisse}(X)$ when X is a smooth and complete algebraic curve, a simpler argument will be given in Sect. 9.7.

5.4.3. Our main application is when $\mathbf{H} = \mathrm{QLisse}(X)$, and so $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) = \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

Denote by $\mathrm{LocSys}^{\mathrm{restr,coarse}}_{\mathsf{G}}(X)$ the disjoint union of the formal affine schemes $\mathcal{Z}^{\mathrm{coarse}}$ over the connected components $\mathcal{Z}$ of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, and consider the corresponding map

$$\mathbf{r} : \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \to \mathrm{LocSys}^{\mathrm{restr,coarse}}_{\mathsf{G}}(X).$$

In Sect. 6.1 we will show that in the Betti context, this map can be obtained as a formal completion of the map

$$\mathbf{r} : \mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X) \to \mathrm{LocSys}^{\mathrm{Betti,coarse}}_{\mathsf{G}}(X)$$

of (4.15) at the disjoint union of e-points of $\mathrm{LocSys}^{\mathrm{Betti,coarse}}_{\mathsf{G}}(X)$.

5.4.4. For a connected component $\mathcal{Z}$ of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$, set

$$\mathcal{Z}^{\mathrm{rigid}} := \mathcal{Z} \underset{\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})}{\times} \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}.$$

A consequence of Theorem 5.4.2 of particular importance for the sequel is:

Corollary 5.4.5. *The fiber product*

$$\mathrm{pt} \underset{\mathcal{Z}^{\mathrm{coarse}}}{\times} \mathcal{Z}$$

is an algebraic stack[18].

From here we obtain:

Corollary 5.4.6. *The fiber product*

$$\mathrm{pt} \underset{\mathcal{Z}^{\mathrm{coarse}}}{\times} \mathcal{Z}^{\mathrm{rigid}}$$

is an affine scheme.

(Indeed, it is easy to see that a prestack that is simultaneously an ind-affine ind-scheme and an algebraic stack is actually an affine scheme.)

Remark 5.4.7. We emphasize that the assertion of Corollary 5.4.6 (resp., Corollary 5.4.5) is that the corresponding fiber products do *not* have ind-directions.

They may be non-reduced, but the point is that they are (locally) schemes, as opposed to formal schemes.

6. The formal coarse moduli space

This section is devoted to the proof of Theorem 5.4.2, and we continue to assume that G is reductive.

In the course of the proof we will encounter another fundamental feature of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ (Theorem 6.7.8):

Recall that at the classical level, when we can think of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ as the prestack of homomorphisms $\mathsf{H} \to \mathsf{G}$, where H is the pro-algebraic Tannakian group attached to $(\mathbf{H}^{\heartsuit}, \mathbf{oblv}_{\mathbf{H}})$, see Proposition 2.5.9. The claim is that on each component of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$, these homomorphisms factor via a particular quotient of H which is *topologically finitely generated.*

6.1. **The coarse moduli space in the Betti setting.** In this subsection we return to the context of Sect. 4.5. We will illustrate what Theorem 5.4.2 says in this case.

6.1.1. Let X be a compact connected CW complex.

Recall the setting of Sect. 4.6.2: we have the affine scheme $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti,coarse}}(X)$ and a map

$$\mathbf{r} : \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X) \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti,coarse}}(X). \tag{6.1}$$

Let

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr,coarse}}(X)$$

be the disjoint union of formal completions of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti,coarse}}(X)$ at its e-points.

Note that Corollary 4.8.5 can be reformulated as saying that we have a Cartesian diagram

$$\begin{array}{ccc} \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) & \longrightarrow & \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X) \\ \downarrow & & \downarrow \\ \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr,coarse}}(X) & \longrightarrow & \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti,coarse}}(X). \end{array} \tag{6.2}$$

[18]It follows automatically that it is quasi-compact and locally almost of finite type

6.1.2. For a fixed $\sigma \in \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti,coarse}}(X)$, let $\mathcal{Z}_\sigma \subset \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$ be the corresponding connected component of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)$.

It is clear from (6.2) that

$$(\mathcal{Z}_\sigma)^{\mathrm{coarse}} \simeq (\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti,coarse}}(X))^\wedge_\sigma, \tag{6.3}$$

where the right-hand side is the formal completion of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti,coarse}}(X)$ at σ.

The isomorphism (6.3) makes both assertions of Theorem 5.4.2 manifest. Indeed, point (a) follows from the fact that the projection

$$\mathcal{Z}_\sigma \overset{r}{\to} (\mathcal{Z}_\sigma)^{\mathrm{coarse}}$$

is a base change of the map (6.1), while $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X)$ is an algebraic stack.

6.2. **Property W.** The rest of this section is devoted to the proof of Theorem 5.4.2.

6.2.1. Let $\mathcal{Z}$ be a prestack of the form $\mathcal{Z}^{\mathrm{rigid}}/\mathsf{G}$, where $\mathcal{Z}^{\mathrm{rigid}}$ is an ind-affine ind-scheme and G is a reductive group.

Assume that ${}^{\mathrm{red}}\mathcal{Z}$ is connected and mock-proper, so that Lemma 5.3.8 applies. In particular, ${}^{\mathrm{red}}\mathcal{Z}^{\mathrm{rigid}}$ has a unique closed G-orbit, which corresponds to a unique closed point of $\mathcal{Z}$,

$$\mathrm{pt} \to \mathcal{Z}. \tag{6.4}$$

Consider the corresponding map

$$\mathbf{r} : \mathcal{Z} \to \mathcal{Z}^{\mathrm{coarse}},$$

and the unique point

$$\mathrm{pt} \to \mathcal{Z}^{\mathrm{coarse}}.$$

6.2.2. We shall say that $\mathcal{Z}$ has Property W if the prestack

$$\mathcal{W} := \mathrm{pt} \underset{\mathcal{Z}^{\mathrm{coarse}}}{\times} \mathcal{Z}$$

is an algebraic stack (as opposed to an ind-algebraic stack).

This is equivalent to requiring that

$$\mathcal{W}^{\mathrm{rigid}} := \mathrm{pt} \underset{\mathcal{Z}^{\mathrm{coarse}}}{\times} \mathcal{Z}^{\mathrm{rigid}} \tag{6.5}$$

is an affine scheme (as opposed to an ind-affine ind-scheme).

6.2.3. We claim:

Lemma 6.2.4. *The following conditions are equivalent:*

(i) *The map* $\mathbf{r}$ *makes* $\mathcal{Z}$ *into a relative algebraic stack over* $\mathcal{Z}^{\mathrm{coarse}}$*;*

(ii) $\mathcal{Z}$ *has Property W.*

Proof. Clearly, we have (i) $\Rightarrow$ (ii). For the opposite implication, it suffices to show that if $\mathcal{Z}^{\mathrm{coarse}}$ is written as

$$\underset{i}{\mathrm{colim}}\ \mathrm{Spec}(A_i)$$

with A_i Artinian, then each

$$\mathrm{Spec}(A_i) \underset{\mathcal{Z}^{\mathrm{coarse}}}{\times} \mathcal{Z}^{\mathrm{rigid}}$$

is an affine scheme.

However, since A_i is Artinian, this would follow once we know that the further base change

$$\mathrm{pt} \underset{\mathrm{Spec}(A_i)}{\times} \left(\mathrm{Spec}(A_i) \underset{\mathcal{Z}^{\mathrm{coarse}}}{\times} \mathcal{Z}^{\mathrm{rigid}}\right) \tag{6.6}$$

is an affine scheme (indeed, given an ind-scheme $\mathcal{Y}$ over $\mathrm{Spec}(A)$ with A an Artinian ring, if $\mathcal{Y} \underset{\mathrm{Spec}(A)}{\times} \mathrm{pt}$ is a scheme, then $\mathcal{Y}$ is a scheme). However the fiber product (6.6) is the same as $\mathcal{W}^{\mathrm{rigid}}$.

□

6.2.5. Here is how Property W will be used. Let $\mathcal{Z}$ be as in Sect. 6.2.1.

Proposition 6.2.6. *Assume that the ind-affine ind-scheme $\mathcal{Z}^{\text{rigid}}$ is a formal affine scheme, and assume that $\mathcal{Z}$ has property W. Then $\mathcal{Z}^{\text{coarse}}$ is also a formal affine scheme.*

Proof. By Theorem 3.1.4, it suffices to show that the tangent space to $\mathcal{Z}^{\text{coarse}}$ at its unique closed point, viewed as an object of $\text{Vect}_{\mathsf{e}}^{\geq 0}$, is finite-dimensional in each degree. For that it suffices to check that the !-pullback of $T_{\text{pt}}(\mathcal{Z}^{\text{coarse}})$ to $\mathcal{W}^{\text{rigid}}$, viewed as an object of $\text{IndCoh}(\mathcal{W}^{\text{rigid}})$, is such that all its cohomologies are in $\text{Coh}(\mathcal{W}^{\text{rigid}})^{\heartsuit}$.

We have a fiber sequence

$$T(\mathcal{W}^{\text{rigid}}) \to T(\mathcal{Z}^{\text{rigid}})|_{\mathcal{W}^{\text{rigid}}} \to T_{\text{pt}}(\mathcal{Z}^{\text{coarse}})|_{\mathcal{W}^{\text{rigid}}}.$$

The cohomologies of $T(\mathcal{W}^{\text{rigid}}) \in \text{IndCoh}(\mathcal{W}^{\text{rigid}})$ lie in $\text{Coh}(\mathcal{W}^{\text{rigid}})^{\heartsuit}$ because $\mathcal{W}^{\text{rigid}}$ is an affine scheme (locally almost of finite type). Now, $T(\mathcal{Z}^{\text{rigid}})|_{\mathcal{W}^{\text{rigid}}}$ also has cohomologies lying in $\text{Coh}(\mathcal{W}^{\text{rigid}})^{\heartsuit}$ since $\mathcal{Z}^{\text{rigid}}$ is formal affine scheme.

□

6.2.7. Let us return to the setting of Theorem 5.4.2. As in Sect. 3.6.1, we will refer to points of $\mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathbf{H})$ as "local systems".

Let σ be a semi-simple local system, and let $\mathcal{Z}_\sigma$ be the sconnected component of $\mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathbf{H})$ corresponding to σ.

Combining Lemma 6.2.4 and Proposition 6.2.6, we obtain that in order to prove Theorem 5.4.2, it suffices to show that $\mathcal{Z}_\sigma$ has Property W.

6.3. **Property A.** Let $\mathcal{Z}$ be as in Sect. 6.2.1.

6.3.1. We shall say that $\mathcal{Z}$ has Property A if there exists a classical affine scheme $\text{Spec}(A)$ (*not* necessarily almost of finite type) and a map

$$\mathbf{r}_A : {}^{\text{cl}}\mathcal{Z} \to \text{Spec}(A)$$

such that the following holds:

The classical prestack underlying the fiber product

$$\text{pt} \underset{\text{Spec}(A)}{\times} {}^{\text{cl}}\mathcal{Z}^{\text{rigid}}, \tag{6.7}$$

where $\text{pt} \to \text{Spec}(A)$ is the map

$$\text{pt} \overset{(6.4)}{\longrightarrow} \mathcal{Z}^{\text{rigid}} \to \text{Spec}(A), \tag{6.8}$$

is a (classical) affine scheme (as opposed to an ind-affine ind-scheme).

6.3.2. We claim:

Proposition 6.3.3. *If $\mathcal{Z}$ has property A, then it has property W.*

6.3.4. Assuming Proposition 6.3.3 for a moment, we obtain that in order to prove Theorem 5.4.2, it suffices to show that the prestack $\mathcal{Z}_\sigma$ as in Sect. 6.2.7 has Property A.

6.3.5. The rest of this subsection is devoted to the proof of Proposition 6.3.3.

Let $\operatorname{Spec}(A)$ and $\mathbf{r}_A$ be as above. First, we claim that we can extend $\mathbf{r}_A$ to a map at the derived level,

$$\mathcal{Z} \to \operatorname{Spec}(A)$$

which we will denote by the same symbol $\mathbf{r}_A$.

Indeed, with no restriction of generality, we can assume that A is a classical polynomial algebra, so the datum of $\mathbf{r}_A$ amounts to a collection G-invariant elements in $\Gamma({}^{\mathrm{cl}}\mathcal{Z}^{\mathrm{rigid}}, \mathcal{O}_{{}^{\mathrm{cl}}\mathcal{Z}^{\mathrm{rigid}}})$ or $\Gamma(\mathcal{Z}^{\mathrm{rigid}}, \mathcal{O}_{\mathcal{Z}^{\mathrm{rigid}}})$ for the classical and derived versions of $\mathbf{r}_A$, respectively.

Now, since $\mathcal{Z}^{\mathrm{rigid}}$ is a formal affine scheme, the map

$$\Gamma(\mathcal{Z}^{\mathrm{rigid}}, \mathcal{O}_{\mathcal{Z}^{\mathrm{rigid}}}) \to \Gamma({}^{\mathrm{cl}}\mathcal{Z}^{\mathrm{rigid}}, \mathcal{O}_{{}^{\mathrm{cl}}\mathcal{Z}^{\mathrm{rigid}}})$$

is an isomorphism on H^0. Hence, so is the map

$$\Gamma(\mathcal{Z}^{\mathrm{rigid}}_\phi, \mathcal{O}_{\mathcal{Z}^{\mathrm{rigid}}})^{\mathsf{G}} \to \Gamma({}^{\mathrm{cl}}\mathcal{Z}^{\mathrm{rigid}}_\phi, \mathcal{O}_{{}^{\mathrm{cl}}\mathcal{Z}^{\mathrm{rigid}}})^{\mathsf{G}},$$

since G is reductive. Hence every element can be lifted.

6.3.6. We now claim that the fiber product

$$\mathrm{pt} \underset{\operatorname{Spec}(A)}{\times} \mathcal{Z}^{\mathrm{rigid}}$$

itself is an affine scheme.

Indeed, its underlying classical prestack is a classical affine scheme, by assumption. Further, it has a connective co-representable deformation theory, because $\mathcal{Z}^{\mathrm{rigid}}$ has this property. Hence, it is indeed an affine scheme by [Lu3, Theorem 18.1.0.1].

6.3.7. We are now ready to prove that $\mathcal{W}^{\mathrm{rigid}}$ is an affine scheme.

Note that for $\operatorname{Spec}(A)$ as above, the map

$$\mathbf{r}_A : \mathcal{Z} \to \operatorname{Spec}(A)$$

canonically factors as

$$\mathcal{Z} \overset{\mathbf{r}}{\to} \mathcal{Z}^{\mathrm{coarse}} \to \operatorname{Spec}(A).$$

Let us base change these maps by (6.8). Thus, from $\mathbf{r}$, we obtain a map

$$\mathrm{pt} \underset{\operatorname{Spec}(A)}{\times} \mathcal{Z} \to \mathrm{pt} \underset{\operatorname{Spec}(A)}{\times} \mathcal{Z}^{\mathrm{coarse}}. \tag{6.9}$$

The map (6.9) realizes $\mathrm{pt} \underset{\operatorname{Spec}(A)}{\times} \mathcal{Z}^{\mathrm{coarse}}$ as

$$\left(\mathrm{pt} \underset{\operatorname{Spec}(A)}{\times} \mathcal{Z}\right)^{\mathrm{coarse}}.$$

The left-hand side in (6.9) is

$$(\mathrm{pt} \underset{\operatorname{Spec}(A)}{\times} \mathcal{Z}^{\mathrm{rigid}})/\mathsf{G},$$

and hence, by Sect. 6.3.6, is a mock-affine *algebraic stack* (as opposed to ind-algebraic stack).

From here, we obtain that the right-hand side in (6.9) is an affine *scheme* (as opposed to ind-scheme). Therefore, the map

$$\mathrm{pt} \underset{\operatorname{Spec}(A)}{\times} \mathcal{Z}^{\mathrm{rigid}} \to \mathrm{pt} \underset{\operatorname{Spec}(A)}{\times} \mathcal{Z}^{\mathrm{coarse}}.$$

is a map between affine schemes. Hence, its further pullback with respect to

$$\mathrm{pt} \to \mathrm{pt} \underset{\operatorname{Spec}(A)}{\times} \mathcal{Z}^{\mathrm{coarse}}$$

is still an affine scheme. But the latter pullback is the prestack $\mathcal{W}^{\mathrm{rigid}}$ of (6.5).

Thus, $\mathcal{W}^{\mathrm{rigid}}$ is an affine scheme, as required.

□[Proposition 6.3.3]

6.4. **A digression: the case of algebraic groups.** In this subsection we will establish a particular case of Theorem 5.4.2. Namely, we will show that it holds for $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$, where H be a (finite-dimensional) algebraic group.

6.4.1. Let H be an affine algebraic group of finite type, and consider the prestacks

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G}) \text{ and } \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})/\operatorname{Ad}(\mathsf{G}).$$

In Proposition 2.6.2 we have already established that $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})$ is an ind-affine ind-scheme. Furthermore, if H is reductive, we know by Proposition 3.5.4 that $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})$ is a disjoint union of (classical smooth) affine schemes.

6.4.2. Choose a Levi splitting of H, i.e.,

$$\mathsf{H} := \mathsf{H}_{\mathrm{red}} \ltimes \mathsf{H}_u.$$

We have a natural projection

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G}) \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\mathrm{red}},\mathsf{G}).$$

Fix a point $\phi \in \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\mathrm{red}},\mathsf{G})$, and set

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})_\phi := \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G}) \underset{\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\mathrm{red}},\mathsf{G})}{\times} \{\phi\}.$$

This is an ind-scheme, equipped with an action of $\mathrm{Stab}_{\mathsf{G}}(\phi)$. We are going to exhibit $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})_\phi$ as the completion of an affine scheme along a Zariski closed subset, such that the entire situation carries an action of $\mathrm{Stab}_{\mathsf{G}}(\phi)$.

6.4.3. Note that $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})_\phi$ identifies with

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u,\mathsf{G})^{\mathsf{H}_{\mathrm{red}}},$$

where $\mathsf{H}_{\mathrm{red}}$ acts on G via ϕ and on H_u by conjugation.

Consider the affine scheme

$$\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h}_u,\mathfrak{g})$$

(see Sect. 10.2.1 below), and its closed subscheme

$$\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h}_u,\mathfrak{g})^{\mathsf{H}_{\mathrm{red}}}.$$

We have a naturally defined map

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u,\mathsf{G})^{\mathsf{H}_{\mathrm{red}}} \to \mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h}_u,\mathfrak{g})^{\mathsf{H}_{\mathrm{red}}}. \tag{6.10}$$

We claim:

Proposition 6.4.4. *The map* (6.10) *realizes* $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u,\mathsf{G})^{\mathsf{H}_{\mathrm{red}}}$ *as the formal completion of the affine scheme* $\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h}_u,\mathfrak{g})^{\mathsf{H}_{\mathrm{red}}}$ *along the closed subset* $\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h}_u,\mathfrak{g})^{\mathsf{H}_{\mathrm{red}}}_{\mathrm{nilp.im.}}$ *consisting of those maps*

$$\mathfrak{h}_u \to \mathfrak{g}$$

whose image is contained in the nilpotent cone of $\mathfrak{g}$.

Proof. We interpret

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u,\mathsf{G})$$

as $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathrm{Rep}(\mathsf{H}_u))^{\mathrm{rigid}}$, and $\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h}_u,\mathfrak{g})$ as

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathfrak{h}_u\text{-mod})^{\mathrm{rigid}},$$

see Proposition 10.2.3.

The fact that (6.10) is an ind-closed embedding and a formal isomorphism follows now from the fact that the restriction functor

$$\mathrm{Rep}(\mathsf{H}_u) \to \mathfrak{h}_u\text{-mod}$$

is fully faithful, whose essential image consists of objects, all of whose cohomologies are such that the action of $\mathfrak{h}_u$ on them is locally nilpotent.

This description also implies the stated description of the essential image at the reduced level. ☐

Corollary 6.4.5. *The formal affine scheme* $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})_\phi$ *is connected.*

Proof. The action of $\mathbb{G}_m$ by dilations contracts $\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h}_u,\mathfrak{g})^{\mathsf{H}_{\mathrm{red}}}$ to a single point, and this action preserves the closed subset $\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h}_u,\mathfrak{g})^{\mathsf{H}_{\mathrm{red}}}_{\mathrm{nilp.im.}}$. ☐

6.4.6. Let $\mathcal{Z}$ be a connected component of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})/\operatorname{Ad}(\mathsf{G})$. From Corollary 6.4.5 we obtain that $\mathcal{Z}$ has the form

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})_\phi/\operatorname{Ad}(\operatorname{Stab}_{\mathsf{G}}(\phi))$$

for some $\phi:\mathsf{H}_{\mathrm{red}}\to\mathsf{G}$. Denote such $\mathcal{Z}$ by $\mathcal{Z}_\phi$.

Note that the unique closed point of $\mathcal{Z}_\phi$ identifies with

$$\mathrm{pt}\to\mathrm{pt}/\operatorname{Stab}_{\mathsf{G}}(\phi)\hookrightarrow\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u,\mathsf{G})^{\mathsf{H}_{\mathrm{red}}}/\operatorname{Ad}(\operatorname{Stab}_{\mathsf{G}}(\phi))\simeq\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})_\phi/\operatorname{Ad}(\operatorname{Stab}_{\mathsf{G}}(\phi)),$$

where the middle arrow corresponds to the trivial homomorphism $\mathsf{H}_u\to\mathsf{G}$.

In other other words, it corresponds to the locus of homomorphisms $\mathsf{H}\to\mathsf{G}$ that factor as

$$\mathsf{H}\twoheadrightarrow\mathsf{H}_{\mathrm{red}}\xrightarrow{\phi}\mathsf{G}.$$

6.4.7. Let us show that the stack $\mathcal{Z}_\phi$ has Property A, thereby establishing that Theorem 5.4.2 holds for $\mathbf{H}=\operatorname{Rep}(\mathsf{H})$, see Sect. 6.3.4.

Let

$$\mathfrak{a}:=\mathfrak{g}/\!/\operatorname{Ad}(\mathsf{G})$$

be the Chevalley space of $\mathfrak{g}$. This is an affine scheme equipped with an action of $\mathbb{G}_m$.

We let $\operatorname{Spec}(A)$ be the affine scheme

$$\mathbf{Maps}_{\mathrm{Sch}}(\mathfrak{h}_u,\mathfrak{a})^{\mathbb{G}_m},$$

where $\mathbb{G}_m$ acts on $\mathfrak{h}_u$ by dilations, and on $\mathfrak{a}$ via its action on $\mathfrak{g}$ (also by dilations). It is easy to see that this is indeed an affine scheme.

We define map $\mathbf{r}_A$ as the composition

$$\mathcal{Z}_\phi\to\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})/\operatorname{Ad}(\mathsf{G})\to\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u,\mathsf{G})/\operatorname{Ad}(\mathsf{G})\to\mathbf{Maps}_{\mathrm{Lie}}(\mathfrak{h}_u,\mathfrak{g})/\operatorname{Ad}(\mathsf{G})\to$$
$$\to\mathbf{Maps}_{\mathrm{Sch}}(\mathfrak{h}_u,\mathfrak{g})^{\mathbb{G}_m}/\operatorname{Ad}(\mathsf{G})\to\mathbf{Maps}_{\mathrm{Sch}}(\mathfrak{h}_u,\mathfrak{a})^{\mathbb{G}_m}$$

Let us show that the fiber product

$$\mathrm{pt}\underset{\mathbf{Maps}_{\mathrm{Sch}}(\mathfrak{h}_u,\mathfrak{a})^{\mathbb{G}_m}}{\times}\mathcal{Z}_\phi$$

is an algebraic stack (as opposed to an ind-algebroac stack). We will do so right away at the derived level.

We will establish an equivalent fact, namely, that

$$\mathrm{pt}\underset{\mathbf{Maps}_{\mathrm{Sch}}(\mathfrak{h}_u,\mathfrak{a})^{\mathbb{G}_m}}{\times}\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})_\phi \tag{6.11}$$

is an affine scheme.

6.4.8. We rewrite (6.11) as

$$\mathrm{pt} \underset{\mathbf{Maps}_{\mathrm{Sch}}(\mathsf{h}_u,\mathfrak{a})^{\mathbb{G}_m}}{\times} \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u,\mathsf{G})^{\mathsf{H}_{\mathrm{red}}},$$

and consider the fiber product

$$\mathrm{pt} \underset{\mathbf{Maps}_{\mathrm{Sch}}(\mathsf{h}_u,\mathfrak{a})^{\mathbb{G}_m}}{\times} \mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}_u,\mathsf{g})^{\mathsf{H}_{\mathrm{red}}},$$

which is an affine scheme, because $\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}_u,\mathsf{g})^{\mathsf{H}_{\mathrm{red}}}$ is such.

Hence, it suffices to show that the map

$$\mathrm{pt} \underset{\mathbf{Maps}_{\mathrm{Sch}}(\mathsf{h}_u,\mathfrak{a})^{\mathbb{G}_m}}{\times} \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u,\mathsf{G})^{\mathsf{H}_{\mathrm{red}}} \to \mathrm{pt} \underset{\mathbf{Maps}_{\mathrm{Sch}}(\mathsf{h}_u,\mathfrak{a})^{\mathbb{G}_m}}{\times} \mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}_u,\mathsf{g})^{\mathsf{H}_{\mathrm{red}}} \tag{6.12}$$

is schematic. We claim that (6.12) is in fact an isomorphism.

6.4.9. By Proposition 6.4.4, a priori, the map (6.12) realizes the left-hand side as the formal completion of the right-hand side along the closed subset

$$\mathrm{pt} \underset{\mathbf{Maps}_{\mathrm{Sch}}(\mathsf{h}_u,\mathfrak{a})^{\mathbb{G}_m}}{\times} \mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}_u,\mathsf{g})^{\mathsf{H}_{\mathrm{red}}}_{\mathrm{nilp.im.}} \tag{6.13}$$

where

$$\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}_u,\mathsf{g})^{\mathsf{H}_{\mathrm{red}}}_{\mathrm{nilp.im.}} \subset \mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}_u,\mathsf{g})^{\mathsf{H}_{\mathrm{red}}}$$

is the locus of maps

$$\mathsf{h}_u \to \mathsf{g} \tag{6.14}$$

whose image is contained in the nilpotent cone.

However, if a map (6.14) is such that the composition

$$\mathsf{h}_u \to \mathsf{g} \to \mathfrak{a}$$

is zero, then this map automatically lands in the nilpotent cone.

This implies that the closed subset (6.13) is all of

$$\mathrm{pt} \underset{\mathbf{Maps}_{\mathrm{Sch}}(\mathsf{h}_u,\mathfrak{a})^{\mathbb{G}_m}}{\times} \mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}_u,\mathsf{g})^{\mathsf{H}_{\mathrm{red}}},$$

and hence (6.12) is an isomorphism.

6.5. **The case of pro-algebraic groups.** In this subsection we will study connected components of the ind-algebraic stack $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})/\operatorname{Ad}(\mathsf{G})$, where H is a pro-algebraic group.

6.5.1. Choose a Levi splitting

$$\mathsf{H} \simeq \mathsf{H}_{\mathrm{red}} \ltimes \mathsf{H}_u,$$

see [HM, Theorem 3.2].

The description of connected components of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})/\operatorname{Ad}(\mathsf{G})$ in the case when H is of finite type given in Sect. 6.4 applies verbatim to the present situation:

The connected components are in bijection with conjugacy classes of homomorphisms $\phi : \mathsf{H}_{\mathrm{red}} \to \mathsf{G}$, and for a given ϕ, the corresponding connected component $\mathcal{Z}_\phi$ identifies with

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})_\phi/\operatorname{Ad}(\mathrm{Stab}_{\mathsf{G}}(\phi))$$

and

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})_\phi \simeq \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u,\mathsf{G})^{\mathsf{H}_{\mathrm{red}}}.$$

However, we do not know, in general, whether such $\mathcal{Z}_\phi$ satisfies Property A.

Remark 6.5.2. Note that in the above discussion, H is an arbitrary pro-algebraic group, so it is *not* true, in general, that its category of representation $\mathrm{Rep}(\mathsf{H})$ is a gentle Tannakian category. Hence, it is *not* true that the ind-affine ind-scheme

$$\mathcal{Z}_\phi^{\mathrm{rigid}} := \mathcal{Z}_\phi \underset{\mathrm{pt}/\mathsf{G}}{\times} \mathrm{pt}$$

is a formal affine scheme.

6.5.3. Let Free_n be the free group on n letters, and let $\mathrm{Free}_n^{\text{Pro-alg}}$ be its pro-algebraic envelope over e, i.e.,

$$\mathrm{Hom}_{\mathrm{Grp}}(\mathrm{Free}_n^{\text{Pro-alg}}, \mathsf{H}') \simeq (\mathsf{H}'(\mathsf{e}))^{\times n}, \quad \mathsf{H}' \in \mathrm{Alg.\,Groups}\,. \tag{6.15}$$

6.5.4. Let H be a pro-algebraic group, written as $\lim\limits_{\alpha} \mathsf{H}_\alpha$ with surjective transition maps. A map $\mathrm{Free}_n^{\text{Pro-alg}} \to \mathsf{H}$ is then the same as an n-tuple $\underline{g}$ of elements in $\mathsf{H}(\mathsf{e})$.

We shall say that an n-tuple $\underline{g}$ *topologically generates* H if the corresponding map $\mathrm{Free}_n^{\text{Pro-alg}} \to \mathsf{H}$ is such that all the composite maps

$$\mathrm{Free}_n^{\text{Pro-alg}} \to \mathsf{H} \to \mathsf{H}_\alpha$$

are surjective.

This is equivalent to the condition that the Zariski closure of the abstract group generated by the images of the elements of $\underline{g}$ in H_α is all of H_α.

6.5.5. We will say that H is *topologically finitely generated* if it admits a finite set of topological generators.

6.5.6. We will prove:

Theorem 6.5.7. *Assume that* H *is topologically finitely generated. Then every connected component of* $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})/\mathrm{Ad}(\mathsf{G})$ *has Property A.*

Remark 6.5.8. Note that since any algebraic group of finite type (over a field of characteristic 0) is topologically finitely generated, Theorem 6.5.7 provides an alternative proof of Theorem 5.4.2 for $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$, where H is an algebraic group of finite type.

6.6. Proof of Theorem 6.5.7.

6.6.1. Write

$$\mathsf{H} \simeq \lim_{\alpha} \mathsf{H}_\alpha.$$

Let $\mathsf{H}' \to \mathsf{H}$ be a homomorphism of pro-algebraic groups, such that for every α the composite map

$$\mathsf{H}' \to \mathsf{H} \to \mathsf{H}_\alpha$$

is surjective.

We claim that if every connected component of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}', \mathsf{G})/\mathrm{Ad}(\mathsf{G})$ has Property A, then so does every connected component of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})/\mathrm{Ad}(\mathsf{G})$ (for a given G).

6.6.2. Let $\mathcal{Z}_\phi$ be a connected component of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})/\mathrm{Ad}(\mathsf{G})$ containing a given map

$$\phi : \mathsf{H}_{\mathrm{red}} \to \mathsf{G}.$$

The map $\mathsf{H}' \to \mathsf{H}$ induces a map

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})/\mathrm{Ad}(\mathsf{G}) \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}', \mathsf{G})/\mathrm{Ad}(\mathsf{G}). \tag{6.16}$$

The surjectivity property of the map of groups implies that (6.16) is a *closed embedding*.

Let $\mathcal{Z}'_\phi$ be the connected component of $\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}', \mathsf{G})/\mathrm{Ad}(\mathsf{G})$ containing the image of $\mathcal{Z}_\phi$. Since $\mathcal{Z}'_\phi$ has Property A, we can find a map

$$\mathbf{r}'_A : \mathcal{Z}'_\phi \to \mathrm{Spec}(A),$$

such that (the classical prestack underlying) $\mathrm{pt} \underset{\mathrm{Spec}(A)}{\times} \mathcal{Z}'_\phi$ is an algebraic stack.

Define a map $\mathbf{r}_A : \mathcal{Z}_\phi \to \operatorname{Spec}(A)$ to be the composition

$$\mathcal{Z}_\phi \to \mathcal{Z}'_\phi \xrightarrow{\mathbf{r}'_A} \operatorname{Spec}(A).$$

Since the map

$$\operatorname{pt} \underset{\operatorname{Spec}(A)}{\times} \mathcal{Z}_\phi \to \operatorname{pt} \underset{\operatorname{Spec}(A)}{\times} \mathcal{Z}'_\phi$$

is a closed embedding, and we obtain that (the classical prestack underlying) $\operatorname{pt} \underset{\operatorname{Spec}(A)}{\times} \mathcal{Z}_\phi$ is also an algebraic stack, as required.

6.6.3. Thus, by the assumption on H and Sect. 6.6.1, we can replace the original H by $\operatorname{Free}_n^{\text{Pro-alg}}$.

Note that

$$\operatorname{Rep}(\operatorname{Free}_n^{\text{Pro-alg}}) \simeq \operatorname{QLisse}(X),$$

where X is the bouquet of n copies of S^1.

Hence, the prestack

$$\mathbf{Maps}_{\operatorname{Grp}}(\operatorname{Free}_n^{\text{Pro-alg}}, \mathsf{G})/\operatorname{Ad}(\mathsf{G})$$

is the same as (the Betti version of) $\operatorname{LocSys}_{\mathsf{G}}^{\text{restr}}(X)$.

Now, the fact that connected components of (the Betti version of) $\operatorname{LocSys}_{\mathsf{G}}^{\text{restr}}(X)$ have Property A follows from Sect. 6.1: we can take

$$\operatorname{Spec}(A) := \operatorname{LocSys}_{\mathsf{G}}^{\text{Betti,coarse}}(X).$$

$\square$[Theorem 6.5.7]

Remark 6.6.4. Let us emphasize that the pro-algebraic group $\operatorname{Free}_n^{\text{Pro-alg}}$ satisfies its universal property (6.15) for individual target groups H', but *not* in families. So, the ind-scheme prestack $\mathbf{Maps}_{\operatorname{Grp}}(\operatorname{Free}_n^{\text{Pro-alg}}, \mathsf{G})$ is $\operatorname{LocSys}_{\mathsf{G}}^{\text{restr,rigid}_x}(X)$ (for X the bouquet of n copies of S^1), which is different from

$$\mathbf{Maps}_{\operatorname{Grp}}(\operatorname{Free}_n, \mathsf{G}) \simeq \operatorname{LocSys}_{\mathsf{G}}^{\text{Betti,rigid}_x}(X) \simeq \mathsf{G}^{\times n}.$$

6.7. **Proof of Theorem 5.4.2.**

6.7.1. Let $\mathcal{Z}_\sigma$ be a connected component of $\mathbf{Maps}(\operatorname{Rep}(\mathsf{G}), \mathbf{H})$. According to Sect. 6.3.4, it suffices to show that $\mathcal{Z}_\sigma$ has Property A.

Recall that, according to Proposition 2.5.9, the prestack ${}^{\text{cl}}\mathbf{Maps}(\operatorname{Rep}(\mathsf{G}), \mathbf{H})$ identifies with the classical prestack underlying

$$\mathbf{Maps}_{\operatorname{Grp}}(\mathsf{H}, \mathsf{G})/\operatorname{Ad}(\mathsf{G}),$$

where H is as in Sect. 2.5.6. The local system σ corresponds to the conjugacy class of a homomorphism $\phi : \mathsf{H}_{\text{red}} \to \mathsf{G}$, so that

$${}^{\text{cl}}\mathcal{Z}_\sigma \simeq {}^{\text{cl}}\mathcal{Z}_\phi.$$

Hence, it suffices to show that $\mathcal{Z}_\phi$ has Property A,

Remark 6.7.2. If we knew that H is topologically finitely generated, then the fact that $\mathcal{Z}_\phi$ has Property A would follow from Theorem 6.5.7.

However, we do not know whether H is topologically finitely generated. Instead, we will show that for every σ, there exists a particular quotient of (the unipotent part of) H that is topologically finitely generated, such that the passage to this quotient does not change ${}^{\text{cl}}\mathcal{Z}_\phi$. This will effectively reduce us to the situation of Theorem 6.5.7.

6.7.3. Let

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})_\phi \simeq \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u,\mathsf{G})^{\mathsf{H}_{\mathrm{red}}}$$

be as in Sect. 6.5.1.

Being a pro-unipotent group, we can write H_u as

$$\lim_\alpha \mathsf{H}_\alpha,$$

where α runs over a filtered family of indices, the groups H_α are finite-dimensional and unipotent and the transition maps

$$\mathsf{H}_{\alpha_2} \to \mathsf{H}_{\alpha_1}$$

are surjective.

With no restriction of generality, we can assume that the $\mathsf{H}_{\mathrm{red}}$-action on the pro-algebraic group H_u comes from a compatible family of actions on the H_α's.

We have:

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u,\mathsf{G})^{\mathsf{H}_{\mathrm{red}}} \simeq \underset{\alpha}{\mathrm{colim}}\, \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_\alpha,\mathsf{G})^{\mathsf{H}_{\mathrm{red}}}.$$

6.7.4. For each index α, let $\mathfrak{h}_{\alpha,\phi\text{-isotyp}}$ be the maximal Lie algebra quotient of

$$\mathfrak{h}_{u,\alpha} := \mathrm{Lie}(\mathsf{H}_\alpha)$$

on which the action of $\mathsf{H}_{\mathrm{red}}$ has only the same isotypic components as those that appear in $\mathfrak{g} := \mathrm{Lie}(\mathsf{G})$, where the latter is acted on by $\mathsf{H}_{\mathrm{red}}$ via ϕ.

Let $\mathsf{H}_{\alpha,\phi\text{-isotyp}}$ denote the corresponding quotient of H_α.

Lemma 6.7.5. *The map*

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\alpha,\phi\text{-isotyp}},\mathsf{G})^{\mathsf{H}_{\mathrm{red}}} \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_\alpha,\mathsf{G})^{\mathsf{H}_{\mathrm{red}}} \tag{6.17}$$

induces an isomorphism of the underlying classical prestacks.

Proof. Follows from Proposition 6.4.4. □

6.7.6. Set

$$\mathsf{H}_{\phi\text{-isotyp}} := \lim_\alpha \mathsf{H}_{\alpha,\phi\text{-isotyp}}.$$

From Lemma 6.7.5 we obtain that the map

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\phi\text{-isotyp}},\mathsf{G})^{\mathsf{H}_{\mathrm{red}}} \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_u,\mathsf{G})^{\mathsf{H}_{\mathrm{red}}}$$

induces an isomorphism of the underlying classical prestacks.

Hence, it is sufficient to show that

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\phi\text{-isotyp}},\mathsf{G})^{\mathsf{H}_{\mathrm{red}}} / \mathrm{Ad}(\mathrm{Stab}_{\mathsf{G}}(\phi))$$

has Property A.

6.7.7. Consider the maps

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\phi\text{-isotyp}},\mathsf{G})^{\mathsf{H}_{\mathrm{red}}} / \mathrm{Ad}(\mathrm{Stab}_{\mathsf{G}}(\phi)) \to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\phi\text{-isotyp}},\mathsf{G}) / \mathrm{Ad}(\mathrm{Stab}_{\mathsf{G}}(\phi)) \to$$
$$\to \mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\phi\text{-isotyp}},\mathsf{G}) / \mathrm{Ad}(\mathsf{G}).$$

In the above composition, the first map is a closed embedding, and the second map is schematic.

Hence, it is sufficient to show that every connected component of

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}_{\phi\text{-isotyp}},\mathsf{G}) / \mathrm{Ad}(\mathsf{G})$$

has Property A.

This follows by combining Theorem 6.5.7 and the following result:

Theorem 6.7.8. *The pro-algebraic group $\mathsf{H}_{\phi\text{-isotyp}}$ is topologically finitely generated.*

6.8. **Proof of Theorem 6.7.8.**

6.8.1. Let H' be a pro-unipotent group

$$\mathsf{H}' \simeq \lim_{\beta} \mathsf{H}'_\beta,$$

where H'_β are unipotent algebraic groups of finite type.

Consider $\mathsf{h}' := \mathrm{Lie}(\mathsf{H}')$ as a pro-finite dimensional vector space. The following is elementary:

Lemma 6.8.2. *Let V be a finite-dimensional subspace of h' such that for every β, the image of V in $\mathsf{h}'_\beta := \mathrm{Lie}(\mathsf{H}'_\beta)$ generates it as a Lie algebra. Then H' is topologically finitely generated.*

6.8.3. Let H' be a pro-unipotent group as above. We claim:

Proposition 6.8.4. *Assume that $\mathsf{h}'/[\mathsf{h}', \mathsf{h}']$ is finite-dimensional. Then H' is topologically finitely generated.*

Proof. Let $\mathsf{V} \subset \mathsf{h}'$ be a finite-dimensional vector space that projects surjectively onto $\mathsf{h}'/[\mathsf{h}', \mathsf{h}']$. By Lemma 6.8.2, it suffices to see that for any β, the image of V in h'_β generates it as a Lie algebra.

But this follows from the next property of nilpotent Lie algebras: if a subspace $\widetilde{\mathsf{V}}$ in a nilpotent finite-dimensional Lie algebra h'' projects surjectively onto $\mathsf{h}''/[\mathsf{h}'', \mathsf{h}'']$, then $\widetilde{\mathsf{V}}$ generates h'' as a Lie algebra.

□

6.8.5. We will prove Theorem 6.7.8 by applying Proposition 6.8.4 to $\mathsf{H}_{\phi\text{-isotyp}}$.

Note that the quotient

$$\mathsf{h}_{\phi\text{-isotyp}}/[\mathsf{h}_{\phi\text{-isotyp}}, \mathsf{h}_{\phi\text{-isotyp}}]$$

is the maximal pro-abelian quotient of $\mathrm{Lie}(\mathsf{H}_u)$ on which $\mathsf{H}_{\mathrm{red}}$ acts via isotypic components that appear in its action on g via σ.

Hence, it is enough to show that the vector space

$$\mathrm{Hom}\left(\mathrm{Lie}(\mathsf{H}_u)/[\mathrm{Lie}(\mathsf{H}_u), \mathrm{Lie}(\mathsf{H}_u)], \mathsf{g}\right)^{\mathsf{H}_{\mathrm{red}}}$$

is finite-dimensional.

Note, however, that the above vector space is the same as

$$H^1\left(\mathrm{Lie}(\mathsf{H}_u), \mathsf{g}\right)^{\mathsf{H}_{\mathrm{red}}},$$

which is the same as

$$H^1\left(\mathbf{inv}_{\mathsf{H}_u}(\mathsf{g})\right)^{\mathsf{H}_{\mathrm{red}}} \simeq H^1\left(\mathbf{inv}_{\mathsf{H}}(\mathsf{g})\right).$$

6.8.6. We have

$$H^1\left(\mathbf{inv}_{\mathsf{H}}(\mathsf{g})\right) \simeq H^0\left(T_\phi(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})/\,\mathrm{Ad}(\mathsf{G}))\right),$$

and we have an exact triangle

$$\mathsf{g} \to T_\phi(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})) \to T_\phi(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G})/\,\mathrm{Ad}(\mathsf{G})),$$

hence it is enough to show that $H^0\left(T_\phi(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G}))\right)$ is finite-dimensional.

6.8.7. We claim that we have a canonical isomorphism

$$H^0\left(T_\phi(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H}, \mathsf{G}))\right) \simeq H^0\left(T_\sigma(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}})\right). \tag{6.18}$$

This would imply the finite-dimensionality claim by Corollary 2.2.6(b').

6.8.8. To prove (6.18), we note that by Corollary 2.2.6(a), both

$$T^*_\phi(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})) \text{ and } T^*_\sigma(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}})$$

belong to $\mathrm{Vect}_{\mathsf{e}}^{\leq 0}$.

Hence, for $V \in \mathrm{Vect}_{\mathsf{e}}^{\heartsuit}$,

$$\mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(T^*_\phi(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})),V) \simeq H^0\left(T_\phi(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G}))\right)\otimes V$$

and

$$\mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(T^*_\sigma(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}),V) \simeq H^0\left(T_\sigma(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}})\right)\otimes V.$$

Now, by deformation theory

$$\mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(T^*_\phi(\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})),V) \simeq \\ \simeq \mathrm{Maps}(\mathrm{Spec}(\mathsf{e}\oplus V),\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G})) \underset{\mathrm{Maps}(\mathrm{pt},\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G}))}{\times} \{\phi\} \tag{6.19}$$

and

$$\mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(T^*_\sigma(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}),V) \simeq \\ \simeq \mathrm{Maps}(\mathrm{Spec}(\mathsf{e}\oplus V),\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}) \underset{\mathrm{Maps}(\mathrm{pt},\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}})}{\times} \{\sigma\}, \tag{6.20}$$

where $\mathsf{e}\oplus V$ is a square-zero extension of e by means of V.

However, since V is classical, in the right-hand sides in (6.19) and (6.20) we can replace

$$\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G}) \text{ and } \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}$$

by

$${}^{\mathrm{cl}}\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G}) \text{ and } {}^{\mathrm{cl}}\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}},$$

respectively. Now, the assertion follows from the fact that

$${}^{\mathrm{cl}}\mathbf{Maps}_{\mathrm{Grp}}(\mathsf{H},\mathsf{G}) \simeq {}^{\mathrm{cl}}\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}},$$

by Proposition 2.5.9.

□[Theorem 6.7.8]

7. Quasi-coherent sheaves on a formal affine scheme

In this section we will study properties of the category of quasi-coherent sheaves on a formal affine scheme, and then apply the results to $\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}))$, where $\mathbf{H}$ is a gentle Tannakian category.

The special feature of formal schemes among general ind-schemes is the following: for an ind-scheme $\mathcal{Y}$, the category $\mathrm{QCoh}(\mathcal{Y})$ is by definition the inverse limit of the categories $\mathrm{QCoh}(Y_i)$ for closed subschemes $Y_i \hookrightarrow \mathcal{Y}$. The functors in this inverse systems are given by *-pullback and they do not generally admit left adjoints. So we do not in general know whether $\mathrm{QCoh}(\mathcal{Y})$ is compactly generated.

However, in the case of formal affine schemes, the situation is much better.

7.1. **Formal affine schemes: basic properties.** Let $\mathcal{Y}$ be a formal affine scheme. I.e., $\mathcal{Y}$ is a prestack that can be written as

$$\underset{n\geq 1}{\operatorname{colim}} \operatorname{Spec}(R_n) \tag{7.1}$$

as in Theorem 1.4.5(d). I.e., R_n are connective commutative e-algebras of the form

$$R_n = R \underset{\mathsf{e}[t_1,...,t_m]}{\otimes} \mathsf{e}[t_1, ..., t_m]/(t_1^n, ..., t_m^n), \quad t_i \mapsto f_i \in R,\ i = 1, ..., m,$$

where R is a connective commutative e-algebra and $f_1, ..., f_m$ is a collection of elements in R.

Equivalently, we can write

$$R_n = R \underset{\mathsf{e}[t_1,...,t_m]}{\otimes} \mathsf{e}, \quad t_i \mapsto f_i^n,$$

In this subsection we will describe some favorable properties enjoyed by $\operatorname{QCoh}(\mathcal{Y})$ for such $\mathcal{Y}$. In general, QCoh of an ind-scheme is unwieldy, but Proposition 7.1.5 below allows one to get one's hand on $\operatorname{QCoh}(\mathcal{Y})$ for $\mathcal{Y}$ a formal affine scheme.

7.1.1. Fix a presentation of $\mathcal{Y}$ as in (7.1); denote by i_∞ the resulting map $\mathcal{Y} \to \operatorname{Spec}(R)$. Set

$$Y_n := \operatorname{Spec}(R_n) \overset{i_n}{\hookrightarrow} \operatorname{Spec}(R).$$

For $n_1 \leq n_2$, let i_{n_1,n_2} denote the corresponding map $Y_{n_1} \to Y_{n_2}$.

Let $U \overset{j}{\hookrightarrow} \operatorname{Spec}(R)$ be the (open) complement of $\operatorname{Spec}(R_1)$.

Remark 7.1.2. Note that by the proof of Theorem 3.1.4, the e-algebra R and the map $i_\infty : \mathcal{Y} \to \operatorname{Spec}(R)$ can be constructed canonically starting from $\mathcal{Y}$, namely

$$R = \Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y}).$$

However, there is a choice involved in choosing the elements $f_1, ..., f_n \in R$, and hence of the subschemes Y_n.

7.1.3. Let

$$\operatorname{QCoh}(\operatorname{Spec}(R))_\mathcal{Y} \overset{(i_\infty)_!}{\hookrightarrow} \operatorname{QCoh}(\operatorname{Spec}(R))$$

be the inclusion of the full subcategory consisting of objects with *set-theoretic* support on Y_1 (i.e., these are objects whose restriction to U vanishes). This inclusion admits a right adjoint, denoted $(i_\infty)^!$; explicitly, for every $\mathcal{F} \in \operatorname{QCoh}(\operatorname{Spec}(R))$ we have the Cousin exact triangle

$$(i_\infty)_! \circ (i_\infty)^!(\mathcal{F}) \to \mathcal{F} \to j_* \circ j^*(\mathcal{F}).$$

Furthermore, we can explicitly write the functor $(i_\infty)_! \circ (i_\infty)^!$ as

$$\underset{n}{\operatorname{colim}}\, (i_n)_* \circ (i_n)^!, \tag{7.2}$$

where we note that each $i_n^!$ is continuous because i_n is a regular embedding. (Note, however, that for fixed n_1, n_2, the functor $(i_{n_1,n_2})^!$, right adjoint to $(i_{n_1,n_2})_*$, is discontinuous.)

7.1.4. Consider the composite functor

$$\operatorname{QCoh}(\operatorname{Spec}(R))_\mathcal{Y} \overset{(i_\infty)_!}{\hookrightarrow} \operatorname{QCoh}(\operatorname{Spec}(R)) \overset{(i_\infty)^*}{\to} \operatorname{QCoh}(\mathcal{Y}). \tag{7.3}$$

The following is established in [GR3, Proposition 7.1.3]:

Proposition 7.1.5. *The functor* (7.3) *is an equivalence.*

From here we formally obtain:

Corollary 7.1.6.

(a) *There exists a (unique) equivalence* $\mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}} \simeq \mathrm{QCoh}(\mathcal{Y})$*, under which the functor*

$$(i_\infty)^! : \mathrm{QCoh}(\mathrm{Spec}(R)) \to \mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}$$

goes over to the functor

$$(i_\infty)^* : \mathrm{QCoh}(\mathrm{Spec}(R)) \to \mathrm{QCoh}(\mathcal{Y}).$$

(b) *The functor* $(i_\infty)^*$ *realizes* $\mathrm{QCoh}(\mathcal{Y})$ *both as a co-localization and a localization of* $\mathrm{QCoh}(\mathrm{Spec}(R))$ *with respect to the essential image of* $\mathrm{QCoh}(U)$ *along* j_**.*

7.1.7. We observe:

Lemma 7.1.8. *Let* $\mathcal{Y}$ *and* Y_n *be as above.*

(a) *For* $\mathcal{F} \in \mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}$*, the map*

$$\underset{n}{\mathrm{colim}}\, (i_n)_* \circ (i_n)^!(\mathcal{F}) \to \mathcal{F}$$

is an isomorphism.

(b) *The category* $\mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}$ *is compactly generated by the objects* $(i_n)_*(\mathcal{O}_{Y_n})$*.*

(c) *The subcategory of compact objects in* $\mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}$ *is closed under the monoidal operation.*

Proof. Point (a) follows from (7.2).

The fact that the objects $(i_n)_*(\mathcal{O}_{Y_n})$ generate $\mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}$ follows from point (a). The fact that they are compact follows from the fact that they are compact as objects of $\mathrm{QCoh}(\mathrm{Spec}(R))$. This proves point (b).

The fact that the subcategory of compact objects is closed under the monoidal operation follows from the corresponding fact for $\mathrm{QCoh}(\mathrm{Spec}(R))$. This proves point (c).

□

7.1.9. Let $i_{n,\infty}$ denote the map $Y_n \to \mathcal{Y}$. Note that by Corollary 7.1.6, the functor

$$(i_{n,\infty})_* : \mathrm{QCoh}(Y_n) \to \mathrm{QCoh}(\mathcal{Y}),$$

right adjoint to

$$(i_{n,\infty})^* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(Y_n),$$

identifies with $(i_\infty)^* \circ (i_n)_*$; in particular, it is continuous.

Furthermore, the above functor $(i_{n,\infty})_*$ admits a right adjoint, to be denoted $(i_{n,\infty})^!$, which under the equivalence of (7.3) corresponds to

$$(i_n)^! : \mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}} \to \mathrm{QCoh}(Y_n).$$

Hence, from Lemma 7.1.8, we obtain:

Corollary 7.1.10.

(a) *For* $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$*, the map*

$$\underset{n}{\mathrm{colim}}\, (i_{n,\infty})_* \circ (i_{n,\infty})^!(\mathcal{F}) \to \mathcal{F}$$

is an isomorphism.

(b) *The category* $\mathrm{QCoh}(\mathcal{Y})$ *is compactly generated by the objects* $(i_{n,\infty})_*(\mathcal{O}_{Y_n})$*.*

(c) *The subcategory of compact objects in* $\mathrm{QCoh}(\mathcal{Y})$ *is closed under the monoidal operation.*

7.1.11. Finally, we claim:

Proposition 7.1.12. *The functor*

$$\operatorname{colim}_n \operatorname{QCoh}(Y_n) \to \operatorname{QCoh}(\operatorname{Spec}(R))_{\mathcal{Y}}, \tag{7.4}$$

given by $\{(i_n)_*\}$*, is an equivalence.*

Combining with Proposition 7.1.5, we obtain:

Corollary 7.1.13. *The functor*

$$\operatorname{colim}_n \operatorname{QCoh}(Y_n) \to \operatorname{QCoh}(\mathcal{Y}),$$

given by $\{(i_{n,\infty})_*\}$*, is an equivalence.*

7.2. Proof of Proposition 7.1.12.

7.2.1. For an index n_0, let

$$\operatorname{ins}_{n_0} : \operatorname{QCoh}(Y_{n_0}) \to \operatorname{colim}_n \operatorname{QCoh}(Y_n)$$

denote the corresponding tautological functor.

For any object

$$\mathcal{F} \in \operatorname{colim}_n \operatorname{QCoh}(Y_n),$$

we have a tautological isomorphism

$$\mathcal{F} \simeq \operatorname{colim}_n \operatorname{ins}_n \circ (\operatorname{ins}_n)^R(\mathcal{F}). \tag{7.5}$$

7.2.2. Denote the functor (7.4) by Ψ and its right adjoint by Φ (note that we do not yet know that Φ is continuous).

Let us rewrite $\operatorname{colim}_n \operatorname{QCoh}(Y_n)$ as

$$\lim_n \operatorname{QCoh}(Y_n), \tag{7.6}$$

where the limit is formed using the *discontinuous* functors

$$(i_{n_1,n_2})^! : \operatorname{QCoh}(Y_{n_2}) \to \operatorname{QCoh}(Y_{n_1}),$$

see [GR1, Chapter 1, Proposition 2.5.7].

In terms of (7.6), the functor Φ is given by the compatible collection of functors

$$\{(i_n)^!\} : \operatorname{QCoh}(\operatorname{Spec}(R)) \to \lim_n \operatorname{QCoh}(Y_n),$$

precomposed with $\operatorname{QCoh}(\operatorname{Spec}(R))_{\mathcal{Y}} \hookrightarrow \operatorname{QCoh}(\operatorname{Spec}(R))$.

In other words,

$$(\operatorname{ins}_n)^R \circ \Phi \simeq (i_n)^!.$$

7.2.3. Using (7.5), we obtain that the composition $\Psi \circ \Phi$ identifies with the functor (7.2). Hence, the counit of the adjunction

$$\Psi \circ \Phi \to \operatorname{Id}$$

is an isomorphism, by Lemma 7.1.8(a).

Hence, Φ is fully faithful.

7.2.4. We now show that the essential image of Φ generates the colimit category. It suffices to show that for every fixed n_0, and $\mathcal{F}_0 \in \mathrm{QCoh}(Y_{n_0})$ the object $\mathrm{ins}_{n_0}(\mathcal{F}_0)$ lies in the essential image of Φ. We will show that

$$\mathrm{ins}_{n_0}(\mathcal{F}_0) \simeq \Phi \circ (i_{n_0})_*(\mathcal{F}_0). \tag{7.7}$$

Using (7.5), the desired isomorphism (7.7) translates as

$$\mathrm{ins}_{n_0}(\mathcal{F}_0) \simeq \underset{n \geq n_0}{\mathrm{colim}}\, \mathrm{ins}_n \circ i_n^! \circ (i_{n_0})_*(\mathcal{F}_0). \tag{7.8}$$

Consider yet another object:

$$\underset{n \geq n_0}{\mathrm{colim}}\, \underset{N \geq n}{\mathrm{colim}}\, \mathrm{ins}_n \circ i_{n,N}^! \circ (i_{n_0,N})_*(\mathcal{F}_0). \tag{7.9}$$

We will show that (7.9) is isomorphic both to the left-hand side and the right-hand side of (7.8).

7.2.5. The isomorphism with the left-hand side follows by replacing the index category in (7.9) by the cofinal category with $N = n$.

7.2.6. For the isomorphism with the right-hand side, we will show that for every fixed $n \geq n_0$, the natural map

$$\underset{N \geq n}{\mathrm{colim}}\, i_{n,N}^! \circ (i_{n_0,N})_*(\mathcal{F}_0) \to i_n^! \circ (i_{n_0})_*(\mathcal{F}_0), \tag{7.10}$$

is an isomorphism (taking place in $\mathrm{QCoh}(Y_n)$).

Since we are dealing with affine schemes, it suffices to show that the isomorphism takes place at the level of global sections. By base change, the latter is equivalent to the fact that the map

$$\underset{N \geq n}{\mathrm{colim}}\, \mathcal{H}om_{\mathrm{QCoh}(Y_{n_0})}(\mathcal{O}_{Y_{n_0} \underset{Y_N}{\times} Y_n}, \mathcal{F}_0) \to \mathcal{H}om_{\mathrm{QCoh}(Y_{n_0})}(\mathcal{O}_{Y_{n_0} \underset{\mathrm{Spec}(R)}{\times} Y_n}, \mathcal{F}_0)$$

is an isomorphism in $\mathrm{Vect}_{\mathsf{e}}$. This follows from the next assertion:

Lemma 7.2.7. *The map from* $\mathcal{O}_{Y_{n_0} \underset{\mathrm{Spec}(R)}{\times} Y_n}$ *to*

$$M \mapsto \mathcal{O}_{Y_{n_0} \underset{Y_N}{\times} Y_n},$$

as a pro-object of $\mathrm{QCoh}(Y_{n_0})$*, is an isomorphism.*

Proof. The assertion immediately reduces to the case when $R = \mathsf{e}[t_1, ..., t_m]$, and further to the case when $m = 1$. In this case, it becomes a calculation similar to [GR3, Lemma 7.1.5].

□

□[Proposition 7.1.12]

7.3. **Mapping affine schemes into a formal affine scheme.** Let $\mathcal{Y}$ be a formal affine scheme.

7.3.1. First, we notice:

Lemma 7.3.2. *The diagonal map* $\Delta_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ *is affine.*

Proof. Fix a presentation of $\mathcal{Y}$ as in (7.1). Then the map $\Delta_{\mathcal{Y}}$ can be obtained as the base change of the diagonal map $\Delta_{\mathrm{Spec}(R)} : \mathrm{Spec}(R) \to \mathrm{Spec}(R) \times \mathrm{Spec}(R)$, i.e., the square

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\Delta_{\mathcal{Y}}} & \mathcal{Y} \times \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \xrightarrow{\Delta_{\mathrm{Spec}(R)}} & \mathrm{Spec}(R) \times \mathrm{Spec}(R) \end{array} \tag{7.11}$$

is Cartesian. □

7.3.3. Let S be an affine scheme, equipped with a map f to $\mathcal{Y}$. Note that f is *affine* as a map of prestacks (by Lemma 7.3.2). Hence, the functor f_*, right adjoint to f^* is continuous.

Corollary 7.3.4. *The functor*

$$\underset{(S,f)}{\operatorname{colim}} \operatorname{QCoh}(S) \to \operatorname{QCoh}(\mathcal{Y}), \tag{7.12}$$

is an equivalence, where:

- *The index category is either of the following:*
$$\operatorname{Sch}^{\mathrm{aff}}_{/\mathcal{Y}},\ \operatorname{Sch}^{\mathrm{aff}}_{/\mathcal{Y},\mathrm{closed}},$$
where the subscript "closed" indicates that we consider only closed embeddings[19] $S \to \mathcal{Y}$*;*
- *The colimit is formed using the pushforward functors* $(f_{1,2})_* : \operatorname{QCoh}(S_1) \to \operatorname{QCoh}(S_2)$ *for*
$$f_{1,2} : S_1 \to S_2, \quad f_2 \circ f_{1,2} = f_1.$$
- *The map in* (7.12) *is given by* $\{\operatorname{QCoh}(S) \overset{f_*}{\to} \operatorname{QCoh}(\mathcal{Y})\}$.

Proof. Fix a presentation of $\mathcal{Y}$ as in (1.8). The assertion follows from Corollary 7.1.13 and the fact that the family $Y_n \overset{i_{n,\infty}}{\longrightarrow} \mathcal{Y}$ is cofinal in any of the above categories. □

7.3.5. Let $S \overset{f}{\to} \mathcal{Y}$ be as above. We shall say that f is a *regular closed embedding* if there exists a map $\mathcal{Y} \to \mathbb{A}^m$, and a Cartesian diagram

$$\begin{array}{ccc} S & \xrightarrow{f} & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \xrightarrow{\mathbf{i}_0} & \mathbb{A}^m. \end{array}$$

In this case, we have
$$\operatorname{QCoh}(S) \simeq \operatorname{QCoh}(\mathrm{pt}) \underset{\operatorname{QCoh}(\mathbb{A}^m)}{\otimes} \operatorname{QCoh}(\mathcal{Y}).$$

Therefore, the adjunction $((\mathbf{i}_0)_*, (\mathbf{i}_0)^!)$ implies that the right adjoint $f^!$ of f_* is continuous and is strictly compatible with the $\operatorname{QCoh}(\mathcal{Y})$-actions. In particular, the functor f_* preserves compactness.

Furthermore, the isomorphism
$$(\mathbf{i}_0)^! \simeq (\mathbf{i}_0)^*[-m]$$
implies that we have an isomorphism

$$f^!(\mathcal{F}) \simeq f^*(\mathcal{F})[-m]. \tag{7.13}$$

7.3.6. Let $\mathcal{Y}$ be realized as
$$\operatorname{Spec}(R)^{\wedge}_{\operatorname{Spec}({}^{\mathrm{cl}}R/I)},$$
where $I \subset {}^{\mathrm{cl}}R$ is a finitely generated ideal. Let us be given a map $\operatorname{Spec}(R) \to \mathbb{A}^m$, such that
$${}^{\mathrm{red}}(\mathrm{pt} \underset{\mathbb{A}^m}{\times} \operatorname{Spec}(R)) = {}^{\mathrm{red}}\mathcal{Y}$$
as subsets of ${}^{\mathrm{red}}\operatorname{Spec}(R)$.

Then

$$\mathrm{pt} \underset{\mathbb{A}^m}{\times} \mathcal{Y} \to \mathrm{pt} \underset{\mathbb{A}^m}{\times} \operatorname{Spec}(R) \tag{7.14}$$

is an isomorphism. Indeed, the left-hand side in (7.14) is a priori the completion of the right-hand side along a closed subset, which is actually the whole thing.

In particular, we obtain that in the above situation, we have a regular closed embedding
$$\mathrm{pt} \underset{\mathbb{A}^m}{\times} \operatorname{Spec}(R) \to \mathcal{Y}.$$

[19] When $\mathcal{Y}$ locally almost of finite type as a prestack, we can further allow $(\operatorname{Sch}^{\mathrm{aff}}_{\mathrm{aft}\ /\mathrm{e}})_{/\mathcal{Y}}$ and $(\operatorname{Sch}^{\mathrm{aff}}_{\mathrm{aft}\ /\mathrm{e}})_{/\mathcal{Y},\mathrm{closed}}$ as index categories in the above colimit.

7.3.7. The situation of Sect. 7.3.6 is realized for $\mathcal{Y}$ written as in (7.1), with the map $\mathrm{Spec}(R) \to \mathbb{A}^m$ given by the m-tuple $(t_1, ..., t_m) \in R$.

Hence, we obtain that the maps $i_{n,\infty} : Y_n \to \mathcal{Y}$ of Sect. 7.1.1 are regular closed embeddings.

7.4. **Semi-rigidity and semi-passable prestacks.**

7.4.1. In Sect. C we introduce the notion of *semi-rigid* symmetric monoidal category. We observe:

Lemma 7.4.2. *Let $\mathcal{Y}$ be a prestack such that:*
(i) *The diagonal morphism $\Delta_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is schematic*[20]*;*
(ii) *The category $\mathrm{QCoh}(\mathcal{Y})$ is dualizable.*

Then $\mathrm{QCoh}(\mathcal{Y})$ is semi-rigid.

Proof. If $\mathrm{QCoh}(\mathcal{Y})$ is dualizable, for any prestack $\mathcal{Z}$, the functor

$$\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Z}) \to \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Z})$$

is an equivalence, [GR1, Chapter 3, Proposition 3.1.7].

In particular, the functor

$$\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y})$$

is an equivalence. Hence, we can identify the functor

$$\mathrm{mult}_{\mathrm{QCoh}(\mathcal{Y})} : \mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y})$$

with $\Delta^*_{\mathcal{Y}}$.

Hence, from the fact that the diagonal morphism of $\mathcal{Y}$ is schematic, we obtain that the functor

$$\mathrm{mult}_{\mathrm{QCoh}(\mathcal{Y})} : \mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}),$$

admits a continuous right adjoint, namely, $(\Delta_{\mathcal{Y}})_*$, see [GR1, Chapter 3, Proposition 2.2.2]. Moreover, by [GR1, Chapter 3, Lemma 3.2.4] the functor $(\Delta_{\mathcal{Y}})_*$ satisfies the projection formula; hence, the structure of right-lax compatibility on $(\mathrm{mult}_{\mathrm{QCoh}(\mathcal{Y})})^R$ with the $\mathrm{QCoh}(\mathcal{Y})$-bimodule structure is strict.

□

7.4.3. Let us say call a prestack $\mathcal{Y}$ *semi-passable* if it satisfies the assumptions of Lemma 7.4.2 (cf. [GR1, Chapter 3, Sect. 3.5.1] for the choice of the terminology).

We obtain:

Corollary 7.4.4. *Let $\mathcal{Y}$ be a formal affine scheme. Then $\mathcal{Y}$ is semi-passable.*

7.4.5. Semi-rigid categories enjoy some very favorable 2-categorical properties. For example, a module category over a semi-rigid category is dualizable if and only if it is dualizable as a plain DG category, see Lemma C.2.12.

7.5. **Duality for semi-passable prestacks.**

7.5.1. Recall that if Y is an affine scheme, the functors

$$\mathrm{QCoh}(Y) \otimes \mathrm{QCoh}(Y) \overset{\otimes}{\to} \mathrm{QCoh}(Y) \overset{\Gamma(Y,-)}{\to} \mathrm{Vect}_{\mathsf{e}} \tag{7.15}$$

and

$$\mathrm{Vect}_{\mathsf{e}} \overset{\mathsf{e} \mapsto \mathcal{O}_Y}{\longrightarrow} \mathrm{QCoh}(Y) \overset{(\Delta_*)_Y}{\longrightarrow} \mathrm{QCoh}(Y \times Y) \simeq \mathrm{QCoh}(Y) \otimes \mathrm{QCoh}(Y) \tag{7.16}$$

define an identification

$$\mathrm{QCoh}(Y) \simeq \mathrm{QCoh}(Y)^\vee.$$

7.5.2. Assume now that $\mathcal{Y}$ is a semi-passable prestack. In this case, $\Gamma(\mathcal{Y}, -)$ may be discontinuous (this happens when $\mathcal{Y}$ is a formal affine scheme). So, (7.15) cannot serve as a counit of a self-duality. Yet, we will see that (7.16) does form the unit of a self-duality.

[20]In this paper, all schemes are assumed quasi-separated and quasi-compact.

7.5.3. We claim:

Proposition 7.5.4. *Let $\mathcal{Y}$ be a semi-passable prestack. Then the object*

$$(\Delta_{\mathcal{Y}})_*(\mathcal{O}_{\mathcal{Y}}) \in \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}) \simeq \mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y})$$

is the unit of a duality.

Proof. This is a special case of Lemma C.2.8. □

For future needs, we observe:

Lemma 7.5.5. *Let $\mathcal{Y}$ be a semi-passable prestack. Then for an affine scheme S and a map $S \xrightarrow{f} \mathcal{Y}$, with respect to the above self-duality on $\mathrm{QCoh}(\mathcal{Y})$ and the canonical self-duality on $\mathrm{QCoh}(S)$, the functor f^* is the dual of the functor f_*.*

Proof. We need to establish an isomorphism

$$(f \times \mathrm{id}_{\mathcal{Y}})^* \circ (\Delta_{\mathcal{Y}})_*(\mathcal{O}_{\mathcal{Y}}) \simeq (\mathrm{id}_S \times f)_* \circ (\Delta_S)_*(\mathcal{O}_S). \tag{7.17}$$

Consider the Cartesian diagram

$$\begin{array}{ccc} S & \xrightarrow{f} & \mathcal{Y} \\ {\scriptstyle \mathrm{Graph}_f}\downarrow & & \downarrow{\scriptstyle \Delta_{\mathcal{Y}}} \\ S \times \mathcal{Y} & \xrightarrow{f \times \mathrm{id}_{\mathcal{Y}}} & \mathcal{Y} \times \mathcal{Y}. \end{array}$$

Since the vertical arrows are schematic, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{QCoh}(S) & \xleftarrow{f^*} & \mathrm{QCoh}(\mathcal{Y}) \\ {\scriptstyle (\mathrm{Graph}_f)_*}\downarrow & & \downarrow{\scriptstyle (\Delta_{\mathcal{Y}})_*} \\ \mathrm{QCoh}(S \times \mathcal{Y}) & \xleftarrow{(f \times \mathrm{id}_{\mathcal{Y}})^*} & \mathrm{QCoh}(\mathcal{Y} \times \mathcal{Y}). \end{array} \tag{7.18}$$

Evaluating the two circuits of (7.18) on $\mathcal{O}_{\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{Y})$, we obtain the desired isomorphism in (7.17). □

7.6. **The functor of !-global sections.** In this subsection, we will let $\mathcal{Y}$ be a semi-passable prestack.

As was mentioned above, for a formal affine scheme, the functor of global sections

$$\Gamma(\mathcal{Y}, -) = \mathcal{H}om_{\mathrm{QCoh}(\mathcal{Y})}(\mathcal{O}_{\mathcal{Y}}, -), \quad \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}}$$

is discontinuous.

In this subsection we will introduce its substitute, denoted $\Gamma_!(\mathcal{Y}, -)$.

7.6.1. Let $\Gamma_!(\mathcal{Y}, -)$ denote the functor

$$\mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}},$$

dual to the functor

$$\mathrm{Vect}_{\mathsf{e}} \xrightarrow{\mathcal{O}_{\mathcal{Y}}} \mathrm{QCoh}(\mathcal{Y})$$

with respect to the self-duality

$$\mathrm{QCoh}(\mathcal{Y})^{\vee} \simeq \mathrm{QCoh}(\mathcal{Y}) \tag{7.19}$$

of Proposition 7.5.4.

Remark 7.6.2. According to Remark C.3.9, the functor $\Gamma_!(\mathcal{Y}, -)$ can be characterized as follows: it is the unique continuous functor

$$\mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}}$$

that restricts to $\Gamma(\mathcal{Y}, -)$ on the subcategory of compact objects.

7.6.3. We claim:

Proposition 7.6.4. *The counit for the self-duality* (7.19) *is given by*

$$\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}) \overset{\otimes}{\to} \mathrm{QCoh}(\mathcal{Y}) \overset{\Gamma_!(\mathcal{Y},-)}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}}\,. \tag{7.20}$$

Proof. This is a special case of Lemma C.2.10. □

7.6.5. Here is one more property of the functor $\Gamma_!(\mathcal{Y},-)$:

Proposition 7.6.6. *For an affine scheme S and a map $S \overset{f}{\to} \mathcal{Y}$, there is a canonical isomorphism*

$$\Gamma_!(\mathcal{Y},-) \circ f_* \simeq \Gamma(S,-) : \mathrm{QCoh}(S) \to \mathrm{Vect}_{\mathsf{e}}\,.$$

Proof. By Lemma 7.5.5, the functors dual to both sides identify with

$$\mathrm{Vect}_{\mathsf{e}} \overset{\mathcal{O}_S}{\to} \mathrm{QCoh}(S).$$

□

7.7. **The functor of !-global sections on a formal affine scheme.** In this subsection we specialize again to the case when $\mathcal{Y}$ is a formal affine scheme.

7.7.1. Note that Proposition 7.6.6 allows us to describe the functor $\Gamma_!(\mathcal{Y},-)$ as follows: in terms of the presentation (7.12), it corresponds to the compatible collection of functors

$$\Gamma(S,-) : \mathrm{QCoh}(S) \to \mathrm{Vect}_{\mathsf{e}}\,.$$

This functor should *not* be confused with the *discontinuous* functor

$$\Gamma(\mathcal{Y},-) : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}},$$

corepresented by $\mathcal{O}_{\mathcal{Y}}$.

7.7.2. For a choice of the presentation of $\mathcal{Y}$ as in (7.1), in terms of the identification

$$\mathrm{QCoh}(\mathcal{Y}) \simeq \mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}},$$

the functor $\Gamma_!(\mathcal{Y},-)$ corresponds to the composition

$$\Gamma(\mathrm{Spec}(R),-) \circ (i_\infty)_!.$$

In other words,

$$\Gamma_!(\mathcal{Y},-) \simeq \Gamma(\mathrm{Spec}(R),-) \circ ((i_\infty)^*)^L. \tag{7.21}$$

7.7.3. Here is an explicit expression for the functor $\Gamma_!(\mathcal{Y},-)$ in terms of Sect. 7.1.9:

For $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$, we have

$$\Gamma_!(\mathcal{Y},\mathcal{F}) \simeq \underset{n}{\mathrm{colim}}\, \Gamma(Y_n, i^!_{n,\infty}(\mathcal{F})), \tag{7.22}$$

where we also note that

$$i^!_{n,\infty}(\mathcal{F}) \simeq i^*_{n,\infty}(\mathcal{F}) \otimes i^!_{n,\infty}(\mathcal{O}_{\mathcal{Y}}),$$

by (7.13).

7.7.4. Finally, we claim:

Proposition 7.7.5. *The category $\mathrm{QCoh}(\mathcal{Y})$ carries a t-structure, uniquely characterized by the requirement that the functor $\Gamma_!(\mathcal{Y},-)$ is t-exact. Furthermore, $\mathrm{QCoh}(\mathcal{Y})$ is left-complete in this t-structure.*

Proof. Choose a presentation as in (7.1). Then the assertion of the proposition follows from Proposition 7.1.5:

The corresponding t-structure on $\mathrm{QCoh}(\mathrm{Spec}(R))_{\mathcal{Y}}$ is the unique one for which the functor $(i_\infty)_!$ is t-exact. □

7.8. **Applications of 1-affineness.**

7.8.1. Recall what it means for a prestack to be 1-affine, see [Ga2, Definition 1.3.7].

From [Ga2, Theorem 2.3.1], we obtain:

Proposition 7.8.2. *A formal affine scheme is 1-affine.*

7.8.3. We also have (see [Ga2, Theorems 1.5.7 and 2.2.2]):

Theorem 7.8.4. *Let* G *be an algebraic group. Then the (pre)stack* pt/G *is 1-affine.*

7.8.5. Here is the concrete meaning of Theorem 7.8.4. It says that the operations

$$\mathbf{C} \mapsto \mathbf{C} \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathrm{Vect}_{\mathsf{e}}, \quad \mathrm{Rep}(\mathsf{G})\text{-mod} \to \mathrm{QCoh}(\mathsf{G})\text{-mod}$$

and

$$\mathbf{C}' \mapsto (\mathbf{C}')^{\mathsf{G}} := \mathrm{Funct}_{\mathrm{QCoh}(\mathsf{G})\text{-mod}}(\mathrm{Vect}_{\mathsf{e}}, \mathbf{C}'), \quad \mathrm{QCoh}(\mathsf{G})\text{-mod} \to \mathrm{Rep}(\mathsf{G})\text{-mod}$$

define mutually inverse equivalences of categories.

In the above formulas, we regard $\mathrm{QCoh}(\mathsf{G})$ as a monoidal DG category with respect to the operation *convolution*, i.e., by means of taking pushfoward along the group law $\mathsf{G} \times \mathsf{G} \to \mathsf{G}$.

7.8.6. In particular, an object $\mathbf{C} \in \mathrm{Rep}(\mathsf{G})$-mod is dualizable (this is equivalent to being dualizable as a plain DG category, since $\mathrm{Rep}(\mathsf{G})$ is rigid, see [GR1, Chapter 1, Proposition 9.4.4]) if and only if

$$\mathbf{C}' := \mathbf{C} \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathrm{Vect}_{\mathsf{e}}$$

is dualizable as an object of $\mathrm{QCoh}(\mathsf{G})$-mod (this is equivalent to being dualizable as a plain DG category, see [Ga2, Proposition 1.4.5]).

As another consequence of Theorem 7.8.4, we obtain that the functor

$$\mathbf{C} \mapsto \mathbf{C} \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathrm{Vect}_{\mathsf{e}}, \quad \mathrm{Rep}(\mathsf{G})\text{-mod} \to \mathrm{DGCat} \tag{7.23}$$

is conservative.

7.8.7. Let $\mathcal{Y}'$ be a prestack acted on by G, and set $\mathcal{Y} = \mathcal{Y}'/\mathsf{G}$. We can regard $\mathrm{QCoh}(\mathcal{Y}')$ as a $\mathrm{QCoh}(\mathsf{G})$-module category and $\mathrm{QCoh}(\mathcal{Y})$ as a $\mathrm{Rep}(\mathsf{G})$-module category so that we have

$$\mathrm{QCoh}(\mathcal{Y}) \simeq \mathrm{QCoh}(\mathcal{Y}')^{\mathsf{G}}$$

and

$$\mathrm{QCoh}(\mathcal{Y}') \simeq \mathrm{Vect}_{\mathsf{e}} \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathrm{QCoh}(\mathcal{Y}).$$

From Theorem 7.8.4, we obtain:

Corollary 7.8.8. *For* $\mathcal{Y}'$ *and* $\mathcal{Y}$ *as above, we have:*

(a) *If* $\mathrm{QCoh}(\mathcal{Y}')$ *is dualizable, then so is* $\mathrm{QCoh}(\mathcal{Y})$*.*

(b) *If* $\mathcal{Y}'$ *is semi-passable, then so is* $\mathcal{Y}$*.*

(c) *If* $\mathcal{Y}'$ *is 1-affine, then so is* $\mathcal{Y}$*.*

As a particular case, we obtain:

Corollary 7.8.9. *If* $\mathcal{Y}$ *is a prestack of the form* $\mathcal{Y}'/\mathsf{G}$*, where* $\mathcal{Y}'$ *is a formal affine scheme, then:*

(a) $\mathcal{Y}$ *is semi-passable;*

(b) $\mathcal{Y}$ *is 1-affine.*

7.8.10. Here is an application of 1-affinenness that we will need:

Lemma 7.8.11. *Let $\mathcal{Y}$ be a 1-affine prestack, and let*

$$\mathcal{Z} \to \mathcal{Y} \leftarrow \mathcal{Z}'$$

be a diagram of prestacks. Assume that $\mathrm{QCoh}(\mathcal{Z}')$ *is dualizable as a* $\mathrm{QCoh}(\mathcal{Y})$*-module. Then the functor*

$$\mathrm{QCoh}(\mathcal{Z}) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathrm{QCoh}(\mathcal{Z}') \to \mathrm{QCoh}(\mathcal{Z} \underset{\mathcal{Y}}{\times} \mathcal{Z}')$$

is an equivalence.

Proof. Write

$$\mathrm{QCoh}(\mathcal{Z}) \simeq \lim_{f:S \to \mathcal{Z}} \mathrm{QCoh}(S),$$

where S are affine schemes. Since $\mathrm{QCoh}(\mathcal{Z}')$ was assumed dualizable as a $\mathrm{QCoh}(\mathcal{Y})$-module, the functor

$$\mathrm{QCoh}(\mathcal{Z}) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathrm{QCoh}(\mathcal{Z}') \to \lim_{f:S \to \mathcal{Z}} \left(\mathrm{QCoh}(S) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathrm{QCoh}(\mathcal{Z}') \right)$$

is an equivalence.

The functor

$$\mathrm{QCoh}(\mathcal{Z} \underset{\mathcal{Y}}{\times} \mathcal{Z}') \to \lim_{f:S \to \mathcal{Z}} \mathrm{QCoh}(S \underset{\mathcal{Y}}{\times} \mathcal{Z}')$$

is tautologically an equivalence.

This reduces the assertion of the lemma to the case when $\mathcal{Z} = S$ is an affine scheme. In this case, it follows from [Ga2, Proposition 3.1.9].

□

Corollary 7.8.12. *Let $\mathcal{Y}$ be of the form $\mathcal{Y}'/\mathsf{G}$, where $\mathcal{Y}'$ is a formal affine scheme. Then for a diagram of prestacks*

$$\mathcal{Z} \to \mathcal{Y} \leftarrow \mathcal{Z}',$$

if either $\mathcal{Z}$ or $\mathcal{Z}'$ is dualizable as a plain DG category, then the functor

$$\mathrm{QCoh}(\mathcal{Z}) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathrm{QCoh}(\mathcal{Z}') \to \mathrm{QCoh}(\mathcal{Z} \underset{\mathcal{Y}}{\times} \mathcal{Z}') \tag{7.24}$$

is an equivalence.

Proof. Follows by combining Lemma 7.8.11, Corollary 7.8.9 and Lemma C.2.12.

□

7.9. Compact generation of $\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))$.

7.9.1. Recall that the prestack $\mathcal{Y} := \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))$ can be written as $\mathcal{Y}'/\mathsf{G}$, where $\mathcal{Y}'$ is a disjoint union of formal affine schemes, equipped with an action of an algebraic group G.

The results of the preceding subsections apply to prestacks of this form as well. In particular, such $\mathcal{Y}$ is semi-passable, 1-affine, and an analog of Corollary 7.8.12 holds.

However, there is one property of $\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))$ that does not follow from the preceding results, namely, that $\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))$ is compactly generated. The goal of this subsection is to establish this.

Remark 7.9.2. Let $\mathcal{Y}'$ be a formal affine scheme acted on by G, and set $\mathcal{Y} \simeq \mathcal{Y}'/\mathsf{G}$.

Recall that according to Remark 7.1.2 we have a canonical choice for an affine scheme $\mathrm{Spec}(R)$ such that $\mathcal{Y}'$ can be obtained as its formal completion. By canonicity, e-points of G act on R. However, we are not guaranteed to have an action on R of G as an algebraic group; this is because the construction of R involves the procedure of passing to the inverse limit.

Hence, it is not clear that we can find a G-equivariant model for a presentation of $\mathcal{Y}'$ as in (7.1).

If we had such a presentation, we could give an easy proof of the fact that $\mathrm{QCoh}(\mathcal{Y})$ is compactly generated.

In the case when $\mathcal{Y}$ is a connected component of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$, we will take a different route, namely, one supplied by Theorem 5.4.2.

7.9.3. Let $\mathcal{Z}$ be a connected component of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$. Our current goal is to prove the following:

Theorem 7.9.4. *The category* $\mathrm{QCoh}(\mathcal{Z})$ *is compactly generated.*

Remark 7.9.5. We will actually prove a slightly more precise version of Theorem 7.9.4, see Theorem 7.9.11, in which we will explicitly describe compact generators of $\mathrm{QCoh}(\mathcal{Z})$.

7.9.6. Consider the coarse moduli space $\mathcal{Z}^{\mathrm{coarse}} =: \mathcal{S}$ corresponding to $\mathcal{Z}$ and the map

$$\mathbf{r} : \mathcal{Z} \to \mathcal{S},$$

see Sect. 5.3.6.

Recall that according to Theorem 5.4.2(b), the ind-scheme $\mathcal{S}$ is actually a formal affine scheme. Write

$$\mathcal{S} \simeq \underset{n}{\mathrm{colim}}\, \mathrm{Spec}(R_n)$$

as in (7.1). Denote by $i_{n,\infty}$ the corresponding maps

$$\mathrm{Spec}(R_n) =: S_n \to \mathcal{S}.$$

By Sects. 7.3.5-7.3.7, for every n, we have an adjunction

$$(i_{n,\infty})_* : \mathrm{QCoh}(S_n) \rightleftarrows \mathrm{QCoh}(\mathcal{S}) : (i_{n,\infty})^! \tag{7.25}$$

as $\mathrm{QCoh}(\mathcal{S})$-module categories. Moreover, by Corollary 7.1.10(a), the map

$$\underset{n}{\mathrm{colim}}\, (i_{n,\infty})_* \circ (i_{n,\infty})^! \to \mathrm{Id}_{\mathrm{QCoh}(\mathcal{S})} \tag{7.26}$$

is an isomorphism.

7.9.7. Set

$$Z_n := S_n \underset{\mathcal{S}}{\times} \mathcal{Z},$$

and let $\widetilde{i}_{n,\infty}$ denote the resulting maps

$$Z_n \to \mathcal{Z}.$$

By Corollary 7.8.12, we have

$$\mathrm{QCoh}(Z_n) \simeq \mathrm{QCoh}(S_n) \underset{\mathrm{QCoh}(\mathcal{S})}{\otimes} \mathrm{QCoh}(\mathcal{Z}).$$

Hence, from (7.25) we obtain an adjunction

$$(\widetilde{i}_{n,\infty})_* : \mathrm{QCoh}(Z_n) \rightleftarrows \mathrm{QCoh}(\mathcal{Z}) : (\widetilde{i}_{n,\infty})^! \tag{7.27}$$

as $\mathrm{QCoh}(\mathcal{Z})$-module categories.

In particular, the functors $(\widetilde{i}_{n,\infty})_*$ preserve compactness. Moreover, from (7.26) we obtain that the map

$$\underset{n}{\mathrm{colim}}\, (\widetilde{i}_{n,\infty})_* \circ (\widetilde{i}_{n,\infty})^! \to \mathrm{Id}_{\mathrm{QCoh}(\mathcal{Z})} \tag{7.28}$$

is an isomorphism.

7.9.8. Let

$$\mathcal{Z}^{\mathrm{rigid}} := \mathcal{Z} \underset{\mathrm{pt}/\mathsf{G}}{\times} \mathrm{pt}$$

be the preimage of $\mathcal{Z}$ in $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$.

Set

$$Z_n^{\mathrm{rigid}} := Z_n \underset{\mathcal{Z}}{\times} \mathcal{Z}^{\mathrm{rigid}} \simeq S_n \underset{\mathcal{S}}{\times} \mathcal{Z}^{\mathrm{rigid}},$$

so that

$$Z_n \simeq Z_n^{\mathrm{rigid}}/\mathsf{G}.$$

Note now that by Theorem 5.4.2, Z_n^{rigid} is an affine scheme. Hence, Z_n is an algebraic stack.

7.9.9. *Proof of Theorem 7.9.4.* Since the functors $(\widetilde{i}_{n,\infty})_*$ preserve compactness, and by (7.28), it suffices to show that each of the categories $\mathrm{QCoh}(Z_n)$ is compactly generated.

However, for any algebraic stack Z equal to the quotient of an affine scheme Z^{rigid} by an action of the algebraic group G, the category $\mathrm{QCoh}(Z)$ is compactly generated by objects of the form

$$\mathcal{O}_Z \otimes p^*(V), \quad V \in \mathrm{Rep}(\mathsf{G})^c,$$

where p denote the map

$$Z \to \mathrm{pt}/\mathsf{G},$$

corresponding to the G-torsor $Z^{\mathrm{rigid}} \to Z$.

□[Theorem 7.9.4]

7.9.10. We will now give a slightly more precise form of the generation assertion. Let p denote the map

$$\mathcal{Z} \to \mathrm{pt}/\mathsf{G},$$

corresponding to the G-torsor $\mathcal{Z}^{\mathrm{rigid}} \to \mathcal{Z}$.

Theorem 7.9.11. *The category* $\mathrm{QCoh}(\mathcal{Z})$ *is compactly generated by a family of objects of the form* $\mathcal{F} \otimes p^*(V)$*, where:*

- $V \in \mathrm{Rep}(\mathsf{G})^c$*;*
- $\mathcal{F}$ *can be expressed as a finite colimit in terms of* $\mathcal{O}_{\mathcal{Z}}$*.*

Remark 7.9.12. We emphasize that the object $\mathcal{O}_{\mathcal{Z}} \in \mathrm{QCoh}(\mathcal{Z})$ itself is *not* compact.

Proof. The proof of Theorem 7.9.4 in Sect. 7.9.9 shows that in order to prove Theorem 7.9.11, we only need to prove that

$$(\widetilde{i}_{n,\infty})_*(\mathcal{O}_{Z_n}) \in \mathrm{QCoh}(\mathcal{Z})$$

can be expressed as a finite colimit in terms of $\mathcal{O}_{\mathcal{Z}}$.

For that, it is sufficient to show that each

$$(i_{n,\infty})_*(\mathcal{O}_{S_n}) \in \mathrm{QCoh}(\mathcal{S})$$

can be expressed as a finite colimit in terms of $\mathcal{O}_{\mathcal{S}}$.

However, this follows from the expression for the ring R_n as

$$R_n \simeq R \underset{\mathsf{e}[t_1,\ldots,t_m]}{\otimes} \mathsf{e}.$$

□

7.10. **Enhanced categorical trace.** In this subsection, we will prove an assertion that will be used in Sect. 23.3.

7.10.1. Recall the set-up of [GKRV, Sects. 3.6-3.8]. We start with a symmetric monoidal category $\mathbf{A}$ (assumed dualizable as a DG category), equipped with a symmetric monoidal endofunctor $F_{\mathbf{A}}$. Let $\mathbf{M}$ be an $\mathbf{A}$-module category (assumed dualizable as such), equipped with an endofunctor $F_{\mathbf{M}}$, compatible with $F_{\mathbf{A}}$.

Consider the category

$$\mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A}),$$

i.e., the category of Hochschild chains on $\mathbf{A}$ twisted by $F_{\mathbf{A}}$, see [GKRV, Sect. 3.7.2]. The fact that the monoidal structure on $(\mathbf{A}, F_{\mathbf{A}})$ is symmetric allows us to define a symmetric monoidal structure on $\mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A})$.

Further, to $(\mathbf{M}, F_{\mathbf{M}})$ we can attach an object

$$\mathrm{Tr}^{\mathrm{enh}}_{\mathbf{A}}(F_{\mathbf{M}}, \mathbf{M}) \in \mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A}),$$

see [GKRV, Sect. 3.8.2].

7.10.2. Under the assumption that $\mathbf{A}$ is rigid, we have the following assertion ([GKRV, Theorem 3.8.5]):

There exists a canonical isomorphism in $\mathrm{Vect}_{\mathsf{e}}$:

$$\mathrm{Tr}(F_{\mathbf{M}}, \mathbf{M}) \simeq \mathcal{H}om_{\mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A})}\left(\mathbf{1}_{\mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A})}, \mathrm{Tr}^{\mathrm{enh}}_{\mathbf{A}}(F_{\mathbf{M}}, \mathbf{M})\right), \tag{7.29}$$

where $\mathbf{1}_{\mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A})}$ is the monoidal unit in $\mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A})$.

For example, if $\mathbf{A} = \mathrm{QCoh}(\mathcal{Y})$, where $\mathcal{Y}$ is an algebraic stack, and $F_{\mathbf{A}}$ is given by ϕ^*, where ϕ is an endomorphism of $\mathcal{Y}$, we have

$$\mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A}) \simeq \mathrm{QCoh}(\mathcal{Y}^\phi),$$

where

$$\mathcal{Y}^\phi := \mathcal{Y} \underset{\Delta_{\mathcal{Y}}, \mathcal{Y}\times\mathcal{Y}, (\mathrm{id}\times\phi)\circ\Delta_{\mathcal{Y}}}{\times} \mathcal{Y},$$

and the right-hand side in (7.29) is

$$\Gamma(\mathcal{Y}^\phi, \mathrm{Tr}^{\mathrm{enh}}_{\mathbf{A}}(F_{\mathbf{M}}, \mathbf{M})). \tag{7.30}$$

7.10.3. Our current goal is to generalize (7.29) when instead of requiring that $\mathbf{A}$ be rigid, we only require that $\mathbf{A}$ be semi-rigid.

The appropriate generalization is stated in Theorem C.6.5, and proved in Sect. C.7.

Here we will formulate its particular case, pertaining to the geometric situation, when $\mathbf{A} := \mathrm{QCoh}(\mathcal{Y})$, for $\mathcal{Y} = \mathcal{Y}'/\mathsf{G}$ where $\mathcal{Y}'$ and G are as in Sect. 7.9.1.

7.10.4. By Corollary 7.8.9(a), $\mathbf{A}$ is semi-rigid. In particular, by Lemma C.2.12, in this case, an $\mathbf{A}$-module category $\mathbf{M}$ is dualizable if and only if the underlying DG category is dualizable.

Assume that $\mathcal{Y}'$ is equipped with an endomorphism that commutes with the G-action, so that ϕ induces an endomorphism of $\mathcal{Y}$. We will denote both these endomorphisms by ϕ.

Note that

$$\mathcal{Y}^\phi \simeq \left((\mathsf{G}\times\mathcal{Y}') \underset{\mathrm{act}, \mathcal{Y}'\times\mathcal{Y}', (\mathrm{id}\times\phi)\circ\Delta'_{\mathcal{Y}}}{\times} \mathcal{Y}'\right)/\mathsf{G},$$

where $(\mathsf{G}\times\mathcal{Y}') \underset{\mathrm{act}, \mathcal{Y}'\times\mathcal{Y}', \Delta'_{\mathcal{Y}}}{\times} \mathcal{Y}'$ is also a formal affine scheme.

Let $F_{\mathbf{A}} := \phi^*$. By definition

$$\mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A}) \simeq \mathrm{QCoh}(\mathcal{Y}) \underset{\mathrm{mult}_{\mathrm{QCoh}(\mathcal{Y})}, \mathrm{QCoh}(\mathcal{Y})\otimes\mathrm{QCoh}(\mathcal{Y}), (\mathrm{id}\otimes\phi^*)\circ\mathrm{mult}_{\mathrm{QCoh}(\mathcal{Y})}}{\otimes} \mathrm{QCoh}(\mathcal{Y}).$$

By Corollary 7.8.12, the latter category maps isomorphically to

$$\mathrm{QCoh}(\mathcal{Y} \underset{\Delta_{\mathcal{Y}}, \mathcal{Y}\times\mathcal{Y}, (\mathrm{id}\times\phi)\circ\Delta_{\mathcal{Y}}}{\times} \mathcal{Y}) = \mathrm{QCoh}(\mathcal{Y}^\phi).$$

Hence, in the setting of Sect. 7.10.1, we can think of $\mathrm{Tr}^{\mathrm{enh}}_{\mathbf{A}}(F_{\mathbf{M}}, \mathbf{M})$ as an object of $\mathrm{QCoh}(\mathcal{Y}^\phi)$.

7.10.5. We claim:

Theorem 7.10.6. *In the setting of Sect. 7.10.4, there is a canonical isomorphism*

$$\mathrm{Tr}(F_{\mathbf{M}}, \mathbf{M}) \simeq \Gamma_! \left(\mathcal{Y}^\phi, \mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathcal{Y})}(F_{\mathbf{M}}, \mathbf{M})\right). \tag{7.31}$$

The proof is a generalization of the argument in [GKRV, Theorem 3.10.6], and will be given in Sect. C (see Sect. C.6.9).

Remark 7.10.7. Note the difference between the assertion of Theorem 7.10.6 and a similar assertion when $\mathcal{Y}$ is an algebraic stack: in the latter case, instead of the right-hand side in (7.31) we have (7.30).

By contrast, in the case of formal schemes/stacks, the (discontinuous) functor $\Gamma(\mathcal{Y}^\phi, -)$ gets replaced by the functor $\Gamma_!(\mathcal{Y}^\phi, -)$.

Remark 7.10.8. Let us take $(\mathbf{M}, F_{\mathbf{M}})$ to be $(\mathbf{A}, F_{\mathbf{A}})$ itself. Then $\mathrm{Tr}^{\mathrm{enh}}_{\mathbf{A}}(F_{\mathbf{A}}, \mathbf{A}) = \mathbf{1}_{\mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A})}$. So, in the setting of Theorem 7.10.6, $\mathrm{Tr}^{\mathrm{enh}}_{\mathbf{A}}(F_{\mathbf{A}}, \mathbf{A}) \simeq \mathcal{O}_{\mathcal{Y}}$.

Note that for any semi-rigid symmetric monoidal category $\mathbf{A}$, the functor $\Gamma_{!,\mathbf{A}}$ (see Sect. C.2.9) is *non-unital* right-lax symmetric monoidal. In particular, for any semi-passable prestack $\widetilde{\mathcal{Y}}$, the functor $\Gamma_!(\widetilde{\mathcal{Y}}, -)$ is *non-unital* right-lax symmetric monoidal. So, $\Gamma_!(\widetilde{\mathcal{Y}}, \mathcal{O}_{\widetilde{\mathcal{Y}}})$ acquires a structure of (not necessarily unital) commutative algebra, and for any $\mathcal{F} \in \mathrm{QCoh}(\widetilde{\mathcal{Y}})$, the object $\Gamma_!(\widetilde{\mathcal{Y}}, \mathcal{F})$ is naturally a module for $\Gamma_!(\widetilde{\mathcal{Y}}, \mathcal{O}_{\widetilde{\mathcal{Y}}})$.

Applying this to $\widetilde{\mathcal{Y}} = \mathcal{Y}^\phi$, we obtain that, on the one hand, $\Gamma_!\left(\mathcal{Y}^\phi, \mathrm{Tr}^{\mathrm{enh}}_{\mathbf{A}}(F_{\mathbf{A}}, \mathbf{A})\right)$ acquires a structure of (not necessarily unital) commutative algebra, and $\Gamma_!\left(\mathcal{Y}^\phi, \mathrm{Tr}^{\mathrm{enh}}_{\mathbf{A}}(F_{\mathbf{M}}, \mathbf{M})\right)$ acquires a structure of module over this commutative algebra.

On the other hand, [GKRV, Sects. 3.3.2 and 3.3.3] implies that $\mathrm{Tr}(F_{\mathbf{A}}, \mathbf{A})$ is naturally also a (not necessarily unital) commutative algebra and $\mathrm{Tr}(F_{\mathbf{M}}, \mathbf{M})$ is a module over $\mathrm{Tr}(F_{\mathbf{A}}, \mathbf{A})$.

As in [GKRV, Theorem 3.8.5], the statement of Theorem 7.10.6 should be complemented as follows:

- The isomorphism
$$\mathrm{Tr}(F_{\mathbf{A}}, \mathbf{A}) \simeq \Gamma_!(\mathcal{Y}^\phi, \mathcal{O}_{\mathcal{Y}^\phi})$$
of (7.31) is compatible with commutative algebra structures on both sides.
- The isomorphism
$$\mathrm{Tr}(F_{\mathbf{M}}, \mathbf{M}) \simeq \Gamma_! \left(\mathcal{Y}^\phi, \mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathcal{Y})}(F_{\mathbf{M}}, \mathbf{M})\right)$$
respects the module structures for these algebras.

The proof of these compatibility assertions follows formally from Theorem 7.10.6 as in [GKRV, Sects. 3.12.7-3.12.8].

Part II: Lisse actions and the spectral decomposition over $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$

Let us make a brief overview of the contents of this Part.

In Sect. 8 we describe the set-up for the following question: what does it take to have an action of the monoidal category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$ on a DG category $\mathbf{C}$? It turns that the appropriate input datum is what one can call *an action of* $\mathrm{Rep}(\mathsf{G})^{\otimes X\text{-lisse}}$ *on* $\mathbf{C}$. In Theorem 8.1.4 we state that these two pieces of data are indeed in bijection. We introduce an abstract framework for this result, where instead of $\mathrm{QLisse}(X)$ we are dealing with a general symmetric monoidal category $\mathbf{H}$, equipped with a t-structure. The object of study becomes the functor between symmetric monoidal categories

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \to \mathrm{QCoh}(\mathrm{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))$$

(see (8.8)). We call a symmetric monoidal category *adapted for spectral decomposition* if the above functor is an equivalence. We state Conjecture 8.3.6 to the effect that any *gentle Tannakian category* (see Sect. 1.7) is adapted for spectral decomposition. A reformulation of Theorem 8.1.4, stated as Theorem 8.3.7, says that this conjecture holds for $\mathbf{H} := \mathrm{QLisse}(X)$.

In Sect. 9 we prove Theorem 8.3.7. We first show that the category $\mathbf{H} := \mathrm{Shv}_{\mathrm{loc.const.}}(X)$, where X is a connected CW complex is adapted for spectral decomposition. Next, we show that if $\mathbf{H}$ is a adapted for spectral decomposition, and $\mathbf{H}' \subset \mathbf{H}$ is a full subcategory, then under certain conditions, $\mathbf{H}'$ is also adapted for spectral decomposition. We then use a series of reductions showing that the category $\mathrm{QLisse}(X_1)$ (where X_2 is a smooth and complete curve over a ground field k of any characteristic) can be realized as a full subcategory in one of the form $\mathrm{Shv}_{\mathrm{loc.const.}}(X_2)$, where X_2 is a curve over $\mathbb{C}$, thereby deducing Theorem 8.3.7 from the Betti case.

Sect. 10 is not needed for the rest of the paper. Here we consider another class of symmetric monoidal categories adapted for spectral decomposition, namely, categories of the form $\mathfrak{h}$-mod, where $\mathfrak{h}$ is a connective Lie algebra. We prove it by a method that we hope can be useful for the proof of Conjecture 8.3.6.

In Sects. 11 and 12 we introduce a tool that will be extensively used in Part III of the paper– Beilinson's spectral projector.

8. The spectral decomposition theorem

In this section we the state the main theorem of Part II, Theorem 8.1.4 that describes what it takes to have an action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$ on a DG category $\mathbf{C}$.

We will introduce an abstract framework in which Theorem 8.1.4 will be proved, and discuss several reformulations.

8.1. Actions of $\mathrm{Rep}(\mathsf{G})^{\otimes X\text{-lisse}}$.

Let X be a smooth, connected and complete curve.

In this subsection we define what it means to have an action $\mathrm{Rep}(\mathsf{G})^{\otimes X\text{-lisse}}$ on $\mathbf{C}$, and state the main theorem of this part, Theorem 8.1.4, which says that the datum of such an action on a category $\mathbf{C}$ is equivalent to the datum of an action on $\mathbf{C}$ of the category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$.

8.1.1. Let $\mathbf{C}$ be a DG category. We define the notion of *action of* $\mathrm{Rep}(\mathsf{G})^{\otimes X\text{-lisse}}$ *on* $\mathbf{C}$ by imitating [GKRV, Sects. C.1.2 and C.2.2]. Namely, this is a natural transformation between the following two functors $\mathrm{fSet} \to \mathrm{DGCat}^{\mathrm{Mon}}$:

From the functor

$$I \mapsto \mathrm{Rep}(\mathsf{G})^{\otimes I}$$

to the functor

$$I \mapsto \mathrm{End}(\mathbf{C}) \otimes \mathrm{QLisse}(X)^{\otimes I}.$$

In other words, informally, for every finite set I we need to specify a monoidal functor

$$\mathrm{Rep}(\mathsf{G})^{\otimes I} \to \mathrm{End}(\mathbf{C}) \otimes \mathrm{QLisse}(X)^{\otimes I},$$

and for every map of finite sets $I \to J$, we need to supply a data of commutativity for

$$\begin{array}{ccc} \operatorname{Rep}(\mathsf{G})^{\otimes I} & \longrightarrow & \operatorname{End}(\mathbf{C}) \otimes \operatorname{QLisse}(X)^{\otimes I} \\ \downarrow & & \downarrow \\ \operatorname{Rep}(\mathsf{G})^{\otimes J} & \longrightarrow & \operatorname{End}(\mathbf{C}) \otimes \operatorname{QLisse}(X)^{\otimes J}, \end{array}$$

along with a homotopy-coherent system of compatibilities for compositions.

8.1.2. Consider the symmetric monoidal category $\operatorname{QCoh}(\operatorname{LocSys}^{\operatorname{restr}}_{\mathsf{G}}(X))$. We claim that there is a canonically defined natural transformation between the following two functors $\operatorname{fSet} \to \operatorname{DGCat}^{\operatorname{SymMon}}$:

From the functor

$$I \mapsto \operatorname{Rep}(\mathsf{G})^{\otimes I} \tag{8.1}$$

to the functor

$$I \mapsto \operatorname{QCoh}(\operatorname{LocSys}^{\operatorname{restr}}_{\mathsf{G}}(X)) \otimes \operatorname{QLisse}(X)^{\otimes I}. \tag{8.2}$$

Indeed, since $\operatorname{QLisse}(X)$ is dualizable (and hence, tensoring by it commutes with limits), a datum of such a natural transformation is equivalent to a compatible system of natural transformations from (8.1) to

$$I \mapsto \operatorname{QCoh}(S) \otimes \operatorname{QLisse}(X)^{\otimes I} \text{ for } S \in \operatorname{Sch}^{\operatorname{aff}}_{/\operatorname{LocSys}^{\operatorname{restr}}_{\mathsf{G}}(X)}.$$

By definition, the datum of a map $S \to \operatorname{LocSys}^{\operatorname{restr}}_{\mathsf{G}}(X)$ is a (right t-exact) symmetric monoidal functor

$$\mathsf{F} : \operatorname{Rep}(\mathsf{G}) \to \operatorname{QCoh}(S) \otimes \operatorname{QLisse}(X).$$

The required functor

$$\operatorname{Rep}(\mathsf{G})^{\otimes I} \to \operatorname{QCoh}(S) \otimes \operatorname{QLisse}(X)^{\otimes I}$$

is then the composition

$$\operatorname{Rep}(\mathsf{G})^{\otimes I} \xrightarrow{\mathsf{F}^{\otimes I}} \operatorname{QCoh}(S)^{\otimes I} \otimes \operatorname{QLisse}(X)^{\otimes I} \to \operatorname{QCoh}(S) \otimes \operatorname{QLisse}(X)^{\otimes I}, \tag{8.3}$$

where

$$\operatorname{QCoh}(S)^{\otimes I} \to \operatorname{QCoh}(S)$$

is the tensor product map.

8.1.3. From Sect. 8.1.2 we obtain that for any DG category $\mathbf{C}$, equipped with an action of $\operatorname{QCoh}(\operatorname{LocSys}^{\operatorname{restr}}_{\mathsf{G}}(X))$, we obtain an action of $\operatorname{Rep}(\mathsf{G})^{\otimes X\text{-lisse}}$ on $\mathbf{C}$. I.e., we obtain a map of spaces

$$\{\text{Actions of } \operatorname{QCoh}(\operatorname{LocSys}^{\operatorname{restr}}_{\mathsf{G}}(X)) \text{ on } \mathbf{C}\} \to \{\text{Actions of } \operatorname{Rep}(\mathsf{G})^{\otimes X\text{-lisse}} \text{ on } \mathbf{C}\}. \tag{8.4}$$

The main result of Part II of this paper is the following:

Main Theorem 8.1.4. *The map* (8.4) *is an isomorphism.*

We can regard this theorem as saying that a category $\mathbf{C}$ equipped with an action of $\operatorname{Rep}(\mathsf{G})^{\otimes X\text{-lisse}}$, admits a spectral decomposition with respect to $\operatorname{LocSys}^{\operatorname{restr}}_{\mathsf{G}}(X)$.

8.1.5. The proof of Theorem 8.1.4 will be given in Sect. 9.

In the next few subsections we will set up an abstract framework for Theorem 8.1.4.

8.2. The coHom symmetric monoidal category.

In this subsection we will make preparations for an abstract framework in which Theorem 8.1.4 can be stated.

8.2.1. Let $\mathbf{H}$ be a symmetric monoidal category. Assume that it is dualizable as a DG category. In this case, there exists a monoidal category, to be denoted $\underline{\operatorname{coHom}}(\operatorname{Rep}(\mathsf{G}), \mathbf{H})$, defined by the universal property that for a target symmetric monoidal category $\mathbf{A}$, we have

$$\operatorname{Maps}_{\operatorname{DGCat}^{\operatorname{SymMon}}}(\underline{\operatorname{coHom}}(\operatorname{Rep}(\mathsf{G}), \mathbf{H}), \mathbf{A}) \simeq \operatorname{Maps}_{\operatorname{DGCat}^{\operatorname{SymMon}}}(\operatorname{Rep}(\mathsf{G}), \mathbf{A} \otimes \mathbf{H}).$$

The construction of $\underline{\operatorname{coHom}}(\operatorname{Rep}(\mathsf{G}), \mathbf{H})$ fits into the following general paradigm.

8.2.2. Let **O** be a symmetric monoidal category. We will assume that **O** admits all colimits and that the monoidal operation commutes with colimits in each variable.

Let A (resp., C) be a unital commutative algebra (resp., cocommutative coalgebra) object in **O**. In this case one can form a unital commutative algebra object

$$\mathrm{coEnd}(A,C) \in \mathbf{O},$$

with the following universal property: for a unital commutative algebra object $A' \in \mathbf{O}$, the space of maps of (unital) commutative algebras $\mathrm{coEnd}(A,C) \to A'$ is the space of maps in **O**

$$\phi : A \otimes C \to A',$$

equipped with a datum of commutativity for the diagrams

$$\begin{array}{ccccc} A\otimes A\otimes C & \xrightarrow{\mathrm{mult}_A\otimes \mathrm{id}_C} & A\otimes C & \xrightarrow{\phi} & A' \\ {\scriptstyle \mathrm{id}_{A\otimes A}\otimes \mathrm{comult}_C}\downarrow & & & & \uparrow{\scriptstyle \mathrm{mult}_{A'}} \\ A\otimes A\otimes C\otimes C & & \xrightarrow{\phi\otimes\phi} & & A'\otimes A', \end{array}$$

and

$$\begin{array}{ccc} C & \xrightarrow{\mathrm{unit}_A} & A\otimes C \\ {\scriptstyle \mathrm{counit}_C}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathsf{e} & \xrightarrow{\mathrm{unit}_{A'}} & A', \end{array}$$

along with a homotopy-coherent system of higher compatibilities.

8.2.3. One can describe $\mathrm{coEnd}(A,C)$ as an object of **O** explicitly.

Let fSet be the category of finite sets, and let TwArr(fSet) be the corresponding twisted arrows category, see [GKRV, Sect. 1.2.2].

Consider the functor $\mathrm{TwArr(fSet)} \to O$ that at the level of objects sends

$$(I \to J) \in \mathrm{TwArr(fSet)} \to A^{\otimes I} \otimes C^{\otimes J}.$$

At the level of 1-morphisms, it sends the morphism

$$\begin{array}{ccc} I_0 & \longrightarrow & J_0 \\ \downarrow & & \uparrow \\ I_1 & \longrightarrow & J_1 \end{array}$$

in TwArr(fSet), to the corresponding map

$$A^{\otimes I_0} \otimes C^{\otimes J_0} \to A^{\otimes I_1} \otimes C^{\otimes J_1}$$

given by the maps $A^{\otimes I_0} \to A^{\otimes I_1}$ (resp., $C^{\otimes J_0} \to C^{\otimes J_1}$), given by the commutative algebra structure on A (resp., cocommutative coalgebra structure on C).

Consider the colimit

$$\underset{(I\to J)\in \mathrm{TwArr(fSet)}}{\mathrm{colim}} A^{\otimes I} \otimes C^{\otimes J}. \tag{8.5}$$

We endow (8.5) with a structure of commutative algebra via the operation of disjoint union on fSet.

Then (8.5) identifies canonically with $\mathrm{coEnd}(A,C)$. The proof of this fact will be given in Sect. B.2.13.

8.2.4. Let A be as above, and let B be another unital commutative algebra, assumed dualizable as a plain object of $\mathbf{O}$. Set $C := B^\vee$; this is a cocommutative coalgebra in $\mathbf{O}$. Set

$$\underline{\mathrm{coHom}}(A, B) := \mathrm{coEnd}(A, C).$$

By construction, $\underline{\mathrm{coHom}}(A, B)$ has the following universal property: for a target unital commutative algebra A', the space of maps of unital commutative algebras

$$\underline{\mathrm{coHom}}(A, B) \to A'$$

is the same as the space of maps of unital commutative algebras

$$A \to A' \otimes B.$$

8.2.5. Thus, the symmetric monoidal category $\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ introduced in Sect. 8.2.1 fits into the above paradigm with $\mathbf{O} := \mathrm{DGCat}$ and $A := \mathrm{Rep}(\mathsf{G})$.

8.2.6. We return to the setting of Sect. 8.2.4. As in [GKRV, Theorem 1.2.4], one shows:

Lemma 8.2.7. *Assume that B be a cocommutative coalgebra that is dualizable as an object of $\mathbf{O}$.*

(a) *For an associative/commutative algebra object $D \in \mathbf{O}$, the space of maps of associative/commutative algebras*

$$\underline{\mathrm{coHom}}(A, B) \to D$$

identifies with the space of compatible collections of maps of associative/commutative algebras

$$A^{\otimes I} \to D \otimes B^{\otimes I}, \quad I \in \mathrm{fSet}\,.$$

(b) *For a* plain *object $D \in \mathbf{O}$, the space* $\mathrm{Maps}_{\mathbf{O}}(\underline{\mathrm{coHom}}(A, B), D)$ *identifies with the space of compatible collections of maps in* $\mathbf{O}$

$$A^{\otimes I} \to D \otimes B^{\otimes I}, \quad I \in \mathrm{fSet}\,.$$

8.2.8. *Sketch of proof of Lemma 8.2.7.* Here we will sketch a proof. A full argument will be given in Sect. B.3.

Consider the colimit (8.5). We first consider it as a plain object of O. Then for $D \in O$, the space of maps from (8.5) to D is, by definition,

$$\lim_{(I\to J)\in \mathrm{TwArr(fSet)}} \mathrm{Maps}_O(A^{\otimes I}, D \otimes B^{\otimes J}).$$

However (see, e.g., [GKRV, Lemma 1.3.12]), the latter expression identifies with the space of natural transformations between the functors

$$\mathrm{fSet} \to \mathbf{O}, \quad (I \mapsto A^{\otimes I}) \Rightarrow (I \mapsto D \otimes B^{\otimes I}).$$

Suppose now that D is an associative/commutative algebra in $\mathbf{O}$. We claim that the space of maps from (8.5) to D that are upgraded to maps of algebras correspond to compatible systems of maps of algebras

$$A^{\otimes I} \to D \otimes B^{\otimes I}, \quad I \in \mathrm{fSet}\,. \tag{8.6}$$

Indeed, for a map from (8.5) to D, the data of compatibility with an associative/commutative algebra structure translates into the data of commutativity of the diagrams

$$\begin{array}{ccc} A^{\otimes I_1} \otimes ... \otimes A^{\otimes I_k} & \longrightarrow & (D \otimes B^{\otimes I_1}) \otimes ... \otimes (D \otimes B^{\otimes I_k}) \\ \sim\downarrow & & \downarrow \\ A^{\otimes (I_1 \sqcup ... \sqcup I_k)} & \longrightarrow & D \otimes B^{\otimes (I_1 \sqcup ... \sqcup I_k)} \end{array} \tag{8.7}$$

for unordered/ordered collections of finite sets $I_1, ..., I_k$.

If the maps in (8.6) are maps of algebras, the commutativity for the diagrams (8.7) arises from

$$\begin{array}{ccc} A^{\otimes I_1} \otimes ... \otimes A^{\otimes I_k} & \longrightarrow & (D \otimes B^{\otimes I_1}) \otimes ... \otimes (D \otimes B^{\otimes I_k}) \\ \downarrow & & \downarrow \\ (A^{\otimes I})^{\otimes k} & \longrightarrow & (D \otimes B^{\otimes I})^{\otimes k} \\ \downarrow & & \downarrow \\ A^{\otimes I} & \longrightarrow & D \otimes B^{\otimes I}, \end{array}$$

where $I = I_1 \sqcup ... \sqcup I_k$, and where the upper vertical maps are given by inclusions $I_j \to I$.

Vice versa, given the commutative diagrams (8.7), we construct the data of compatibility for (8.6) by

$$\begin{array}{ccc} (A^{\otimes I})^{\otimes k} & \longrightarrow & (D \otimes B^{\otimes I})^{\otimes k} \\ \sim\downarrow & & \downarrow \\ A^{\otimes I'} & \longrightarrow & D \otimes B^{\otimes I'} \\ \downarrow & & \downarrow \\ A^{\otimes I} & \longrightarrow & D \otimes B^{\otimes I}, \end{array}$$

where I' is the disjoint union of k copies of I, and the lower vertical maps are given by the natural projection $I' \to I$.

□

8.3. **Maps vs coHom.** In this subsection we study the relationship of the category $\underline{\text{coHom}}(\text{Rep}(\mathsf{G}), \mathbf{H})$ introduced above and its algebro-geometric counterpart, the prestack $\mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathbf{H})$.

8.3.1. Let $\mathbf{H}$ be a symmetric monoidal category, equipped with a t-structure and a fiber functor satisfying the assumptions of Sect. 1.7.1.

From now on we will add the assumption that $\mathbf{H}$ is dualizable as a DG category.

8.3.2. On the one hand, we can consider the symmetric monoidal category $\underline{\text{coHom}}(\text{Rep}(\mathsf{G}), \mathbf{H})$, introduced above. On the other hand, we can consider the prestack $\mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathbf{H})$, see Sect. 1.8.

We claim that we have a canonically defined symmetric monoidal functor

$$\underline{\text{coHom}}(\text{Rep}(\mathsf{G}), \mathbf{H}) \to \text{QCoh}(\mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathbf{H})). \tag{8.8}$$

Indeed, the datum of such a functor is by definition equivalent to the datum of a symmetric monoidal functor

$$\text{Rep}(\mathsf{G}) \to \text{QCoh}(\mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathbf{H})) \otimes \mathbf{H}. \tag{8.9}$$

The functor in (8.9) is obtained by passing to the limit from the tautological functors

$$\text{Rep}(\mathsf{G}) \to \text{QCoh}(S) \otimes \mathbf{H}, \quad S \in \text{Sch}^{\text{aff}}_{/\mathbf{Maps}(\text{Rep}(\mathsf{G}),\mathbf{H})},$$

using the fact that

$$\text{QCoh}(\mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathbf{H})) \otimes \mathbf{H} = \left(\lim_{S \in \text{Sch}^{\text{aff}}_{/\mathbf{Maps}(\text{Rep}(\mathsf{G}),\mathbf{H})}} \text{QCoh}(S) \right) \otimes \mathbf{H} \to$$
$$\to \lim_{S \in \text{Sch}^{\text{aff}}_{/\mathbf{Maps}(\text{Rep}(\mathsf{G}),\mathbf{H})}} (\text{QCoh}(S) \otimes \mathbf{H})$$

is an equivalence, the latter since $\mathbf{H}$ is dualizable as a DG category.

8.3.3. We shall say that $\mathbf{H}$ is *adapted for spectral decomposition* (for a given G) if the functor (8.8) is an equivalence.

Remark 8.3.4. In the course of the next two sections we will see examples of symmetric monoidal categories $\mathbf{H}$ that are adapted for spectral decomposition. These examples include $\mathbf{H} := \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}$, where $\mathcal{X}$ is a connected object of Spc (see Sect. 4.5.6), and $\mathbf{H} := \mathsf{h}\text{-mod}$, where h is a connective Lie algebra.

Note that in the above two examples, $\mathbf{H}$ is not gentle (see Sect. 1.7.2 for what this means).

One can also show it holds for $\mathbf{H} = \mathrm{Rep}(\mathsf{H})$, where H is an affine algebraic group of finite type (but we will not prove this in the present paper).

8.3.5. We propose:

Conjecture 8.3.6. *If* $\mathbf{H}$ *is a gentle Tannakian category, then it is adapted for spectral decomposition.*

We will prove:

Main Theorem 8.3.7. *Conjecture 8.3.6 holds when* $\mathbf{H} = \mathrm{QLisse}(X)$*, where* X *is a smooth and complete algebraic curve.*

We will see shortly that Theorem 8.3.7 is equivalent to Theorem 8.1.4.

8.4. **Spectral decomposition vs actions.** In this subsection we will reformulate the property of being adapted for spectral decomposition in terms of actions on a module category $\mathbf{C}$.

8.4.1. Let $\mathbf{H}$ be a (dualizable) symmetric monoidal category and let $\mathbf{C}$ be a DG category. By an $\mathbf{H}$-family of actions of $\mathrm{Rep}(\mathsf{G})$ on $\mathbf{C}$ we will mean a natural transformation between the following two functors $\mathrm{fSet} \to \mathrm{DGCat}^{\mathrm{Mon}}$:

From the functor

$$I \mapsto \mathrm{Rep}(\mathsf{G})^{\otimes I}$$

to the functor

$$I \mapsto \mathrm{End}(\mathbf{C}) \otimes \mathbf{H}^{\otimes I}.$$

Lemma 8.2.7(a) implies that this data is equivalent to that of an action of $\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$, viewed as a monoidal category, on $\mathbf{C}$.

8.4.2. Taking $\mathbf{H} = \mathrm{QLisse}(X)$, we obtain that the notion of a family of $\mathrm{QLisse}(X)$-actions on $\mathbf{C}$ just defined is the same as a the notion of action of $\mathrm{Rep}(\mathsf{G})^{\otimes X\text{-lisse}}$ on $\mathbf{C}$ from Sect. 8.1.1.

In particular, the symbol $\mathrm{Rep}(\mathsf{G})^{\otimes X\text{-lisse}}$ stands for an actual (symmetric) monoidal category, namely,

$$\mathrm{Rep}(\mathsf{G})^{\otimes X\text{-lisse}} \simeq \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathrm{QLisse}(X)).$$

Thus, we can regard Theorem 8.3.7 as saying that the functor

$$\mathrm{Rep}(\mathsf{G})^{\otimes X\text{-lisse}} \to \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)), \tag{8.10}$$

described in Sect. 8.1.2, is an equivalence.

8.4.3. Let now $\mathbf{H}$ be endowed with a t-structure and a fiber functor satisfying the assumptions of Sect. 1.7.1.

The map (8.8) gives rise to a map of spaces

$$\{\text{Actions of } \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \text{ on } \mathbf{C}\} \to \{\mathbf{H}\text{-families of actions of } \mathrm{Rep}(\mathsf{G}) \text{ on } \mathbf{C}\}. \tag{8.11}$$

From here, we obtain:

Lemma 8.4.4. *The category* $\mathbf{H}$ *is adapted for spectral decomposition if and only if the map* (8.11) *is an equivalence for any* $\mathbf{C}$.

8.4.5. Unwinding the constructions, it is easy to see that for $\mathbf{H} = \mathrm{QLisse}(X)$, the map (8.11) is the same as the map (8.4).

Hence, Lemma 8.4.4 implies that Theorems 8.1.4 and 8.3.7 are logically equivalent.

8.4.6. Let $\mathbf{H}$ be again a (dualizable) symmetric monoidal category and let $\mathbf{C}$ be a DG category. By an $\mathbf{H}$-family of functors $\mathrm{Rep}(\mathsf{G}) \to \mathbf{C}$ we will mean a natural transformation between the following two functors $\mathrm{fSet} \to \mathrm{DGCat}$:

From the functor

$$I \mapsto \mathrm{Rep}(\mathsf{G})^{\otimes I}$$

to the functor

$$I \mapsto \mathbf{C} \otimes \mathbf{H}^{\otimes I}.$$

From Lemma 8.2.7(b) we obtain that this data is equivalent to the data of a functor

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \to \mathbf{C}.$$

8.4.7. Let $\mathbf{H}$ be endowed with t-structure and a fiber functor satisfying the assumptions of Sect. 1.7.1.

The map (8.8) gives rise to a map of spaces

$$\{\text{Functors } \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \to \mathbf{C}\} \to \{\mathbf{H}\text{-families of functors } \mathrm{Rep}(\mathsf{G}) \to \mathbf{C}\}. \tag{8.12}$$

As in Lemma 8.4.4, we have:

Lemma 8.4.8. *The category* $\mathbf{H}$ *is adapted for spectral decomposition if and only if the map* (8.12) *is an equivalence for any* $\mathbf{C}$.

8.4.9. Let us write out the map (8.12) explicitly.

We start with the functors (8.9). Then for $I \in \mathrm{fSet}$ we obtain a functor

$$\mathrm{Rep}(\mathsf{G})^{\otimes I} \to \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))^{\otimes I} \otimes \mathbf{H}^{\otimes I} \to \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \otimes \mathbf{H}^{\otimes I}, \tag{8.13}$$

where the last arrow uses the tensor product functor

$$\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))^{\otimes I} \to \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})).$$

We will denote the functor (8.13) by $\mathcal{E}^I$. For a fixed $V \in \mathrm{Rep}(\mathsf{G})^{\otimes I}$, we denote the resulting object of $\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \otimes \mathbf{H}^{\otimes I}$ by $\mathcal{E}^I_V$.

Now, given a functor

$$\mathcal{S} : \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \to \mathbf{C},$$

the resulting system of functors

$$\mathcal{S}^I : \mathrm{Rep}(\mathsf{G})^{\otimes I} \to \mathbf{C} \otimes \mathbf{H}^{\otimes I}$$

sends

$$V \in \mathrm{Rep}(\mathsf{G})^{\otimes I} \ \mapsto \ (\mathcal{S} \otimes \mathrm{Id})(\mathcal{E}^I_V) \in \mathbf{C} \otimes \mathbf{H}^{\otimes I}.$$

Remark 8.4.10. Note that for $\mathbf{H} = \mathrm{QLisse}(X)$ and $\mathbf{C} = \mathrm{Vect}_{\mathsf{e}}$, a system of functors

$$\mathrm{Rep}(\mathsf{G})^{\otimes I} \to \mathrm{QLisse}(X)^{\otimes I}, \quad I \in \mathrm{fSet}$$

is exactly the structure that arises from the *Shtuka* construction.

We will explore this in Sect. 21.5 to relate Shtukas to objects in $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$.

8.5. **A rigidified version.** Let $\mathbf{H}$ be as in Sect. 1.7.1. Recall that along with the prestack $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ we considered its rigidified version $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$. In this subsection we will introduce a counterpart of this rigidification for $\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$.

8.5.1. Let $\mathbf{H}$ be a (dualizable) symmetric monoidal category, equipped with a symmetric monoidal functor $\mathbf{oblv_H} : \mathbf{H} \to \text{Vect}_\mathsf{e}$.

Composition with $\mathbf{oblv_H}$ defines a symmetric monoidal functor

$$\text{Rep}(\mathsf{G}) \simeq \underline{\text{coHom}}(\text{Rep}(\mathsf{G}), \text{Vect}_\mathsf{e}) \to \underline{\text{coHom}}(\text{Rep}(\mathsf{G}), \mathbf{H}).$$

Denote

$$\underline{\text{coHom}}(\text{Rep}(\mathsf{G}), \mathbf{H})^{\text{rigid}} := \underline{\text{coHom}}(\text{Rep}(\mathsf{G}), \mathbf{H}) \underset{\text{Rep}(\mathsf{G})}{\otimes} \text{Vect}_\mathsf{e}\,.$$

8.5.2. By construction, for a symmetric monoidal category $\mathbf{A}$, the datum of a symmetric monoidal functor

$$\underline{\text{coHom}}(\text{Rep}(\mathsf{G}), \mathbf{H})^{\text{rigid}} \to \mathbf{A}$$

is equivalent to that of a symmetric monoidal functor

$$\text{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathbf{H},$$

equipped with an identification of the composition

$$\text{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathbf{H} \overset{\text{Id}_\mathbf{A} \otimes \mathbf{oblv_H}}{\longrightarrow} \mathbf{A},$$

with the forgetful functor

$$\text{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_\mathsf{G}}{\to} \text{Vect}_\mathsf{e} \overset{\mathbf{1}_\mathbf{A}}{\longrightarrow} \mathbf{A}.$$

8.5.3. As in Sect. 8.3.2, we have a symmetric monoidal functor

$$\underline{\text{coHom}}(\text{Rep}(\mathsf{G}), \mathbf{H})^{\text{rigid}} \to \text{QCoh}(\mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathbf{H})^{\text{rigid}}). \tag{8.14}$$

Since

$$\text{QCoh}(\mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathbf{H})^{\text{rigid}}) \simeq \text{QCoh}(\mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathbf{H})) \underset{\text{Rep}(\mathsf{G})}{\otimes} \text{Vect}_\mathsf{e},$$

we obtain that $\mathbf{H}$ is adapted for spectral decomposition if and only if the functor (8.14) is an equivalence. Indeed, this follows from the fact that the functor (7.23) is conservative.

9. Categories adapted for spectral decomposition

The goal of this section is to prove Theorem 8.3.7. Our strategy will be as follows:

We will first show that the category $\mathbf{H} := \text{Vect}_\mathsf{e}^\mathcal{X}$ is adapted for spectral decomposition (where $\mathcal{X}$ is a connected object of Spc). From this we will then formally deduce that the category $\mathbf{H} := \text{QLisse}(X)$ is also adapted for spectral decomposition, in the particular case when X is a smooth and compete curve.

9.1. **The Betti case.**

In this subsection we let $\mathcal{X}$ be a connected object of Spc.

9.1.1. Consider the symmetric monoidal category $\text{Vect}_\mathsf{e}^\mathcal{X}$, equipped with its natural t-structure and the fiber functor (the latter is given by a choice of a base point $x \in X$).

We claim:

Theorem 9.1.2. *The symmetric monoidal category* $\text{Vect}_\mathsf{e}^\mathcal{X}$ *is adapted for spectral decomposition.*

This result is stated and proved in [GKRV, Theorem 1.5.5]. In fact, the category that we denote $\underline{\text{coHom}}(\text{Rep}(\mathsf{G}), \text{Vect}_\mathsf{e}^\mathcal{X})$ is exactly the category denoted $\text{Rep}(\mathsf{G})^{\otimes \mathcal{X}}$ in *loc.cit.*, and

$$\mathbf{Maps}(\text{Rep}(\mathsf{G}), \text{Vect}_\mathsf{e}^\mathcal{X}) = \text{LocSys}_\mathsf{G}^{\text{Betti}}(\mathcal{X}),$$

see Sect. 4.5.7.

Remark 9.1.3. In Sect. 10.4 we will give another proof of Theorem 9.1.2, which has a potential for generalization for other symmetric monoidal categories $\mathbf{H}$.

9.2. **The heriditary property of being adapted.**

In this subsection we will perform a crucial step towards the proof of Theorem 8.3.7: we will show that the property of being adapted for spectral decomposition is, under certain conditions, inherited by full subcategories.

9.2.1. Let $\mathbf{H}$ be a (dualizable) symmetric monoidal category as in Sect. 1.7.1, and let $\mathbf{H}' \subset \mathbf{H}$ be a full symmetric monoidal subcategory. Assume that $\mathbf{H}'$ inherits a t-structure (i.e., it is preserved by the truncation functors).

We will prove:

Theorem 9.2.2. *Suppose that:*

- *The embedding* $\iota : \mathbf{H}' \hookrightarrow \mathbf{H}$*, considered as a functor between plain DG categories, admits a continuous right adjoint;*
- *The prestack* $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$ *is an eventually coconnective affine scheme almost of finite type;*
- *The map* $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}') \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})$ *is a formal isomorphism and an ind-closed embedding*[21]*.*

Then if $\mathbf{H}$ *is adapted for spectral decomposition, then so is* $\mathbf{H}'$*.*

The rest of this subsection is devoted to the proof of this theorem.

9.2.3. Let $\mathbf{A}$ be a target symmetric monoidal category. We wish to show that the map

$$\mathrm{Maps}_{\mathrm{DGCat}^{\mathrm{SymMon}}}(\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}')), \mathbf{A}) \to \mathrm{Maps}_{\mathrm{DGCat}^{\mathrm{SymMon}}}(\mathrm{Rep}(\mathsf{G}), \mathbf{A} \otimes \mathbf{H}')$$

is an isomorphism of spaces.

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Maps}_{\mathrm{DGCat}^{\mathrm{SymMon}}}(\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})), \mathbf{A}) & \longrightarrow & \mathrm{Maps}_{\mathrm{DGCat}^{\mathrm{SymMon}}}(\mathrm{Rep}(\mathsf{G}), \mathbf{A} \otimes \mathbf{H}) \\ \uparrow & & \uparrow \\ \mathrm{Maps}_{\mathrm{DGCat}^{\mathrm{SymMon}}}(\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}')), \mathbf{A}) & \longrightarrow & \mathrm{Maps}_{\mathrm{DGCat}^{\mathrm{SymMon}}}(\mathrm{Rep}(\mathsf{G}), \mathbf{A} \otimes \mathbf{H}'), \end{array} \tag{9.1}$$

where the top horizontal arrow is an isomorphism by assumption. We will show that both vertical arrows are fully faithful, and that their essential images match up under the equivalence given by the top horizontal arrow.

9.2.4. The right vertical arrow in (9.1) is a fully faithful because the functor

$$\mathbf{A} \otimes \mathbf{H}' \to \mathbf{A} \otimes \mathbf{H}$$

is fully faithful. The latter is true because the inclusion functor $\mathbf{H}' \hookrightarrow \mathbf{H}$ is fully faithful and admits a continuous right adjoint.

9.2.5. Let

$$\mathcal{Y}_1 = (\mathcal{Y}_2)^\wedge_{\mathcal{Z}} \to \mathcal{Y}_2$$

be as in Remark 4.2.5.

A simple colimit argument, combined with [GR3, Proposition 7.1.3], shows that the restriction functor

$$\mathrm{QCoh}(\mathcal{Y}_2) \to \mathrm{QCoh}(\mathcal{Y}_1) \tag{9.2}$$

admits a fully faithful left adjoint, whose essential image is the full subcategory

$$\mathrm{QCoh}(\mathcal{Y}_2)_{\mathcal{Z}} \subset \mathrm{QCoh}(\mathcal{Y}_2)$$

consisting of objects with set-theoretic support on $\mathcal{Z}$, i.e., those objects $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y}_2)$ such that for every affine scheme S mapping to $\mathcal{Y}_2$, the pullback $\mathcal{F}_S$ of $\mathcal{F}$ to S vanishes on the localization of S at every scheme-theoretic point not contained in $S \underset{\mathcal{Y}_2}{\times} \mathcal{Z}$ (equivalently, $\mathcal{F}_S$ is such that its cohomologies are unions of subsheaves supported on closed subsets of S that comprise $S \underset{\mathcal{Y}_2}{\times} \mathcal{Z}$).

[21]See Remark 4.2.5 where it is explained what the combination of these two conditions amounts to.

9.2.6. This implies that in the situation of Sect. 9.2.5, for any DG category $\mathbf{D}$, restriction along (9.2) defines a fully faithful embedding

$$\mathrm{Maps}_{\mathrm{DGCat}}(\mathrm{QCoh}(\mathcal{Y}_1),\mathbf{D}) \to \mathrm{Maps}_{\mathrm{DGCat}}(\mathrm{QCoh}(\mathcal{Y}_2),\mathbf{D}), \tag{9.3}$$

with essential image consisting of those functors that vanish on the full subcategory

$$\{\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y}_2),\ \mathcal{F}|_{\mathcal{Y}_1} = 0\} \tag{9.4}$$

of $\mathrm{QCoh}(\mathcal{Y}_2)$.

This formally implies that for any symmetric monoidal category $\mathbf{A}$ the map

$$\mathrm{Maps}_{\mathrm{DGCat}^{\mathrm{SymMon}}}(\mathrm{QCoh}(\mathcal{Y}_1),\mathbf{A}) \to \mathrm{Maps}_{\mathrm{DGCat}^{\mathrm{SymMon}}}(\mathrm{QCoh}(\mathcal{Y}_2),\mathbf{A})$$

is fully faithful, whose essential image consists of those functors that vanish on the subcategory (9.4) of $\mathrm{QCoh}(\mathcal{Y}_2)$.

9.2.7. We will need the following assertion:

Lemma 9.2.8. *Suppose that $\mathcal{Y}_2$ is an eventually coconnective affine scheme almost of finite type. Then the subcategory* (9.4) *is generated by objects of the form $f_*(\widetilde{\mathsf{e}})$, where $\widetilde{\mathsf{e}}$ is a field extension of e and f is a map* $\mathrm{Spec}(\widetilde{\mathsf{e}}) \to \mathcal{Y}_2$ *that* does not *factor through $\mathcal{Z}$.*

Proof. Since $\mathcal{Y}_2$ is eventually coconnective, every object is a (finite) colimit of objects obtained as direct images along ${}^{\mathrm{cl}}\mathcal{Y}_2 \to \mathcal{Y}_2$.

Hence, we can assume that $\mathcal{Y}_2$ is classical. In this case, the statement follows by a standard Cousin complex argument.

□

9.2.9. We apply the discussion in Sect. 9.2.6 to the embedding

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}') \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}). \tag{9.5}$$

We obtain that the left vertical arrow in (9.1) is fully faithful.

Remark 9.2.10. Note that the above argument shows that in the situation of Theorem 9.2.2, the functor

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}') \to \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}'))$$

is a localization (admits a fully faithful (but not necessarily continuous) right adjoint, even without the assumption that $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}$ is eventually coconnective.

9.2.11. To prove Theorem 9.2.2, it remains to show that the essential images of the vertical arrows in (9.1) match under the equivalence given by the top horizontal arrow.

9.3. Proof of Theorem 9.2.2: identifying the essential image.

9.3.1. Applying base change

$$\mathrm{Vect}_{\mathsf{e}} \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} -,$$

we can assume that we are given a functor

$$\Phi : \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}) \to \mathbf{A},$$

such that the corresponding functor

$$\mathsf{F} : \mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathbf{H}$$

factors as

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathsf{F}'}{\to} \mathbf{A} \otimes \mathbf{H}' \to \mathbf{A} \otimes \mathbf{H}.$$

We wish to show that Φ factors as

$$\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}) \to \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}')^{\mathrm{rigid}}) \to \mathbf{A}.$$

I.e., we wish to show that Φ vanishes on

$$\ker\left(\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})^{\mathrm{rigid}}) \to \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H}')^{\mathrm{rigid}})\right).$$

9.3.2. By Lemma 9.2.8, it suffices to show the following;

Let $\widetilde{\mathsf{e}}$ be a field extension of e, and let us be given a map

$$f : \mathrm{Spec}(\widetilde{\mathsf{e}}) \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}.$$

We wish to show that if $\Phi(f_*(\widetilde{\mathsf{e}})) \neq 0$, then f factors through $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}')^{\mathrm{rigid}}$.

9.3.3. Consider the tensor product category

$$\widetilde{\mathbf{A}} := \mathrm{Vect}_{\widetilde{\mathsf{e}}} \underset{\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}})}{\otimes} \mathbf{A}.$$

Since the morphism f is affine, we have

$$\mathrm{Vect}_{\widetilde{\mathsf{e}}} \simeq f_*(\widetilde{\mathsf{e}})\text{-mod}(\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}})).$$

Hence,

$$\widetilde{\mathbf{A}} \simeq \Phi(f_*(\widetilde{\mathsf{e}}))\text{-mod}(\mathbf{A}).$$

Therefore, if $\Phi(f_*(\widetilde{\mathsf{e}})) \neq 0$, then $\widetilde{\mathbf{A}} \neq 0$.

9.3.4. Denote by $\widetilde{\Phi}$ the composite functor

$$\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}) \overset{\Phi}{\to} \mathbf{A} \to \widetilde{\mathbf{A}}$$

and by $\widetilde{\mathsf{F}}$ the corresponding functor

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathsf{F}}{\to} \mathbf{A} \otimes \mathbf{H} \to \widetilde{\mathbf{A}} \otimes \mathbf{H}.$$

By assumption, the functor $\widetilde{\mathsf{F}}$ takes values in the full subcategory

$$\widetilde{\mathbf{A}} \otimes \mathbf{H}' \subset \widetilde{\mathbf{A}} \otimes \mathbf{H}.$$

Note, however, that $\widetilde{\Phi}$ factors as

$$\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}) \overset{f^*}{\to} \mathrm{Vect}_{\widetilde{\mathsf{e}}} \to \widetilde{\mathbf{A}},$$

and $\widetilde{\mathsf{F}}$ factors as

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathsf{F}_f}{\to} \mathrm{Vect}_{\widetilde{\mathsf{e}}} \otimes \mathbf{H} \to \widetilde{\mathbf{A}} \otimes \mathbf{H},$$

where F_f is the functor corresponding to f in the definition of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}}$.

9.3.5. Thus, we wish to show that F_f takes values in

$$\mathrm{Vect}_{\widetilde{\mathsf{e}}} \otimes \mathbf{H}' \subset \mathrm{Vect}_{\widetilde{\mathsf{e}}} \otimes \mathbf{H}.$$

9.3.6. Recall that ι denotes the embedding $\mathbf{H}' \hookrightarrow \mathbf{H}$. For an object $V \in \mathrm{Rep}(\mathsf{G})$ consider the counit of the adjunction

$$(\mathrm{Id}_{\mathrm{Vect}_{\widetilde{\mathsf{e}}}} \otimes (\iota \circ \iota^R))(\mathsf{F}_f(V)) \to \mathsf{F}_f(V).$$

We wish to show that it is an isomorphism. We know that this map becomes an isomorphism after applying the functor

$$\mathrm{Vect}_{\widetilde{\mathsf{e}}} \otimes \mathbf{H} \to \widetilde{\mathbf{A}} \otimes \mathbf{H}.$$

Hence, it is enough to show that the latter functor is conservative. Since $\mathbf{H}$ is dualizable, it suffices to show that the functor

$$\mathrm{Vect}_{\widetilde{\mathsf{e}}} \to \widetilde{\mathbf{A}}$$

is conservative.

However, the latter is evident: up to a cohomological shift, a non-zero object of $\mathrm{Vect}_{\widetilde{\mathsf{e}}}$ has a copy of $\widetilde{\mathsf{e}}$ as a retract, and $\widetilde{\mathsf{e}} \mapsto \mathbf{1}_{\widetilde{\mathbf{A}}}$, which is non-zero, since $\widetilde{\mathbf{A}}$ was assumed non-zero.

☐[Theorem 9.2.2]

9.4. **Proof of Theorem 8.3.7, Betti and de Rham contexts.** In this subsection we will prove Theorem 8.3.7 in the Betti and de Rham contexts.

Our method will consist of combining Theorems 9.1.2 and 9.2.2. Throughout this section, X will be a smooth, complete and connected curve over a ground field k.

9.4.1. We will first consider the case when $k = \mathbb{C}$ and our sheaf-theoretic context is Betti (see Sect. 1.1.1 for what this means). In this case, our curve X does not need to be complete.

We take $\mathbf{H} := \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(X)$ and $\mathbf{H}' := \mathrm{QLisse}(X)$. We know that the category $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(X)$ is adapted for spectral decomposition by Theorem 9.1.2, which would imply Theorem 8.3.7 in this case.

The corresponding functor

$$\iota : \mathrm{QLisse}(X) \hookrightarrow \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(X) \tag{9.6}$$

is fully faithful by Proposition 4.7.2. We will show that it satisfies the requirements of Theorem 9.2.2.

9.4.2. We first show that ι admits a continuous right adjoint. We will distinguish two cases:

Case 1: X has genus[22] 0. In this case ι is an equivalence.

Case 2: X has genus ≥ 1. In this case, by Theorem E.2.8(a) and Corollary E.2.3, the functor

$$\mathrm{IndLisse}(X) \to \mathrm{QLisse}(X)$$

is an equivalence. Now, for any X, the composite functor

$$\mathrm{IndLisse}(X) \to \mathrm{QLisse}(X) \to \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(X)$$

sends compacts to compacts (indeed, for a finite CW complex, objects from $\mathrm{Lisse}(X)$ are compact in $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(X)$), and since $\mathrm{IndLisse}(X)$ is compactly generated, it admits a continuous right adjoint.

9.4.3. We now show that

$$\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})^{\mathrm{rigid}} \simeq \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti},\mathrm{rigid}_x}(X)$$

is eventually coconnective and almost of finite type. The aft condition holds for any X that is homotopy-equivalent to a finite CW complex. To show that it is eventually coconnective, we will show that it is quasi-smooth.

Let $\overset{\circ}{X}$ be obtained by removing from X one point (different from x). Then $\overset{\circ}{X}$ is homotopy-equivalent to a bouquet of n circles, and

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti},\mathrm{rigid}_x}(\overset{\circ}{X}) \simeq \mathsf{G}^{\times n}$$

We have an isomorphism of homotopy types

$$X \simeq \overset{\circ}{X} \underset{S^1}{\sqcup} \mathrm{pt},$$

hence

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti},\mathrm{rigid}_x}(X) \simeq \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti},\mathrm{rigid}_x}(\overset{\circ}{X}) \underset{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti},\mathrm{rigid}_x}(S^1)}{\times} \mathrm{pt} \simeq \mathsf{G}^{\times n} \underset{\mathsf{G}}{\times} \mathrm{pt},$$

and hence is manifestly quasi-smooth.

9.4.4. Finally, the fact that the map

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X) \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X)$$

is a formal isomorphism and an ind-closed embedding is the content of Proposition 4.7.8 and Theorem 4.7.9.

9.4.5. Next we consider the de Rham context. We wish to show that the category $\mathrm{QLisse}(X)$, which is a symmetric monoidal category over k is adapted for spectral decomposition.

By Lefschetz principle, we can assume that $k = \mathbb{C}$. In this case, by Riemann-Hilbert, the de Rham version of $\mathrm{QLisse}(X)$ is equivalent as a symmetric monoidal category to its Betti counterpart with $\mathsf{e} = \mathbb{C}$.

Hence, the assertion follows from Sect. 9.4.1.

[22]If X not complete, then we stipulate that we are in Case 2 below.

9.5. **Proof of Theorem 8.3.7, étale context over a field of characteristic** 0. In this subsection we will prove Theorem 8.3.7 in the étale context, but for k being an algebraically closed field of characteristic 0.

9.5.1. We wish to show that the symmetric monoidal category $\mathrm{QLisse}(X)$ is adapted for spectral decomposition. We will consider separately the cases when X has genus 0, and when the genus of X is ≥ 1.

When X has genus 0, the statement follows from the description of the category $\mathrm{QLisse}(X)$ in Sect. E.2.6.

Hence, from now on we will assume that the genus of X is ≥ 1. In this case, by Theorem E.2.8(a) and Corollary E.2.3,

$$\mathrm{QLisse}(X) \simeq \mathrm{IndLisse}(X) \text{ and } \mathrm{Lisse}(X) \simeq D^b(\mathrm{Lisse}(X)^{\heartsuit}). \tag{9.7}$$

9.5.2. Recall (see [SGA1, Exposé X, Corollary 1.8]) that if $k \hookrightarrow k'$ is an extension of algebraically closed fields, for a proper[23] scheme Y over k and its base change Y' to k', the assignment

$$(\widetilde{Y} \to Y) \mapsto (\widetilde{Y}' \to Y'), \quad \widetilde{Y}' := \widetilde{Y} \underset{\mathrm{Spec}(k)}{\times} \mathrm{Spec}(k')$$

is an equivalence of categories between finite étale covers of Y and those of Y'. Hence, pullback defines an equivalence of categories

$$\mathrm{Lisse}(Y)^{\heartsuit} \simeq \mathrm{Lisse}(Y')^{\heartsuit}.$$

Taking $Y = X$, by (9.7) we obtain

$$\mathrm{QLisse}(X) \simeq \mathrm{QLisse}(X').$$

Thus, embedding k into a larger algebraically closed field that also contains $\mathbb{C}$, we can assume that $k = \mathbb{C}$.

Remark 9.5.3. The fact that pullback defines an isomorphism

$$\mathrm{C}^{\cdot}_{\mathrm{et}}(Y, \mathcal{F}) \to \mathrm{C}^{\cdot}_{\mathrm{et}}(Y', \mathcal{F}|_{Y'}), \quad \mathcal{F} \in \mathrm{Lisse}(Y)$$

implies that the pullback functor

$$\mathrm{QLisse}(Y) \to \mathrm{QLisse}(Y')$$

is fully faithful, and being an equivalence at the abelian level, is actually an equivalence for any Y (not just a curve of genus ≥ 1).

9.5.4. For $k = \mathbb{C}$, on the one hand, we can consider the symmetric monoidal category

$$\mathrm{QLisse}_{\mathrm{et}}(X),$$

and on the other hand

$$\mathrm{QLisse}_{\mathrm{Betti}}(X),$$

for $\mathsf{e} = \overline{\mathbb{Q}}_\ell$.

We will construct a (symmetric monoidal) functor

$$\jmath : \mathrm{QLisse}_{\mathrm{et}}(X) \to \mathrm{QLisse}_{\mathrm{Betti}}(X), \tag{9.8}$$

which is fully faithful, admits a continuous right adjoint. Moreover, the induced map

$$\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}_{\mathrm{et}}}(X) \to \mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}_{\mathrm{Betti}}}(X) \tag{9.9}$$

will be an isomorphism on each connected component of the source (i.e., this map identifies the source with the union of some of the connected components of the target).

Once we show this, composing with the functor (9.6), we will establish that $\mathrm{QLisse}_{\mathrm{et}}(X)$ is adapted for spectral decomposition in view of what we have shown already in Sect. 9.4, combined with Theorems 9.1.2 and 9.2.2.

[23]When k has characteristic 0 (which is the case here), the assertion formulated below holds without the properness assumption.

9.5.5. Since the genus of X is ≥ 1, in addition to (9.7), we also have the equivalences

$$\mathrm{QLisse}_{\mathrm{Betti}}(X) \simeq \mathrm{IndLisse}_{\mathrm{Betti}}(X) \text{ and } \mathrm{Lisse}_{\mathrm{Betti}}(X) \simeq D^b(\mathrm{Lisse}_{\mathrm{Betti}}(X)^{\heartsuit}). \tag{9.10}$$

Hence, in order to construct (9.8), it suffices to construct the corresponding functor

$$\mathrm{Lisse}_{\mathrm{et}}(X)^{\heartsuit} \to \mathrm{Lisse}_{\mathrm{Betti}}(X)^{\heartsuit}. \tag{9.11}$$

The sought-for functor (9.11) is obtained from the equivalence

$$\{\text{Finite étale covers of } X\} \leftrightarrow \{\text{Finite covers of the topological space underlying } X\}.$$

The functor (9.11) is fully faithful. In fact, its essential image consists of those those representations of the fundamental group of (the topological space underlying) X on finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces that admit a $\overline{\mathbb{Z}}_\ell$-lattice.

9.5.6. The resulting functor $\jmath$ in (9.8) maps

$$\mathrm{Lisse}_{\mathrm{et}}(X) \to \mathrm{Lisse}_{\mathrm{Betti}}(X), \tag{9.12}$$

by construction. Hence, it preserves compactness, and hence admits a continuous right adjoint.

We claim that $\jmath$ is fully faithful. This is standard, but we will give an elementary proof for completeness:

It is enough to show that the functor (9.12) is fully faithful. I.e., we have to show that for $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Lisse}_{\mathrm{et}}(X)^{\heartsuit}$, the maps

$$\mathrm{Hom}_{\mathrm{Lisse}_{\mathrm{et}}(X)^{\heartsuit}}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Hom}_{\mathrm{Lisse}_{\mathrm{Betti}}(X)^{\heartsuit}}(\jmath(\mathcal{F}_1), \jmath(\mathcal{F}_2)) \tag{9.13}$$

$$\mathrm{Ext}^1_{\mathrm{Lisse}_{\mathrm{et}}(X)^{\heartsuit}}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Ext}^1_{\mathrm{Lisse}_{\mathrm{Betti}}(X)^{\heartsuit}}(\jmath(\mathcal{F}_1), \jmath(\mathcal{F}_2)) \tag{9.14}$$

and

$$\mathrm{Ext}^2_{\mathrm{Lisse}_{\mathrm{et}}(X)^{\heartsuit}}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Ext}^2_{\mathrm{Lisse}_{\mathrm{Betti}}(X)^{\heartsuit}}(\jmath(\mathcal{F}_1), \jmath(\mathcal{F}_2)) \tag{9.15}$$

are isomorphisms.

The map (9.13) is an isomorphism since (9.11) is fully faithful.

To prove that (9.15) is an isomorphism, replacing $\mathcal{F}_1$ by $\mathcal{F}_1 \otimes \mathcal{F}_2^\vee$, we can assume that $\mathcal{F}_2 = \underline{\mathsf{e}}_X$. For $\mathcal{F} \in \mathrm{QLisse}(X)$ (in either context) let $\mathcal{F}_0$ be its maximal trivial quotient. Verdier duality implies that the map

$$\mathcal{F} \to \mathcal{F}_0 \simeq V \otimes \underline{\mathsf{e}}_X$$

defines an isomorphism

$$V^* \otimes H^2(X, \mathsf{e}) \simeq \mathrm{Hom}_{\mathrm{Lisse}(X)}(\mathcal{F}_0, \underline{\mathsf{e}}_X[2]) \simeq \mathrm{Hom}_{\mathrm{Lisse}(X)}(\mathcal{F}, \underline{\mathsf{e}}_X[2]) \simeq \mathrm{Ext}^2_{\mathrm{Lisse}(X)^{\heartsuit}}(\mathcal{F}, \underline{\mathsf{e}}_X).$$

Since the functor (9.11) is fully faithful, we have $\jmath(\mathcal{F})_0 \simeq \jmath(\mathcal{F}_0)$. This implies that (9.15) is an isomorphism, since the functor (9.8) induces an isomorphism

$$H^2_{\mathrm{et}}(X, \mathsf{e}) \to H^2_{\mathrm{Betti}}(X, \mathsf{e}).$$

(Indeed, both sides are 1-dimensional vector spaces, and the above map is easily seen to be non-zero.)

The map (9.14) is injective again because (9.11) is fully faithful. Hence, in order to show that it is surjective, it suffices to show that both sides have the same dimension. However, the latter follows from the Grothendieck-Ogg-Shafarevich formula.

Remark 9.5.7. One can show that for an algebraic variety Y over $\mathbb{C}$, we have a well-defined fully faithful functor

$$\jmath : \mathrm{Shv}_{\mathrm{et}}(Y)^{\heartsuit} \to \mathrm{Shv}_{\mathrm{Betti}}(Y)^{\heartsuit}.$$

9.5.8. Finally, let us show that the map (9.9) is an isomorphism on every connected component of the source.

Given that we already know that (9.8) is fully faithful, the arguments proving Proposition 4.1.8 and Theorem 4.1.10 imply that (9.9) is a formal isomorphism and an ind-closed embedding. Hence, it suffices to show that (9.9) defines a surjection at the level of e-points from a connected component of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}_{\mathrm{et}}}(X)$ to the corresponding connected component of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}_{\mathrm{Betti}}}(X)$.

By Sect. 3.4, it suffices to show that if M is a Levi quotient of a parabolic and σ_{M} is an irreducible étale M-local system on X, then the corresponding map

$$\mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}^{\mathrm{restr}_{\mathrm{et}}}(X) \to \mathrm{LocSys}_{\mathsf{P},\sigma_{\mathsf{M}}}^{\mathrm{restr}_{\mathrm{Betti}}}(X) \tag{9.16}$$

is surjective at the level of e-points.

However, as in Proposition 4.3.3, the map (9.16) is in fact an isomorphism.

Remark 9.5.9. The last argument shows the mechanism by which the map (9.9) fails to be an isomorphism: it misses those connected components that correspond to semi-simple Betti local systems that do not come from the étale ones.

9.6. **Proof of Theorem 8.3.7, étale context over a field of positive characteristic.** In this subsection we will show that Theorem 8.3.7 holds in the étale context, when k is an algebraically closed field of positive characteristic.

9.6.1. Let R denote the ring of Witt vectors on k, and let k' be an algebraic closure of the field of fractions of R.

Since X is proper and smooth of dimension 1, it can be lifted to a smooth proper relative curve $X_{R^\wedge}$ over $\mathrm{Spf}(R)$.

By GAGA, $X_{R^\wedge}$ comes from a uniquely defined curve X_R over $\mathrm{Spec}(R)$. Let X' denote the base change of X_R to $\mathrm{Spec}(k')$.

9.6.2. We claim that there exists a symmetric monoidal functor

$$\mathrm{QLisse}(X) \to \mathrm{QLisse}(X'),$$

with the same properties as the functor (9.8).

Once we show this, composing with the functors (9.8) and (9.6), we will know that $\mathrm{QLisse}(X)$ is adapted for spectral decomposition by Sect. 9.4.1.

If X has genus 0, the assertion follows from the explicit description of the category $\mathrm{QLisse}(X)$ in this case, see Sect. E.2.6. Hence, we can assume that the genus of X is ≥ 1. In this case, it suffices to construct an exact symmetric monoidal functor at the abelian level

$$\mathrm{Lisse}(X)^{\heartsuit} \to \mathrm{Lisse}(X')^{\heartsuit}, \tag{9.17}$$

and show that it is fully faithful (the fully faithfulness at the derived level will then follow by the argument in Sect. 9.5.6).

9.6.3. Now, the existence of the functor (9.17) with the required properties follows from the fact that we have a canonically defined surjection at the level of étale fundamental groups

$$\pi_{1,\mathrm{et}}(X') \to \pi_{1,\mathrm{et}}(X).$$

This follows from [SGA1, Exposé X, Corollary 2.3].

□[Theorem 8.3.7]

9.7. **A simple proof of Theorem 5.4.2.** In this subsection we let X be a smooth and complete algebraic curve. We will revisit Theorem 5.4.2 in the case $\mathbf{H} = \mathrm{QLisse}(X)$.

9.7.1. The assertion of Theorem 5.4.2 in the Betti context was established in Sect. 6.1.

The assertion of Theorem 5.4.2 in the de Rham context follows from the Betti case by Riemann-Hilbert.

9.7.2. Thus, it remains to treat the étale context.

However, as we have seen in Sects. 9.5 and 9.6, there exists a curve X' over $\mathbb{C}$, such that every connected component of the étale $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ is isomorphic to a connected component of the Betti $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X')$ for $\mathsf{e} = \overline{\mathbb{Q}}_\ell$.

□[Theorem 5.4.2]

9.8. **Complements: de Rham and Betti spectral actions.** In this subsection we will make a brief digression, and consider the de Rham or Betti contexts, in which the "usual" (i.e., not restricted) $\mathrm{LocSys}_{\mathsf{G}}(X)$ is defined. Let us be given a category $\mathbf{C}$ equipped with an action of $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))$.

Let X be a smooth and complete curve. We will explicitly describe the full subcategory

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))}{\otimes} \mathbf{C} \subset \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))}{\otimes} \mathbf{C} = \mathbf{C},$$

where we view

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)}$$

as a co-localization of $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))$, see Sect. 9.2.5.

9.8.1. Let is first specialize to the Betti context. Consider the algebraic stack $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X))$.

For a given $V \in \mathrm{Rep}(\mathsf{G})$, let

$$\mathcal{E}_V \in \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X)) \otimes \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(X)$$

be as in Sect. 8.4.9.

Let $\mathbf{C}$ be a DG category, equipped with an action of $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))$. In particular, for $V \in \mathrm{Rep}(\mathsf{G})$, we have the functor

$$\mathrm{H}(V, -) : \mathbf{C} \to \mathbf{C} \otimes \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(X),$$

corresponding to the action of the object $\mathcal{E}_V$ above.

Let

$$\mathbf{C}^{\mathrm{fin.mon.}} \subset \mathbf{C}$$

be the full subcategory consisting of objects $\mathbf{c} \in \mathbf{C}$, for which

$$\mathrm{H}(V, \mathbf{c}) \in \mathbf{C} \otimes \mathrm{QLisse}(X) \subset \mathbf{C} \otimes \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(X).$$

As in [GKRV, Proposition C.2.5], one shows that the category $\mathbf{C}^{\mathrm{fin.mon.}}$ is stable under the action of $\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))$ and it carries an action of $\mathrm{Rep}(\mathsf{G})^{\otimes X\text{-lisse}}$.

9.8.2. We claim:

Proposition 9.8.3. *The full subcategory* $\mathbf{C}^{\mathrm{fin.mon.}} \subset \mathbf{C}$ *equals*

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X))}{\otimes} \mathbf{C} \subset \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X))}{\otimes} \mathbf{C} = \mathbf{C}.$$

Proof. The inclusion

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X))}{\otimes} \mathbf{C} \subset \mathbf{C}^{\mathrm{fin.mon.}}$$

is clear.

For the opposite inclusion, we can assume that $\mathbf{C}^{\mathrm{fin.mon.}} = \mathbf{C}$, and we will need to show that

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X))}{\otimes} \mathbf{C} \to \mathbf{C}$$

is an equivalence.

The assumption on $\mathbf{C}$ implies that the action of $\mathrm{Rep}(\mathsf{G})^{\otimes X}$ factors through an action of $\mathrm{Rep}(\mathsf{G})^{\otimes X\text{-lisse}}$. Hence, by Theorem 8.3.7, the action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X))$ on $\mathbf{C}$ factors through $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$.

Hence,

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X))}{\otimes} \mathbf{C} \simeq$$

$$\simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X))}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))}{\otimes} \mathbf{C},$$

while

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X))}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)),$$

since $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$ is a monoidal co-localization of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X))$.

□

9.8.4. Let us now specialize to the de Rham context.

For a given $V \in \mathrm{Rep}(\mathsf{G})$ consider the corresponding object

$$\mathcal{E}_V \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)) \otimes \text{D-mod}(X),$$

see Sect. 8.4.9.

Let $\mathbf{C}$ be a DG category, equipped with an action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X))$. In particular, for $V \in \mathrm{Rep}(\mathsf{G})$, we have the functor

$$\mathrm{H}(V,-) : \mathbf{C} \to \mathbf{C} \otimes \text{D-mod}(X),$$

corresponding to the action of the object $\mathcal{E}_V$ above.

Let

$$\mathbf{C}^{\mathrm{Lisse}} \subset \mathbf{C}$$

be the full subcategory consisting of objects $\mathbf{c} \in \mathbf{C}$, for which

$$\mathrm{H}(V,\mathbf{c}) \in \mathbf{C} \otimes \mathrm{QLisse}(X) \subset \mathbf{C} \otimes \text{D-mod}(X).$$

As in [GKRV, Proposition C.2.5], one shows that the category $\mathbf{C}^{\mathrm{Lisse}}$ is stable under the action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X))$ and it carries an action of $\mathrm{Rep}(\mathsf{G})^{\otimes X\text{-lisse}}$.

9.8.5. We claim:

Proposition 9.8.6. *The full subcategory* $\mathbf{C}^{\mathrm{Lisse}} \subset \mathbf{C}$ *equals*

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}(X))}{\otimes} \mathbf{C} \subset \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X))}{\otimes} \mathbf{C} = \mathbf{C}.$$

The proof repeats that of Proposition 9.8.3.

Remark 9.8.7. A statement somewhat weaker than Proposition 9.8.6 appeared in [GKRV] as Conjecture C.5.5 of *loc. cit.*

Remark 9.8.8. Consider the category

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \text{D-mod}(X)).$$

This is the category that appears, e.g., in [Ga7, Sect. 4.2.7]; in this paper we denote it[24]

$$\mathrm{Rep}(\mathsf{G})^{\mathrm{dR}}_{\mathrm{Ran}},$$

see Remark 11.1.9.

The corresponding functor

$$\mathrm{Rep}(\mathsf{G})^{\mathrm{dR}}_{\mathrm{Ran}} \simeq \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \text{D-mod}(X)) \to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X))$$

is a localization (i.e., admits fully faithful right adjoint, which is, moreover, continuous), see [Ga7, Proposition 4.3.4] .

---

[24] Our version of $\mathrm{Rep}(\mathsf{G})^{\mathrm{dR}}_{\mathrm{Ran}}$ is a slightly different from the one in [Ga7, Sect. 4.2.7] in that it is the unital version of the category considered in *loc.cit.*

However, it is *not* an equivalence. Hence, the arrow

$$\{\text{Actions of QCoh}(\text{LocSys}^{\text{dR}}_{\mathsf{G}}(X)) \text{ on } \mathbf{C}\} \to \{\text{Actions of Rep}(\mathsf{G})_{\text{Ran}} \text{ on } \mathbf{C}\}$$

is fully faithful, but not an equivalence.

A key result of [Ga7, Corollary 4.5.5] says that for $\mathbf{C} = \text{D-mod}(\text{Bun}_G)$, the action of $\text{Rep}(\check{G})_{\text{Ran}}$ given by Hecke functors lies in the essential image of the above map.

10. Other examples of categories adapted for spectral decomposition

The contents of this section are *not* needed for the rest of the paper.

We will find another class of symmetric monoidal categories adapted for spectral decomposition, namely, categories of modules over connective Lie algebras.

The method of proof will allow us to give an alternative argument also for the proof Theorem 9.1.2, and potentially sheds some light on the nature of the "adapted for spectral decomposition" condition.

10.1. The case of Lie algebras.

We are going to establish a variant of Theorem 9.1.2, where instead of an object $\mathcal{X} \in \text{Spc}$ we have a Lie algebra $\mathsf{h} \in \text{LieAlg}(\text{Vect}^{\leq 0}_{\mathsf{e}})$.

10.1.1. Consider the category

$$\mathbf{H} := \mathsf{h}\text{-mod}.$$

It carries a symmetric monoidal structure (given by tensor product of modules over h), and a fiber functor

$$\mathbf{oblv}_{\mathsf{h}} : \mathsf{h}\text{-mod} \to \text{Vect}_{\mathsf{e}},$$

given by forgetting the action of the Lie algebra. Since h was assumed connective, the category h-mod carries a t-structure, for which $\mathbf{oblv}_{\mathsf{h}}$ is t-exact. This t-structure is left-complete: indeed

$$\mathsf{h}\text{-mod} \simeq U(\mathsf{h})\text{-mod},$$

and it is is known that the category of modules over a connective associative algebra is left-complete in its t-structure.

Hence, h-mod is a category that satisfies the requirements of Sect. 1.7.1. Furthermore, h-mod is dualizable (in fact, compactly generated).

10.1.2. We will prove:

Theorem 10.1.3. *The category* h-mod *is adapted for spectral decomposition.*

The proof will occupy the next two subsections. As a first step, we will reinterpret the prestack $\mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathsf{h}\text{-mod})$.

Remark 10.1.4. Note that when $H^0(\mathsf{h})$ is nilpotent, Theorem 10.1.3 is a particular case of Theorem 9.1.2: indeed, rational homotopy type theory implies that there exists a pointed space $\mathcal{X}$ such that the pair $(\text{Vect}^{\mathcal{X}}_{\mathsf{e}}, \text{ev}_x)$ is equivalent to $(\mathsf{h}\text{-mod}, \mathbf{oblv}_{\mathsf{h}})$.

10.2. The space of maps of Lie algebras.

As a first step towards the proof of Theorem 10.1.3, we will reinterpret the prestack $\mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathsf{h}\text{-mod})$, or rather its version $\mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathsf{h}\text{-mod})^{\text{rigid}}$, as the space of maps of Lie algebras.

10.2.1. Let g denote the Lie algebra of G. Consider the prestack, denoted $\mathbf{Maps}_{\text{Lie}}(\mathsf{h}, \mathsf{g})$, that sends an affine scheme $S = \text{Spec}(A)$ to the space of maps of Lie algebras in A-mod.

$$\mathsf{h} \otimes A \to \mathsf{g} \otimes A.$$

10.2.2. Let S be an affine scheme, and let us be given an S-point of $\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h},\mathsf{g})$. It gives rise to a symmetric monoidal functor

$$\mathsf{g}\text{-mod} \to \mathrm{QCoh}(S) \otimes \mathsf{h}\text{-mod},$$

such that the composition

$$\mathsf{g}\text{-mod} \to \mathrm{QCoh}(S) \otimes \mathsf{h}\text{-mod} \overset{\mathrm{Id}\otimes \mathbf{oblv}_{\mathsf{h}}}{\longrightarrow} \mathrm{QCoh}(S)$$

identifies with

$$\mathsf{g}\text{-mod} \overset{\mathbf{oblv}_{\mathsf{g}}}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}} \overset{\mathrm{unit}}{\longrightarrow} \mathrm{QCoh}(S).$$

We have the (symmetric monoidal) restriction functor

$$\mathrm{Rep}(\mathsf{G}) \to \mathsf{g}\text{-mod}, \tag{10.1}$$

which commutes with the forgetful functors to $\mathrm{Vect}_{\mathsf{e}}$.

Composing, we obtain a map of prestacks

$$\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h},\mathsf{g}) \to \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathsf{h}\text{-mod})^{\mathrm{rigid}}. \tag{10.2}$$

We claim:

Proposition 10.2.3. *The map* (10.2) *is an isomorphism.*

10.2.4. For the proof of Proposition 10.2.3 (as well as that of Proposition 10.2.8 below) we recall that any object in $\mathrm{LieAlg}(\mathrm{Vect}_{\mathsf{e}}^{\leq 0})$ can be written as a *sifted* colimit of objects of the form

$$\mathbf{free}_{\mathrm{Lie}}(V), \quad V \in \mathrm{Vect}_{\mathsf{e}}^{\leq 0}. \tag{10.3}$$

Note also that for h as in (10.3), we have:

$$\mathbf{Maps}_{\mathrm{Lie}}(\mathbf{free}_{\mathrm{Lie}}(V),\mathsf{g}) \simeq \mathrm{Spec}(\mathrm{Sym}(V\otimes \mathsf{g}^{\vee})) \tag{10.4}$$

is an affine scheme.

We also note the following lemma:

Lemma 10.2.5. *The assignment*

$$\mathsf{h} \mapsto \mathsf{h}\text{-mod}, \quad \mathrm{LieAlg}(\mathrm{Vect}_{\mathsf{e}}) \to \mathrm{DGCat}$$

sends sifted colimits to limits.

Proof. Consider the functor of universal enveloping algebra

$$\mathrm{LieAlg}(\mathrm{Vect}_{\mathsf{e}}) \to \mathrm{AssocAlg}(\mathrm{Vect}_{\mathsf{e}}).$$

Being a left adjoint, this functor sends colimits to colimits. We have

$$\mathsf{h}\text{-mod} \simeq U(\mathsf{h})\text{-mod}.$$

Now, the assertion follows from [GKRV, Lemma 2.5.5].

□

10.2.6. *Proof Proposition 10.2.3.* We need to show that for an affine scheme S, the map

$$\mathrm{Maps}(S, \mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h},\mathsf{g})) \to \mathrm{Maps}(S, \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathsf{h}\text{-mod})^{\mathrm{rigid}}) \tag{10.5}$$

is an isomorphism.

The left-hand side in (10.5) sends colimits in h to limits in Spc. The right-hand side in (10.5), sends sifted colimits in h to limits in Spc, by Lemma 10.2.5. Hence, we can assume that h is of the form (10.3). Moreover, we can assume that

$$V \in \mathrm{Vect}_{\mathsf{e}}^{\leq 0} \cap \mathrm{Vect}_{\mathsf{e}}^{c}.$$

Note that, by (10.4), for $S = \mathrm{Spec}(A)$,

$$\mathrm{Maps}(S, \mathbf{Maps}_{\mathrm{Lie}}(\mathbf{free}_{\mathrm{Lie}}(V),\mathsf{g})) \simeq \mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(V, A\otimes \mathbf{oblv}_{\mathrm{Lie}}(\mathsf{g})).$$

Let A' be the split square-zero extension of A equal to

$$A' = A \otimes (\mathsf{e} \oplus \epsilon \cdot V^*), \quad \epsilon^2 = 0.$$

Note that the category

$$\mathrm{QCoh}(S) \otimes \mathbf{free}_{\mathrm{Lie}}(V)\text{-mod}$$

identifies as a symmetric monoidal category with the category of triples $(\mathcal{M}, t, \alpha)$, where:

- $\mathcal{M} \in A$-mod;
- t is an automorphism of $\mathcal{M}' := A' \underset{A}{\otimes} \mathcal{M}$;
- α is a trivialization of the induced automorphism on $\mathcal{M} \simeq A \underset{A'}{\otimes} \mathcal{M}'$ induced by t.

Hence, the space

$$\mathrm{Maps}\left(S, \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{free}_{\mathrm{Lie}}(V)\text{-mod})^{\mathrm{rigid}}\right)$$

identifies with the space of automorphisms of the symmetric monoidal functor

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}} \overset{A'}{\longrightarrow} A'\text{-mod},$$

equipped with the trivialization of the automorphism of the composite functor

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}} \overset{A'}{\longrightarrow} A'\text{-mod} \to A\text{-mod}.$$

By Tannaka duality, the latter is the same as the space of maps

$$\mathrm{Spec}(A') \to \mathsf{G},$$

equipped with the trivialization of the composite

$$S = \mathrm{Spec}(A) \to \mathrm{Spec}(A') \to \mathsf{G}.$$

By deformation theory, we rewrite the latter as

$$\mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(\mathsf{g}^\vee, A \otimes V^*) \simeq \mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(V, A \otimes \mathbf{oblv}_{\mathrm{Lie}}(\mathsf{g})).$$

Unwinding the above identifications, it is easy to see that the map (10.5) corresponds to the identity map on $\mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(V, A \otimes \mathbf{oblv}_{\mathrm{Lie}}(\mathsf{g}))$; in particular, it is an isomorphism.

□[Proposition 10.2.3]

10.2.7. We now record the following:

Proposition 10.2.8.

(a) *The prestack* $\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})$ *is an affine scheme.*

(b) *The affine scheme* $\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})$ *is almost of finite type if* h *is finite-dimensional in each degree.*

Proof. Since a limit of affine schemes is an affine scheme, for the proof of point (a) we can assume that h is of the form (10.3). Then the statement follows from (10.4).

For the proof of point (b) we argue as follows. Assume that h is finite-dimensional in each degree. Then ${}^{\mathrm{cl}}\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})$ is the classical scheme that classifies maps of (classical) Lie algebras $H^0(\mathsf{h}) \to \mathsf{g}$; in particular it is a closed subscheme in the affine space of the vector space Hom from $H^0(\mathsf{h})$ to $\mathfrak{g}$, i.e.,

$$\mathrm{Tot}(\mathrm{Hom}_{\mathrm{Vect}_{\mathsf{e}}}(\mathbf{oblv}_{\mathrm{Lie}}(H^0(\mathsf{h})), \mathbf{oblv}_{\mathrm{Lie}}(\mathsf{g}))),$$

and so is of finite type. By [GR2, Chapter 1, Theorem 9.1.2], it remains to show that for a classical scheme S of finite type and an S-point F of $\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})$, the cotangent space

$$T^*_{\mathsf{F}}(\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})) \in \mathrm{QCoh}(S)^{\leq 0}$$

has coherent cohomologies.

Using Proposition 10.2.3 above and Corollary 2.2.6(b), we obtain that for $S = \mathrm{Spec}(A)$ and an S-point $\phi \in \mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})$,

$$T^*_\phi(\mathbf{Maps}_{\mathrm{Lie}}(\mathsf{h}, \mathsf{g})) \simeq \mathrm{Fib}\left(\mathsf{g}^\vee \otimes A \to \mathrm{C}_\cdot(\mathsf{h}, \mathsf{g}^\vee \otimes A)\right),$$

where $\mathsf{g}^\vee \otimes A$ is viewed as a h-module via ϕ. This implies the required assertion as $\mathrm{C}_\cdot(\mathsf{h}, -)$ can be computed by the standard Chevalley complex.

□

Combining with Proposition 10.2.3, we obtain:

Corollary 10.2.9.
(a) *The prestack* $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathsf{h}\text{-mod})^{\mathrm{rigid}})$ *is an affine scheme.*
(b) *The affine scheme* $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathsf{h}\text{-mod})^{\mathrm{rigid}})$ *is almost of finite type if* h *is finite-dimensional in each degree.*

10.3. **Proof of Theorem 10.1.3.** We are now ready to prove Theorem 10.1.3.

10.3.1. We are going to prove an equivalent statement, namely that the functor

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathsf{h}\text{-mod})^{\mathrm{rigid}} \to \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathsf{h}\text{-mod})^{\mathrm{rigid}})$$

is an equivalence.

I.e., we have to show that for a target symmetric monoidal category $\mathbf{A}$, restriction along (8.14) defines an equivalence from the space of symmetric monoidal functors

$$\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathsf{h}\text{-mod})^{\mathrm{rigid}}) \to \mathbf{A} \tag{10.6}$$

to the space of symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathsf{h}\text{-mod}, \tag{10.7}$$

equipped with an identification of the composition

$$\mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathsf{h}\text{-mod} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathbf{oblv}_{\mathsf{h}}}{\longrightarrow} \mathbf{A}$$

with

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}} \overset{\mathbf{1}_{\mathbf{A}}}{\to} \mathbf{A}.$$

10.3.2. *Step 1.* We will first show that we can assume that $\mathbf{A}$ is of the form A-mod for some $A \in \mathrm{ComAlg}(\mathrm{Vect}_{\mathsf{e}})$. Namely, we will show that both (10.6) and (10.7) factor canonically via A-mod, where

$$A := \mathcal{E}nd_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}).$$

For (10.6) this follows from Corollary 10.2.9(a): for any affine scheme $Y = \mathrm{Spec}(R)$, symmetric monoidal functors

$$\mathrm{QCoh}(Y) = R\text{-mod} \to \mathbf{A}$$

are in bijection with maps of commutative algebras $R \to \mathcal{E}nd_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}) =: A$, and the latter are the same as symmetric monoidal functors

$$R\text{-mod} \to A\text{-mod}.$$

For (10.7) we argue as follows. We have a tautological (symmetric monoidal) functor

$$A\text{-mod} =: \mathbf{A}' \to \mathbf{A}, \tag{10.8}$$

(it is not necessarily fully faithful because $\mathbf{1}_{\mathbf{A}} \in \mathbf{A}$ is not necessarily compact). We claim that (10.8) induces an isomorphism between the data of (10.7) for $\mathbf{A}'$ and $\mathbf{A}$, respectively.

Indeed, the datum of (10.7) for $\mathbf{A}$ (or $\mathbf{A}'$) amounts to defining an action of h on

$$\underline{V} \otimes \mathbf{1}_{\mathbf{A}} \tag{10.9}$$

(viewed as an object of either $\mathbf{A}$ or $\mathbf{A}'$) for every $V \in \mathrm{Rep}(\mathsf{G})^c$, in a way compatible with the tensor structure (here $V \to \underline{V}$ denotes the forgetful functor $\mathrm{Rep}(\mathsf{G}) \to \mathrm{Vect}_{\mathsf{e}}$).

The datum of such action consists of a compatible family of diagrams

$$U(\mathsf{h})^{\otimes I} \otimes \underline{V} \otimes \mathbf{1}_{\mathbf{A}} \to \underline{V} \otimes \mathbf{1}_{\mathbf{A}}, \quad V \in \mathrm{Rep}(\mathsf{G}^J)^c,\ I, J \in \mathrm{fSet}\,. \tag{10.10}$$

The assertion that (10.8) induces an isomorphism on the data (10.7) follows now from the fact that the functor (10.8) *does* induce an isomorphisms on the mapping space from objects of the form

$$W_1 \otimes \mathbf{1}_{\mathbf{A}}, \quad W_1 \in \mathrm{Vect}_{\mathsf{e}}$$

to objects of the form

$$W_2 \otimes \mathbf{1}_{\mathbf{A}}, \quad W_2 \in \mathrm{Vect}_{\mathsf{e}}^c.$$

10.3.3. *Step 2.* Thus, we can assume that $\mathbf{A} = A\text{-mod}$ for $A \in \mathrm{ComAlg}(\mathrm{Vect}_{\mathsf{e}})$. Next we claim that we can assume that A is connective. More precisely, we claim that (10.6) and (10.7) factor canonically via A'-mod, where $A' := \tau^{\leq 0}(A)$.

This is again obvious for (10.6): for $R \in \mathrm{ComAlg}(\mathrm{Vect}_{\mathsf{e}}^{\leq 0})$, a map $R \to A$ factors canonically through a map $R \to A'$.

For (10.7) we argue as follows: since $\mathrm{Rep}(\mathsf{G})$ is the derived category of its heart and the tensor product operation is t-exact, in (10.9) we can assume that $V \in \mathrm{Rep}(\mathsf{G})^{\heartsuit} \cap \mathrm{Rep}(\mathsf{G})^c$. Hence, in (10.10) we can also assume that

$$V \in \mathrm{Rep}(\mathsf{G}^J)^{\heartsuit} \cap \mathrm{Rep}(\mathsf{G}^J)^c.$$

Now, in this case, maps in (10.10), which correspond to points in

$$\mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(U(\mathsf{h})^{\otimes I} \otimes \underline{V}, A \otimes \underline{V})$$

factor canonically via

$$\mathrm{Maps}_{\mathrm{Vect}_{\mathsf{e}}}(U(\mathsf{h})^{\otimes I} \otimes \underline{V}, A' \otimes \underline{V}).$$

10.3.4. *Step 3.* Thus, we can assume that $\mathbf{A} = \mathrm{QCoh}(S)$, where S is an affine scheme. However, in this case, the spaces (10.6) and (10.7) are just the same.

□[Theorem 10.1.3]

10.4. **Back to the Betti case.** We will now show how to prove Theorem 9.1.2, along the lines of the proof of Theorem 10.1.3.

10.4.1. Let $\mathcal{X}$ be a connected object of Spc, and let $x \in \mathcal{X}$ be a base point.

It suffices to show that functor

$$\mathrm{Rep}(\mathsf{G})_{\mathsf{e}}^{\otimes \mathcal{X}} \underset{\mathrm{Rep}(\mathsf{G})}{\otimes} \mathrm{Vect}_{\mathsf{e}} \to \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X)^{\mathrm{rigid}_x})$$

is an equivalence.

I.e., we have to show that for a target symmetric monoidal category $\mathbf{A}$, the space of symmetric monoidal functors

$$\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X)^{\mathrm{rigid}_x}) \to \mathbf{A} \tag{10.11}$$

maps isomorphically to the space of symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}} \tag{10.12}$$

equipped with an identification of the composition

$$\mathrm{Rep}(\mathsf{G}) \to \mathbf{A} \otimes \mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}} \overset{\mathrm{Id} \otimes \mathrm{ev}_x}{\longrightarrow} \mathbf{A}$$

with

$$\mathrm{Rep}(\mathsf{G}) \overset{\mathbf{oblv}_{\mathsf{G}}}{\to} \mathrm{Vect}_{\mathsf{e}} \overset{\mathbf{1}_{\mathbf{A}}}{\to} \mathbf{A}.$$

10.4.2. As in the proof of Theorem 10.1.3, it suffices to show that we can replace the category $\mathbf{A}$ by a category A-mod, where A is a connective commutative algebra.

For (10.11), this follows by the same argument as in Sect. 10.3, since $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X)^{\mathrm{rigid}_x}$ is an affine scheme (by Proposition 4.5.4).

For (10.12) we argue as follows.

10.4.3. Let

$$\Omega(\mathcal{X}, x) \in \mathrm{Grp}(\mathrm{Spc})$$

be the loop space of $\mathcal{X}$ based at x. We interpret $\mathrm{Vect}_{\mathsf{e}}^{\mathcal{X}}$ as

$$\Omega(\mathcal{X}, x)\text{-mod}(\mathrm{Vect}_{\mathsf{e}}).$$

Then the datum in (10.12) amounts to the datum of tensor-compatible collection of actions of $\Omega(\mathcal{X}, x)$ on the objects

$$\underline{V} \otimes \mathbf{1}_{\mathbf{A}}, \quad V \in \mathrm{Rep}(\mathsf{G}).$$

Then the arguments in Sects. 10.3.2-10.3.3 apply verbatim, allowing to replace

$$\mathbf{A} \rightsquigarrow \mathcal{E}nd_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}})$$

and

$$A \rightsquigarrow \tau^{\leq 0}(A).$$

□

Remark 10.4.4. Note that an analog of Proposition 10.2.3 holds also in the present context: we can interpret (the affine scheme) $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{Betti}}(X)^{\mathrm{rigid}_x}$ as the affine scheme

$$\mathbf{Maps}_{\mathrm{Grp}}(\Omega(\mathcal{X}, x), \mathsf{G}),$$

where

$$\mathrm{Maps}(S, \mathbf{Maps}_{\mathrm{Grp}}(\Omega(\mathcal{X}, x), \mathsf{G})) := \mathrm{Maps}_{\mathrm{Grp}}(\Omega(\mathcal{X}, x), \mathrm{Maps}_{\mathrm{Sch}^{\mathrm{aff}}}(S, \mathsf{G})).$$

This follows from the fact that for $S = \mathrm{Spec}(A)$ and $\mathcal{Y} \in \mathrm{Spc}$, the datum of a system tensor-compatible maps

$$\mathcal{Y} \mapsto \mathrm{Aut}_{A\text{-mod}}(\underline{V} \otimes A), \quad V \in \mathrm{Rep}(\mathsf{G}),$$

is equivalent to the datum of a map

$$\mathcal{Y} \to \mathrm{Maps}_{\mathrm{Sch}^{\mathrm{aff}}}(S, \mathsf{G}),$$

by Tannaka duality.

11. Ran version of $\mathrm{Rep}(\mathsf{G})$ and Beilinson's spectral projector

In this section and the next sections we develop a tool that we will use in the sequel in order to produce Hecke eigensheaves.

This tool is Beilinson's spectral projector, which is an object of the *Ran version* of the category $\mathrm{Rep}(\mathsf{G})$.

11.1. **The category** $\mathrm{Rep}(\mathsf{G})_{\mathrm{Ran}}$. In this subsection we introduce the Ran version of the category $\mathrm{Rep}(\mathsf{G})$, to be denoted $\mathrm{Rep}(\mathsf{G})_{\mathrm{Ran}}$.

11.1.1. Let X be an arbitrary scheme, and let $\mathcal{C}$ be a symmetric monoidal category.

Recall that fSet denotes the category of finite sets, and TwArr(fSet) its twisted arrows category.

Consider the functor

$$\mathrm{TwArr}(\mathrm{fSet}) \to \mathrm{DGCat} \tag{11.1}$$

that at the level of objects sends

$$(I \to J) \mapsto \mathcal{C}^{\otimes I} \otimes \mathrm{Shv}(X^J).$$

At the level of morphisms, for a map

$$\begin{array}{ccc} I_1 & \longrightarrow & J_1 \\ {\scriptstyle \phi_I}\downarrow & & \uparrow{\scriptstyle \phi_J} \\ I_2 & \longrightarrow & J_2, \end{array} \tag{11.2}$$

in TwArr(fSet), the corresponding functor

$$\mathcal{C}^{\otimes I_1} \otimes \mathrm{Shv}(X^{J_1}) \to \mathcal{C}^{\otimes I_2} \otimes \mathrm{Shv}(X^{J_2}) \tag{11.3}$$

is given by the tensor product functor along the fibers of ϕ_I

$$\mathrm{mult}_{\mathcal{C}}^{\phi_I} : \mathcal{C}^{\otimes I_1} \to \mathcal{C}^{\otimes I_2} \tag{11.4}$$

and the functor

$$(\Delta_{\phi_J})_* : \mathrm{Shv}(X^{J_1}) \to \mathrm{Shv}(X^{J_2}), \tag{11.5}$$

where $\Delta_{\phi_J} : X^{J_1} \to X^{J_2}$ is the diagonal map induced by ϕ_J.

11.1.2. We define the key actor in section, the category $\mathcal{C}_{\mathrm{Ran}}$, as the colimit of the functor (11.1).

Our main example of interest is when $\mathcal{C} = \mathrm{Rep}(\mathsf{G})$, where G is an algebraic group.

Remark 11.1.3. Note that the definition of $\mathcal{C}_{\mathrm{Ran}}$ makes sense without the assumption that X be proper.

11.1.4. Let $(I \to J) \in \mathrm{TwArr}(\mathrm{fSet})$ be given. We will denote by

$$\mathrm{ins}_{I \to J} : \mathcal{C}^{\otimes I} \otimes \mathrm{Shv}(X^J) \to \mathcal{C}_{\mathrm{Ran}}$$

the corresponding functor.

11.1.5. The functor (11.1) is naturally right-lax symmetric monoidal. Therefore, the colimit $\mathcal{C}_{\mathrm{Ran}}$ carries a natural symmetric monoidal structure. Explicitly, this symmetric monoidal structure can be described as follows. For

$$V_1 \otimes \mathcal{F}_1 \in \mathcal{C}^{\otimes I_1} \otimes \mathrm{Shv}(X^{J_1}) \text{ and } V_2 \otimes \mathcal{F}_2 \in \mathcal{C}^{\otimes I_2} \otimes \mathrm{Shv}(X^{J_2}),$$

the tensor product of their images in $\mathcal{C}_{\mathrm{Ran}}$ is the image of the object

$$(V_1 \otimes V_2) \otimes (\mathcal{F}_1 \boxtimes \mathcal{F}_2) \in \mathcal{C}^{\otimes (I_1 \sqcup I_2)} \otimes \mathrm{Shv}(X^{J_1 \sqcup J_2}).$$

We will denote the resulting monoidal operation on $\mathcal{C}_{\mathrm{Ran}}$ by

$$\mathcal{V}_1, \mathcal{V}_2 \mapsto \mathcal{V}_1 \star \mathcal{V}_2.$$

We denote the unit object by $\mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}}$. It is given by

$$\mathrm{ins}_{\emptyset \to \emptyset}(\mathsf{e}), \quad \mathsf{e} \in \mathrm{Vect}_{\mathsf{e}} \simeq \mathcal{C}^{\emptyset} \otimes \mathrm{Shv}(X^{\emptyset}).$$

11.1.6. *Example.* Let $\mathcal{C} = \mathrm{Vect}_{\mathsf{e}}$. Then $\mathcal{C}_{\mathrm{Ran}} \simeq \mathrm{Vect}$. For example, an object of the form

$$\mathrm{ins}_{\psi}(\mathsf{e} \otimes \mathcal{F}), \quad \mathcal{F} \in \mathrm{Shv}(X^J)$$

is canonically isomorphic to

$$\mathrm{C}^{\cdot}(X^J, \mathcal{F}) \otimes \mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}}$$

via the following morphisms in TwArr(fSet):

$$\begin{array}{ccc} I & \xrightarrow{\psi} & J \\ \uparrow & & \downarrow \mathrm{id} \\ \emptyset & \longrightarrow & J \\ \downarrow & & \uparrow \\ \emptyset & \longrightarrow & \emptyset. \end{array}$$

11.1.7. As in Sects. 8.2.3 and 8.2.8, given a target symmetric monoidal/monoidal/plain DG category $\mathbf{A}$ there is a naturally defined map

- From the space of natural transformation of functors from fSet to
$$\mathrm{DGCat}^{\mathrm{SymMon}} / \mathrm{DGCat}^{\mathrm{Mon}} / \mathrm{DGCat},$$
$$(I \mapsto \mathcal{C}^{\otimes I}) \to (I \mapsto \mathbf{A} \otimes \mathrm{Shv}(X^I)),$$
- To the space of monoidal/monoidal/plain continuous functors $\mathcal{C}_{\mathrm{Ran}} \to \mathbf{A}$.

(Here we use the fact that the DG category $\mathrm{Shv}(X^I)$ (or in fact $\mathrm{Shv}(Y)$ on any scheme Y) is canonically self-dual by means of Verdier duality, see Sect. F.4.1.)

As in Lemma 8.2.7(b), the above map is an isomorphism for plain DG categories, However, unlike Lemma 8.2.7(a), this map fails to be an isomorphism in the associative and commutative cases.

Remark 11.1.8. Along with $\mathcal{C}_{\mathrm{Ran}}$, one can also consider the category
$$\underline{\mathrm{coHom}}(\mathcal{C}, \mathrm{Shv}(X)),$$
where $\mathrm{Shv}(X)$ is viewed as a symmetric monoidal category via the $\overset{!}{\otimes}$ operation.

The duals (which are also the right adjoints) of the functors
$$\mathrm{Shv}(X)^{\otimes J} \overset{\boxtimes}{\to} \mathrm{Shv}(X^J)$$
define a symmetric monoidal functor
$$\mathcal{C}_{\mathrm{Ran}} \to \underline{\mathrm{coHom}}(\mathcal{C}, \mathrm{Shv}(X)). \tag{11.6}$$

Remark 11.1.9. We can apply the construction of $\mathcal{C}_{\mathrm{Ran}}$ verbatim, when instead of $\mathrm{Shv}(X)$ we use the category $\mathrm{D\text{-}mod}(X)$ (when the ground field k has characteristic 0); denote the resulting symmetric monoidal category by $\mathcal{C}^{\mathrm{dR}}_{\mathrm{Ran}}$.

However, in this case, the functors
$$\mathrm{D\text{-}mod}(X)^{\otimes J} \overset{\boxtimes}{\to} \mathrm{D\text{-}mod}(X^J)$$
are equivalences. Hence, the counterpart of the functor (11.6)
$$\mathcal{C}^{\mathrm{dR}}_{\mathrm{Ran}} \to \underline{\mathrm{coHom}}(\mathcal{C}, \mathrm{D\text{-}mod}(X))$$
is an equivalence.

In a constructible de Rham context, the (fully faithful) functors $\mathrm{Shv}(-) \to \mathrm{D\text{-}mod}(-)$ induce a fully faithful symmetric monoidal functor
$$\mathcal{C}_{\mathrm{Ran}} \to \mathcal{C}^{\mathrm{dR}}_{\mathrm{Ran}}.$$

Remark 11.1.10. Similarly to Remark 11.1.9, we can also consider the category $\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}$, where now instead of $\mathrm{Shv}(X)$ we use $\mathrm{Shv}^{\mathrm{all}}(X)$ and instead of the functors $(\Delta_\phi)_*$ (which fail to be continuous unless X is proper), we use the functors $(\Delta_\phi)_!$.

In this case, the functors
$$\mathrm{Shv}^{\mathrm{all}}(X)^{\otimes J} \overset{\boxtimes}{\to} \mathrm{Shv}^{\mathrm{all}}(X^J)$$
are also equivalences (see Sect. G.1.2).

The categories $\mathrm{Shv}^{\mathrm{all}}(X^J)$ (or more generally, $\mathrm{Shv}^{\mathrm{all}}(Y)$ on any finite CW complex Y) are also canonically self-dual via the pairing
$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathrm{C}^\cdot_c(Y, \mathcal{F}_1 \overset{*}{\otimes} \mathcal{F}_2).$$

With respect to this duality, the functors $(\Delta_\phi)_!$ are dual to the functors $(\Delta_\phi)^*$. So, the (symmetric monoidal) category $\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}$ identifies with
$$\underline{\mathrm{coHom}}(\mathcal{C}, \mathrm{Shv}^{\mathrm{all}}(X)),$$
where $\mathrm{Shv}^{\mathrm{all}}(X)$ is viewed as a symmetric monoidal category via the $\overset{*}{\otimes}$ operation.

11.2. **Relation to the lisse version.** In this subsection, we will relate the category $\mathcal{C}_{\mathrm{Ran}}$ to its lisse counterpart.

11.2.1. Recall the category

$$\mathrm{Rep}(\mathsf{G})^{\otimes X\text{-lisse}} \simeq \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathrm{QLisse}(X)),$$

see Sect. 8.4.2. By the same token, we can consider the category

$$\mathcal{C}^{\otimes X\text{-lisse}} := \underline{\mathrm{coHom}}(\mathcal{C}, \mathrm{QLisse}(X))$$

for an arbitrary symmetric monoidal $\mathcal{C}$.

11.2.2. Let us note the difference between $\mathcal{C}^{\otimes X\text{-lisse}}$ and $\mathcal{C}_{\mathrm{Ran}}$. In the former the terms of the colimit are

$$\mathcal{C}^{\otimes I} \otimes (\mathrm{QLisse}(X)^{\vee})^{\otimes J}$$

and in the latter

$$\mathcal{C}^{\otimes I} \otimes \mathrm{Shv}(X^J).$$

11.2.3. We claim that there is a naturally defined symmetric monoidal functor

$$\mathcal{C}_{\mathrm{Ran}} \to \mathcal{C}^{\otimes X\text{-lisse}}. \tag{11.7}$$

In fact, there is a natural transformation between the corresponding right-lax symmetric monoidal functors $\mathrm{TwArr}(\mathrm{fSet}) \to \mathrm{DGCat}$.

Namely, for every $I \to J$, the corresponding functor is induced by the functor

$$\mathrm{Shv}(X^J) \to (\mathrm{QLisse}(X)^{\vee})^{\otimes J},$$

dual with respect to the Verdier self-duality on $\mathrm{Shv}(X^J)$ to the functor

$$\mathrm{QLisse}(X)^{\otimes J} \to \mathrm{Shv}(X)^{\otimes J} \overset{\boxtimes}{\to} \mathrm{Shv}(X^J),$$

where

$$\mathrm{QLisse}(X) \to \mathrm{Shv}(X)$$

is the embedding (1.4).

Remark 11.2.4. Note that the functor (11.7) factors as

$$\mathcal{C}_{\mathrm{Ran}} \overset{(11.6)}{\longrightarrow} \underline{\mathrm{coHom}}(\mathcal{C}, \mathrm{Shv}(X)) \to \underline{\mathrm{coHom}}(\mathcal{C}, \mathrm{QLisse}(X)),$$

where the second arrow comes from the symmetric monoidal embedding $\mathrm{QLisse}(X) \to \mathrm{Shv}(X)$ of (1.4).

11.2.5. We claim:

Proposition 11.2.6. *Let* $\mathbf{A}$ *be a dualizable DG category. Then the functor*

$$\mathrm{Funct}_{\mathrm{cont}}(\mathcal{C}^{\otimes X\text{-lisse}}, \mathbf{A}) \to \mathrm{Funct}_{\mathrm{cont}}(\mathcal{C}_{\mathrm{Ran}}, \mathbf{A}), \tag{11.8}$$

given by precomposition with (11.7) *is fully faithful.*

Proof. By Sect. 11.1.7, it suffices to show that for every $I \in \mathrm{fSet}$, the functor

$$\mathbf{A} \otimes \mathrm{QLisse}(X)^{\otimes I} \to \mathbf{A} \otimes \mathrm{Shv}(X^I)$$

is fully faithful. However, this follows from the fact that $\mathrm{QLisse}(X)^{\otimes I} \to \mathrm{Shv}(X^I)$ is fully faithful, since $\mathbf{A}$ is dualizable.

□

Corollary 11.2.7. *For any DG category* $\mathbf{A}'$*, a natural number* n *and a dualizable* $\mathbf{A}$*, the functor*

$$\mathrm{Funct}_{\mathrm{cont}}((\mathcal{C}^{\otimes X\text{-lisse}})^{\otimes n} \otimes \mathbf{A}', \mathbf{A}) \to \mathrm{Funct}_{\mathrm{cont}}((\mathcal{C}_{\mathrm{Ran}})^{\otimes n} \otimes \mathbf{A}', \mathbf{A})$$

is fully faithful.

Proof. The assertion for $n = 1$ follows from Proposition 11.2.6 by taking $\mathrm{Funct}_{\mathrm{cont}}(\mathbf{A}', -)$ into both sides of (11.8).

The assertion for $n > 1$ follows by iteration.

□

11.2.8. From Corollary 11.2.7 we obtain:

Corollary 11.2.9.
(a) *For any monoidal category* $\mathbf{A}$*, precomposition with* (11.7) *defines a* monomorphism

$$\mathrm{Maps}_{\mathrm{DGCat}^{\mathrm{Mon}}}(\mathcal{C}^{\otimes X\text{-lisse}}, \mathbf{A}) \to \mathrm{Maps}_{\mathrm{DGCat}^{\mathrm{Mon}}}(\mathcal{C}_{\mathrm{Ran}}, \mathbf{A}),$$

provided $\mathbf{A}$ *is* dualizable *as a DG category.*
(b) *For any symmetric monoidal category* $\mathbf{A}$*, precomposition with* (11.7) *defines a* monomorphism

$$\mathrm{Maps}_{\mathrm{DGCat}^{\mathrm{SymMon}}}(\mathcal{C}^{\otimes X\text{-lisse}}, \mathbf{A}) \to \mathrm{Maps}_{\mathrm{DGCat}^{\mathrm{SymMon}}}(\mathcal{C}_{\mathrm{Ran}}, \mathbf{A}),$$

provided $\mathbf{A}$ *is* dualizable *as a DG category.*
(c) *For a pair of* $\mathcal{C}^{\otimes X\text{-lisse}}$*-module categories* $\mathbf{M}_1, \mathbf{M}_2$*, the map*

$$\mathrm{Funct}_{\mathcal{C}^{\otimes X\text{-lisse}}\text{-mod}}(\mathbf{M}_1, \mathbf{M}_2) \to \mathrm{Funct}_{\mathcal{C}_{\mathrm{Ran}}\text{-mod}}(\mathbf{M}_1, \mathbf{M}_2)$$

is an isomorphism, provided $\mathbf{M}_2$ *is dualizable as a DG category.*

Remark 11.2.10. An analog of the situation described in this subsection takes place for $\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}$. In this case, we have a tautological embedding

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(X) \hookrightarrow \mathrm{Shv}^{\mathrm{all}}(X), \tag{11.9}$$

which gives rise to a symmetric monoidal functor

$$\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}} \simeq \underline{\mathrm{coHom}}(\mathcal{C}, \mathrm{Shv}^{\mathrm{all}}(X)) \to \underline{\mathrm{coHom}}(\mathcal{C}, \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(X)) \simeq \mathcal{C}^{\otimes X}.$$

The assertions and proofs of Proposition 11.2.6 and Corollary 11.2.9 remain valid in this context as well.

11.3. **Rigidity.** In this subsection we will show that $\mathcal{C}_{\mathrm{Ran}}$ is *rigid* as a monoidal category, and as a result, is canonically self-dual. We will also describe the resulting datum of self-duality explicitly.

11.3.1. We now reimpose the condition that X be proper, for the duration of this section.

We will also assume that $\mathcal{C}$ is compactly generated and *rigid*. Given compact generation, the latter condition means that compact generators of $\mathcal{C}$ are dualizable (in the sense of the symmetric monoidal structure).

11.3.2. We claim that the above conditions imply that $\mathcal{C}_{\mathrm{Ran}}$ is also compactly generated and rigid.

First, since the transition functors (11.3) preserve compactness, a set of compact generators of $\mathcal{C}_{\mathrm{Ran}}$ is provided by objects of the form

$$\mathrm{ins}_{I\to J}(V \otimes \mathcal{F}), \quad V \in (\mathcal{C}^{\otimes I})^c,\ \mathcal{F} \in \mathrm{Shv}(X^J)^c, \tag{11.10}$$

11.3.3. We now show that $\mathcal{C}_{\mathrm{Ran}}$ is rigid. To do so, it is enough to show that its compact generators are dualizable (in the sense of the symmetric monoidal structure on $\mathcal{C}_{\mathrm{Ran}}$). We will exhibit the duality data for compact generators explicitly.

Namely, for an object (11.10), its monoidal dual is given by

$$\mathrm{ins}_{I\to J}(V^\vee \otimes \mathbb{D}(\mathcal{F})),$$

where $\mathbb{D}$ denotes Verdier duality on $\mathrm{Shv}(X^J)^c$.

The unit and counit maps are defined as follows.

11.3.4. The counit is:

$$\begin{gathered}\mathrm{ins}_{I\to J}(V\otimes \mathcal{F})\otimes \mathrm{ins}_{I\to J}(V^\vee\otimes \mathbb{D}(\mathcal{F}))\simeq \mathrm{ins}_{I\sqcup I\to J\sqcup J}((V\boxtimes V^\vee)\otimes (\mathcal{F}\boxtimes \mathbb{D}(\mathcal{F})))\to \\ \to \mathrm{ins}_{I\sqcup I\to J\sqcup J}((V\boxtimes V^\vee)\otimes (\Delta_{X^J})_*(\omega_{X^J}))\simeq \mathrm{ins}_{I\sqcup I\to J}((V\boxtimes V^\vee)\otimes \omega_{X^J})\simeq \\ \simeq \mathrm{ins}_{I\to J}((V\otimes V^\vee)\otimes \omega_{X^J})\to \mathrm{ins}_{I\to J}(\mathbf{1}_{\mathcal{C}^{\otimes I}}\otimes \omega_{X^J})\simeq \mathrm{ins}_{\emptyset\to J}(\mathsf{e}\otimes \omega_{X^J})\simeq \\ \simeq \mathrm{ins}_{\emptyset\to\emptyset}(\mathsf{e}\otimes \mathrm{C}^\cdot(X^J,\omega_{X^J}))\simeq \mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}}\otimes \mathrm{C}^\cdot(X^J,\omega_{X^J})\to \mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}},\end{gathered}$$

where the last arrow is the trace map, well-defined due to the fact that X is proper.

Remark 11.3.5. In the above formula, we have used the notation

$$V\boxtimes V^\vee\in \mathcal{C}^{\otimes I}\otimes \mathcal{C}^{\otimes I},$$

to be distinguished from

$$V\otimes V^\vee\in \mathcal{C}^{\otimes I}.$$

I.e., the latter object is obtained from the former by applying the monoidal operation

$$\mathcal{C}^{\otimes I}\otimes \mathcal{C}^{\otimes I}\to \mathcal{C}^{\otimes I}.$$

11.3.6. The unit is given by

$$\begin{gathered}\mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}}\to \mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}}\otimes \mathrm{C}^\cdot(X^J,\underline{\mathsf{e}}_{X^J})\simeq \mathrm{ins}_{\emptyset\to\emptyset}(\mathsf{e}\otimes \mathrm{C}^\cdot(X^J,\underline{\mathsf{e}}_{X^J}))\simeq \mathrm{ins}_{\emptyset\to J}(\mathsf{e}\otimes \underline{\mathsf{e}}_{X^J})\simeq \\ \simeq \mathrm{ins}_{I\to J}(\mathbf{1}_{\mathcal{C}^{\otimes I}}\otimes \underline{\mathsf{e}}_{X^J})\to \mathrm{ins}_{I\to J}((V\otimes V^\vee)\otimes \underline{\mathsf{e}}_{X^J})\simeq \mathrm{ins}_{I\sqcup I\to J}((V\boxtimes V^\vee)\otimes \underline{\mathsf{e}}_{X^J})\simeq \\ \simeq \mathrm{ins}_{I\sqcup I\to J\sqcup J}((V\boxtimes V^\vee)\otimes (\Delta_{X^J})_*(\underline{\mathsf{e}}_{X^J}))\simeq \mathrm{ins}_{I\sqcup I\to J\sqcup J}((V\boxtimes V^\vee)\otimes (\Delta_{X^J})_!(\underline{\mathsf{e}}_{X^J}))\to \\ \to \mathrm{ins}_{I\sqcup I\to J\sqcup J}((V\boxtimes V^\vee)\otimes (\mathcal{F}\boxtimes \mathbb{D}(\mathcal{F})))\simeq \mathrm{ins}_{I\to J}(V\otimes \mathcal{F})\otimes \mathrm{ins}_{I\to J}(V^\vee\otimes \mathbb{D}(\mathcal{F})).\end{gathered}$$

11.3.7. Recall that if $\mathbf{A}$ is a compactly generated rigid symmetric monoidal category, then it is canonically self-dual as a DG category. Namely, the corresponding anti-equivalence

$$(\mathbf{A}^c)^{\mathrm{op}}\to \mathbf{A}^c$$

is given by monoidal dualization. (For another description of this self-duality see Sect. 11.5.1 below.)

In the next subsection we will describe explicitly the resulting self-duality on $\mathcal{C}_{\mathrm{Ran}}$.

Remark 11.3.8. The material in this subsection can be applied "as-is" to $\mathcal{C}_{\mathrm{Ran}}$ replaced by $\mathcal{C}^{\mathrm{dR}}_{\mathrm{Ran}}$.

However, the situation is different for $\mathrm{Shv}^{\mathrm{all}}(-)$ in that $\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}$ is *not* rigid (the unit object is no longer compact). Yet, it retains some features, which will make the key construction work, see Sect. 11.8.2.

11.4. **Self-duality.**

11.4.1. Let I be an index category, and let

$$i\mapsto \mathcal{C}_i,\quad (i_1\to i_2)\rightsquigarrow \mathcal{C}_{i_1}\overset{\phi_{i_1,i_2}}{\longrightarrow}\mathcal{C}_{i_2}.$$

is a functor $I\to \mathrm{DGCat}$. Denote

$$\mathcal{D}:=\underset{i\in I}{\mathrm{colim}}\,\mathcal{C}_i.$$

For $i\in I$, let ins_i denote the tautological functor $\mathcal{C}_i\to\mathcal{D}$.

11.4.2. Assume that for every 1-morphism $i_1\to i_2$ in I, the transition functor $\phi_{i_1,i_2}:\mathcal{C}_{i_1}\to\mathcal{C}_{i_2}$ admits a continuous right adjoint.

In this case we can form a functor

$$I^{\mathrm{op}}\to \mathrm{DGCat},\quad i\mapsto \mathcal{C}_i,\quad (i_1\to i_2)\rightsquigarrow \mathcal{C}_{i_2}\overset{\phi^R_{i_1,i_2}}{\longrightarrow}\mathcal{C}_{i_1}.$$

According to [GR1, Chapter 1, Proposion 2.5.7], the functors ins_i also admit continuous right adjoints. Furthermore, the resulting functor

$$\mathcal{D}\to \lim_{i\in I^{\mathrm{op}}}\mathcal{C}_i, \tag{11.11}$$

whose components are the right adjoints $(\mathrm{ins}_i)^R$, is an equivalence.

11.4.3. For future reference note that the equivalence (11.11) implies that for $d \in \mathcal{D}$, the canonical map

$$\underset{i\in I}{\operatorname{colim}}\ \operatorname{ins}_i \circ \operatorname{ins}_i^R(d) \to d \tag{11.12}$$

is an isomorphism.

11.4.4. Assume now that each $\mathcal{C}_i$ is dualizable. We can form a new functor

$$I \to \mathrm{DGCat}, \qquad i \mapsto \mathcal{C}_i^\vee, \quad (i_1 \to i_2) \rightsquigarrow \mathcal{C}_{i_1}^\vee \overset{(\phi^R_{i_1,i_2})^\vee}{\longrightarrow} \mathcal{C}_{i_1}^\vee.$$

Note that if the $\mathcal{C}_i$ are compactly generated, the functor $(\phi^R_{i_1,i_2})^\vee$, when restricted to compact objects, viewed as the functor $(\mathcal{C}^c_{i_1})^{\mathrm{op}} \to (\mathcal{C}^c_{i_2})^{\mathrm{op}}$, is the opposite of $\phi_{i_1,i_2} : \mathcal{C}^c_{i_1} \to \mathcal{C}^c_{i_2}$, see [GR1, Chapter 1, Proposition 7.3.5].

According to [DrGa2, Proposition 1.8.3], the category $\mathcal{D}$ is also dualizable, and the functor

$$\underset{i\in I}{\operatorname{colim}}\ \mathcal{C}_i^\vee \to \mathcal{D}^\vee$$

comprised of the functors

$$(\operatorname{ins}_i^R)^\vee : \mathcal{C}_i^\vee \to \mathcal{D}^\vee$$

is an equivalence.

11.4.5. To summarize, we obtain that there is a canonical duality between

$$\mathcal{D} := \underset{i\in I}{\operatorname{colim}}\ \mathcal{C}_i \text{ and } \mathcal{D}' := \underset{i\in I}{\operatorname{colim}}\ \mathcal{C}_i^\vee,$$

under which, the functor

$$\operatorname{ins}_i' : \mathcal{C}_i^\vee \to \mathcal{D}'$$

identifies with the dual of

$$\operatorname{ins}_i^R : \mathcal{D} \to \mathcal{C}_i.$$

In particular, if $\mathrm{u}_{\mathcal{D}} \in \mathcal{D} \otimes \mathcal{D}'$ denote the unit of the duality, formula (11.12) implies that we have a canonical isomorphism

$$\mathrm{u}_{\mathcal{D}} \simeq \underset{i}{\operatorname{colim}}\ (\operatorname{ins}_i \otimes \operatorname{ins}_i')(\mathrm{u}_{\mathcal{C}_i}), \tag{11.13}$$

where $\mathrm{u}_{\mathcal{C}_i} \in \mathcal{C}_i \otimes \mathcal{C}_i^\vee$ is the unit of the $(\mathcal{C}_i, \mathcal{C}_i^\vee)$ duality.

11.4.6. Suppose now that for every $i \in I$ we are given a data of self-duality

$$\mathcal{C}_i^\vee \simeq \mathcal{C}_i,$$

so that the functor

$$I^{\mathrm{op}} \to \mathrm{DGCat}, \quad i \mapsto \mathcal{C}_i^\vee, \quad (i_1 \to i_2) \rightsquigarrow \mathcal{C}_{i_2}^\vee \overset{\phi^\vee_{i_1,i_2}}{\longrightarrow} \mathcal{C}_{i_1}^\vee$$

is identifies with the functor

$$I^{\mathrm{op}} \to \mathrm{DGCat}, \quad i \mapsto \mathcal{C}_i, \quad (i_1 \to i_2) \rightsquigarrow \mathcal{C}_{i_2} \overset{\phi^R_{i_1,i_2}}{\longrightarrow} \mathcal{C}_{i_1}.$$

We obtain that in this case there is a canonical self-duality

$$\mathcal{D}^\vee \simeq \mathcal{D},$$

with respect to which we have

$$\operatorname{ins}_i^R \simeq \operatorname{ins}_i^\vee.$$

11.4.7. Applying this to $I := \text{TwArr(fSet)}$ and the functor (11.1), we obtain a self-duality

$$(\mathcal{C}_{\text{Ran}})^{\vee} \simeq \mathcal{C}_{\text{Ran}}. \tag{11.14}$$

Indeed, for an individual object $(I \overset{\psi}{\to} J) \in \text{TwArr(fSet)}$, the category

$$\mathcal{C}^{\otimes I} \otimes \text{Shv}(X^J)$$

is canonically self-dual due to:

- The canonical self-duality on $\mathcal{C}$ arising from the fact that $\mathcal{C}$ is rigid (i.e., it acts on compact objects as monoidal dualization);
- Verdier self-duality on $\text{Shv}(X^J)$.

For a 1-morphism (11.2), the functor (11.3) identifies with the dual of its right adjoint because:

- The functor (11.4) is monoidal and hence commutes with monoidal dualization on dualizable (hence, compact) objects;
- The functor $(\Delta_{\phi_J})_*$ commutes with Verdier duality, due to the assumption that X is proper.

11.4.8. According to Sect. 11.4.6, with respect to the identification

$$(\mathcal{C}_{\text{Ran}})^{\vee} \simeq \mathcal{C}_{\text{Ran}}$$

of Sect. 11.14, the dual of the functor

$$\text{ins}_{I\to J} : \mathcal{C}^{\otimes I} \otimes \text{Shv}(X^J) \to \mathcal{C}_{\text{Ran}}$$

identifies with

$$(\mathcal{C}_{\text{Ran}})^{\vee} \simeq \mathcal{C}_{\text{Ran}} \overset{(\text{ins}_{I\to J})^R}{\longrightarrow} \mathcal{C}^{\otimes I} \otimes \text{Shv}(X^J) \simeq (\mathcal{C}^{\otimes I} \otimes \text{Shv}(X^J))^{\vee}. \tag{11.15}$$

11.4.9. Unwinding the definitions, one can see that the self-duality (11.14) coincides with the one in Sect. 11.3.7.

11.5. **The progenitor of the projector.** In this subsection we introduce an object

$$\mathsf{R}_{\mathcal{C},\text{Ran}} \in \mathcal{C}_{\text{Ran}} \otimes \mathcal{C}_{\text{Ran}},$$

which will ultimately give rise to Beilinson's spectral projector.

11.5.1. Recall again that if $\mathbf{A}$ is a rigid symmetric monoidal category, it is canonically self-dual, see [GR1, Chapter 1, Sect. 9.2].

Namely, the counit is given by

$$\mathbf{A} \otimes \mathbf{A} \overset{\text{mult}_{\mathbf{A}}}{\to} \mathbf{A} \overset{\mathcal{H}om_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}},-)}{\longrightarrow} \text{Vect}_{\mathsf{e}},$$

and the unit is given by

$$\text{Vect}_{\mathsf{e}} \overset{\mathbf{1}_{\mathbf{A}}}{\to} \mathbf{A} \overset{\text{comult}_{\mathbf{A}}}{\to} \mathbf{A} \otimes \mathbf{A},$$

where the functor

$$\text{comult}_{\mathbf{A}} : \mathbf{A} \to \mathbf{A} \otimes \mathbf{A} \tag{11.16}$$

is the right adjoint to $\text{mult}_{\mathbf{A}}$.

Let $\mathsf{R}_{\mathbf{A}} \in \mathbf{A} \otimes \mathbf{A}$ denote the unit of the above self-duality on $\mathbf{A}$. I.e.,

$$\mathsf{R}_{\mathbf{A}} := \text{comult}_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}).$$

11.5.2. Let us apply the above discussion to $\mathcal{C}_{\text{Ran}}$. Let

$$\mathsf{R}_{\mathcal{C},\text{Ran}} \in \mathcal{C}_{\text{Ran}} \otimes \mathcal{C}_{\text{Ran}}$$

denote the unit of the self-duality.

This object will play a key role in the sequel. Our next goal is to describe $\mathsf{R}_{\mathcal{C},\text{Ran}}$ explicitly as a colimit.

11.5.3. Recall that if $\mathbf{A}$ is a rigid symmetric monoidal category, then the functor $\text{comult}_{\mathbf{A}}$ of (11.16) is *strictly* compatible with the $\mathbf{A}$-bimodule structure (being a right adjoint of a map of bimodules, the functor $\text{comult}_{\mathbf{A}}$ is a priori right-lax compatible with the bimodule structure).

This means that the object $\mathsf{R}_{\mathbf{A}} \in \mathbf{A} \otimes \mathbf{A}$ naturally lifts to an object of

$$\text{HC}^\bullet(\mathbf{A}, \mathbf{A} \otimes \mathbf{A}) := \text{Funct}_{(\mathbf{A}\otimes\mathbf{A})\text{-mod}}(\mathbf{A}, \mathbf{A} \otimes \mathbf{A})$$

(here $\text{HC}^\bullet(\mathbf{A}, -)$ stands for the "Hochschild cohomology" category with coefficients in a given $\mathbf{A}$-bimodule category).

In other words, we have a canonical system of isomorphisms

$$(\mathbf{a} \otimes \mathbf{1}_{\mathbf{A}}) \otimes \mathsf{R}_{\mathbf{A}} \simeq \mathsf{R}_{\mathbf{A}} \otimes (\mathbf{1}_{\mathbf{A}} \otimes \mathbf{a}), \quad \mathbf{a} \in \mathbf{A},$$

compatible with the monoidal structure on $\mathbf{A}$.

11.5.4. Applying this for $\mathbf{A} := \mathcal{C}_{\text{Ran}}$, we obtain a system of isomorphisms

$$(\mathcal{V} \otimes \mathbf{1}_{\mathcal{C}_{\text{Ran}}}) \star \mathsf{R}_{\mathcal{C},\text{Ran}} \simeq \mathsf{R}_{\mathcal{C},\text{Ran}} \star (\mathbf{1}_{\mathcal{C}_{\text{Ran}}} \otimes \mathcal{V}), \quad \mathcal{V} \in \mathcal{C}_{\text{Ran}}. \tag{11.17}$$

As we shall see, the system of isomorphisms (11.17) is the source of Hecke eigen-property of various objects that we will establish in the sequel.

Remark 11.5.5. The system of isomorphisms (11.17) is equally valid when we work with $\mathcal{C}^{\text{dR}}_{\text{Ran}}$.

11.6. **The progenitor as a colimit.**

11.6.1. Applying (11.11) to $\mathcal{C}_{\text{Ran}}$, we obtain that it can also be written as a *limit*

$$\mathcal{C}_{\text{Ran}} \simeq \lim_{(I \to J) \in (\text{TwArr(fSet)})^{\text{op}}} \mathcal{C}^{\otimes I} \otimes \text{Shv}(X^J), \tag{11.18}$$

where the transition functor corresponding to (11.2) is the tensor product of

$$(\text{mult}^{\phi_I}_{\mathcal{C}})^R : \mathcal{C}^{\otimes I_2} \to \mathcal{C}^{\otimes I_1}$$

and

$$(\Delta_{\phi_J})^! : \text{Shv}(X^{J_2}) \to \text{Shv}(X^{J_1}).$$

11.6.2. Let us apply (11.13) to the object

$$\mathsf{R}_{\mathcal{C},\text{Ran}} \in \mathcal{C}_{\text{Ran}} \otimes \mathcal{C}_{\text{Ran}}.$$

We claim:

$$\mathsf{R}_{\mathcal{C},\text{Ran}} \simeq \underset{(I \to J) \in \text{TwArr(fSet)}}{\text{colim}} (\text{ins}_{I\to J} \otimes \text{ins}_{I\to J})(\mathsf{R}_{\mathcal{C}}^{\boxtimes I} \otimes \text{u}_{\text{Shv}(X^J)}), \tag{11.19}$$

where:

- $\mathsf{R}_{\mathcal{C}} \in \mathcal{C} \otimes \mathcal{C}$ denotes the unit of the self-duality on $\mathcal{C}$, arising from the fact that $\mathcal{C}$ is a rigid symmetric monoidal category;
- $\mathsf{R}_{\mathcal{C}}^{\boxtimes I}$ denotes the I-tensor power of $\mathsf{R}_{\mathcal{C}}$, viewed as an object of $\mathcal{C}^{\otimes I} \otimes \mathcal{C}^{\otimes I}$;
- For a scheme Y, we denote by $\text{u}_{\text{Shv}(Y)} \in \text{Shv}(Y) \otimes \text{Shv}(Y)$ is the unit of the Verdier self-duality on $\text{Shv}(Y)$.

Indeed, this follows from (11.13) using the identification

$$(\text{ins}_{I\to J})^R \overset{(11.15)}{\simeq} (\text{ins}_{I\to J})^\vee$$

and the fact that $\mathsf{R}_{\mathcal{C}}^{\boxtimes I}$ is the unit of the self-duality on $\mathcal{C}^{\otimes I}$, induced by the self-duality of $\mathcal{C}$.

Remark 11.6.3. For future use, let us observe that the object $\mathrm{u}_{\mathrm{Shv}(Y)} \in \mathrm{Shv}(Y) \otimes \mathrm{Shv}(Y)$ introduced above can also be interpreted as the value on $(\Delta_Y)_*(\omega_Y)$ of the right adjoint $\boxtimes^R$ to the external tensor product functor

$$\mathrm{Shv}(Y) \otimes \mathrm{Shv}(Y) \overset{\boxtimes}{\to} \mathrm{Shv}(Y \times Y) \tag{11.20}$$

on $(\Delta_Y)_*(\omega_Y)$, see Sect. 21.2.4.

By a slight abuse of notation, we will sometimes denote by the same symbol $\mathrm{u}_{\mathrm{Shv}(Y)}$ the image of this object along the (fully faithful) functor (11.20). This is done in order to avoid the somewhat awkward notation $\boxtimes(\mathrm{u}_{\mathrm{Shv}(Y)})$.

The counit of the adjunction

$$\mathrm{u}_{\mathrm{Shv}(Y)} \to (\Delta_Y)_*(\omega_Y)$$

has the following basic property: for $\mathcal{F} \in \mathrm{Shv}(Y)$, the induced map

$$(p_2)_*(\mathrm{u}_{\mathrm{Shv}(Y)} \overset{!}{\otimes} (p_1)^!(\mathcal{F})) \to (p_2)_*((\Delta_Y)_*(\omega_Y) \overset{!}{\otimes} (p_1)^!(\mathcal{F})) \simeq \mathcal{F} \tag{11.21}$$

is an isomorphism.

Remark 11.6.4. Formula (11.19) holds also for the unit of the self-duality of $\mathcal{C}^{\mathrm{dR}}_{\mathrm{Ran}}$, with the difference that now instead of the object $\mathrm{u}_{\mathrm{Shv}(X^J)}$ we use

$$(\Delta_{X^J})_*(\omega_{X^J}) \in \text{D-mod}(X^J \times X^J) \simeq \text{D-mod}(X^J) \otimes \text{D-mod}(X^J).$$

11.6.5. We will now describe explicitly particular values of the unit and counit of the adjunction

$$\mathrm{mult}_{\mathcal{C}_{\mathrm{Ran}}} : \mathcal{C}_{\mathrm{Ran}} \otimes \mathcal{C}_{\mathrm{Ran}} \rightleftarrows \mathcal{C}_{\mathrm{Ran}} : \mathrm{comult}_{\mathcal{C}_{\mathrm{Ran}}},$$

in terms of formula (11.19).

The unit of the adjunction, when evaluated on $\mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}} \otimes \mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}} \in \mathcal{C}_{\mathrm{Ran}} \otimes \mathcal{C}_{\mathrm{Ran}}$ is a map

$$\mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}} \otimes \mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}} \to \mathsf{R}_{\mathcal{C},\mathrm{Ran}}.$$

It corresponds to the term $(I \to J) = (\emptyset \to \emptyset)$ in the colimit (11.19).

The counit of the adjunction, when evaluated on $\mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}}$, is a map

$$\mathrm{mult}_{\mathcal{C}_{\mathrm{Ran}}}(\mathsf{R}_{\mathcal{C},\mathrm{Ran}}) \to \mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}}. \tag{11.22}$$

Here is an explicit description of this map in terms of (11.19).

11.6.6. In order to describe (11.22), we need to specify a compatible system of maps

$$\mathrm{ins}_{I \sqcup I \to J \sqcup J}(\mathsf{R}_{\mathcal{C}}^{\boxtimes I} \otimes \mathrm{u}_{\mathrm{Shv}(X^J)}) \to \mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}}, \quad (I \to J) \in \mathrm{TwArr(fSet)}, \tag{11.23}$$

The map in (11.23) is the following composition:

- Using the counit of the adjunction $\mathrm{u}_{\mathrm{Shv}(X^J)} \to (\Delta_{X^J})_*(\omega_{X_J})$ we map the left-hand side in (11.23) to

$$\mathrm{ins}_{I \sqcup I \to J \sqcup J}(\mathsf{R}_{\mathcal{C}}^{\boxtimes I} \otimes (\Delta_{X^J})_*(\omega_{X_J})). \tag{11.24}$$

- The expression in (11.24) is isomorphic to

$$\mathrm{ins}_{I \sqcup I \to J}(\mathsf{R}_{\mathcal{C}}^{\boxtimes I} \otimes \omega_{X_J}). \tag{11.25}$$

- The expression in (11.25) is isomorphic to

$$\mathrm{ins}_{I \to J}((\mathrm{mult}_{\mathcal{C}}(\mathsf{R}_{\mathcal{C}}))^{\boxtimes I} \otimes \omega_{X_J}), \tag{11.26}$$

  where $\mathrm{mult}_{\mathcal{C}}(\mathsf{R}_{\mathcal{C}}) \in \mathcal{C}$ and $(\mathrm{mult}_{\mathcal{C}}(\mathsf{R}_{\mathcal{C}}))^{\boxtimes I} \in \mathcal{C}^{\otimes I}$.
- Using the counit of the adjunction $\mathrm{mult}_{\mathcal{C}}(\mathsf{R}_{\mathcal{C}}) \to \mathbf{1}_{\mathcal{C}}$, we map (11.26) to

$$\mathrm{ins}_{I \to J}((\mathbf{1}_{\mathcal{C}})^{\boxtimes I} \otimes \omega_{X_J}). \tag{11.27}$$

- The expression in (11.27) is isomorphic to

$$\mathrm{ins}_{\emptyset \to J}(\mathsf{e} \otimes \omega_{X_J}). \tag{11.28}$$

- The expression in (11.28) is isomorphic to

$$\mathrm{ins}_{\emptyset\to\emptyset}(\mathsf{e}\otimes \mathrm{C}^{\cdot}(X^J,\omega_{X^J})). \tag{11.29}$$

- Using the trace map $\mathrm{C}^{\cdot}(X^J,\omega_{X^J})\to \mathsf{e}$, we map (11.29) to

$$\mathrm{ins}_{\emptyset\to\emptyset}(\mathsf{e}\otimes \mathsf{e})=\mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}}.$$

11.7. **Explicit construction of the Hecke isomorphisms.** We have deduced the system of isomorphisms (11.17) from the rigidity property of $\mathcal{C}_{\mathrm{Ran}}$. However, one can prove it by a direct computation if we take formula (11.19) as the *definition* of $\mathsf{R}_{\mathcal{C},\mathrm{Ran}}$.

11.7.1. Let $\mathcal{V}\in\mathcal{C}_{\mathrm{Ran}}$ be of the form

$$\mathrm{ins}_{I_0\to J_0}(V\otimes\mathcal{F}),\quad V\in\mathcal{C}^{\otimes I_0},\ \mathcal{F}\in\mathrm{Shv}(X^{J_0}).$$

Let us construct the corresponding isomorphism

$$(\mathcal{V}\otimes\mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}})\star\mathsf{R}_{\mathcal{C},\mathrm{Ran}}\simeq \mathsf{R}_{\mathcal{C},\mathrm{Ran}}\star(\mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}}\otimes\mathcal{V}),\quad \mathcal{V}\in\mathcal{C}_{\mathrm{Ran}}. \tag{11.30}$$

Namely, we claim that each side in (11.30) can be identified with the corresponding side in

$$\underset{(I\to J)\in\mathrm{TwArr(fSet)}}{\mathrm{colim}}\,(\mathrm{ins}_{I_0\sqcup I\to J_0\sqcup J}\otimes\mathrm{ins}_{I_0\sqcup I\to J_0\sqcup J})\left((V\otimes\mathsf{R}_{\mathcal{C}}^{\boxtimes I_0\sqcup I})\otimes(\mathcal{F}\overset{!}{\otimes}\mathrm{u}_{\mathrm{Shv}(X^{J_0\sqcup J})})\right)\simeq \tag{11.31}$$
$$\simeq \underset{(I\to J)\in\mathrm{TwArr(fSet)}}{\mathrm{colim}}\,(\mathrm{ins}_{I_0\sqcup I\to J_0\sqcup J}\otimes\mathrm{ins}_{I_0\sqcup I\to J_0\sqcup J})\left((\mathsf{R}_{\mathcal{C}}^{\boxtimes I_0\sqcup I}\otimes V)\otimes(\mathrm{u}_{\mathrm{Shv}(X^{J_0\sqcup J})}\overset{!}{\otimes}\mathcal{F})\right),$$

where:

- $V\otimes\mathsf{R}_{\mathcal{C}}^{\boxtimes I_0\sqcup I}$ and $\mathsf{R}_{\mathcal{C}}^{\boxtimes I_0\sqcup I}\otimes V$ are (isomorphic) objects of $\mathcal{C}^{\otimes I_0\sqcup I}\otimes\mathcal{C}^{\otimes I_0\sqcup I}$ obtained by tensoring the object $\mathsf{R}_{\mathcal{C}}^{\boxtimes I_0\sqcup I}$ by

  $V\otimes\mathsf{e}\in\mathcal{C}^{\otimes I_0}\otimes\mathcal{C}^{\otimes I_0}\to\mathcal{C}^{\otimes I_0\sqcup I}\otimes\mathcal{C}^{\otimes I_0\sqcup I}$ and $\mathsf{e}\otimes V\in\mathcal{C}^{\otimes I_0}\otimes\mathcal{C}^{\otimes I_0}\to\mathcal{C}^{\otimes I_0\sqcup I}\otimes\mathcal{C}^{\otimes I_0\sqcup I}$,

  respectively.
- $\mathcal{F}\overset{!}{\otimes}\mathrm{u}_{\mathrm{Shv}(X^{J_0\sqcup J})}$ and $\mathrm{u}_{\mathrm{Shv}(X^{J_0\sqcup J})}\overset{!}{\otimes}\mathcal{F}$ are (isomorphic) objects of $\mathrm{Shv}(X^{J_0\sqcup J})\otimes\mathrm{Shv}(X^{J_0\sqcup J})$ obtained by tensoring the object $\mathrm{u}_{\mathrm{Shv}(X^{J_0\sqcup J})}$ by the !-pullback of $\mathcal{F}$ along

  $$X^{J_0\sqcup J}\to X^{J_0}$$

  on the left and right factor, respectively.

Let us show how to identify the left-hand side of (11.30) with the left-hand side of (11.31) (the right-hand sides are handled by symmetry).

11.7.2. Let us construct a map from the left-hand side of (11.30) to the left-hand side of (11.31). Fix $(I\to J)\in\mathrm{TwArr(fSet)}$.

Step 0. We start with

$$\mathrm{ins}_{I_0\to J_0}(V\otimes\mathcal{F})\star\left((\mathrm{ins}_{I\to J}\otimes\mathrm{ins}_{I\to J})(\mathsf{R}_{\mathcal{C}}^{\boxtimes I}\otimes\mathrm{u}_{\mathrm{Shv}(X^J)})\right), \tag{11.32}$$

which is a term corresponding to $(I\to J)$ in the colimit expression in the left-hand side of (11.30) .

Step 1. The object (11.32) is canonically isomorphic to the object

$$(\mathrm{ins}_{I_0\sqcup I\to J_0\sqcup J}\otimes\mathrm{ins}_{I\to J_0\sqcup J})\left((V\boxtimes\mathsf{R}_{\mathcal{C}}^{\boxtimes I})\otimes(\mathcal{F}\overset{!}{\otimes}\mathrm{u}_{\mathrm{Shv}(X^{J_0\sqcup J})})\right), \tag{11.33}$$

where we regard $V\boxtimes\mathsf{R}_{\mathcal{C}}^{\boxtimes I}$ as an object of $\mathcal{C}^{\otimes I_0\sqcup I}\otimes\mathcal{C}^{\otimes I}$.

Indeed, this isomorphism is induced by the 1-morphism in TwArr(fSet) $\times$ TwArr(fSet), which is identity along the first factor, and

$$\begin{array}{ccc} I & \longrightarrow & J_0 \sqcup J \\ {\scriptstyle \mathrm{id}}\downarrow & & \uparrow \\ I & \longrightarrow & J \end{array}$$

along the second factor.

Step 2. Next, we consider the object

$$(\mathrm{ins}_{I_0\sqcup I_0\sqcup I\to J_0\sqcup J}\otimes \mathrm{ins}_{I_0\sqcup I\to J_0\sqcup J})\left(((V\otimes \mathsf{e}\otimes \mathsf{e})\boxtimes \mathsf{R}_{\mathcal{C}}^{\boxtimes I})\otimes (\mathcal{F}\overset{!}{\otimes} \mathrm{u}_{\mathrm{Shv}(X^{J_0\sqcup J})})\right), \tag{11.34}$$

where we view $V\otimes \mathsf{e}\otimes \mathsf{e}$ as an object of $\mathcal{C}^{\otimes I_0\sqcup I_0}\otimes \mathcal{C}^{\otimes I_0}$, and $(V\otimes \mathsf{e}\otimes \mathsf{e})\boxtimes \mathsf{R}_{\mathcal{C}}^{\boxtimes I}$ as an object of $\mathcal{C}^{\otimes I_0\sqcup I_0\sqcup I}\otimes \mathcal{C}^{\otimes I_0\sqcup I}$.

We have a canonical isomorphism from (11.33) to (11.34), induced by the inclusions

$$I_0\sqcup I\hookrightarrow I_0\sqcup I_0\sqcup I \text{ and } I\hookrightarrow I_0\sqcup I.$$

Step 3. Consider the object

$$(\mathrm{ins}_{I_0\sqcup I_0\sqcup I\to J_0\sqcup J}\otimes \mathrm{ins}_{I_0\sqcup I\to J_0\sqcup J})\left((V\boxtimes \mathsf{R}_{\mathcal{C}}^{\boxtimes I_0\sqcup I})\otimes (\mathcal{F}\overset{!}{\otimes} \mathrm{u}_{\mathrm{Shv}(X^{J_0\sqcup J})})\right), \tag{11.35}$$

where we regard $V\boxtimes \mathsf{R}_{\mathcal{C}}^{\boxtimes I_0\sqcup I}$ as an object of $\mathcal{C}^{\otimes I_0\sqcup I_0\sqcup I}\otimes \mathcal{C}^{\otimes I_0\sqcup I}$.

We have a canonically defined map from (11.34) to (11.35), given by

$$\mathsf{e}\otimes \mathsf{e}\to \mathsf{R}_{\mathcal{C}}^{\boxtimes I_0}.$$

Step 4. The object (11.35) admits a canonical isomorphism to

$$(\mathrm{ins}_{I_0\sqcup I\to J_0\sqcup J}\otimes \mathrm{ins}_{I_0\sqcup I\to J_0\sqcup J})\left((V\otimes \mathsf{R}_{\mathcal{C}}^{\boxtimes I_0\sqcup I})\otimes (\mathcal{F}\overset{!}{\otimes} \mathrm{u}_{\mathrm{Shv}(X^{J_0\sqcup J})})\right). \tag{11.36}$$

This isomorphism is induced by the 1-morphism in TwArr(fSet) $\times$ TwArr(fSet), which is identity along the second factor and

$$\begin{array}{ccc} I_0\sqcup I_0\sqcup I & \longrightarrow & J_0 \sqcup J \\ \downarrow & & \uparrow{\scriptstyle \mathrm{id}} \\ I_0\sqcup I & \longrightarrow & J_0\sqcup J \end{array}$$

along the first factor.

Final step. Finally, the object (11.36) is the term corresponding to $(I\to J)$ in the colimit expression in the left-hand side of (11.31).

11.7.3. Let us now construct a map from the left-hand side of (11.31) to the left-hand side of (11.30).

Step 0. By Step 4 in Sect. 11.7.2, the term corresponding to $(I\to J)$ in the colimit expression in the left-hand side of (11.31) is isomorphic to (11.35).

Step 1. We have a canonical isomorphism between (11.35) and

$$(\mathrm{ins}_{I_0\sqcup I_0\sqcup I\to J_0\sqcup J_0\sqcup J}\otimes \mathrm{ins}_{I_0\sqcup I\to J_0\sqcup J}) \left((V\boxtimes \mathsf{R}_{\mathcal{C}}^{\boxtimes I_0\sqcup I})\otimes ((\Delta_{X^{J_0}}\times \mathrm{id}_{X^J})_*\otimes \mathrm{Id})(\mathcal{F}\overset{!}{\otimes} \mathrm{u}_{\mathrm{Shv}(X^{J_0\sqcup J})})\right), \tag{11.37}$$

where $(\Delta_{X^{J_0}}\times \mathrm{id}_{X^J})_*\otimes \mathrm{Id}$ is the functor

$$\mathrm{Shv}(X^{J_0\sqcup J})\otimes \mathrm{Shv}(X^{J_0\sqcup J})\to \mathrm{Shv}(X^{J_0\sqcup J_0\sqcup J})\otimes \mathrm{Shv}(X^{J_0\sqcup J}).$$

This isomorphism is defined using the 1-morphism in $\mathrm{TwArr}(\mathrm{fSet})\times\mathrm{TwArr}(\mathrm{fSet})$, which is is identity along the second factor and

$$\begin{array}{ccc} I_0\sqcup I_0\sqcup I & \longrightarrow & J_0\sqcup J \\ {\scriptstyle \mathrm{id}}\downarrow & & \uparrow \\ I_0\sqcup I_0\sqcup I & \longrightarrow & J_0\sqcup J_0\sqcup J \end{array}$$

along the first factor.

Step 2. We have a canonically defined map (11.37) to

$$(11.38)\qquad (\mathrm{ins}_{I_0\sqcup I_0\sqcup I\to J_0\sqcup J_0\sqcup J}\otimes \mathrm{ins}_{I_0\sqcup I\to J_0\sqcup J})\left((V\boxtimes \mathsf{R}_{\mathcal{C}}^{\boxtimes I_0\sqcup I})\otimes(\mathcal{F}\boxtimes \mathrm{u}_{\mathrm{Shv}(X^{J_0\sqcup J})})\right),$$

where we regard $\mathcal{F}\boxtimes \mathrm{u}_{\mathrm{Shv}(X^{J_0\sqcup J})}$ as an object of $\mathrm{Shv}(X^{J_0\sqcup J_0\sqcup J})\otimes\mathrm{Shv}(X^{J_0\sqcup J})$.

This map is induced by the morphism

$$((\Delta_{X^{J_0}}\times \mathrm{id}_{X^J})_*\otimes \mathrm{Id})(\mathcal{F}\overset{!}{\otimes} \mathrm{u}_{\mathrm{Shv}(X^{J_0\sqcup J})})\to(\mathcal{F}\boxtimes \mathrm{u}_{\mathrm{Shv}(X^{J_0\sqcup J})}),$$

arising by adjunction from the isomorphism

$$\mathcal{F}\overset{!}{\otimes} \mathrm{u}_{\mathrm{Shv}(X^{J_0\sqcup J})}\simeq((\Delta_{X^{J_0}}\times \mathrm{id}_{X^J})^!\otimes \mathrm{Id})(\mathcal{F}\boxtimes \mathrm{u}_{\mathrm{Shv}(X^{J_0\sqcup J})}).$$

Final step. The object (11.38) is the term corresponding to $(I_0\sqcup I\to J_0\sqcup J)$ in the colimit expression in the right-hand side of (11.30).

11.7.4. One shows that by a routine diagram chase that the two maps between the left-hand side of (11.30) and the left-hand side of (11.31) are mutually inverse.

11.8. **The progenitor for coHom.**

11.8.1. Let $\mathcal{C}$ be a rigid symmetric monoidal category, and let $\mathbf{H}$ be another symmetric monoidal category, assumed dualizable as a plain DG category.

Assume that the functor $\mathbf{1}_{\mathbf{H}}:\mathrm{Vect}_{\mathsf{e}}\to\mathbf{H}$ admits a left adjoint, to be denoted $\mathbf{coinv}_{\mathbf{H}}$. Let us view $\mathbf{coinv}_{\mathbf{H}}$ as an object of $\mathbf{H}^\vee$. Let $\mathsf{R}_{\mathbf{H}^\vee}\in\mathbf{H}^\vee\otimes\mathbf{H}^\vee$ denote the image of $\mathbf{coinv}_{\mathbf{H}}$ under the dual of the monoidal operation $\mathrm{mult}_{\mathbf{H}}:\mathbf{H}\otimes\mathbf{H}\to\mathbf{H}$.

Consider the category $\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})$ (see Sect. 8.2.1). We will show that the functor

$$\mathrm{mult}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})}:\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})\otimes\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})\to\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})$$

admits a continuous right adjoint, to be denoted $\mathrm{comult}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})}$.

Moreover, we will show the structure on $\mathrm{comult}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})}$ of right-lax compatibility with the $\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})$-bimodule structure structure is strict. Denote

$$(11.39)\qquad \mathsf{R}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})}:=\mathrm{comult}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})}(\mathbf{1}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})})\in\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})\otimes\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H}).$$

Recall that the category $\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})$ identifies with

$$\underset{(I\to J)\in\mathrm{TwArr}(\mathrm{fSet})}{\mathrm{colim}}\ \mathcal{C}^{\otimes I}\otimes(\mathbf{H}^\vee)^{\otimes J},$$

see Sect. 8.2.3. Denote by

$$\mathrm{ins}_{I\to J}:\mathcal{C}^{\otimes I}\otimes(\mathbf{H}^\vee)^{\otimes J}\to\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})$$

the resulting tautological functors.

We will also show that

$$(11.40)\qquad \mathsf{R}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})}\simeq\underset{(I\to J)\in\mathrm{TwArr}(\mathrm{fSet})}{\mathrm{colim}}(\mathrm{ins}_{I\to J}\otimes\mathrm{ins}_{I\to J})\left((\mathsf{R}_{\mathcal{C}})^{\boxtimes I}\otimes(\mathsf{R}_{\mathbf{H}^\vee})^{\boxtimes J}\right)\in$$
$$\in\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})\otimes\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H}),$$

where we view $(\mathsf{R}_{\mathcal{C}})^{\boxtimes I}\otimes(\mathsf{R}_{\mathbf{H}^\vee})^{\boxtimes J}$ as an object of

$$(\mathcal{C}\otimes\mathcal{C})^{\otimes I}\otimes(\mathbf{H}^\vee\otimes\mathbf{H}^\vee)^{\otimes J}\simeq(\mathcal{C}^{\otimes I}\otimes(\mathbf{H}^\vee)^{\otimes J})\otimes(\mathcal{C}^{\otimes I}\otimes(\mathbf{H}^\vee)^{\otimes J}).$$

11.8.2. Let us apply the above discussion to $\mathbf{H} = \mathrm{Shv}^{\mathrm{all}}(X)$. We obtain that (although the category $\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}$ is not rigid), the functor

$$\mathrm{mult}_{\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}} : \mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}} \otimes \mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}} \to \mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}$$

admits a continuous right adjoint, to be denoted $\mathrm{comult}_{\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}}$.

Moreover, we obtain that the structure on $\mathrm{comult}_{\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}}$ of right-lax compatibility with the $\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}$-bimodule structure structure is strict. Denote

$$\mathsf{R}_{\mathcal{C},\mathrm{Ran}} := \mathrm{comult}_{\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}}(\mathbf{1}_{\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}}) \in \mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}} \otimes \mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}.$$

We obtain that $\mathsf{R}_{\mathcal{C},\mathrm{Ran}}$ naturally lifts to an object of

$$\mathrm{HC}^\bullet(\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}, \mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}} \otimes \mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}),$$

i.e., it is equipped with Hecke isomorphisms (11.17).

11.8.3. Finally, note that with respect to the canonical self-duality of $\mathrm{Shv}^{\mathrm{all}}(X)$ (see Sect. G.1.3), the object

$$\mathsf{R}_{\mathrm{Shv}^{\mathrm{all}}(X)^\vee} \in \mathrm{Shv}^{\mathrm{all}}(X)^\vee \otimes \mathrm{Shv}^{\mathrm{all}}(X)^\vee \simeq \mathrm{Shv}^{\mathrm{all}}(X) \otimes \mathrm{Shv}^{\mathrm{all}}(X) \simeq \mathrm{Shv}^{\mathrm{all}}(X \times X)$$

identifies with

$$(\Delta_X)_!(\omega_X).$$

Hence, we obtain that $\mathsf{R}_{\mathcal{C},\mathrm{Ran}}$ is given by the formula similar to (11.19), namely

$$\mathsf{R}_{\mathcal{C},\mathrm{Ran}} \simeq \underset{(I\to J)\in \mathrm{TwArr}(\mathrm{fSet})}{\mathrm{colim}} (\mathrm{ins}_{I\to J} \otimes \mathrm{ins}_{I\to J})(\mathsf{R}_{\mathcal{C}}^{\boxtimes I} \otimes (\Delta_{X^J})_!(\omega_{X^J})). \tag{11.41}$$

Remark 11.8.4. Note the difference between formulas (11.19) and (11.41): in the latter we have the objects

$$(\Delta_{X^J})_!(\omega_{X^J}) \simeq (\Delta_{X^J})_*(\omega_{X^J}) \in \mathrm{Shv}(X^J) \subset \mathrm{Shv}^{\mathrm{all}}(X^J),$$

while in the former we have

$$\mathrm{u}_{\mathrm{Shv}(X^J)} \in \mathrm{Shv}(X^J) \otimes \mathrm{Shv}(X^J).$$

Note also that, unlike the constructible contexts and that of D-mod$(-)$, the object

$$(\Delta_X)_!(\omega_X) \in \mathrm{Shv}^{\mathrm{all}}(X) \otimes \mathrm{Shv}^{\mathrm{all}}(X)$$

is *not* the unit of the canonical self-duality on $\mathrm{Shv}^{\mathrm{all}}(X)$. The unit is given by $(\Delta_X)_!(\underline{\mathsf{e}}_X)$.

11.8.5. Let us apply the discussion in Sect. 11.8.1 to $\mathbf{H} = \mathrm{QLisse}(X)$, and consider the corresponding object

$$\mathsf{R}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathrm{QLisse}(X))} \in \underline{\mathrm{coHom}}(\mathcal{C}, \mathrm{QLisse}(X)) \otimes \underline{\mathrm{coHom}}(\mathcal{C}, \mathrm{QLisse}(X)) \simeq \mathcal{C}^{\otimes X\text{-lisse}} \otimes \mathcal{C}^{\otimes X\text{-lisse}}.$$

Recall the functor

$$\mathcal{C}_{\mathrm{Ran}} \to \mathcal{C}^{\otimes X\text{-lisse}} \tag{11.42}$$

of (11.7), and let us denote by

$$\widetilde{\mathsf{R}}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathrm{QLisse}(X))} \in \mathcal{C}^{\otimes X\text{-lisse}} \otimes \mathcal{C}^{\otimes X\text{-lisse}}$$

the image of

$$\mathsf{R}_{\mathcal{C},\mathrm{Ran}} \in \mathcal{C}_{\mathrm{Ran}} \otimes \mathcal{C}_{\mathrm{Ran}}$$

along the tensor square of (11.42)

$$\mathcal{C}_{\mathrm{Ran}} \otimes \mathcal{C}_{\mathrm{Ran}} \to \mathcal{C}^{\otimes X\text{-lisse}} \otimes \mathcal{C}^{\otimes X\text{-lisse}}.$$

By adjunction, we obtain a map of commutative algebras in $\mathcal{C}^{\otimes X\text{-lisse}} \otimes \mathcal{C}^{\otimes X\text{-lisse}}$

$$\widetilde{\mathsf{R}}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathrm{QLisse}(X))} \to \mathsf{R}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathrm{QLisse}(X))}. \tag{11.43}$$

We will prove:

Proposition 11.8.6. *The map* (11.43) *is an isomorphism.*

Proof. We will show that in terms of presentations of $\widetilde{\mathsf{R}}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathrm{QLisse}(X))}$ and $\mathsf{R}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathrm{QLisse}(X))}$ as colimits, given by formulas (11.19) and (11.40), respectively, the map (11.43) is a term-wise isomorphism.

For the latter, we need to show that for a given finite set J, the functor dual to

$$\mathrm{QLisse}(X)^{\otimes J} \otimes \mathrm{QLisse}(X)^{\otimes J} \to \mathrm{Shv}(X^J) \otimes \mathrm{Shv}(X^J)$$

sends

$$\mathrm{u}_{\mathrm{Shv}(X^J)} \in \mathrm{Shv}(X^J) \otimes \mathrm{Shv}(X^J) \simeq \mathrm{Shv}(X^J)^\vee \otimes \mathrm{Shv}(X^J)^\vee$$

to the object

$$(\mathsf{R}_{\mathrm{QLisse}(X)^\vee})^{\boxtimes J} \in (\mathrm{QLisse}(X)^\vee)^{\otimes J} \otimes (\mathrm{QLisse}(X)^\vee)^{\otimes J}.$$

Note that the object $\mathrm{u}_{\mathrm{Shv}(X^J)} \in \mathrm{Shv}(X^J)^\vee \otimes \mathrm{Shv}(X^J)^\vee$ equals the value of the functor dual to

$$\mathrm{Shv}(X^J) \otimes \mathrm{Shv}(X^J) \overset{\overset{!}{\otimes}}{\to} \mathrm{Shv}(X^J)$$

on $\mathrm{C}^\cdot(X^J,-)$, viewed as an object of $\mathrm{Shv}(X^J)^\vee$.

The required assertion follows now by passing to dual functors in the commutative diagram

$$\begin{array}{ccc}
\mathrm{Shv}(X^J) \otimes \mathrm{Shv}(X^J) & \xrightarrow[\overset{!}{\otimes}]{} & \mathrm{Shv}(X^J) \\
{\scriptstyle (1.4)\otimes(1.4)}\uparrow & & \uparrow{\scriptstyle (1.4)} \\
\mathrm{QLisse}(X)^{\otimes J} \otimes \mathrm{QLisse}(X)^{\otimes J} & \longrightarrow & \mathrm{QLisse}(X)^{\otimes J},
\end{array}$$

using the fact that the composition

$$\mathrm{QLisse}(X)^{\otimes J} \overset{(1.4)}{\longrightarrow} \mathrm{Shv}(X^J) \overset{\mathrm{C}^\cdot(X^J,-)}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}}$$

identifies with $(\mathbf{coinv}_{\mathrm{QLisse}(X)})^{\otimes J}$.

□

Remark 11.8.7. An analog of the construction in Sect. 11.8.5 applies when instead of the pair $(\mathcal{C}_{\mathrm{Ran}}, \underline{\mathrm{coHom}}(\mathcal{C}, \mathrm{QLisse}(X))$ we take $(\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}, \underline{\mathrm{coHom}}(\mathcal{C}, \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const}}(X))$.

An assertion parallel to Proposition 11.8.6 continues to hold in this context, with the same proof.

11.9. **Abstract version of factorization homology.** In this subsection we prove the claims made in Sect. 11.8.1.

11.9.1. Consider the following paradigm. Let $\mathbf{A}$ and $\mathbf{A}'$ be a pair of symmetric monoidal categories, and let $\Phi : \mathbf{A}' \to \mathbf{A}$ be a symmetric monoidal functor that admits a left adjoint, denoted Φ^L, as a functor of plain DG categories.

Then the induced functor

$$\Phi : \mathrm{ComAlg}(\mathbf{A}') \to \mathrm{ComAlg}(\mathbf{A})$$

admits a left adjoint, to be denoted $\Phi^{L,\mathrm{ComAlg}}$, which is described as follows.

Define the functor

$$\widetilde{\Phi}^{L,\mathrm{ComAlg}} : \mathrm{ComAlg}(\mathbf{A}) \to \mathrm{ComAlg}(\mathbf{A}')$$

as follows:

Its value on $R \in \mathrm{ComAlg}(\mathbf{A})$, viewed as a plain object of $\mathbf{A}'$, is given by the colimit over TwArr(fSet) of the functor that sends

$$(I \overset{\psi}{\to} J) \in \mathrm{TwArr}(\mathrm{fSet}) \tag{11.44}$$

to

$$\mathrm{mult}^J_{\mathbf{A}'} \circ (\Phi^L)^{\otimes J} \circ \mathrm{mult}^\psi_{\mathbf{A}}(R^{\otimes I}),$$

where $\mathrm{mult}^J_{\mathbf{A}'}$ is the J-fold tensor product functor

$$(\mathbf{A}')^{\otimes J} \to \mathbf{A}'.$$

The structure on $\widetilde{\Phi}^{L,\mathrm{ComAlg}}$ of commutative algebra is induced by the operation of disjoint union on fSet.

Lemma 11.9.2. *The functor* $\widetilde{\Phi}^{L,\mathrm{ComAlg}}$ *is canonically isomorphic to the left adjoint, denoted* $\Phi^{L,\mathrm{ComAlg}}$, *of*

$$\Phi : \mathrm{ComAlg}(\mathbf{A}') \to \mathrm{ComAlg}(\mathbf{A}).$$

The proof will be given in Sect. B.2.

Remark 11.9.3. The above description of the left adjoint to $\Phi : \mathrm{ComAlg}(\mathbf{A}') \to \mathrm{ComAlg}(\mathbf{A})$ is most familiar in the context of *factorization homology.* Namely, take

$$\mathbf{A} = (\mathrm{Shv}(X), \overset{!}{\otimes}),\ \mathbf{A}' = \mathrm{Vect}_{\mathsf{e}},\ \Phi(\mathsf{e}) = \omega_X.$$

Then the functor $\widetilde{\Phi}^{L,\mathrm{ComAlg}}$ evaluated on $R \in \mathrm{ComAlg}^!(\mathrm{Shv}(X))$ sends to

$$\underset{(I \overset{\psi}{\to} J) \in \mathrm{TwArr(fSet)}}{\mathrm{colim}} \mathrm{C}^{\cdot}_c \left(X^J, \underset{j \in J}{\boxtimes} R^{\otimes \psi^{-1}(j)} \right),$$

which is the formula for the facorization homology of R along X.

11.9.4. We now make the following observation: let

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\Psi} & \mathbf{A}_1 \\ {\scriptstyle\Phi}\uparrow & & \uparrow{\scriptstyle\Phi_1} \\ \mathbf{A}' & \xrightarrow{\Psi'} & \mathbf{A}'_1 \end{array}$$

be a commutative diagram of symmetric monoidal categories. Note that we have natural transformations

$$\Phi_1^L \circ \Psi \to \Psi' \circ \Phi^L, \tag{11.45}$$

and

$$\Phi_1^{L,\mathrm{ComAlg}} \circ \Psi \to \Psi' \circ \Phi^{L,\mathrm{ComAlg}}. \tag{11.46}$$

Lemma 11.9.5. *If the natural transformation* (11.45) *is an isomorphism, then so is* (11.46).

Proof. The category $\mathrm{ComAlg}(\mathbf{A})$ is generated under sifted colimits by free objects, i.e., objects of the form $\mathrm{Sym}(\mathbf{a})$, for $\mathbf{a} \in \mathbf{A}$. Since all functors in (11.46) preserve colimits, it suffices to show that the map (11.46) is an isomorphism when evaluated on objects of the above form.

We have, tautologically:

$$\Phi^{L,\mathrm{ComAlg}}(\mathrm{Sym}(\mathbf{a})) \simeq \mathrm{Sym}(\Phi^L(\mathbf{a})).$$

And similarly, $\Phi_1^{L,\mathrm{ComAlg}}(\mathrm{Sym}(\mathbf{a}_1)) \simeq \mathrm{Sym}(\Phi_1^L(\mathbf{a}_1))$. Hence, the map (11.46), evaluated on $\mathrm{Sym}(\mathbf{a})$ identifies with

$$\mathrm{Sym}(\Phi_1^L(\Psi(\mathbf{a}_1))) \to \Psi'(\mathrm{Sym}(\Phi^L(\mathbf{a}))) \simeq \mathrm{Sym}(\Psi(\Phi^L(\mathbf{a}))),$$

which is an isomorphism by assumption.

□

11.9.6. Let $\mathbf{A}$ be a symmetric monoidal DG category, and let $R_{\mathbf{A}} \in \mathbf{A}$ be a commutative algebra object. Let $\mathbf{H}$ be a symmetric monoidal DG category as in Sect. 11.8.1.

Consider the symmetric monoidal DG categories $\underline{\mathrm{coHom}}(\mathbf{A}, \mathbf{H})$ and $\underline{\mathrm{coHom}}(R_{\mathbf{A}}\text{-mod}(\mathbf{A}), \mathbf{H})$. Define the commutative algebra object $R_{\underline{\mathrm{coHom}}(\mathbf{A},\mathbf{H})} \in \underline{\mathrm{coHom}}(\mathbf{A}, \mathbf{H})$ as follows:

It is the value of the functor

$$(\mathrm{Id} \otimes \mathbf{1_H})^{L,\mathrm{ComAlg}} : \mathrm{ComAlg}(\underline{\mathrm{coHom}}(\mathbf{A}, \mathbf{H}) \otimes \mathbf{H}) \to \mathrm{ComAlg}(\underline{\mathrm{coHom}}(\mathbf{A}, \mathbf{H}))$$

(see Lemma 11.9.2 for the notations) on the image of $R_{\mathbf{A}}$ along the tautological symmetric monoidal functor

$$\mathbf{A} \to \underline{\mathrm{coHom}}(\mathbf{A}, \mathbf{H}) \otimes \mathbf{H}.$$

11.9.7. Unwinding the definitions and using Lemma 11.9.5, we obtain:

Lemma 11.9.8. *There is a canonical equivalence*

$$\underline{\mathrm{coHom}}(R_{\mathbf{A}}\text{-mod}(\mathbf{A}), \mathbf{H}) \simeq R_{\underline{\mathrm{coHom}}(\mathbf{A},\mathbf{H})}\text{-mod}(\underline{\mathrm{coHom}}(\mathbf{A}, \mathbf{H})),$$

so that the symmetric monoidal functor

$$\underline{\mathrm{coHom}}(\mathbf{A}, \mathbf{H}) \to \underline{\mathrm{coHom}}(R_{\mathbf{A}}\text{-mod}(\mathbf{A}), \mathbf{H}),$$

attached by the funcoriality of $\underline{\mathrm{coHom}}(-, \mathbf{H})$ *to the symmetric monoidal functor* $\mathbf{A} \to R_{\mathbf{A}}\text{-mod}(\mathbf{A})$ *corresponds to the symmetric monoidal functor*

$$\underline{\mathrm{coHom}}(\mathbf{A}, \mathbf{H}) \to R_{\underline{\mathrm{coHom}}(\mathbf{A},\mathbf{H})}\text{-mod}(\underline{\mathrm{coHom}}(\mathbf{A}, \mathbf{H})).$$

11.9.9. Note that by Lemma 11.9.2 we obtain the following explicit description for the object $R_{\underline{\mathrm{coHom}}(\mathbf{A},\mathbf{H})}$. Let us identify $\underline{\mathrm{coHom}}(\mathbf{A}, \mathbf{H})$ with

$$\underset{(I\to J)\in \mathrm{TwArr}(\mathrm{fSet})}{\mathrm{colim}} \mathbf{A}^{\otimes I} \otimes (\mathbf{H}^\vee)^{\otimes J},$$

see Sect. 8.2.3. Denote by

$$\mathrm{ins}_{I\to J} : \mathbf{A}^{\otimes I} \otimes (\mathbf{H}^\vee)^{\otimes J} \to \underline{\mathrm{coHom}}(\mathbf{A}, \mathbf{H})$$

the resulting tautological functors.

Unwinding the definitions, we obtain:

$$R_{\underline{\mathrm{coHom}}(\mathbf{A},\mathbf{H})} \simeq \underset{(I\to J)\in \mathrm{TwArr}(\mathrm{fSet})}{\mathrm{colim}} \mathrm{ins}_{I\to J}(R_{\mathbf{A}}^{\otimes I} \otimes (\mathbf{coinv}_{\mathbf{H}})^{\otimes J}), \tag{11.47}$$

where we regard $\mathbf{coinv}_H$ as an object of $\mathbf{H}^\vee$.

11.9.10. We return to the setting of Sect. 11.8.1. Set $\mathbf{A} := \mathcal{C} \otimes \mathcal{C}$ and $R_{\mathbf{A}} = \mathsf{R}_{\mathcal{C}}$. By Barr-Beck-Lurie, the functor $\mathrm{comult}_{\mathcal{C}}$ identifies

$$\mathcal{C} \simeq \mathsf{R}_{\mathcal{C}}\text{-mod}(\mathcal{C} \otimes \mathcal{C}).$$

Denote the resulting commutative algebra object of $\underline{\mathrm{coHom}}(\mathcal{C} \otimes \mathcal{C}, \mathbf{H})$, constructed in Sect. 11.9.6, by $\widetilde{\mathsf{R}}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})}$. By Lemma 11.9.8, we obtain an equivalence

$$\underline{\mathrm{coHom}}(\mathcal{C}, \mathbf{H}) \simeq \widetilde{\mathsf{R}}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})}\text{-mod}(\underline{\mathrm{coHom}}(\mathcal{C} \otimes \mathcal{C}, \mathbf{H})),$$

so that the symmetric monoidal functor

$$\underline{\mathrm{coHom}}(\mathcal{C} \otimes \mathcal{C}, \mathbf{H}) \overset{\mathrm{mult}_{\mathcal{C}}}{\longrightarrow} \underline{\mathrm{coHom}}(\mathcal{C}, \mathbf{H}) \tag{11.48}$$

corresponds to the symmetric monoidal functor

$$\underline{\mathrm{coHom}}(\mathcal{C} \otimes \mathcal{C}, \mathbf{H}) \to \widetilde{\mathsf{R}}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})}\text{-mod}(\underline{\mathrm{coHom}}(\mathcal{C} \otimes \mathcal{C}, \mathbf{H})).$$

In particular, we obtain that the right adjoint of the functor (11.48) identifies with the forgetful functor

$$\widetilde{\mathsf{R}}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})}\text{-mod}(\underline{\mathrm{coHom}}(\mathcal{C} \otimes \mathcal{C}, \mathbf{H})) \to \underline{\mathrm{coHom}}(\mathcal{C} \otimes \mathcal{C}, \mathbf{H}),$$

and hence is continuous, respects the $\underline{\mathrm{coHom}}(\mathcal{C} \otimes \mathcal{C}, \mathbf{H})$-module structure, and its value on $\mathbf{1}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})}$ is given by

$$\widetilde{\mathsf{R}}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})} \simeq \underset{(I\to J)\in \mathrm{TwArr}(\mathrm{fSet})}{\mathrm{colim}} \mathrm{ins}_{I\to J}((\mathsf{R}_{\mathcal{C}})^{\otimes I} \otimes \mathbf{coinv}_{\mathbf{H}}^{\otimes J}) \in \underline{\mathrm{coHom}}(\mathcal{C} \otimes \mathcal{C}, \mathbf{H}).$$

11.9.11. Note that for a pair of symmetric monoidal categories $\mathcal{C}_1$ and $\mathcal{C}_2$ we have a natural identification

$$\underline{\mathrm{coHom}}(\mathcal{C}_1, \mathbf{H}) \otimes \underline{\mathrm{coHom}}(\mathcal{C}_2, \mathbf{H}) \simeq \underline{\mathrm{coHom}}(\mathcal{C}_1 \otimes \mathcal{C}_2, \mathbf{H}).$$

Let us take $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$. We obtain an equivalence

$$\underline{\mathrm{coHom}}(\mathcal{C}, \mathbf{H}) \otimes \underline{\mathrm{coHom}}(\mathcal{C}, \mathbf{H}) \simeq \underline{\mathrm{coHom}}(\mathcal{C} \otimes \mathcal{C}, \mathbf{H}), \tag{11.49}$$

so that the functor (11.48) identifies with

$$\mathrm{mult}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})} : \underline{\mathrm{coHom}}(\mathcal{C}, \mathbf{H}) \otimes \underline{\mathrm{coHom}}(\mathcal{C}, \mathbf{H}) \to \underline{\mathrm{coHom}}(\mathcal{C}, \mathbf{H}).$$

Hence, we obtain that its right adjoint has the properties specified in Sect. 11.8.1, and under the equivalence (11.49), the object

$$\widetilde{\mathsf{R}}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})} \in \mathrm{ComAlg}(\underline{\mathrm{coHom}}(\mathcal{C} \otimes \mathcal{C}, \mathbf{H}))$$

corresponds to the object

$$\mathsf{R}_{\underline{\mathrm{coHom}}(\mathcal{C},\mathbf{H})} \in \mathrm{ComAlg}(\underline{\mathrm{coHom}}(\mathcal{C}, \mathbf{H}) \otimes \underline{\mathrm{coHom}}(\mathcal{C}, \mathbf{H})).$$

Finally, it is easy to see that under the equivalence (11.49), the colimit expression

$$\underset{(I \to J) \in \mathrm{TwArr(fSet)}}{\mathrm{colim}} \mathrm{ins}_{I \to J}((\mathsf{R}_{\mathcal{C}})^{\otimes I} \otimes \mathbf{coinv}_{\mathbf{H}}^{\otimes J})$$

coincides term-wise with

$$\underset{(I \to J) \in \mathrm{TwArr(fSet)}}{\mathrm{colim}} (\mathrm{ins}_{I \to J} \otimes \mathrm{ins}_{I \to J}) \left((\mathsf{R}_{\mathcal{C}})^{\boxtimes I} \otimes (\mathsf{R}_{\mathbf{H}^\vee})^{\boxtimes J} \right).$$

11.10. **Identification of the diagonal.**

11.10.1. In the setting of Sect. 11.8.1, let us take $\mathcal{C} = \mathrm{Rep}(\mathsf{G})$, and let $\mathbf{H}$ be as in Sect. 8.3.1. Consider the functor

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \to \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \tag{11.50}$$

of (8.8).

Consider the object

$$\mathsf{R}_{\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})} \in \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \otimes \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}),$$

see (11.39).

Let us denote by

$$\mathsf{R}_{\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})} \in \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \otimes \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))$$

the image of $\mathsf{R}_{\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})}$ along the tensor square of the functor (11.50)

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \otimes \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \to \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \otimes \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})).$$

By adjunction, we obtain a map

$$\mathsf{R}_{\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})} \to (\Delta_{\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})})_*(\mathcal{O}_{\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}),\mathbf{H})}) \tag{11.51}$$

of commutative algebras in

$$\mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \otimes \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \simeq$$
$$\simeq \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \times \mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})).$$

The goal of this subsection is to prove the following result:

Theorem 11.10.2. *The map* (11.51) *is an isomorphism.*

Remark 11.10.3. Note that if $\mathbf{H}$ is adapted for spectral decomposition, the statement of Theorem 11.10.2 is tautological.

In general, we hope that Theorem 11.10.2 goes some way in the direction of the proof of Conjecture 8.3.6.

Remark 11.10.4. Our main interest is the case when $\mathbf{H} = \text{QLisse}(X)$ (provided that $\text{QLisse}(X)$ is dualizable). Note, however, that by the previous remark, if X is a complete algebraic curve, the assertion of Theorem 11.10.2 in this case is already known, due to Theorem 8.3.7.

Remark 11.10.5. Note that Theorem 11.10.2 is applicable also to $\mathbf{H}$ being $\text{D-mod}(X)$ or $\text{Shv}_{\text{loc.const}}(X)$, and hence it gives rise to a description of

$$(\Delta_{\text{LocSys}^?_{\mathsf{G}}(X)})_*(\mathcal{O}_{\text{LocSys}^?_{\mathsf{G}}(X)}) \in \text{QCoh}(\text{LocSys}^?_{\mathsf{G}}(X)) \otimes \text{QCoh}(\text{LocSys}^?_{\mathsf{G}}(X)$$

for ? being dR or Betti in terms of $\mathsf{R}_{\text{Rep}(\mathsf{G}),\text{Ran}}$, viewed as an object in $\text{Rep}(\mathsf{G})^{\text{dR}}_{\text{Ran}}$ or $\text{Rep}(\mathsf{G})^{\text{Betti}}_{\text{Ran}}$. respectively.

11.10.6. The rest of this subsection is devoted to the proof of Theorem 11.10.2.

Let S be an affine scheme, equipped with two maps

$$\sigma_i : S \to \mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathbf{H}),$$

corresponding to symmetric monoidal functors

$$\mathsf{F}_i : \text{Rep}(\mathsf{G}) \to \text{QCoh}(S) \otimes \mathbf{H}.$$

Let us denote by

$$\mathsf{R}_{\sigma_1,\sigma_2} \in \text{QCoh}(S)$$

the commutative object equal to the pullback by means of

$$S \overset{\sigma_1,\sigma_2}{\longrightarrow} \mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathbf{H})) \times \mathbf{Maps}(\text{Rep}(\mathsf{G}), \mathbf{H})$$

of the object $\mathsf{R}_{\mathbf{Maps}(\text{Rep}(\mathsf{G}),\mathbf{H})}$.

We need to show that the space of homomorphisms of commutative algebras

$$\mathsf{R}_{\sigma_1,\sigma_2} \to \mathcal{O}_S$$

identifies canonically with the space of isomorphisms of symmetric monoidal functors $\mathsf{F}_1 \simeq \mathsf{F}_2$.

11.10.7. Consider the following general situation. Let $\mathbf{A}$ be a symmetric monoidal category, and let us be given a pair of symmetric monoidal functors

$$\mathsf{F}_1, \mathsf{F}_2 : \text{Rep}(\mathsf{G}) \to \mathbf{A}.$$

Consider the commutative algebra object $\mathsf{R}_{\mathsf{F}_1,\mathsf{F}_2} \in \mathbf{A}$, obtained by applying the symmetric monoidal functor

$$\text{Rep}(\mathsf{G}) \otimes \text{Rep}(\mathsf{G}) \overset{\mathsf{F}_1 \otimes \mathsf{F}_2}{\longrightarrow} \mathbf{A} \otimes \mathbf{A} \overset{\text{mult}_{\mathbf{A}}}{\longrightarrow} \mathbf{A}$$

to the regular representation $\mathsf{R}_{\text{Rep}(\mathsf{G})} \in \text{Rep}(\mathsf{G}) \otimes \text{Rep}(\mathsf{G})$.

Then it is easy to see that the space of isomorphisms between F_1 and F_2 identifies canonically with the space of maps of commutative algebras

$$\mathsf{R}_{\mathsf{F}_1,\mathsf{F}_2} \to \mathbf{1}_{\mathbf{A}}.$$

11.10.8. Hence, we need to show that the space of homomorphisms of commutative algebras in $\text{QCoh}(S)$

$$\mathsf{R}_{\sigma_1,\sigma_2} \to \mathcal{O}_S$$

is canonically isomorphic to the space of maps of commutative algebras in $\mathbf{H} \otimes \text{QCoh}(S)$

$$\text{mult}_{\text{QCoh}(S)\otimes\mathbf{H}} \circ (\mathsf{F}_1 \otimes \mathsf{F}_2)(\mathsf{R}_{\text{Rep}(\mathsf{G})}) \to \mathcal{O}_S \otimes \mathbf{1}_{\mathbf{H}}.$$

11.10.9. Recall the setting of Sect. 11.9.4. Set

$$\mathbf{A}' := \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}) \otimes \mathrm{Rep}(\mathsf{G}), \mathbf{H}), \quad \mathbf{A} := \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}) \otimes \mathrm{Rep}(\mathsf{G}), \mathbf{H}) \otimes \mathbf{H}, \quad \Phi = - \otimes \mathbf{1}_{\mathbf{H}},$$

$$\mathbf{A}'_1 = \mathrm{QCoh}(S), \quad \mathbf{A}_1 := \mathrm{QCoh}(S) \otimes \mathbf{H}, \quad \Phi_1 = - \otimes \mathbf{1}_{\mathbf{H}},$$

with Ψ' being the functor

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}) \otimes \mathrm{Rep}(\mathsf{G}), \mathbf{H}) \simeq \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \otimes \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}) \to$$
$$\to \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \otimes \mathrm{QCoh}(\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H})) \overset{(\sigma_1,\sigma_2)^*}{\longrightarrow} \mathrm{QCoh}(S),$$

and $\Psi = \Psi' \otimes \mathrm{Id}_{\mathbf{H}}$.

Note that the functor

$$\mathrm{Rep}(\mathsf{G}) \otimes \mathrm{Rep}(\mathsf{G}) \to \mathrm{QCoh}(S) \otimes \mathbf{H},$$

corresponding to Ψ' by adjunction, is the functor $\mathrm{mult}_{\mathrm{QCoh}(S)\otimes\mathbf{H}} \circ (\mathsf{F}_1 \otimes \mathsf{F}_2)$.

Take

$$R \in \mathrm{ComAlg}(\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}) \otimes \mathrm{Rep}(\mathsf{G}), \mathbf{H}) \otimes \mathbf{H})$$

to be equal to the image of $\mathsf{R}_{\mathrm{Rep}(\mathsf{G})}$ under the tautological functor

$$\mathrm{Rep}(\mathsf{G}) \otimes \mathrm{Rep}(\mathsf{G}) \to \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}) \otimes \mathrm{Rep}(\mathsf{G}), \mathbf{H}) \otimes \mathbf{H}.$$

Then

$$\Psi(R) \simeq \mathrm{mult}_{\mathrm{QCoh}(S)\otimes\mathbf{H}} \circ (\mathsf{F}_1 \otimes \mathsf{F}_2)(\mathsf{R}_{\mathrm{Rep}(\mathsf{G})}).$$

The required assertion follows now evaluating both sides of Lemma 11.9.5 on the above object R. $\square$[Theorem 11.10.2]

11.11. **Localization on** $\mathrm{LocSys}_{\mathsf{G}}(X)$**.**

11.11.1. Consider the symmetric monoidal functors

$$\mathrm{Rep}(\mathsf{G})_{\mathrm{Ran}} \to \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathrm{QLisse}(X))$$

of (11.42) and

$$\underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathrm{QLisse}(X)) \to \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))$$

of (11.50).

Let us denote their composition by

$$\mathrm{Loc} : \mathrm{Rep}(\mathsf{G})_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)).$$

11.11.2. Explicitly, the functor Loc sends an object of $\mathrm{Rep}(\mathsf{G})_{\mathrm{Ran}}$ of the form

$$\mathrm{ins}_{I \overset{\psi}{\to} J}(V \otimes \mathcal{F}), \quad V \in \mathrm{Rep}(\mathsf{G})^{\otimes I},\ \mathcal{F} \in \mathrm{Shv}(X^J)$$

to

$$(\mathrm{Id}_{\mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))} \otimes \mathrm{C}^{\cdot}(X^I, -)) \left(\mathcal{E}_V^I \otimes (\Delta_\psi)_*(\mathcal{F}) \right),$$

where

$$\mathcal{E}_V^I \in \mathrm{QCoh}(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)) \otimes \mathrm{QLisse}(X)^{\otimes I}$$

is as in Sect. 8.4.9.

11.11.3. As in Sect. 11.10.1, by adjunction, we obtain a map

$$(\mathrm{Loc} \otimes \mathrm{Loc})(\mathsf{R}_{\mathrm{Rep}(\mathsf{G}),\mathrm{Ran}}) \to (\Delta_{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)})_*(\mathcal{O}_{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)}). \tag{11.52}$$

Combining Proposition 11.8.6 and Theorem 11.10.2, we obtain:

Theorem 11.11.4. *The map*

$$(\mathrm{Loc} \otimes \mathrm{Loc})(\mathsf{R}_{\mathrm{Rep}(\mathsf{G}),\mathrm{Ran}}) \to (\Delta_{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)})_*(\mathcal{O}_{\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)})$$

is an isomorphism.

11.11.5. We now claim:

Proposition 11.11.6. *The tensor product functor*

$$\text{QCoh}(\text{LocSys}_{\mathsf{G}}^{\text{restr}}(X)) \underset{\text{Rep}(\mathsf{G})_{\text{Ran}}}{\otimes} \text{QCoh}(\text{LocSys}_{\mathsf{G}}^{\text{restr}}(X)) \to \text{QCoh}(\text{LocSys}_{\mathsf{G}}^{\text{restr}}(X)) \tag{11.53}$$

is an equivalence.

Proof. This is a formal consequence of Theorem 11.11.4. Namely, let $\mathbf{A}, \mathbf{A}'$ be a pair of symmetric monoidal categories such that the functors

$$\text{mult}_{\mathbf{A}} : \mathbf{A} \otimes \mathbf{A} \to \mathbf{A} \text{ and } \text{mult}_{\mathbf{A}'} : \mathbf{A}' \otimes \mathbf{A}' \to \mathbf{A}'$$

both admit continuous right adjoints, which also respect the bimodule structure.

Let $\Phi : \mathbf{A} \to \mathbf{A}'$ be a symmetric monoidal functor, such that the resulting map

$$(\Phi \otimes \Phi)(\text{mult}_{\mathbf{A}}^R(\mathbf{1}_{\mathbf{A}})) \to \text{mult}_{\mathbf{A}'}^R(\mathbf{1}_{\mathbf{A}'}),$$

obtained by adjunction, is an isomorphism.

Then we claim that the functor

$$(\mathbf{A}' \otimes \mathbf{A}') \underset{\mathbf{A} \otimes \mathbf{A}}{\otimes} \mathbf{A} \simeq \mathbf{A}' \underset{\mathbf{A}}{\otimes} \mathbf{A}' \to \mathbf{A}'$$

is an equivalence.

Indeed, the pairs

$$\mathbf{A}' \otimes \mathbf{A}' \simeq (\mathbf{A}' \otimes \mathbf{A}') \underset{\mathbf{A} \otimes \mathbf{A}}{\otimes} (\mathbf{A} \otimes \mathbf{A}) \rightleftarrows (\mathbf{A}' \otimes \mathbf{A}') \underset{\mathbf{A} \otimes \mathbf{A}}{\otimes} \mathbf{A}$$

and

$$\mathbf{A}' \otimes \mathbf{A}' \rightleftarrows \mathbf{A}'$$

are monadic, and the corresponding monads are given by tensoring with

$$(\Phi \otimes \Phi)(\text{mult}_{\mathbf{A}}^R(\mathbf{1}_{\mathbf{A}})) \text{ and } \text{mult}_{\mathbf{A}'}^R(\mathbf{1}_{\mathbf{A}'}),$$

respectively.

□

11.11.7. As a formal consequence of Proposition 11.11.6, we obtain:

Corollary 11.11.8. *Let* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *be module categories over* $\text{QCoh}(\text{LocSys}_{\mathsf{G}}^{\text{restr}}(X))$.
(a) *The functor*

$$\mathbf{M}_1 \underset{\text{Rep}(\mathsf{G})_{\text{Ran}}}{\otimes} \mathbf{M}_2 \to \mathbf{M}_1 \underset{\text{QCoh}(\text{LocSys}_{\mathsf{G}}^{\text{restr}}(X))}{\otimes} \mathbf{M}_2$$

is an equivalence.
(b) *The functor*

$$\text{Funct}_{\text{QCoh}(\text{LocSys}_{\mathsf{G}}^{\text{restr}}(X))\text{-mod}}(\mathbf{M}_1, \mathbf{M}_2) \to \text{Funct}_{\text{Rep}(\mathsf{G})_{\text{Ran}}\text{-mod}}(\mathbf{M}_1, \mathbf{M}_2)$$

is an equivalence.

Proof. Both assertions hold in the general context in which we proved Proposition 11.11.6:

For a pair of $\mathbf{A}'$-module categories $\mathbf{M}_1, \mathbf{M}_2$, we have

$$\mathbf{M}_1 \underset{\mathbf{A}}{\otimes} \mathbf{M}_2 \simeq (\mathbf{M}_1 \otimes \mathbf{M}_2) \underset{\mathbf{A} \otimes \mathbf{A}}{\otimes} \mathbf{A} \simeq (\mathbf{M}_1 \otimes \mathbf{M}_2) \underset{\mathbf{A}' \otimes \mathbf{A}'}{\otimes} (\mathbf{A}' \otimes \mathbf{A}') \underset{\mathbf{A} \otimes \mathbf{A}}{\otimes} \mathbf{A} \overset{\text{Proposition 11.11.6}}{\simeq}$$
$$\simeq (\mathbf{M}_1 \otimes \mathbf{M}_2) \underset{\mathbf{A}' \otimes \mathbf{A}'}{\otimes} \mathbf{A}' \simeq \mathbf{M}_1 \underset{\mathbf{A}'}{\otimes} \mathbf{M}_2.$$

$$\text{Funct}_{\mathbf{A}\text{-mod}}(\mathbf{M}_1, \mathbf{M}_2) \simeq \text{Funct}_{(\mathbf{A} \otimes \mathbf{A})\text{-mod}}(\mathbf{A}, \text{Funct}(\mathbf{M}_1, \mathbf{M}_2)) \simeq$$
$$\simeq \text{Funct}_{(\mathbf{A}' \otimes \mathbf{A}')\text{-mod}}((\mathbf{A}' \otimes \mathbf{A}') \underset{\mathbf{A} \otimes \mathbf{A}}{\otimes} \mathbf{A}, \text{Funct}(\mathbf{M}_1, \mathbf{M}_2)) \overset{\text{Proposition 11.11.6}}{\simeq}$$
$$\simeq \text{Funct}_{(\mathbf{A}' \otimes \mathbf{A}')\text{-mod}}(\mathbf{A}', \text{Funct}(\mathbf{M}_1, \mathbf{M}_2)) \simeq \text{Funct}_{\mathbf{A}'\text{-mod}}(\mathbf{M}_1, \mathbf{M}_2).$$

□

Remark 11.11.9. An analog of the functor Loc exists also in the context of D-mod(−), in which case this is the functor

$$\mathrm{Rep}(\mathsf{G})^{\mathrm{dR}}_{\mathrm{Ran}} \simeq \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \text{D-mod}(X)) \overset{(11.50)}{\longrightarrow} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)).$$

Note that in this context, the functor Loc admits a continuous and fully faithful right adjoint.

A counterpart of Theorem 11.11.4 in this case is Theorem 11.10.2. Hence, analogs of Proposition 11.11.6 and Corollary 11.11.8 continue to hold in this context as well.

Remark 11.11.10. An analog of the functor Loc exists also in the context of $\mathrm{Shv}^{\mathrm{all}}(-)$, in which case this is the functor

$$\mathrm{Rep}(\mathsf{G})^{\mathrm{Betti}}_{\mathrm{Ran}} \simeq \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathrm{Shv}^{\mathrm{all}}(X)) \to \underline{\mathrm{coHom}}(\mathrm{Rep}(\mathsf{G}), \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const}}(X)) \overset{(11.50)}{\longrightarrow}$$
$$\to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)).$$

where the last arrow is an equivalence by Theorem 9.1.2.

An analog of Theorem 11.11.4 continues to hold in this context, see Remark 11.8.7. Hence, analogs of Proposition 11.11.6 and Corollary 11.11.8 continue to hold in this context as well.

12. Beilinson's spectral projector–continuation

Having introduced in the previous section the progenitor of the projector, here we proceed to the definition of the projector itself.

12.1. **Beilinson's spectral projector–abstract form.** In this subsection we will finally define what we mean by the category of Hecke eigen-objects, and introduce Beilinson's spectral projector.

12.1.1. Let $\mathcal{Z}$ be a prestack (over the field of coefficients e). Let us be given a symmetric monoidal functor

$$\mathsf{F} : \mathcal{C} \to \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{QLisse}(X). \tag{12.1}$$

12.1.2. *Example.* Note that if $\mathcal{C} = \mathrm{Rep}(\mathsf{G})$, and if F is right t-exact[25], the above datum is equivalent to that of a map

$$\mathcal{Z} \to \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X).$$

12.1.3. We can interpret F as a symmetric monoidal functor

$$\mathcal{C}^{\otimes X\text{-lisse}} := \underline{\mathrm{coHom}}(\mathcal{C}, \mathrm{QLisse}(X)) \to \mathrm{QCoh}(\mathcal{Z}).$$

Precomposing with (11.7), we obtain a functor

$$\widetilde{\mathsf{F}} : \mathcal{C}_{\mathrm{Ran}} \to \mathrm{QCoh}(\mathcal{Z}). \tag{12.2}$$

Let us describe the functor $\widetilde{\mathsf{F}}$ explicitly. Its value on an object of $\mathcal{C}_{\mathrm{Ran}}$ of the form

$$\mathrm{ins}_{I \overset{\psi}{\to} J}(V \otimes \mathcal{F}), \quad V \in \mathcal{C}^{\otimes I},\ \mathcal{F} \in \mathrm{Shv}(X^J)$$

is

$$\left(\mathrm{Id}_{\mathrm{QCoh}(\mathcal{Z})} \otimes \mathrm{C}^{\cdot}(X^J, -)\right) \left(\mathsf{F}^J(\mathrm{mult}^{\psi}_{\mathcal{C}}(V)) \otimes \mathcal{F}\right),$$

where:

- $\mathrm{mult}^{\psi}_{\mathcal{C}}$ is the tensor product functor $\mathcal{C}^{\otimes I} \to \mathcal{C}^{\otimes J}$ along the fibers of ψ;
- F^J is the functor $\mathcal{C}^{\otimes J} \to \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{QLisse}(X)^{\otimes J}$ obtained from F.
- $- \otimes \mathcal{F}$ refers to the action of QLisse(−) on Shv(−) by tensor products[26].

---

[25] For any prestack $\mathcal{Z}$, the category $\mathrm{QCoh}(\mathcal{Z})$ carries a canonically defined t-structure in which an object is connective if and only if its pullback to any affine scheme is connected.

[26] This is either the $\overset{*}{\otimes}$ tensor product or, equivalently, the $\overset{!}{\otimes}$ tensor product precomposed with (1.4).

12.1.4. Let $\mathbf{M}$ be a module category over $\mathcal{C}_{\mathrm{Ran}}$. We will denote the action functor

$$\mathcal{C}_{\mathrm{Ran}} \otimes \mathbf{M} \to \mathbf{M}$$

by

$$\mathcal{V}, \mathbf{m} \mapsto \mathcal{V} \star \mathbf{m}.$$

We define the category of Hecke eigen-objects in $\mathbf{M}$ with respect to (12.1), to be denoted

$$\mathrm{Hecke}(\mathcal{Z}, \mathbf{M})_{\mathsf{F}}$$

(or simply $\mathrm{Hecke}(\mathcal{Z}, \mathbf{M})$ if no ambiguity is likely to occur), to be

$$\mathrm{HC}^{\bullet}(\mathcal{C}_{\mathrm{Ran}}, \mathbf{M} \otimes \mathrm{QCoh}(\mathcal{Z})) := \mathrm{Funct}_{(\mathcal{C}_{\mathrm{Ran}} \otimes \mathcal{C}_{\mathrm{Ran}})\text{-mod}}(\mathcal{C}_{\mathrm{Ran}}, \mathbf{M} \otimes \mathrm{QCoh}(\mathcal{Z})),$$

where we regard $\mathbf{M} \otimes \mathrm{QCoh}(\mathcal{Z})$ as a bimodule over $\mathcal{C}_{\mathrm{Ran}}$.

By definition, we can also interpret $\mathrm{Hecke}(\mathcal{Z}, \mathbf{M})$ as the category of objects $\mathbf{m} \in \mathbf{M} \otimes \mathrm{QCoh}(\mathcal{Z})$ equipped with a tensor-compatible system of isomorphisms

$$\mathcal{V} \star \mathbf{m} \simeq \mathbf{m} \otimes \widetilde{\mathsf{F}}(\mathcal{V}), \quad \mathcal{V} \in \mathcal{C}_{\mathrm{Ran}}. \tag{12.3}$$

We can regard the system of isomorphisms (12.3) as an abstract form of the Hecke eigen-property.

12.1.5. The adjunction

$$\mathrm{mult}_{\mathcal{C}_{\mathrm{Ran}}} : \mathcal{C}_{\mathrm{Ran}} \otimes \mathcal{C}_{\mathrm{Ran}} \rightleftarrows \mathcal{C}_{\mathrm{Ran}} : \mathrm{comult}_{\mathcal{C}_{\mathrm{Ran}}} \tag{12.4}$$

as $\mathcal{C}_{\mathrm{Ran}}$-bimodule categories induces an adjunction

$$\mathbf{ind}_{\mathrm{Hecke},\mathcal{Z}} : \mathbf{M} \otimes \mathrm{QCoh}(\mathcal{Z}) \rightleftarrows \mathrm{Hecke}(\mathcal{Z}, \mathbf{M})_{\mathsf{F}} : \mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}, \tag{12.5}$$

where $\mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}$ is the tautological forgetful functor.

The functor $\mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}$ is conservative, and hence monadic.

12.1.6. Let $\mathsf{R}_{\mathcal{Z},\mathsf{F}}$ (or simply $\mathsf{R}_{\mathcal{Z}}$ if no ambiguity is likely to occur) denote the object

$$(\mathrm{Id} \otimes \widetilde{\mathsf{F}})(\mathsf{R}_{\mathcal{C},\mathrm{Ran}}) \in \mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathcal{Z}).$$

We obtain that the monad on $\mathbf{M} \otimes \mathrm{QCoh}(\mathcal{Z})$ corresponding to the adjunction (12.5) is given by the action of $\mathsf{R}_{\mathcal{Z},\mathsf{F}}$.

12.1.7. Let $g : \mathcal{Z}' \to \mathcal{Z}$ be a map of prestacks, and let F' denote the composite functor

$$\mathcal{C} \xrightarrow{\mathsf{F}} \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{QLisse}(X) \xrightarrow{g^* \otimes \mathrm{Id}} \mathrm{QCoh}(\mathcal{Z}') \otimes \mathrm{QLisse}(X).$$

It follows from the definitions that in this case we have a naturally defined functor

$$g^* : \mathrm{Hecke}(\mathcal{Z}, \mathbf{M})_{\mathsf{F}} \to \mathrm{Hecke}(\mathcal{Z}', \mathbf{M})_{\mathsf{F}'}$$

that makes both diagrams

$$\begin{array}{ccc} \mathrm{Hecke}(\mathcal{Z}', \mathbf{M})_{\mathsf{F}'} & \xrightarrow{\mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}'}} & \mathbf{M} \otimes \mathrm{QCoh}(\mathcal{Z}') \\ {\scriptstyle g^*}\uparrow & & \uparrow{\scriptstyle \mathrm{Id} \otimes g^*} \\ \mathrm{Hecke}(\mathcal{Z}, \mathbf{M})_{\mathsf{F}} & \xrightarrow{\mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}} & \mathbf{M} \otimes \mathrm{QCoh}(\mathcal{Z}) \end{array}$$

and

$$\begin{array}{ccc} \mathbf{M} \otimes \mathrm{QCoh}(\mathcal{Z}') & \xrightarrow{\mathbf{ind}_{\mathrm{Hecke},\mathcal{Z}'}} & \mathrm{Hecke}(\mathcal{Z}', \mathbf{M})_{\mathsf{F}'} \\ {\scriptstyle \mathrm{Id} \otimes g^*}\uparrow & & \uparrow{\scriptstyle g^*} \\ \mathbf{M} \otimes \mathrm{QCoh}(\mathcal{Z}) & \xrightarrow{\mathbf{ind}_{\mathrm{Hecke},\mathcal{Z}}} & \mathrm{Hecke}(\mathcal{Z}, \mathbf{M})_{\mathsf{F}} \end{array}$$

commute.

12.1.8. Let us denote by $\mathbf{oblv}_{\mathrm{Hecke}}$ the (not necessarily continuous) functor

$$\mathrm{Hecke}(\mathcal{Z},\mathbf{M}) \overset{\mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}}{\longrightarrow} \mathbf{M}\otimes \mathrm{QCoh}(\mathcal{Z}) \to \mathbf{M},$$

where the second arrow is the (not necessarily continuous) right adjoint to

$$\mathbf{M} \overset{\mathrm{Id}\otimes \mathcal{O}_{\mathcal{Z}}}{\longrightarrow} \mathbf{M}\otimes \mathrm{QCoh}(\mathcal{Z}).$$

Remark 12.1.9. Suppose for a moment that $\mathcal{Z}$ is such that $\mathcal{O}_{\mathcal{Z}} \in \mathrm{QCoh}(\mathcal{Z})$ is compact (e.g., $\mathcal{Z}$ is an algebraic stack), so that the functor

$$\Gamma(\mathcal{Z},-) : \mathrm{QCoh}(\mathcal{Z}) \to \mathrm{Vect}_{\mathsf{e}}$$

is continuous. Then the functor $\mathbf{oblv}_{\mathrm{Hecke}}$ is continuous. Indeed, in the case, the corresponding functor $\mathbf{M}\otimes \mathrm{QCoh}(\mathcal{Z}) \to \mathbf{M}$ is given by

$$\mathbf{M}\otimes \mathrm{QCoh}(\mathcal{Z}) \overset{\mathrm{Id}_{\mathbf{M}}\otimes\Gamma(\mathcal{Z},-)}{\longrightarrow} \mathbf{M}.$$

12.1.10. Consider the functor, to be denoted $\mathsf{P}_{\mathcal{Z},\mathsf{F}}$ (or simply $\mathsf{P}_{\mathcal{Z}}$ if no ambiguity is likely to occur),

$$\mathbf{M} \overset{\mathrm{Id}\otimes \mathcal{O}_{\mathcal{Z}}}{\longrightarrow} \mathbf{M}\otimes \mathrm{QCoh}(\mathcal{Z}) \overset{\mathsf{R}_{\mathcal{Z},\mathsf{F}}\star -}{\longrightarrow} \mathbf{M}\otimes \mathrm{QCoh}(\mathcal{Z}).$$

We obtain that the functor $\mathsf{P}_{\mathcal{Z}}$ naturally upgrades to a functor

$$\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}} : \mathbf{M} \to \mathrm{Hecke}(\mathcal{Z},\mathbf{M}), \tag{12.6}$$

where

$$\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}} = \mathbf{ind}_{\mathrm{Hecke},\mathcal{Z}} \circ (\mathrm{Id}\otimes\mathcal{O}_{\mathcal{Z}}).$$

By Sect. 12.1.6, the functor $\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}}$ provides a left adjoint to $\mathbf{oblv}_{\mathrm{Hecke}}$.

The functor (12.6) is the abstract form of Beilinson's spectral projector: it produces Hecke eigen-objects from plain objects of $\mathbf{M}$.

12.1.11. Let us write down the object

$$\mathsf{R}_{\mathcal{Z}} \in \mathcal{C}_{\mathrm{Ran}}\otimes \mathrm{QCoh}(\mathcal{Z})$$

explicitly as a colimit. By (11.19) and the description of the functor $\widetilde{\mathsf{F}}$ in Sect. 12.1.3, it identifies with

$$\underset{(I\overset{\psi}{\to}J)\in\mathrm{TwArr}(\mathrm{fSet})}{\mathrm{colim}} \; \mathrm{ins}_{I\to J}\left(\left(\mathrm{Id}\otimes(\mathsf{F}^J\circ \mathrm{mult}^{\psi}_{\mathcal{C}})\right)(\mathsf{R}^{\boxtimes I}_{\mathcal{C}})\right), \tag{12.7}$$

where:

- $\mathsf{R}^{\boxtimes I}_{\mathcal{C}} \in (\mathcal{C}\otimes\mathcal{C})^{\otimes I} \simeq \mathcal{C}^{\otimes I}\otimes\mathcal{C}^{\otimes I}$;
- $\mathsf{F}^J\circ \mathrm{mult}^{\psi}_{\mathcal{C}} : \mathcal{C}^{\otimes I} \to \mathrm{QCoh}(\mathcal{Z})\otimes \mathrm{QLisse}(X)^{\otimes J} \simeq \mathrm{QLisse}(X)^{\otimes J}\otimes \mathrm{QCoh}(\mathcal{Z})$ is as in Sect. 12.1.3;
- We view $\mathrm{QLisse}(X)^{\otimes J}$ as a full subcategory of $\mathrm{Shv}(X^J)$ via the embedding (1.4),

so that $\left(\mathrm{Id}\otimes(\mathsf{F}^J\circ \mathrm{mult}^{\psi}_{\mathcal{C}})\right)(\mathsf{R}^{\boxtimes I}_{\mathcal{C}})$ is an object of

$$\mathcal{C}^{\otimes I}\otimes \mathrm{Shv}(X^J)\otimes \mathrm{QCoh}(\mathcal{Z}).$$

12.1.12. The material in this subsection applies "as-is" to $(\mathcal{C}_{\mathrm{Ran}}, \mathrm{QLisse}(X))$ replaced by either $(\mathcal{C}^{\mathrm{dR}}_{\mathrm{Ran}}, \mathrm{D\text{-}mod}(X))$ or $(\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}, \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(X))$.

12.2. **A multiplicativity property of the projector.** In this subsection we will establish a certain multiplicativity property of Beilinson's projector, namely, Proposition 12.2.2, that will be needed in Sect. 15.3.

12.2.1. Let us be given a pair symmetric monoidal categories $\mathcal{C}_i$, $i = 1, 2$ and symmetric monoidal functors

$$\mathsf{F}_i : \mathcal{C}_i \to \mathrm{QCoh}(\mathcal{Z}_i) \otimes \mathrm{QLisse}(X),$$

consider

$$\mathcal{C} := \mathcal{C}_1 \otimes \mathcal{C}_2$$

and the corresponding functor

$$\mathsf{F} : \mathcal{C} \to \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{QLisse}(X), \quad \mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2.$$

Consider the corresponding objects

$$\mathsf{R}_{\mathcal{Z}_i} \in \mathcal{C}_{i,\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathcal{Z}_i), \quad \mathsf{R}_{\mathcal{Z}} \in \mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathcal{Z}).$$

Note that we have a naturally defined functor

$$\mathcal{C}_{1,\mathrm{Ran}} \otimes \mathcal{C}_{2,\mathrm{Ran}} \to \mathcal{C}_{\mathrm{Ran}} \otimes \mathcal{C}_{\mathrm{Ran}} \overset{\mathrm{mult}_{\mathcal{C}_{\mathrm{Ran}}}}{\longrightarrow} \mathcal{C}_{\mathrm{Ran}}, \tag{12.8}$$

We claim:

Proposition 12.2.2. *Under the above circumstances, the functor*

$$\mathcal{C}_{1,\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathcal{Z}_1) \otimes \mathcal{C}_{2,\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathcal{Z}_2) \to \mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathcal{Z}), \tag{12.9}$$

induced by (12.8) *sends*

$$\mathsf{R}_{\mathcal{Z}_1} \otimes \mathsf{R}_{\mathcal{Z}_2} \mapsto \mathsf{R}_{\mathcal{Z}}.$$

The rest of this subsection is devoted to the proof of Proposition 12.2.2.

12.2.3. First, given a pair of rigid symmetric monoidal categories $\mathbf{A}_1$ and $\mathbf{A}_2$ and

$$\mathbf{A} := \mathbf{A}_1 \otimes \mathbf{A}$$

note that we have a canonical isomorphism

$$\mathsf{R}_{\mathbf{A}} \simeq \mathsf{R}_{\mathbf{A}_1} \otimes \mathsf{R}_{\mathbf{A}_2} \tag{12.10}$$

as objects in

$$\mathbf{A}^{\otimes 2} \simeq \mathbf{A}_1^{\otimes 2} \otimes \mathbf{A}_2^{\otimes 2}.$$

Further, for a symmetric monoidal functor

$$\phi : \mathbf{A} \to \mathbf{A}'$$

we have a canonical map

$$(\phi \otimes \phi)(\mathsf{R}_{\mathbf{A}}) \to \mathsf{R}_{\mathbf{A}'}.$$

12.2.4. Hence, we obtain a map from the image of

$$\mathsf{R}_{\mathcal{C}_1,\mathrm{Ran}} \otimes \mathsf{R}_{\mathcal{C}_2,\mathrm{Ran}} \in (\mathcal{C}_{1,\mathrm{Ran}} \otimes \mathcal{C}_{2,\mathrm{Ran}})^{\otimes 2}$$

to

$$\mathsf{R}_{\mathcal{C},\mathrm{Ran}} \in (\mathcal{C}_{\mathrm{Ran}})^{\otimes 2}$$

along the tensor square of the map (12.8). (Note, however, that this map itself is *not* an isomorphism; cf. Remark 12.2.5 below.)

The latter map induces a map from the image of $\mathsf{R}_{\mathcal{Z}_1} \otimes \mathsf{R}_{\mathcal{Z}_2}$ along (12.9) to $\mathsf{R}_{\mathcal{Z}}$. We will show that this map is an isomorphism.

Remark 12.2.5. Note that in the case of

$$\mathcal{C}^{\mathrm{dR}}_{\mathrm{Ran}} \simeq \underline{\mathrm{coHom}}(\mathcal{C}, \mathrm{D\text{-}mod}(X)),$$

the corresponding functor

$$\mathcal{C}^{\mathrm{dR}}_{1,\mathrm{Ran}} \otimes \mathcal{C}^{\mathrm{dR}}_{2,\mathrm{Ran}} \to \mathcal{C}^{\mathrm{dR}}_{\mathrm{Ran}}$$

is already an equivalence.

By (12.10), this implies that the image of

$$\mathsf{R}_{\mathcal{C}_1,\mathrm{Ran}} \otimes \mathsf{R}_{\mathcal{C}_2,\mathrm{Ran}} \in (\mathcal{C}^{\mathrm{dR}}_{1,\mathrm{Ran}} \otimes \mathcal{C}^{\mathrm{dR}}_{2,\mathrm{Ran}})^{\otimes 2}$$

in $(\mathcal{C}^{\mathrm{dR}}_{\mathrm{Ran}})^{\otimes 2}$ is canonically isomorphic to $\mathsf{R}_{\mathcal{C},\mathrm{Ran}}$.

This immediately implies the assertion of Proposition 12.2.2 in this context. A similar observation holds also for $\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}$.

12.2.6. By Sect. 12.1.11, the object $\mathsf{R}_{\mathcal{Z}} \in \mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathcal{Z})$ is the colimit

$$\underset{(I \overset{\psi}{\to} J)}{\mathrm{colim}}\ \mathrm{ins}_{I\to J}\left(\left(\mathrm{Id} \otimes (\mathsf{F}^J \circ \mathrm{mult}^{\psi}_{\mathcal{C}})\right)(\mathsf{R}^{\boxtimes I}_{\mathcal{C}})\right), \tag{12.11}$$

Similarly, the image of $\mathsf{R}_{\mathcal{Z}_1} \otimes \mathsf{R}_{\mathcal{Z}_2}$ in $\mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathcal{Z})$ is the colimit

$$\underset{(I_1 \overset{\psi_1}{\to} J_1),(I_2 \overset{\psi_2}{\to} J_2)}{\mathrm{colim}}\ \mathrm{ins}_{I_1\sqcup I_2\to J_1\sqcup J_2}\left(\left(\mathrm{Id} \otimes (\mathsf{F}^{J_1\sqcup J_2} \circ \mathrm{mult}^{\psi_1\sqcup\psi_2}_{\mathcal{C}})\right)(\mathsf{R}^{\boxtimes I_1}_{\mathcal{C}_1} \otimes \mathsf{R}^{\boxtimes I_2}_{\mathcal{C}_2})\right), \tag{12.12}$$

where we regard $\mathsf{R}^{\boxtimes I_1}_{\mathcal{C}_1} \otimes \mathsf{R}^{\boxtimes I_2}_{\mathcal{C}_2}$ as an object of $\mathcal{C}^{\otimes(I_1\sqcup I_2)}$.

The map from (12.12) to (12.11) constructed in Sect. 12.2.4 is given by the functor

$$\mathrm{TwArr(fSet)} \times \mathrm{TwArr(fSet)} \to \mathrm{TwArr(fSet)}, \quad (I_1 \to J_1) \times (I_2 \to J_2) \mapsto (I_1 \sqcup I_2 \to J_1 \sqcup J_2)$$

and the maps

$$\mathsf{R}^{\boxtimes I_1}_{\mathcal{C}_1} \otimes \mathsf{R}^{\boxtimes I_2}_{\mathcal{C}_2} \to \mathsf{R}^{\boxtimes I_1}_{\mathcal{C}} \otimes \mathsf{R}^{\boxtimes I_2}_{\mathcal{C}} \simeq \mathsf{R}^{\boxtimes (I_1\sqcup I_2)}_{\mathcal{C}}.$$

We will now construct an inverse map.

12.2.7. Consider the object

$$\underset{(I \overset{\psi}{\to} J)\in \mathrm{TwArr(fSet)}}{\mathrm{colim}}\ \mathrm{ins}_{I\sqcup I\overset{\psi,\psi}{\to} J}\left(\left(\mathrm{Id} \otimes (\mathsf{F}^J \circ \mathrm{mult}^{\psi,\psi}_{\mathcal{C}})\right)(\mathsf{R}^{\boxtimes I}_{\mathcal{C}_1} \otimes \mathsf{R}^{\boxtimes I}_{\mathcal{C}_2})\right), \tag{12.13}$$

where we regard $\mathsf{R}^{\boxtimes I}_{\mathcal{C}_1} \otimes \mathsf{R}^{\boxtimes I}_{\mathcal{C}_2}$ as an object of $\mathcal{C}^{\otimes(I\sqcup I)}$.

The maps in TwArr(fSet) given by the diagrams

$$\begin{array}{ccc} I\sqcup I & \xrightarrow{\psi,\psi} & J \\ {\scriptstyle \mathrm{id},\mathrm{id}}\downarrow & & \uparrow{\scriptstyle \mathrm{id}} \\ I & \xrightarrow{\psi} & J \end{array}$$

define an isomorphism from (12.13) to (12.11).

We now define a map from (12.13) to (12.12) to be given by mapping

$$\mathrm{ins}_{I\sqcup I\overset{\psi,\psi}{\to} J}\left(\left(\mathrm{Id} \otimes (\mathsf{F}^J \circ \mathrm{mult}^{\psi,\psi}_{\mathcal{C}})\right)(\mathsf{R}^{\boxtimes I}_{\mathcal{C}_1} \otimes \mathsf{R}^{\boxtimes I}_{\mathcal{C}_2})\right) \to$$
$$\to \mathrm{ins}_{I\sqcup I\overset{\psi\sqcup\psi}{\to} J\sqcup J}\left(\left(\mathrm{Id} \otimes (\mathsf{F}^{J\sqcup J} \circ \mathrm{mult}^{\psi\sqcup\psi}_{\mathcal{C}})\right)(\mathsf{R}^{\boxtimes I}_{\mathcal{C}_1} \otimes \mathsf{R}^{\boxtimes I}_{\mathcal{C}_2})\right)$$

using the diagram

$$\begin{array}{ccc} I\sqcup I & \xrightarrow{\psi,\psi} & J \\ {\scriptstyle \mathrm{id}\sqcup\mathrm{id}}\downarrow & & \uparrow{\scriptstyle \mathrm{id},\mathrm{id}} \\ I\sqcup I & \xrightarrow{\psi\sqcup\psi} & J\sqcup J \end{array}$$

and the natural transformation

$$(\mathrm{Id}\otimes(\Delta_{X^J})_*)\circ \mathsf{F}^J\circ \mathrm{mult}_{\mathcal{C}^{\otimes J}}\to \mathsf{F}^{J\sqcup J}.$$

12.2.8. It is a straightforward verification that the two maps

$$(12.12)\leftrightarrow(12.11),$$

constructed above, are mutually inverse.

□[Proposition 12.2.2]

12.3. **Beilinson's projector and** $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$**.**

12.3.1. Let $\mathcal{Z}$ be a prestack equipped with a map $f:\mathcal{Z}\to\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, and let $\mathbf{M}$ be a module category over $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$.

The map f gives rise to a functor F as in (12.1), and we can regard $\mathbf{M}$ as a module category over $\mathrm{Rep}(\mathsf{G})_{\mathrm{Ran}}$ via symmetric monoidal functor Loc of Sect. 11.11, so we find ourselves in the setting of Sect. 12.1.4.

Consider the category

$$\mathrm{Hecke}(\mathcal{Z},\mathbf{M}),$$

which, since $\mathrm{Rep}(\mathsf{G})_{\mathrm{Ran}}$ is rigid, we can interpret as

$$\mathbf{M}\underset{\mathrm{Rep}(\mathsf{G})_{\mathrm{Ran}}}{\otimes}\mathrm{QCoh}(\mathcal{Z})$$

(see [GR1, Chapter 1, Proposition 9.4.4] or Proposition C.2.3), and further, by Corollary 11.11.8 as

$$\mathbf{M}\underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))}{\otimes}\mathrm{QCoh}(\mathcal{Z}).$$

Unwinding the definitions, we obtain that the forgetful functor

$$\mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}:\mathrm{Hecke}(\mathcal{Z},\mathbf{M})\to\mathbf{M}\otimes\mathrm{QCoh}(\mathcal{Z})$$

identifies canonically with

$$\mathbf{M}\underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))}{\otimes}\mathrm{QCoh}(\mathcal{Z})\simeq$$

$$\simeq(\mathbf{M}\otimes\mathrm{QCoh}(\mathcal{Z}))\underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))\otimes\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))}{\otimes}\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))\overset{\mathrm{Id}\otimes(\Delta_{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)})_*}{\longrightarrow}$$

$$\simeq(\mathbf{M}\otimes\mathrm{QCoh}(\mathcal{Z}))\underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))\otimes\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))}{\otimes}\big(\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)\otimes\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))\big)\simeq$$

$$\simeq\mathbf{M}\otimes\mathrm{QCoh}(\mathcal{Z}).$$

12.3.2. Let us be in the situation of Sect. 12.1.7. It is easy to see that in this case the functor

$$g^*:\mathrm{Hecke}(\mathcal{Z},\mathbf{M})\to\mathrm{Hecke}(\mathcal{Z}',\mathbf{M})$$

identifies with

$$\mathbf{M}\underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))}{\otimes}\mathrm{QCoh}(\mathcal{Z})\overset{\mathrm{Id}\otimes g^*}{\longrightarrow}\mathbf{M}\underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))}{\otimes}\mathrm{QCoh}(\mathcal{Z}').$$

12.3.3. Assume now that $\mathcal{O}_{\mathcal{Z}}$ is compact as an object of $\mathrm{QCoh}(\mathcal{Z})$. Factoring the morphism f as

$$\mathcal{Z} \simeq \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X) \underset{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)\times \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)}{\times} (\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)\times \mathcal{Z}) \to$$

$$\to \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)\times \mathcal{Z} \to \mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$$

and using the fact that $\Delta_{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)}$ is an affine morphism, we obtain that in this case the functor f_* is continuous and compatible with $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))$-module structures.

Hence, in this case, using the above expression for $\mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}$, we obtain that the functor

$$\mathbf{oblv}_{\mathrm{Hecke}} \simeq (\mathrm{Id}_{\mathbf{M}} \otimes \Gamma(\mathcal{Z},-)) \circ \mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}$$

identifies with

$$\mathbf{M} \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))}{\otimes} \mathrm{QCoh}(\mathcal{Z}) \overset{\mathrm{Id}\otimes f_*}{\longrightarrow} \mathbf{M} \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \simeq \mathbf{M}.$$

Hence, we obtain that the left of adjoint of $\mathbf{oblv}_{\mathrm{Hecke}}$, i.e., the functor $\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}}$, identifies with

$$\mathbf{M} \simeq \mathbf{M} \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \overset{\mathrm{Id}\otimes f^*}{\longrightarrow} \mathbf{M} \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))}{\otimes} \mathrm{QCoh}(\mathcal{Z}).$$

Remark 12.3.4. The material of this subsection applies equally well when instead of $\mathcal{C}_{\mathrm{Ran}}$ and $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ we consider $\mathcal{C}^{\mathrm{dR}}_{\mathrm{Ran}}$ and $\mathrm{LocSys}^{\mathrm{dR}}_{\mathsf{G}}(X)$ or $\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}$ and $\mathrm{LocSys}^{\mathrm{Betti}}_{\mathsf{G}}(X)$.

12.4. **A version with parameters.** When we work in a constructible sheaf-theoretic context (as opposed to D-mod$(-)$ or $\mathrm{Shv}^{\mathrm{all}}(-)$), the formalism of the (symmetric) monoidal category $\mathrm{Rep}(\mathsf{G})_{\mathrm{Ran}}$ is not sufficient to encode the pattern of the Hecke action, to be studied in Part III of the paper (there G will be the Langlands dual $\check{G}$ of "our" group G).

The reason for this is that the Hecke functors map $\mathrm{Shv}(\mathrm{Bun}_G)$ to $\mathrm{Shv}(\mathrm{Bun}_G \times X^I)$, which contains, but is not equivalent to $\mathrm{Shv}(\mathrm{Bun}_G)\otimes \mathrm{Shv}(X^I)$.

In order to account for this, we will need to introduce a version of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, where we allow an additional scheme as a parameter. We will continue to work in an abstract setting, when instead of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ we have an arbitrary rigid symmetric monoidal category $\mathcal{C}$.

12.4.1. Let X and $\mathcal{C}$ be as in Sect. 11.3.

Let Y be a scheme over k. We introduce the category $\mathcal{C}_{\mathrm{Ran}\times Y}$ by the same colimit procedure as in the case of $\mathcal{C}_{\mathrm{Ran}}$, with the difference that instead of $\mathrm{Shv}(X^J)$ we now use

$$\mathrm{Shv}(X^J\times Y).$$

We endow $\mathcal{C}_{\mathrm{Ran}\times Y}$ with a symmetric monoidal structure using the operation of disjoint of finite sets, where we now use the functors

$$\mathrm{Shv}(X^{J_1}\times Y)\otimes \mathrm{Shv}(X^{J_2}\times Y) \overset{\boxtimes}{\to} \mathrm{Shv}((X^{J_1}\times Y)\times (X^{J_2}\times Y)) \overset{!\text{-pullback}}{\longrightarrow} \mathrm{Shv}(X^{J_1\sqcup J_2}\times Y).$$

Tensoring by !-pullbacks of objects of $\mathrm{Shv}(Y)$, we obtain a (unital) symmetric monoidal functor

$$\mathrm{Shv}(Y)\to \mathcal{C}_{\mathrm{Ran}\times Y}. \tag{12.14}$$

12.4.2. Let $f: Y_1\to Y_2$ be a map of schemes. Then !-pullback along f defines a (unital) symmetric monoidal functor

$$\mathcal{C}_{\mathrm{Ran}\times Y_2}\to \mathcal{C}_{\mathrm{Ran}\times Y_1}.$$

12.4.3. In particular for any Y, we have a (unital) symmetric monoidal functor

$$\mathcal{C}_{\mathrm{Ran}} \simeq \mathcal{C}_{\mathrm{Ran}\times\mathrm{pt}} \to \mathcal{C}_{\mathrm{Ran}\times Y}. \tag{12.15}$$

Combining with (12.14), we obtain a symmetric monoidal functor

$$\mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{Shv}(Y) \to \mathcal{C}_{\mathrm{Ran}\times Y}. \tag{12.16}$$

Since the individual functors

$$\mathrm{Shv}(X^J) \otimes \mathrm{Shv}(Y) \to \mathrm{Shv}(X^J \times Y)$$

are fully faithful and the category $\mathrm{Shv}(Y)$ is dualizable, it follows from (11.18) (and a similar presentation for $\mathcal{C}_{\mathrm{Ran}\times Y}$) that the functor (12.16) is fully faithful.

Remark 12.4.4. Similar definitions apply when instead of $\mathcal{C}_{\mathrm{Ran}}$ we use $\mathcal{C}^{\mathrm{dR}}_{\mathrm{Ran}}$. However, in this case, the corresponding functor

$$\mathcal{C}^{\mathrm{dR}}_{\mathrm{Ran}} \otimes \mathrm{Shv}(Y) \to \mathcal{C}^{\mathrm{dR}}_{\mathrm{Ran}\times Y}$$

is an equivalence. So in this case, there is no point of introducing $\mathcal{C}^{\mathrm{dR}}_{\mathrm{Ran}\times Y}$ as a separate entity.

Similar definitions also apply to $\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}}$ (but the !-pullbacks replaced by *-pullbacks). Here again, the corresponding functor

$$\mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}} \otimes \mathrm{Shv}(Y) \to \mathcal{C}^{\mathrm{Betti}}_{\mathrm{Ran}\times Y}$$

is an equivalence.

12.4.5. Let us be given a symmetric monoidal functor F as in (12.1). From F we produce a symmetric monoidal functor

$$\widetilde{\mathsf{F}}_Y : \mathcal{C}_{\mathrm{Ran}\times Y} \to \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(Y).$$

Namely, $\widetilde{\mathsf{F}}_Y$ sends an object

$$\mathrm{ins}_{I \overset{\psi}{\to} J}(V \otimes \mathcal{F}_Y), \quad V \in \mathcal{C}^{\otimes I},\ \mathcal{F}_Y \in \mathrm{Shv}(X^J \times Y)$$

to

$$\big(\mathrm{Id}_{\mathrm{QCoh}(\mathcal{Z})} \otimes (p_Y)_*\big)\,(\mathsf{F}^J(\mathrm{mult}^{\psi}_{\mathcal{C}}(V)) \otimes \mathcal{F}_Y),$$

where p_Y denotes the projection

$$X^J \times Y \to Y.$$

12.4.6. Let $\mathbf{M}$ be a module category over $\mathcal{C}_{\mathrm{Ran}\times Y}$. Let us regard

$$\mathbf{M} \otimes \mathrm{QCoh}(\mathcal{Z}) \simeq \mathbf{M} \underset{\mathrm{Shv}(Y)}{\otimes} (\mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(Y))$$

as a module category over

$$\mathcal{C}_{\mathrm{Ran}\times Y} \otimes \mathrm{QCoh}(\mathcal{Z}) \simeq \mathcal{C}_{\mathrm{Ran}\times Y} \underset{\mathrm{Shv}(Y)}{\otimes} (\mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(Y)).$$

We can also view it as a module over $\mathcal{C}_{\mathrm{Ran}\times Y} \underset{\mathrm{Shv}(Y)}{\otimes} \mathcal{C}_{\mathrm{Ran}\times Y}$ via $\widetilde{\mathsf{F}}_Y$.

12.4.7. We define

$$\mathrm{Hecke}_Y(\mathcal{Z}, \mathbf{M})$$

as the category of functors

$$\mathcal{C}_{\mathrm{Ran}\times Y} \to \mathbf{M} \otimes \mathrm{QCoh}(\mathcal{Z}).$$

of modules categories over $\mathcal{C}_{\mathrm{Ran}\times Y} \underset{\mathrm{Shv}(Y)}{\otimes} \mathcal{C}_{\mathrm{Ran}\times Y}$.

Equivalently, we can view $\mathrm{Hecke}_Y(\mathcal{Z}, \mathbf{M})$ as the category of functors

$$\mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(Y) \to \mathbf{M} \otimes \mathrm{QCoh}(\mathcal{Z}) \tag{12.17}$$

of modules categories over $\mathcal{C}_{\mathrm{Ran}\times Y} \otimes \mathrm{QCoh}(\mathcal{Z})$, where $\mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(Y)$ in the left-hand side of (12.17) is regarded as a module over $\mathcal{C}_{\mathrm{Ran}\times Y}$ via $\widetilde{\mathsf{F}}_Y$.

We have a naturally defined forgetful functor

$$\mathbf{oblv}_{\mathrm{Hecke}_Y, \mathcal{Z}} : \mathrm{Hecke}_Y(\mathcal{Z}, \mathbf{M}) \to \mathbf{M} \otimes \mathrm{QCoh}(\mathcal{Z}). \tag{12.18}$$

12.4.8. Note that given $\mathbf{M}$ as above, we can regard it as a module category over $\mathcal{C}_{\mathrm{Ran}}$ via the symmetric monoidal functor (12.15). We have a naturally defined forgetful functor

$$\mathrm{Hecke}_Y(\mathcal{Z},\mathbf{M}) \to \mathrm{Hecke}(\mathcal{Z},\mathbf{M}) \tag{12.19}$$

that makes the diagram

$$\begin{array}{ccc} \mathrm{Hecke}_Y(\mathcal{Z},\mathbf{M}) & \xrightarrow{\mathbf{oblv}_{\mathrm{Hecke}_Y,\mathcal{Z}}} & \mathbf{M}\otimes \mathrm{QCoh}(\mathcal{Z}) \\ {\scriptstyle (12.19)}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ \mathrm{Hecke}(\mathcal{Z},\mathbf{M}) & \xrightarrow{\mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}} & \mathbf{M}\otimes \mathrm{QCoh}(\mathcal{Z}) \end{array}$$

commute.

We will prove:

Theorem 12.4.9. *The functor* (12.19) *is an equivalence.*

12.4.10. Let us denote by

$$\mathsf{R}_{\mathcal{Z},Y} \in \mathcal{C}_{\mathrm{Ran}\times Y}\otimes \mathrm{QCoh}(\mathcal{Z})$$

the image of $\mathsf{R}_{\mathcal{Z}} \in \mathcal{C}_{\mathrm{Ran}}\otimes \mathrm{QCoh}(\mathcal{Z})$ along

$$\mathcal{C}_{\mathrm{Ran}}\otimes \mathrm{QCoh}(\mathcal{Z}) \to \mathcal{C}_{\mathrm{Ran}\times Y}\otimes \mathrm{QCoh}(\mathcal{Z}).$$

From Theorem 12.4.9 we obtain:

Corollary 12.4.11. *The functor* $\mathbf{oblv}_{\mathrm{Hecke}_Y,\mathcal{Z}}$ *is monadic, and the resulting monad on* $\mathbf{M}\otimes\mathrm{QCoh}(\mathcal{Z})$ *is given by the action of* $\mathsf{R}_{\mathcal{Z},Y}$.

12.5. Proof of Theorem 12.4.9.

12.5.1. Consider the functor

$$\mathcal{C}_{\mathrm{Ran}\times Y} \underset{\mathrm{Shv}(Y)}{\otimes} (\mathrm{QCoh}(\mathcal{Z})\otimes \mathrm{Shv}(Y)) \overset{\widetilde{\mathsf{F}}_Y\otimes \mathrm{Id}}{\longrightarrow} \tag{12.20}$$

$$\to (\mathrm{QCoh}(\mathcal{Z})\otimes \mathrm{Shv}(Y)) \underset{\mathrm{Shv}(Y)}{\otimes} (\mathrm{QCoh}(\mathcal{Z})\otimes \mathrm{Shv}(Y)) \overset{\mathrm{mult}}{\to} \mathrm{QCoh}(\mathcal{Z})\otimes \mathrm{Shv}(Y).$$

It is enough to show that

$$\mathbf{1}_{\mathrm{QCoh}(\mathcal{Z})\otimes \mathrm{Shv}(Y)} \mapsto \mathsf{R}_{\mathcal{Z},Y}$$

extends to a map of $\mathcal{C}_{\mathrm{Ran}\times Y} \underset{\mathrm{Shv}(Y)}{\otimes} (\mathrm{QCoh}(\mathcal{Z})\otimes \mathrm{Shv}(Y))$-module categories

$$\mathrm{QCoh}(\mathcal{Z})\otimes \mathrm{Shv}(Y) \to \mathcal{C}_{\mathrm{Ran}\times Y}\otimes \mathrm{QCoh}(\mathcal{Z}) \simeq \mathcal{C}_{\mathrm{Ran}\times Y} \underset{\mathrm{Shv}(Y)}{\otimes} (\mathrm{QCoh}(\mathcal{Z})\otimes \mathrm{Shv}(Y)), \tag{12.21}$$

which is the right adjoint of (12.20).

12.5.2. Let us denote the functor (12.16) by Φ.

Note that the individual functors

$$\boxtimes : \mathrm{Shv}(X^J)\otimes \mathrm{Shv}(Y) \to \mathrm{Shv}(X^J\times Y)$$

preserve compactness, and hence admit continuous right adjoints, to be denoted $\boxtimes^R$. Since $\boxtimes$ commutes with Verdier duality, the functor $\boxtimes^R$ also identifies with the dual of $\boxtimes$. The latter observation implies that for $\phi : J_2 \to J_1$, the diagram

$$\begin{array}{ccc} \mathrm{Shv}(X^{J_1}\times Y) & \xrightarrow{(\Delta_\phi\times \mathrm{id})_*} & \mathrm{Shv}(X^{J_2}\times Y) \\ {\scriptstyle \boxtimes^R}\downarrow & & \downarrow{\scriptstyle \boxtimes^R} \\ \mathrm{Shv}(X^{J_1})\otimes \mathrm{Shv}(Y) & \xrightarrow{(\Delta_\phi)_*\otimes \mathrm{Id}} & \mathrm{Shv}(X^{J_2})\otimes \mathrm{Shv}(Y), \end{array}$$

which a priori commutes up to a natural transformation, commutes strictly.

This implies that the functors $\boxtimes^R$ assemble to a functor

$$\Psi : \mathcal{C}_{\mathrm{Ran}\times Y} \to \mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{Shv}(Y),$$

right adjoint to Φ. The unit map $\mathrm{Id} \to \Psi \circ \Phi$ is an isomorphism since Φ is fully faithful.

Being a right adjoint to a symmetric monoidal functor, the functor Ψ carries a canonically defined right-lax symmetric monoidal structure. It is easy to see, however, that Ψ is strictly linear with respect to $\mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{Shv}(Y)$.

12.5.3. Let us observe that the functor $\widetilde{\mathsf{F}}_Y$ identifies with

$$\mathcal{C}_{\mathrm{Ran}\times Y} \overset{\Psi}{\to} \mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{Shv}(Y) \overset{\widetilde{\mathsf{F}}\otimes\mathrm{Id}}{\longrightarrow} \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(Y),$$

as a right-lax symmetric monoidal functor (however, the functor $\widetilde{\mathsf{F}}_Y$ itself is strict).

12.5.4. By (11.17), we have a tensor-compatible system of isomorphisms

$$\mathcal{V} \star \mathsf{R}_{\mathcal{Z}} \simeq \mathsf{R}_{\mathcal{Z}} \otimes \widetilde{\mathsf{F}}(\mathcal{V}), \quad \mathcal{V} \in \mathcal{C}_{\mathrm{Ran}}.$$

From here, the (Φ, Ψ)-adjunction gives rise to a tensor-compatible system of *morphisms*

$$\mathcal{V}_Y \star \mathsf{R}_{\mathcal{Z},Y} \leftarrow \mathsf{R}_{\mathcal{Z},Y} \otimes \widetilde{\mathsf{F}}_Y(\mathcal{V}_Y), \quad \mathcal{V}_Y \in \mathcal{C}_{\mathrm{Ran}\times Y}. \tag{12.22}$$

The key observation is the following:

Proposition 12.5.5. *The maps* (12.22) *are isomorphisms.*

The proof will be given in Sects. 12.6 and 12.7 (we will give two proofs, each in the corresponding section).

Let us accept this proposition temporarily and finish the proof of Theorem 12.4.9.

12.5.6. By Proposition 12.5.5, the object $\mathsf{R}_{\mathcal{Z},Y}$, defines a map

$$\mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(Y) \to \mathcal{C}_{\mathrm{Ran}\times Y} \underset{\mathrm{Shv}(Y)}{\otimes} (\mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(Y)) \simeq \mathcal{C}_{\mathrm{Ran}\times Y} \otimes \mathrm{QCoh}(\mathcal{Z}) \tag{12.23}$$

of module categories over $\mathcal{C}_{\mathrm{Ran}\times Y} \otimes \mathrm{QCoh}(\mathcal{Z})$.

We now construct the adjunction datum between (12.20) and (12.23).

Unit: Since (12.20) and (12.23) are functors of module categories over $\mathcal{C}_{\mathrm{Ran}\times Y} \otimes \mathrm{QCoh}(\mathcal{Z})$, it suffices to specify the value of the unit on the object $\mathbf{1}_{\mathcal{C}_{\mathrm{Ran}\times Y}\otimes \mathrm{QCoh}(\mathcal{Z})} \in \mathcal{C}_{\mathrm{Ran}\times Y} \otimes \mathrm{QCoh}(\mathcal{Z})$.

It is obtained by applying the functor Φ to the unit of the adjunction

$$\mathrm{mult}_{\mathrm{QCoh}(\mathcal{Z})} \circ (\widetilde{\mathsf{F}} \otimes \mathrm{Id}) : \mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathcal{Z}) \rightleftarrows \mathrm{QCoh}(\mathcal{Z}) : \mathsf{R}_{\mathcal{Z}} \tag{12.24}$$

on $\mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}\otimes \mathrm{QCoh}(\mathcal{Z})} \in \mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathcal{Z})$.

Counit: it suffices to specify the value of the counit on $\mathbf{1}_{\mathrm{QCoh}(\mathcal{Z})\otimes \mathrm{Shv}(Y)} \in \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(Y)$. I.e., we have to specify a map

$$\mathrm{mult}_{\mathrm{QCoh}(\mathcal{Z})\otimes \mathrm{Shv}(Y)} \circ (\widetilde{\mathsf{F}}_Y \otimes \mathrm{Id}_{\mathrm{QCoh}(\mathcal{Z})\otimes \mathrm{Shv}(Y)})(\mathsf{R}_{\mathcal{Z},Y}) \to \mathbf{1}_{\mathrm{QCoh}(\mathcal{Z})\otimes \mathrm{Shv}(Y)}. \tag{12.25}$$

Note that

$$\widetilde{\mathsf{F}}_Y \circ \Phi \simeq (\widetilde{\mathsf{F}} \otimes \mathrm{Id}_{\mathrm{Shv}(Y)}) \circ \Psi \circ \Phi \simeq \widetilde{\mathsf{F}} \otimes \mathrm{Id}_{\mathrm{Shv}(Y)}.$$

Hence, the left-hand side in (12.25) identifies with

$$(\widetilde{\mathsf{F}} \otimes \mathrm{Id}_{\mathrm{QCoh}(\mathcal{Z})})(\mathsf{R}_{\mathcal{Z}}) \otimes \mathbf{1}_{\mathrm{Shv}(Y)}.$$

The required map in (12.25) is obtained from the counit of the adjunction (12.24) by tensoring with $\mathbf{1}_{\mathrm{Shv}(Y)} = \omega_Y$.

$\square$[Theorem 12.4.9]

12.6. Direct proof of Proposition 12.5.5.

12.6.1. It is sufficient to prove that (12.22) is an isomorphism for $\mathcal{V}_Y$ of the form

$$\mathrm{ins}_{I_0\to J_0}(V \otimes \mathcal{F}), \quad V \in \mathcal{C}^{\otimes I_0}, \quad \mathcal{F} \in \mathrm{Shv}(X^{J_0} \times Y).$$

12.6.2. Let us denote by

$$\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}} \text{ and } \mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}\times Y}$$

the categories defined as

$$\underset{(I_1\to J_1),(I_2\to J_2)}{\mathrm{colim}} \mathcal{C}^{\otimes I_1}\otimes\mathcal{C}^{\otimes I_2}\otimes\mathrm{Shv}(X^{J_1}\times X^{J_2})$$

and

$$\underset{(I_1\to J_1),(I_2\to J_2)}{\mathrm{colim}} \mathcal{C}^{\otimes I_1}\otimes\mathcal{C}^{\otimes I_2}\otimes\mathrm{Shv}(X^{J_1}\times X^{J_2}\times Y),$$

respectively, where

$$(I_1\to J_1),(I_2\to J_2)\in\mathrm{TwArr}(\mathrm{fSet})\times\mathrm{TwArr}(\mathrm{fSet}).$$

Denote by $\mathrm{ins}_{(I_1\to J_1),(I_2\to J_2)}$ the corresponding tautological functors

$$\mathcal{C}^{\otimes I_1}\otimes\mathcal{C}^{\otimes I_2}\otimes\mathrm{Shv}(X^{J_1}\times X^{J_2})\to\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}}$$

and

$$\mathcal{C}^{\otimes I_1}\otimes\mathcal{C}^{\otimes I_2}\otimes\mathrm{Shv}(X^{J_1}\times X^{J_2}\times Y)\to\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}\times Y},$$

respectively.

12.6.3. The operation of disjoint union on finite sets makes $\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}}$ and $\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}\times Y}$ into symmetric monoidal categories.

We have naturally defined symmetric monoidal functors

$$\Upsilon:\mathcal{C}_{\mathrm{Ran}}\otimes\mathcal{C}_{\mathrm{Ran}}\to\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}} \text{ and } \Upsilon_Y:\mathcal{C}_{\mathrm{Ran}\times Y}\underset{\mathrm{Shv}(Y)}{\otimes}\mathcal{C}_{\mathrm{Ran}\times Y}\to\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}\times Y}.$$

We also have a symmetric monoidal functor

$$\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}}\to\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}\times Y}, \tag{12.26}$$

given by pullback along $Y\to\mathrm{pt}$.

Remark 12.6.4. Note the difference between $\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}}$ and $\mathcal{C}_{\mathrm{Ran}}\otimes\mathcal{C}_{\mathrm{Ran}}$. In the former the terms of the colimit have factors $\mathrm{Shv}(X^{J_1}\times X^{J_2})$, and in the latter $\mathrm{Shv}(X^{J_1})\otimes\mathrm{Shv}(X^{J_2})$.

12.6.5. Let $\mathsf{R}_{\mathrm{geom},\mathcal{C},\mathrm{Ran}}$ be the object of $\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}}$ defined as

$$\underset{(I\to J)\in\mathrm{TwArr}(\mathrm{fSet})}{\mathrm{colim}} \mathrm{ins}_{(I\to J),(I\to J)}\left((\mathsf{R}_{\mathcal{C}})^{\boxtimes I}\otimes(\Delta_{X^J})_*(\omega_{X^J})\right).$$

The maps

$$\mathrm{u}_{\mathrm{Shv}(X^J)}\to(\Delta_{X^J})_*(\omega_{X^J})$$

gives rise to a map

$$\Upsilon(\mathsf{R}_{\mathcal{C},\mathrm{Ran}})\to\mathsf{R}_{\mathrm{geom},\mathcal{C},\mathrm{Ran}}. \tag{12.27}$$

Let $\mathsf{R}_{\mathrm{geom},\mathcal{C},\mathrm{Ran},Y}$ be the object of $\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}\times Y}$ equal to the image of $\mathsf{R}_{\mathrm{geom},\mathcal{C},\mathrm{Ran}}$ along (12.26).

12.6.6. Let $\mathcal{V}$ be an object of $\mathcal{C}^{\otimes I_0}\otimes\mathrm{Shv}(X^{J_0})$ for some $(I_0\to J_0)\in\mathrm{TwArr}(\mathrm{fSet})$. Let $\mathcal{V}^l$ and $\mathcal{V}^r$ denote its images in $\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}}$ along

$$\mathcal{C}^{\otimes I_0}\otimes\mathrm{Shv}(X^{J_0})\overset{\mathrm{ins}_{I_0\to J_0}}{\longrightarrow}\mathcal{C}_{\mathrm{Ran}}\overset{\mathrm{Id}\otimes\mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}}}{\longrightarrow}\mathcal{C}_{\mathrm{Ran}}\otimes\mathcal{C}_{\mathrm{Ran}}\to\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}}$$

and

$$\mathcal{C}^{\otimes I_0}\otimes\mathrm{Shv}(X^{J_0})\overset{\mathrm{ins}_{I_0\to J_0}}{\longrightarrow}\mathcal{C}_{\mathrm{Ran}}\overset{\mathbf{1}_{\mathcal{C}_{\mathrm{Ran}}}\otimes\mathrm{Id}}{\longrightarrow}\mathcal{C}_{\mathrm{Ran}}\otimes\mathcal{C}_{\mathrm{Ran}}\to\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}},$$

respectively.

The calculation performed in Sect. 11.7 shows that we have a canonical isomorphism

$$\mathcal{V}^l\star\mathsf{R}_{\mathrm{geom},\mathcal{C},\mathrm{Ran}}\simeq\mathsf{R}_{\mathrm{geom},\mathcal{C},\mathrm{Ran}}\star\mathcal{V}^r$$

in $\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}}$.

Let $\mathcal{V}_Y$ be an object of $\mathcal{C}^{\otimes I_0}\otimes\mathrm{Shv}(X^{J_0}\times Y)$, and let

$$\mathcal{V}^l_Y,\mathcal{V}^r_Y\in\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}\times Y}$$

be defined in a way similar to the above.

Then the same calculation shows that we have a canonical isomorphism

$$\mathcal{V}^l_Y \star \mathsf{R}_{\mathrm{geom},\mathcal{C},\mathrm{Ran},Y} \simeq \mathsf{R}_{\mathrm{geom},\mathcal{C},\mathrm{Ran},Y} \star \mathcal{V}^r_Y \tag{12.28}$$

in $\mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}\times Y}$.

12.6.7. Let $(\mathcal{Z},\mathsf{F})$ be as in Sect. 12.4.5. We claim that we have a naturally defined symmetric monoidal functor

$$\widetilde{\mathsf{F}}_{\mathrm{Ran}} : \mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}} \to \mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathcal{Z}).$$

Namely, the functor $\widetilde{\mathsf{F}}_{\mathrm{Ran}}$ sends an object

$$\mathrm{ins}_{(I_1 \overset{\psi_1}{\to} J_1),(I_2 \overset{\psi_2}{\to} J_2)}(V_1 \otimes V_2 \otimes \mathcal{F}), \quad V_1 \in \mathcal{C}^{\otimes I_1}, \quad V_2 \in \mathcal{C}^{\otimes I_2}, \quad \mathcal{F} \in \mathrm{Shv}(X^{J_1} \times X^{J_2})$$

to

$$\mathrm{ins}_{I_1 \to J_1}(V_1 \otimes \mathcal{F}_1),$$

where $\mathcal{F}_1$ is the object of $\mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(X^{J_1})$ equal to

$$(\mathrm{Id}_{\mathrm{QCoh}(\mathcal{Z})} \otimes (p_{J_1})_*)(\mathsf{F}^{J_2}(\mathrm{mult}^{\psi_2}_{\mathcal{C}}(V_2)) \otimes \mathcal{F}),$$

where p_{J_1} is the projection $X^{J_1} \times X^{J_2} \to X^{J_1}$.

12.6.8. Note that the composition

$$\mathcal{C}_{\mathrm{Ran}} \otimes \mathcal{C}_{\mathrm{Ran}} \overset{\Upsilon}{\to} \mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}} \overset{\widetilde{\mathsf{F}}_{\mathrm{Ran}}}{\to} \mathcal{C}_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathcal{Z})$$

identifies with the functor $\mathrm{Id}_{\mathcal{C}_{\mathrm{Ran}}} \otimes \widetilde{\mathsf{F}}$, where $\widetilde{\mathsf{F}}$ is an in (12.2).

We claim:

Lemma 12.6.9. *The map*

$$\mathsf{R}_{\mathcal{Z}} = (\mathrm{Id}_{\mathcal{C}_{\mathrm{Ran}}} \otimes \widetilde{\mathsf{F}})(\mathsf{R}_{\mathcal{C},\mathrm{Ran}}) \simeq \widetilde{\mathsf{F}}_{\mathrm{Ran}} \circ \Upsilon(\mathsf{R}_{\mathcal{C},\mathrm{Ran}}) \overset{(12.27)}{\longrightarrow} \widetilde{\mathsf{F}}_{\mathrm{Ran}}(\mathsf{R}_{\mathrm{geom},\mathcal{C},\mathrm{Ran}})$$

is an isomorphism.

Proof. Follows from the isomorphism (11.21). □

12.6.10. By a similar token we construct a map

$$\widetilde{\mathsf{F}}_{\mathrm{Ran},Y} : \mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}\times Y} \to \mathcal{C}_{\mathrm{Ran}\times Y} \otimes \mathrm{QCoh}(\mathcal{Z})$$

such that the composition

$$\mathcal{C}_{\mathrm{Ran}\times Y} \underset{\mathrm{Shv}(Y)}{\otimes} \mathcal{C}_{\mathrm{Ran}\times Y} \overset{\Upsilon_Y}{\to} \mathcal{C}_{\mathrm{Ran}\times\mathrm{Ran}\times Y} \overset{\widetilde{\mathsf{F}}_{\mathrm{Ran},Y}}{\to} \mathcal{C}_{\mathrm{Ran}\times Y} \otimes \mathrm{QCoh}(\mathcal{Z})$$

identifies with the map

$$\mathcal{C}_{\mathrm{Ran}\times Y} \underset{\mathrm{Shv}(Y)}{\otimes} \mathcal{C}_{\mathrm{Ran}\times Y} \overset{\mathrm{Id}\otimes\widetilde{\mathsf{F}}_Y}{\longrightarrow} \mathcal{C}_{\mathrm{Ran}\times Y} \underset{\mathrm{Shv}(Y)}{\otimes} (\mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(Y)) \simeq \mathcal{C}_{\mathrm{Ran}\times Y} \otimes \mathrm{QCoh}(\mathcal{Z}).$$

As in Lemma 12.6.9, we obtain that the resulting map

$$\mathsf{R}_{\mathcal{Z},Y} \to \widetilde{\mathsf{F}}_{\mathrm{Ran},Y}(\mathsf{R}_{\mathrm{geom},\mathcal{C},\mathrm{Ran},Y})$$

is an isomorphism.

12.6.11. Combining the latter isomorphism with the isomorphisms (12.28), we obtain isomorphisms

$$\mathcal{V}_Y \star \mathsf{R}_{\mathcal{Z},Y} \simeq \mathsf{R}_{\mathcal{Z},Y} \otimes \widetilde{\mathsf{F}}_Y(\mathcal{V}_Y).$$

Unwinding the definitions, we obtain that the resulting morphisms

$$\mathcal{V}_Y \star \mathsf{R}_{\mathcal{Z},Y} \leftarrow \mathsf{R}_{\mathcal{Z},Y} \otimes \widetilde{\mathsf{F}}_Y(\mathcal{V}_Y)$$

are equal to those in (12.22).

□[Proposition 12.5.5]

12.7. **An indirect proof of Proposition 12.5.5.** We will give a proof that works in the étale and constructible de Rham contexts; the constructible Betti case will follow from the de Rham case by Riemann-Hilbert.

12.7.1. We need to show that the maps (12.22) are isomorphisms in

$$\mathcal{C}_{\mathrm{Ran}\times Y} \otimes \mathrm{QCoh}(\mathcal{Z}).$$

We can rewrite this category as a *limit* with terms

$$\mathcal{C}^{\otimes I} \otimes \mathrm{Shv}(X^J \times Y) \otimes \mathrm{QCoh}(\mathcal{Z}).$$

So, we need to show that the map in question becomes an isomorphism in each of the above terms.

12.7.2. Note that for a scheme W, an object $\mathcal{F} \in \mathrm{Shv}(W \times Y)$ is zero if and only if for every geometric point $\mathrm{Spec}(k') \to Y$, the pullback $\mathcal{F}$ to

$$W' := \mathrm{Spec}(k') \underset{Y}{\times} (W \times Y)$$

is zero.

The same remains true for $\mathrm{Shv}(W \times Y) \otimes \mathbf{C}$ for any DG category $\mathbf{C}$.

Thus, it is sufficient to show that the map (12.22) becomes an isomorphism after the base change $k \rightsquigarrow k'$ for $Y' = \mathrm{Spec}(k')$.

12.7.3. However, the base change of the map (12.22) is a similar map over the ground field k' for F' being the following functor:

In the étale context, F' is

$$\mathcal{C} \overset{\mathsf{F}}{\to} \mathrm{QCoh}(\mathcal{Z}) \underset{\mathsf{e}}{\otimes} \mathrm{QLisse}(X) \to \mathrm{QCoh}(\mathcal{Z}) \underset{\mathsf{e}}{\otimes} \mathrm{QLisse}(X'),$$

and in the de Rham context, F' is

$$\mathcal{C} \overset{\mathsf{F}}{\to} \mathrm{QCoh}(\mathcal{Z}) \underset{k}{\otimes} \mathrm{QLisse}(X) \to \mathrm{QCoh}(\mathcal{Z}') \underset{k'}{\otimes} \mathrm{QLisse}(X'),$$

where in both cases

$$X' := \mathrm{Spec}(k') \underset{\mathrm{Spec}(k)}{\times} X,$$

and in the de Rham context

$$\mathcal{Z}' := \mathrm{Spec}(k') \underset{\mathrm{Spec}(k)}{\times} \mathcal{Z}.$$

□[Proposition 12.5.5]

12.7.4. Thus, we have reduced the verification of fact that (12.22) is an isomorphism to the case when $Y = \mathrm{pt}$. However, in this case, the assertion is already known by (11.17).

Part III: The category of automorphic sheaves with nilpotent singular support

Let us make a brief overview of the contents of this Part.

In Sect. 13 we introduce and study the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. First, we state a key technical result, Theorem 13.1.5, which says that Bun_G can be covered by quasi-compact open substacks, such that the functor of !-extension from each of them preserves the nilpotence of singular support. Next we observe that the action of Hecke functors on the entire category $\mathrm{Shv}(\mathrm{Bun}_G)$ gives rise to an action of $\mathrm{Rep}(\check{G})^{\otimes X\text{-lisse}}$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. Applying our Spectral Decomposition theorem, we obtain $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ carries a monoidal action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$. We also state the second main result of this paper, Theorem 13.4.4, which says that if an object $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$ is such that the Hecke action on it is lisse, then $\mathcal{F}$ belongs to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. This implies, in particular, that Hecke eigensheaves have nilpotent singular support.

In Sect. 14 we introduce yet another tool in the study of $\mathrm{Shv}(\mathrm{Bun}_G)$–Beilinson's spectral projector, denoted $\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}}$, which is defined for a prestack $\mathcal{Z}$ equipped with a map $f : \mathcal{Z} \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$. This is a functor, given by an explicit Hecke operator, and it provides a left adjoint to the forgetful functor

$$\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G).$$

Using our Spectral Decomposition theorem and Theorem 13.4.4, we interpret $\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}}$ as the left adjoint to the functor[27]

$$\mathrm{QCoh}(\mathcal{Z}) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{f_*\otimes \mathrm{Id}}{\longrightarrow}$$
$$\to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G).$$

In Sect. 15 we use Beilinson's spectral projector to prove an array of structural results about $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$: we will show that the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is compactly generated, that the external tensor product functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G_1}) \otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G_2}) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_{G_1\times G_2})$$

is an equivalence and that, in the de Rham context, all objects in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ have regular singularities.

In Sect. 16 we will make several observations regarding a conjecture, initially formulated in Sect. 13, which can be stated as saying that the right adjoint of the embedding $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G)$ is continuous.

In Sect. 17, we establish analogs of the results of the preceding sections in Part III, when work over the ground field $k = \mathbb{C}$ and instead of $\mathrm{Shv}(-)$ we consider the category $\mathrm{Shv}^{\mathrm{all}}(-)$ of all sheaves in the classical topology.

In Sect. 18 we prove Theorem 13.1.5 about the preservation of nilpotence of singular support under the functor of direct image for certain open embeddings $\mathcal{U} \overset{j}{\hookrightarrow} \mathrm{Bun}_G$. The proof follows closely the strategy of [DrGa2]: by the same method as in *loc.cit.*, it turns out that we can control the singular support of the extension in a *contractive* situation.

In Sect. 19 we prove Theorem 13.4.4. We first consider the case of $G = GL_2$, which explains the main idea of the argument. We then implement this idea in a slightly more involved case of $G = GL_n$ (where it is sufficient consider the minuscule Hecke functors). Finally, we treat the case of an arbitrary G; the proof reduces to the analysis of the local Hitchin map and affine Springer fibers.

[27]Provided $\mathrm{QCoh}(\mathcal{Z})$ is dualizable and $\mathcal{O}_{\mathcal{Z}} \in \mathrm{QCoh}(\mathcal{Z})$ is compact.

13. Automorphic sheaves with nilpotent singular support and spectral decomposition

In this section we introduce and study the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

The central results of this section are:

–Theorem 13.1.5, which expresses a locality property of the nilpotence of singular support condition;

–Theorem 13.3.2, which says that $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ carries a monoidal action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$;

–Theorem 13.4.4 that any object of $\mathrm{Shv}(\mathrm{Bun}_G)$ on which the Hecke action is lisse, belongs to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

13.1. **Definition and basic properties.** In this subsection we define the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ and formulate a key result (Theorem 13.1.5) that ensures that it is, in a certain sense, local with respect to Bun_G.

13.1.1. From now on we let X be a smooth, connected and complete curve and G a reductive group, over a ground field k (assumed algebraically closed).

Consider Bun_G, the moduli space of principal G-bundles on X. Our object of study is the category

$$\mathrm{Shv}(\mathrm{Bun}_G)$$

of sheaves on Bun_G. (The basics of the theory of sheaves on algebraic stacks are reviewed in Sect. F.)

13.1.2. Recall that $T^*(\mathrm{Bun}_G)$ can be identified with the moduli space of pairs $(\mathcal{P}_G, A)$, where $\mathcal{P}_G$ is a G-bundle on X, and A is a global section of $\mathfrak{g}^\vee_{\mathcal{P}_G} \otimes \omega_X$.

Let $\mathrm{Nilp} \subset T^*(\mathrm{Bun}_G)$ be the nilpotent cone, i.e., the closed subset consisting of those $(\mathcal{P}_G, A)$, for which A is nilpotent (at the generic point of X).

When $\mathrm{char}(k) = 0$, it is well-known that Nilp is half-dimensional (and even Lagrangian).

When $\mathrm{char}(k)$ is positive, but a "very good" prime for G and $\mathfrak{g}$ admits a non-degenerate G-equivariant bilinear form, the corresponding assertion seems to have been known in the folklore, but we were not able to find a proof in the literature. For completeness, we will supply a proof in Sect. D.

From now on, we will assume that the above restrictions on $\mathrm{char}(k)$ are satisfied, so that Nilp is half-dimensional.

13.1.3. The main object of study in this part is the subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G),$$

see Sect. F.8.1.

We denote the tautological embedding $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G)$ by ι.

13.1.4. For an open substack $\mathcal{U} \subset \mathrm{Shv}(\mathrm{Bun}_G)$, we can consider the full subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U}) \subset \mathrm{Shv}(\mathcal{U}).$$

We have the following result, which insures that the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ can be obtained as a colimit of the corresponding categories on quasi-compact open substacks of Bun_G, see Sect. F.8.6:

Main Theorem 13.1.5. *The stack* $\mathrm{Shv}(\mathrm{Bun}_G)$ *can be written as a filtered union of quasi-compact open substacks*

$$\mathcal{U}_i \overset{j_i}{\hookrightarrow} \mathrm{Bun}_G$$

such that the extension functors

$$(j_i)_!, (j_i)_* : \mathrm{Shv}(\mathcal{U}_i) \to \mathrm{Shv}(\mathrm{Bun}_G)$$

send $\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U}_i)^{\mathrm{constr}} \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{constr}}$.

The proof will be given in Sect. 18.

Remark 13.1.6. In the terminology of Sect. F.8.6, Theorem 13.1.5 says that the pair $(\mathrm{Bun}_G, \mathrm{Nilp})$ is *truncatable.*

By Sect. F.8.7, the statement of Theorem 13.1.5 can be reformulated as the assertion that that for $(\mathcal{U}_i, j_i)$ as above, the functors $(j_i)_!$ and $(j_i)_*$ send

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U}_i) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

13.1.7. Here is one property of the category $\mathrm{Shv}(\mathrm{Bun}_G)$ that we expect to hold, but at the moment are unable to prove in general (but we can prove it in the de Rham and Betti contexts, see Theorems 15.4.3 and 15.4.10):

Conjecture 13.1.8. *The category* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ *is generated by objects that are compact in the ambient category* $\mathrm{Shv}(\mathrm{Bun}_G)$*.*

In the sequel, we will give several (equivalent) reformulations of Conjecture 13.1.8.

Remark 13.1.9. In the terminology of Sect. F.8.4, the above conjecture says that the pair $(\mathrm{Bun}_G, \mathrm{Nilp})$ is *renormalization-adapted* and *constraccessible.*

Given Theorem 13.1.5, the property of being *renormalization-adapted* is known to hold since Bun_G is locally a quotient, see Corollary F.8.11 (and is in fact expected to hold for any pair $(\mathcal{Y}, \mathcal{N})$ of an algebraic stack (with an affine diagonal) and a subset of its cotangent bundle, see Conjecture F.7.10).

The property of being *constraccessible* is much more mysterious: it reflects a particular feature of the pair $(\mathrm{Bun}_G, \mathrm{Nilp})$ (for example, it fails for $(\mathbb{P}^1, \{0\})$, see Remark E.5.6).

13.2. **Hecke action on the category with nilpotent singular support.** In this subsection we recall the pattern of Hecke action on $\mathrm{Shv}(\mathrm{Bun}_G)$, and the particular feature that the subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$$

has with respect to this action.

13.2.1. Let $\check{G}$ denote the Langlands dual group of G. The following result encodes the Hecke action of $\mathrm{Rep}(\check{G})$ on $\mathrm{Shv}(\mathrm{Bun}_G)$ (see [GKRV, Proposition B.2.3]):

Theorem 13.2.2. *The Hecke functors combine to a compatible family of actions of*

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}(X^I) \text{ on } \mathrm{Shv}(\mathrm{Bun}_G \times X^I), \quad I \in \mathrm{fSet}, \tag{13.1}$$

extending the tautological action of

$$\mathrm{Shv}(X^I) \text{ on } \mathrm{Shv}(\mathrm{Bun}_G \times X^I), \quad I \in \mathrm{fSet}\,.$$

13.2.3. We are going to combine Theorem 13.2.2 with the following result, established in [NY1, Theorem 5.2.1] (see also [GKRV, Theorem B.5.2]):

Theorem 13.2.4. *The Hecke functor*

$$\mathrm{H}(-,-) : \mathrm{Rep}(\check{G}) \otimes \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times X) \tag{13.2}$$

sends

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$$

to the full subcategory

$$\mathrm{Shv}_{\mathrm{Nilp}\times\{0\}}(\mathrm{Bun}_G \times X) \subset \mathrm{Shv}(\mathrm{Bun}_G \times X).$$

13.2.5. Note that by Theorem F.9.8 and Corollary E.4.7, the external tensor product functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\otimes \mathrm{QLisse}(X)\to \mathrm{Shv}_{\mathrm{Nilp}\times\{0\}}(\mathrm{Bun}_G\times X)$$

is an equivalence.

Hence, the Hecke functor restricted to the category of sheaves with nilpotent singular support can be viewed as a functor

$$\mathrm{Rep}(\check{G})\otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\otimes \mathrm{QLisse}(X). \tag{13.3}$$

Remark 13.2.6. In Remark 13.3.9 we will see that (13.3) can be somewhat refined: the essential image of the functor (13.3) actually belongs to

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\otimes \mathrm{IndLisse}(X)\subset \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\otimes \mathrm{QLisse}(X).$$

13.2.7. Iterating, from Theorem 13.2.4, we obtain that for any $I\in \mathrm{fSet}$, the Hecke functors

$$\mathrm{H}(-,-):\mathrm{Rep}(\check{G})^{\otimes I}\otimes \mathrm{Shv}(\mathrm{Bun}_G)\to \mathrm{Shv}(\mathrm{Bun}_G\times X^I),\quad I\in \mathrm{fSet} \tag{13.4}$$

define a system of functors

$$\mathrm{Rep}(\check{G})^{\otimes I}\otimes \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\otimes \mathrm{QLisse}(X)^{\otimes I}. \tag{13.5}$$

Combining with Theorem 13.2.2, we obtain:

Corollary 13.2.8. *The Hecke action gives rise to a compatible family of monoidal functors*

$$\mathrm{Rep}(\check{G})^{\otimes I}\to \mathrm{End}(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))\otimes \mathrm{QLisse}(X)^{\otimes I},\quad I\in \mathrm{fSet}\,.$$

13.2.9. Thus, in the terminology of Sect. 8.1.1, we obtain that the Hecke action gives rise to an action of $\mathrm{Rep}(\check{G})^{\otimes X\text{-lisse}}$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

13.3. **Spectral decomposition of the category with nilpotent singular support.** We now come to the first main point of this paper: the spectral decomposition of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ over $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$.

13.3.1. Combining Corollary 13.2.8 with Theorem 8.1.4, we obtain:

Main Theorem 13.3.2. *The action of* $\mathrm{Rep}(\check{G})^{\otimes X\text{-lisse}}$ *on* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ *(arising from the Hecke action) factors via a (uniquely defined) action of* $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

13.3.3. Let us emphasize the main feature of the action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

For a finite set I and $V\in \mathrm{Rep}(\check{G})^{\otimes I}$, let

$$\mathcal{E}^I_V\in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))\otimes \mathrm{QLisse}(X)^{\otimes I} \tag{13.6}$$

be as in Sect. 8.4.9.

Then the action of $\mathcal{E}^I_V$ on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, viewed as a functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\otimes \mathrm{QLisse}(X)^{\otimes I}$$

equals the Hecke functor

$$\mathrm{H}(V,-):\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\otimes \mathrm{QLisse}(X)^{\otimes I}$$

of (13.5).

13.3.4. As a first corollary of Theorem 13.3.2 (combined with Proposition 3.7.2), we obtain:

Corollary 13.3.5. *The category* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ *splits canonically as a direct sum*

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\simeq \underset{\sigma}{\oplus}\,\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)_\sigma,$$

where σ runs over the set of isomorphism classes of semi-simple $\check{G}$-local systems on X.

13.3.6. *Example.* Let us explain what Corollary 13.3.5 says in concrete terms for $G = \mathbb{G}_m$.

Recall that the geometric class field theory attaches to a 1-dimensional local system σ on X a local system E_σ on Pic. Then Corollary 13.3.5 is the assertion that QLisse(Pic) splits as a direct sum

$$\mathrm{QLisse}(\mathrm{Pic}) \simeq \underset{\sigma}{\oplus} \mathrm{QLisse}(\mathrm{Pic})_\sigma,$$

where each $\mathrm{QLisse}(\mathrm{Pic})_\sigma$ is generated by E_σ.

In the particular case of Pic, such a decomposition is not difficult to establish directly: it follows from the fact that every lisse irreducible object in Shv(Pic) is isomorphic to one of the E_σ (this is the assertion that the étale fundamental group of Pic is the abelianization of the étale fundamental group of X) and the different E_σ are mutually orthogonal.

13.3.7. From Corollary 13.3.5 (combined with Corollary 13.4.10 below) we obtain the following result:

Corollary 13.3.8. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Hecke eigensheaves corresponding to G-local systems σ_1 and σ_2 with non-isomorphic semi-simplifications. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are mutually orthogonal, i.e.,*

$$\mathcal{M}aps(\mathcal{F}_1, \mathcal{F}_2) = 0.$$

Remark 13.3.9. Let us show that the functor (13.3) has essential image in

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{IndLisse}(X) \subset \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X).$$

This follows from the fact that the objects $\mathcal{E}_V$ above in fact belong to

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{IndLisse}(X) \subset \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QLisse}(X).$$

To prove this, it is enough to show that for a *cofinal* family of maps $S \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$, the objects

$$\mathcal{E}_V|_S \in \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X)$$

belong to

$$\mathrm{QCoh}(S) \otimes \mathrm{IndLisse}(X) \subset \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X).$$

We now use the fact that for X a curve, the prestack $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ is *eventually coconnective* (see [GR1, Chapter 2, Sect. 1.3.5]). This follows from the fact that the connected components of $\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\check{G}}(X)$ are *quasi-smooth formal affine schemes*, see Sect. 20.1.1.

Hence, inside the category $\mathrm{Sch}^{\mathrm{aff}}_{/\,\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$, a cofinal family is formed by those S that are eventually coconnective. Now, for S eventually coconnective, the fact that $\mathcal{E}_V|_S$ belongs to the subcategory $\mathrm{QCoh}(S) \otimes \mathrm{IndLisse}(X)$ is a reformulation of Proposition 2.1.7.

13.4. **A converse to Theorem 13.2.4.** In this subsection we state the second main result of this paper, Theorem 13.4.4, which says that the statement of Theorem 13.2.4 is "if and only if". This theorem implies, among the rest, that Hecke eigensheaves have nilpotent singular support.

13.4.1. For the validity of Theorem 13.4.4 we will have to make the following assumptions on char(k):

- There exists a non-degenerate G-equivariant symmetric bilinear form on $\mathfrak{g}$, whose restriction to the center of any Levi subalgebra remains non-degenerate;
- The order of the Weyl group W of G is not divisible by char(k);
- The centralizer in G of a semi-simple element in $\mathfrak{g}$ is a Levi subgroup.

From now on, we will assume that the above assumptions on char(k) are satisfied.

For example, if $G = GL_n$, these assumptions imply that char$(k) > n$.

13.4.2. Let

$$\mathrm{Shv}(\mathrm{Bun}_G)^{\text{Hecke-lisse}} \subset \mathrm{Shv}(\mathrm{Bun}_G)$$

be the full subcategory consisting of objects $\mathcal{F}$ such that for all $V \in \mathrm{Rep}(\check{G})$, we have

$$\mathrm{H}(V, \mathcal{F}) \in \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X) \subset \mathrm{Shv}(\mathrm{Bun}_G \times X),$$

where $\mathrm{H}(V, \mathcal{F})$ is the Hecke functor of (13.2).

We can phrase Theorem 13.2.4 as saying that

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)^{\text{Hecke-lisse}}. \tag{13.7}$$

The following was proposed as a conjecture in [GKRV] (it appears as Conjecture C.2.8 in *loc.cit.*):

Main Theorem 13.4.3. *The inclusion* (13.7) *is an equality.*

In fact, we will prove a stronger result:

Main Theorem 13.4.4. *Let $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$ be such that for all $V \in \mathrm{Rep}(\check{G})$, the singular support of the object*

$$\mathrm{H}(V, \mathcal{F}) \in \mathrm{Shv}(\mathrm{Bun}_G \times X)$$

is contained in $T^*(\mathrm{Bun}_G) \times \{0\} \subset T^*(\mathrm{Bun}_G \times X)$. Then $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Remark 13.4.5. When $\mathrm{char}(k) = 0$, the assertion of Theorem 13.4.4 is actually equivalent to that of Theorem 13.4.3, by Corollary E.9.7.

Remark 13.4.6. Recall that in our notation $\mathrm{Shv}(\mathrm{Bun}_G)$ refers to a constructible sheaf theory. However, the statement of Theorem 13.4.3 remains valid, when instead of $\mathrm{Shv}(-)$ we consider $\mathrm{D\text{-}mod}(-)$ (when $\mathrm{char}(k) = 0$ and $\mathsf{e} = k$). We will prove this in Sect. 19.7.

Note that in the case of D-modules, the inclusion

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{D\text{-}mod}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \tag{13.8}$$

is an equality: since Nilp is Lagrangian, every object of $\mathrm{D\text{-}mod}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is necessarily ind-holonomic.

Remark 13.4.7. We conjecture that the statement of (the stronger) Theorem 13.4.4 also remains valid for $\mathrm{D\text{-}mod}(-)$. In fact, it would follow if we knew that Theorem 19.1.3 holds for $\mathrm{D\text{-}mod}(-)$, see Remark 19.1.6.

13.4.8. From Theorem 13.4.4 we obtain[28]:

Main Corollary 13.4.9. *Let $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$ be a* loose *Hecke eigensheaf, i.e., for every $V \in \mathrm{Rep}(\check{G})^{\heartsuit}$, the object*

$$\mathrm{H}(V, \mathcal{F}) \in \mathrm{Shv}(\mathrm{Bun}_G \times X)$$

is of the form $\mathcal{F} \boxtimes E_V$ for some $E_V \in \mathrm{QLisse}(X)$. Then $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

As a particular case, we obtain the following statement, which was conjectured by G. Laumon ([Lau, Conjecture 6.3.1]):

Main Corollary 13.4.10. *Hecke eigensheaves in* $\mathrm{Shv}(\mathrm{Bun}_G)$ *have nilpotent singular support.*

---

[28]The conclusion of Corollary 13.4.9 appears in [GKRV] as Conjecture C.2.10.

13.4.11. *Example.* The assertion of Theorem 13.4.3 is easy for $G = \mathbb{G}_m$. Note that in this case $\mathrm{Bun}_G = \mathrm{Pic}$, and

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) = \mathrm{QLisse}(\mathrm{Pic}).$$

Proof. The Hecke functor for the standard character of $\check{G} = \mathbb{G}_m$ is the pullback functor with respect to the addition map

$$\mathrm{add} : \mathrm{Pic} \times X \to \mathrm{Pic}\,.$$

Let us be given an object $\mathcal{F} \in \mathrm{Shv}(\mathrm{Pic})$ such that

$$\mathrm{add}^!(\mathcal{F}) \in \mathrm{Shv}(\mathrm{Pic} \times X)$$

belongs to

$$\mathrm{Shv}(\mathrm{Pic}) \otimes \mathrm{QLisse}(X) \subset \mathrm{Shv}(\mathrm{Pic} \times X)$$

.

We wish to show that $\mathcal{F}$ belongs to QLisse(Pic). It is easy to see that it is enough to prove that $\mathcal{F}|_{\mathrm{Pic}^d}$ belongs to $\mathrm{QLisse}(\mathrm{Pic}^d)$ for some/any d.

By [GKRV, Proposition C.2.5] quoted above, for any integer d we have

$$\mathrm{add}_d^!(\mathcal{F}) \in \mathrm{Shv}(\mathrm{Pic}) \otimes \mathrm{QLisse}(X^d),$$

where add_d is the d-fold addition map

$$\mathrm{add}_d : \mathrm{Pic} \times X^d \to \mathrm{Pic}\,.$$

In particular, the !-pullback of $\mathcal{F}$ along

$$X^d \simeq \mathbf{1}_{\mathrm{Pic}} \times X^d \to \mathrm{Pic} \times X^d \to \mathrm{Pic}^d \tag{13.9}$$

belongs to $\mathrm{QLisse}(X^d)$.

Note that the map (13.9) factors as

$$X^d \overset{\mathrm{sym}^d}{\longrightarrow} X^{(d)} \overset{\mathrm{AJ}_d}{\to} \mathrm{Pic}^d,$$

where AJ_d is the Abel-Jacobi map. For $d > 2g-2$, the map AJ_d is smooth and surjective. Let $\overset{\circ}{X}{}^d \subset X^d$ be the complement of the diagonal divisor. For $d \gg 0$, the composite map

$$\overset{\circ}{X}{}^d \hookrightarrow X^d \overset{(13.9)}{\longrightarrow} \mathrm{Pic}^d,$$

is also surjective. It is smooth because the map sym^d is étale when restricted to $\overset{\circ}{X}{}^d$.

Hence, for such d, if $\mathcal{F}|_{\overset{\circ}{X}{}^d}$ is lisse, then so is $\mathcal{F}$.

□

13.4.12. Let us now prove (the stronger) Theorem 13.4.4 for $\mathbb{G}_m$. In fact, the proof is even simpler (and will be the prototype of the proof of Theorem 13.4.4 for any G):

Consider again the map

$$\mathrm{add} : \mathrm{Pic} \times X \to \mathrm{Pic}\,.$$

We identify the cotangent space to Pic at any $\mathcal{L} \in \mathrm{Pic}$ with $\Gamma(X, \omega_X)$. Then the codifferential of add at any $(\mathcal{L}, x) \in \mathrm{Pic} \times X$ is the map

$$\Gamma(X, \omega_X) \to \Gamma(X, \omega_X) \oplus T^*_x(X),$$

whose first component is the identity map and the second component is the evaluation map at x.

Now, if for $\mathcal{F} \in \mathrm{Shv}(\mathrm{Pic})$ its singular support does not lie in $\mathrm{Nilp} = \{0\}$, we can find $\mathcal{L} \in \mathrm{Pic}$ and non-zero $\xi \in H^0(T^*_{\mathcal{L}}(\mathrm{Pic}))$ such that

$$(\xi, \mathcal{L}) \in \mathrm{SingSupp}(\mathcal{F}).$$

Let $x \in X$ be such that the value $\xi|_x \in T^*_x(X)$ of ξ at x is non-zero. Since the map add is smooth, the element

$$((\xi, \xi|_x), (\mathcal{L}, x)) \in T^*(\mathrm{Pic} \times X)$$

belongs to $\mathrm{SingSupp}(\mathrm{add}^!(\mathcal{F}))$.

However, $\xi|_x \neq 0$ by assumption, and we have obtained a contradiction with the fact that

$$\mathrm{SingSupp}(\mathrm{add}^!(\mathcal{F})) \in T^*(\mathrm{Bun}_G) \times \{0\}.$$

13.5. **Spectral decomposition in the de Rham context.** In this subsection we will assume that our ground field k has characteristic 0, and we will work with the entire category of D-modules, i.e., D-mod$(-)$ instead of Shv$(-)$.

13.5.1. Hecke action in the context of D-modules is a compatible family of functors

$$\mathrm{Rep}(\check{G})^{\otimes I} \otimes \text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_G \times X^I) \simeq \text{D-mod}(\mathrm{Bun}_G) \otimes \text{D-mod}(X)^{\otimes I}, \quad I \in \mathrm{fSet}.$$

It was shown in [Ga5, Corollary 4.5.5] that the above family of functors comes from a (uniquely defined) action of the category $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))$ on D-mod(Bun_G).

Here again, for a fixed $V \in \mathrm{Rep}(\check{G})$, the corresponding functor

$$\mathrm{H}(V, -) : \text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(\mathrm{Bun}_G \times X) \simeq \text{D-mod}(\mathrm{Bun}_G) \otimes \text{D-mod}(X)$$

is given by the action of the object

$$\mathcal{E}_V \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X)) \otimes \text{D-mod}(X),$$

see Sect. 9.8.4.

13.5.2. We now claim:

Proposition 13.5.3. *The full subcategory*

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \text{D-mod}(\mathrm{Bun}_G)$$

equals

$$\text{D-mod}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \subset$$
$$\subset \text{D-mod}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X)) = \text{D-mod}(\mathrm{Bun}_G),$$

where we view

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$$

as a co-localization of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))$.

Proof. This is obtained by combining Proposition 9.8.6 with Theorem 13.4.3, applied for all D-modules (see Remark 13.4.6).

□

13.5.4. From Proposition 13.5.3 we deduce:

Corollary 13.5.5. *In the de Rham context, the embedding* $\iota : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \text{D-mod}(\mathrm{Bun}_G)$ *admits a* continuous *right adjoint.*

Proof. The right adjoint is obtained by tensoring $- \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))}{\otimes} \text{D-mod}(\mathrm{Bun}_G)$ from the right adjoint to

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \hookrightarrow \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X)).$$

□

14. Beilinson's spectral projector and Hecke eigensheaves

In this section we recall the notion of Hecke eigensheaf and introduce Beilinson's spectral projector as a tool of constructing them.

14.1. **Ran version of the Hecke action and Hecke eigensheaves.** In this subsection, we will show how the category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ of Sect. 11.1 (and its version with a scheme of parameters, see Sect. 12.4) encode the formalism of Hecke action.

14.1.1. Recall the setting of Sect. 12.1. We claim that the category $\mathrm{Shv}(\mathrm{Bun}_G)$ carries an action of the (symmetric) monoidal category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$.

Namely, for

$$(I \overset{\psi}{\to} J) \in \mathrm{TwArr}(\mathrm{fSet}), \quad V \in \mathrm{Rep}(\check{G})^{\otimes I}, \quad \mathcal{S} \in \mathrm{Shv}(X^J), \tag{14.1}$$

and

$$\mathcal{V} := \mathrm{ins}_{I\to J}(V \otimes \mathcal{S}) \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}, \tag{14.2}$$

we let the action of $\mathcal{V}$ on $\mathrm{Shv}(\mathrm{Bun}_G)$ be given by

$$\mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathrm{H}(V,-)}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G \times X^I) \overset{(\mathrm{id} \times \Delta_\psi)^!}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G \times X^J) \overset{\overset{!}{\otimes}\mathcal{S}}{\to}$$
$$\to \mathrm{Shv}(\mathrm{Bun}_G \times X^J) \overset{(p_{\mathrm{Bun}_G})_*}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G),$$

where:

- The symbol $\overset{!}{\otimes}\mathcal{S}$ means !-tensor product by the !-pullback of $\mathcal{S}$;
- p_{Bun_G} denotes the projection $\mathrm{Bun}_G \times X^J \to X^J$.

14.1.2. Let now $\mathcal{Z}$ be a prestack over e, equipped with a map

$$f : \mathcal{Z} \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X).$$

Let F denote the resulting functor

$$\mathrm{Rep}(\check{G}) \to \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{QLisse}(X).$$

We will study the resulting category of Hecke eigen-objects

$$\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)),$$

see Sect. 12.1.4.

We can think of $\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G))$ as objects $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z})$ equipped with a system of isomorphisms

$$\mathcal{V} \star \mathcal{F} \simeq \mathcal{F} \otimes \widetilde{\mathsf{F}}(\mathcal{V}), \quad \mathcal{V} \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}}. \tag{14.3}$$

14.1.3. We will now rerun the above story, in the presence of a scheme of parameters Y.

Recall the setting of Sect. 12.4. We claim that for any test k-scheme Y, the category $\mathrm{Shv}(\mathrm{Bun}_G \times Y)$ carries an action of the (symmetric) monoidal category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}\times Y}$.

Namely, for

$$(I \overset{\psi}{\to} J) \in \mathrm{TwArr}(\mathrm{fSet}), \quad V \in \mathrm{Rep}(\check{G})^{\otimes I}, \quad \mathcal{S} \in \mathrm{Shv}(X^J \times Y), \tag{14.4}$$

and

$$\mathcal{V}_Y := \mathrm{ins}_{I\to J}(V \otimes \mathcal{S}) \in \mathrm{Rep}(\check{G})_{\mathrm{Ran}\times Y}, \tag{14.5}$$

we let the action of $\mathcal{V}_Y$ on $\mathrm{Shv}(\mathrm{Bun}_G \times Y)$ be given by

$$\mathrm{Shv}(\mathrm{Bun}_G \times Y) \overset{\mathrm{H}(V,-)}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G \times X^I \times Y) \overset{(\mathrm{id} \times \Delta_\psi \times \mathrm{id})^!}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G \times X^J \times Y) \overset{\overset{!}{\otimes}\mathcal{S}}{\to}$$
$$\to \mathrm{Shv}(\mathrm{Bun}_G \times X^J \times Y) \overset{(p_{\mathrm{Bun}_G \times Y})_*}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G \times Y),$$

where:

- We denote by $\mathrm{H}(V,-)$ the corresponding version of the Hecke functor, where we have a scheme Y of parameters;
- The symbol $\overset{!}{\otimes}\mathcal{S}$ means !-tensor product by the !-pullback of $\mathcal{S}$;
- $p_{\mathrm{Bun}_G\times Y}$ denotes the projection $\mathrm{Bun}_G\times X^J\times Y\to \mathrm{Bun}_G\times Y$.

14.1.4. For a pair $(\mathcal{Z}, f:\mathcal{Z}\to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ as above, we can consider the resulting categories

$$\mathrm{Hecke}_Y(\mathcal{Z},\mathrm{Shv}(\mathrm{Bun}_G\times Y)) \text{ and } \mathrm{Hecke}(\mathcal{Z},\mathrm{Shv}(\mathrm{Bun}_G\times Y)).$$

see Sect. 12.4.7.

Note, however, that thanks to Theorem 12.4.9, the forgetful functor

$$\mathrm{Hecke}_Y(\mathcal{Z},\mathrm{Shv}(\mathrm{Bun}_G\times Y))\to \mathrm{Hecke}(\mathcal{Z},\mathrm{Shv}(\mathrm{Bun}_G\times Y))$$

is an equivalence.

14.1.5. For a map $Y_1\to Y_2$, the functor of !-pullback

$$\mathrm{Shv}(\mathrm{Bun}_G\times Y_2)\to \mathrm{Shv}(\mathrm{Bun}_G\times Y_1)$$

is compatible with the action of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$. Hence, it induces a functor

$$\mathrm{Hecke}(\mathcal{Z},\mathrm{Shv}(\mathrm{Bun}_G\times Y_2))\to \mathrm{Hecke}(\mathcal{Z},\mathrm{Shv}(\mathrm{Bun}_G\times Y_1)).$$

In particular, for every Y, we have a canonically defined functor

$$\mathrm{Hecke}(\mathcal{Z},\mathrm{Shv}(\mathrm{Bun}_G))\to \mathrm{Hecke}_Y(\mathcal{Z},\mathrm{Shv}(\mathrm{Bun}_G\times Y))$$

that fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Hecke}(\mathcal{Z},\mathrm{Shv}(\mathrm{Bun}_G)) & \longrightarrow & \mathrm{Hecke}_Y(\mathcal{Z},\mathrm{Shv}(\mathrm{Bun}_G\times Y)) \\ \downarrow & & \downarrow \\ \mathrm{Shv}(\mathrm{Bun}_G)\otimes \mathrm{QCoh}(\mathcal{Z}) & \longrightarrow & \mathrm{Shv}(\mathrm{Bun}_G\times Y)\otimes \mathrm{QCoh}(\mathcal{Z}), \end{array}$$

where the vertical arrows are forgetful functors, and the bottom horizontal arrow is the !-pullback functor.

Hence, for $\mathcal{F}\in \mathrm{Hecke}(\mathcal{Z},\mathrm{Shv}(\mathrm{Bun}_G))$ and an object $\mathcal{V}_Y\in \mathrm{Rep}(\check{G})_{\mathrm{Ran}\times Y}$, we have a Hecke isomorphism in $\mathrm{Shv}(\mathrm{Bun}_G\times Y)\otimes \mathrm{QCoh}(\mathcal{Z})$:

$$\mathcal{V}_Y\star(\mathcal{F}\boxtimes\omega_Y)\simeq(\mathcal{F}\boxtimes\omega_Y)\overset{!}{\otimes}\widetilde{\mathsf{F}}_Y(\mathcal{V}_Y),\quad \mathcal{V}_Y\in \mathrm{Rep}(\check{G})_{\mathrm{Ran}\times Y}, \tag{14.6}$$

for $\widetilde{\mathsf{F}}_Y$ as in Sect. 12.4.5, where $\overset{!}{\otimes}$ denotes the natural action of $\mathrm{QCoh}(\mathcal{Z})\otimes\mathrm{Shv}(Y)$ on

$$\mathrm{Shv}(\mathrm{Bun}_G\times Y)\otimes \mathrm{QCoh}(\mathcal{Z}).$$

14.1.6. All of the above discussion is equally applicable when instead of $\mathrm{Shv}(-)$ we consider D-mod$(-)$ (and f is a map $\mathcal{Z}\to \mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X)$ or $\mathrm{Shv}^{\mathrm{all}}(-)$ and (and f is a map $\mathcal{Z}\to \mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)$).

14.2. **Another notion of Hecke eigensheaf.** In this subsection we relate the category

$$\mathrm{Hecke}(\mathcal{Z},\mathrm{Shv}(\mathrm{Bun}_G))$$

to another, probably more familiar, notion of Hecke eigensheaf.

14.2.1. We will show that the category $\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G))$ can be identified with the category of objects

$$\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z}),$$

equipped with a system of isomorphisms

$$\mathrm{H}(V, \mathcal{F}) \overset{\alpha_V}{\simeq} \mathcal{F} \underset{\mathcal{O}_{\mathcal{Z}}}{\boxtimes} \mathsf{F}^I(V), \quad V \in \mathrm{Rep}(\check{G})^{\otimes I}, \quad I \in \mathrm{fSet}, \tag{14.7}$$

where:

- F^I is the functor $\mathrm{Rep}(\check{G})^{\otimes I} \to \mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{QLisse}(X)^{\otimes I}$, corresponding to f;
- Both sides in (14.7) are viewed as objects of $\mathrm{Shv}(\mathrm{Bun}_G \times X^I) \otimes \mathrm{QCoh}(\mathcal{Z})$;
- $\underset{\mathcal{O}_{\mathcal{Z}}}{\boxtimes}$ denotes the external tensor product functor

$$(\mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z})) \otimes (\mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{QLisse}(X)^{\otimes I}) \to \mathrm{Shv}(\mathrm{Bun}_G \times X^I) \otimes \mathrm{QCoh}(\mathcal{Z}).$$

The isomorphisms (14.7) are required to be equipped with a homotopy-coherent system of compatibilities for maps between finite sets:

For $I_1 \overset{\phi}{\to} I_2$ and $V_1 \in \mathrm{Rep}(\check{G})^{\otimes I_1}$, we must be given a data of commutativity for the diagram

$$\begin{array}{ccc} ((\mathrm{id}_{\mathrm{Bun}_G} \times \Delta_\phi)^! \otimes \mathrm{Id}_{\mathrm{QCoh}(\mathcal{Z})})(\mathrm{H}(V_1, \mathcal{F})) & \xrightarrow{\alpha_{V_1}} & ((\mathrm{id}_{\mathrm{Bun}_G} \times \Delta_\phi)^! \otimes \mathrm{Id}_{\mathrm{QCoh}(\mathcal{Z})})(\mathcal{F} \underset{\mathcal{O}_{\mathcal{Z}}}{\boxtimes} \mathsf{F}^{I_1}(V_1)) \\ \sim \downarrow & & \downarrow \sim \\ \mathrm{H}(V_2, \mathcal{F}) & \xrightarrow{\alpha_{V_2}} & \mathcal{F} \underset{\mathcal{O}_{\mathcal{Z}}}{\boxtimes} \mathsf{F}^{I_2}(V_2), \end{array}$$

where

$$V_2 := \mathrm{mult}^\phi_{\mathrm{Rep}(\check{G})}(V_1) \in \mathrm{Rep}(\check{G})^{\otimes I_2}.$$

14.2.2. Indeed, let $\mathcal{F}$ be an object of $\mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z})$ equipped with a lift to an object of $\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G))$ in the definition of Sect. 14.1.2, and let us construct the isomorphisms (14.7).

For a finite set I and $V \in \mathrm{Rep}(\check{G})^{\otimes I}$, take $Y = X^I$, and take in (14.4)

$$\mathcal{S} := (\Delta_{X^I})_*(\omega_{X^I}) \in \mathrm{Shv}(X^I \times X^I), \quad \mathcal{V}_Y := \mathrm{ins}_{I \to I}(V \otimes \mathcal{S}).$$

Then the isomorphism (14.6) for the above $\mathcal{V}_Y$ amounts to the isomorphism (14.7).

14.2.3. Vice versa, let $\mathcal{F}$ be an object of $\mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z})$ equipped with a system of Hecke isomorphisms (14.7). We recover the isomorphism (14.3) for

$$\mathcal{V} = \mathrm{ins}_{I \overset{\psi}{\to} J}(V \otimes \mathcal{S}), \quad V \in \mathrm{Rep}(\check{G})^{\otimes I}, \ \mathcal{S} \in \mathrm{Shv}(X^J),$$

by applying the functor

$$\left((p_{\mathrm{Bun}_G})_* \circ \left((\mathrm{id}_X \times \Delta_\psi)^!(-) \overset{!}{\otimes} \mathcal{S} \right) \right) \otimes \mathrm{Id}_{\mathrm{QCoh}(\mathcal{Z})}$$

to the two sides of (14.7).

14.2.4. All of the above discussion is equally applicable when instead of $\mathrm{Shv}(-)$ we consider $\mathrm{D\text{-}mod}(-)$ (and f is a map $\mathcal{Z} \to \mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X)$ or $\mathrm{Shv}^{\mathrm{all}}(-)$ and (and f is a map $\mathcal{Z} \to \mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)$).

14.3. **Beilinson's spectral projector on** $\mathrm{Shv}(\mathrm{Bun}_G)$**.** Beilinson's projector is a functor that manifactures Hecke eigen-objects from arbitrary objects of $\mathrm{Shv}(\mathrm{Bun}_G)$. In this subsection we will define it as the left adjoint of the forgetful functor. But its crucial feature is that it can also be constructed as an explicit integral Hecke functor.

14.3.1. Let $f : \mathcal{Z} \to \operatorname{LocSys}^{\text{restr}}_{\check{G}}(X)$ be as above. Recall that we denote by $\mathbf{oblv}_{\text{Hecke},\mathcal{Z}}$ the forgetful functor

$$\operatorname{Hecke}(\mathcal{Z}, \operatorname{Shv}(\operatorname{Bun}_G)) \to \operatorname{Shv}(\operatorname{Bun}_G) \otimes \operatorname{QCoh}(\mathcal{Z}), \tag{14.8}$$

and by $\mathbf{ind}_{\text{Hecke},\mathcal{Z}}$ its left adjoint

$$\operatorname{Shv}(\operatorname{Bun}_G) \otimes \operatorname{QCoh}(\mathcal{Z}) \to \operatorname{Hecke}(\mathcal{Z}, \operatorname{Shv}(\operatorname{Bun}_G)).$$

Recall also that the resulting monad on $\operatorname{Shv}(\operatorname{Bun}_G) \otimes \operatorname{QCoh}(\mathcal{Z})$ is given by the action of the (commutative) algebra object

$$\mathsf{R}_{\mathcal{Z}} \in \operatorname{Rep}(\check{G})_{\text{Ran}} \otimes \operatorname{QCoh}(\mathcal{Z}),$$

see Sect. 12.1.6.

14.3.2. Let $\mathbf{oblv}_{\text{Hecke}}$ denote the (not necessarily continuous) functor equal to the composition

$$\operatorname{Hecke}(\mathcal{Z}, \operatorname{Shv}(\operatorname{Bun}_G)) \overset{\mathbf{oblv}_{\text{Hecke},\mathcal{Z}}}{\longrightarrow} \operatorname{Shv}(\operatorname{Bun}_G) \otimes \operatorname{QCoh}(\mathcal{Z}) \to \operatorname{Shv}(\operatorname{Bun}_G),$$

where the second arrow is the right adjoint to

$$\operatorname{Shv}(\operatorname{Bun}_G) \overset{-\otimes \mathcal{O}_{\mathcal{Z}}}{\to} \operatorname{Shv}(\operatorname{Bun}_G) \otimes \operatorname{QCoh}(\mathcal{Z}). \tag{14.9}$$

Let $\mathsf{P}_{\mathcal{Z}}$ denote the functor

$$\operatorname{Shv}(\operatorname{Bun}_G) \to \operatorname{Shv}(\operatorname{Bun}_G) \otimes \operatorname{QCoh}(\mathcal{Z})$$

equal to the composition of (14.9) and the functor given by the action of $\mathsf{R}_{\mathcal{Z}}$.

We obtain that the functor $\mathsf{P}_{\mathcal{Z}}$ naturally upgrades to a functor

$$\mathsf{P}^{\text{enh}}_{\mathcal{Z}} : \operatorname{Shv}(\operatorname{Bun}_G) \to \operatorname{Hecke}(\mathcal{Z}, \operatorname{Shv}(\operatorname{Bun}_G)),$$

which provides a left adjoint to $\mathbf{oblv}_{\text{Hecke}}$, i.e., it identifies with

$$\operatorname{Shv}(\operatorname{Bun}_G) \overset{-\otimes \mathcal{O}_{\mathcal{Z}}}{\longrightarrow} \operatorname{Shv}(\operatorname{Bun}_G) \otimes \operatorname{QCoh}(\mathcal{Z}) \overset{\mathbf{ind}_{\text{Hecke},\mathcal{Z}}}{\longrightarrow} \operatorname{Hecke}(\mathcal{Z}, \operatorname{Shv}(\operatorname{Bun}_G)).$$

We will refer to $\mathsf{P}^{\text{enh}}_{\mathcal{Z}}$ as *Beilinson's spectral projector*.

14.3.3. Note that the functor $\mathsf{P}_{\mathcal{Z}}$ can written down explicitly as a colimit, via the presentation of $\mathsf{R}_{\mathcal{Z}}$ as a colimit, see Sect. 12.1.11.

Namely,

$$\mathsf{P}_{\mathcal{Z}} \simeq \underset{(I \overset{\psi}{\to} J) \in \operatorname{TwArr}(\text{fSet})}{\operatorname{colim}} \mathsf{P}^{I \overset{\psi}{\to} J}_{\mathcal{Z}},$$

where $\mathsf{P}^{I \overset{\psi}{\to} J}_{\mathcal{Z}}$ equals the composition

$$\operatorname{Shv}(\operatorname{Bun}_G) \overset{\mathsf{R}^{\otimes I}_{\operatorname{Rep}(\check{G})}}{\longrightarrow} (\operatorname{Rep}(\check{G}) \otimes \operatorname{Rep}(\check{G}))^{\otimes I} \otimes \operatorname{Shv}(\operatorname{Bun}_G) \overset{\operatorname{mult}^{\psi}_{\operatorname{Rep}(\check{G}) \otimes \operatorname{Rep}(\check{G})} \otimes \operatorname{Id}}{\longrightarrow}$$
$$\to (\operatorname{Rep}(\check{G}) \otimes \operatorname{Rep}(\check{G}))^{\otimes J} \otimes \operatorname{Shv}(\operatorname{Bun}_G) \simeq \operatorname{Rep}(\check{G})^{\otimes J} \otimes \operatorname{Shv}(\operatorname{Bun}_G) \otimes \operatorname{Rep}(\check{G})^{\otimes J} \overset{\operatorname{H}(-,-) \otimes \operatorname{Id}}{\longrightarrow}$$
$$\to \operatorname{Shv}(\operatorname{Bun}_G \times X^J) \otimes \operatorname{Rep}(\check{G})^{\otimes J} \overset{\operatorname{Id} \otimes \mathsf{F}^J}{\longrightarrow} \operatorname{Shv}(\operatorname{Bun}_G \times X^J) \otimes \operatorname{QLisse}(X)^{\otimes J} \otimes \operatorname{QCoh}(\mathcal{Z}) \to$$
$$\to \operatorname{Shv}(\operatorname{Bun}_G \times X^J \times X^J) \otimes \operatorname{QCoh}(\mathcal{Z}) \overset{(\operatorname{id} \times \Delta_{X^J})^! \otimes \operatorname{Id}}{\longrightarrow}$$
$$\to \operatorname{Shv}(\operatorname{Bun}_G \times X^J) \otimes \operatorname{QCoh}(\mathcal{Z}) \overset{(p_{\operatorname{Bun}_G})_* \otimes \operatorname{Id}}{\longrightarrow} \operatorname{Shv}(\operatorname{Bun}_G) \otimes \operatorname{QCoh}(\mathcal{Z}),$$

where:

- $\mathsf{R}_{\operatorname{Rep}(\check{G})}$ denotes the regular representation of $\check{G}$, regarded as an object of $\operatorname{Rep}(\check{G}) \otimes \operatorname{Rep}(\check{G})$;
- F^J denotes the functor
$$\operatorname{Rep}(\check{G})^{\otimes J} \to \operatorname{QCoh}(\mathcal{Z}) \otimes \operatorname{QLisse}(X)^{\otimes J}$$
corresponding to the given map $f : \mathcal{Z} \to \operatorname{LocSys}^{\text{restr}}_{\check{G}}(X)$;

14.3.4. Let $g : \mathcal{Z}' \to \mathcal{Z}$ be a map of prestacks, and let $f' : \mathcal{Z}' \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ denote the composite $f \circ g$. Then by Sect. 12.1.7 we have a naturally defined functor

$$g^* : \mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{Hecke}(\mathcal{Z}', \mathrm{Shv}(\mathrm{Bun}_G))$$

that makes the diagram

$$\begin{array}{ccc} \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z}') & \xrightarrow{\mathbf{ind}_{\mathrm{Hecke},\mathcal{Z}'}} & \mathrm{Hecke}(\mathcal{Z}', \mathrm{Shv}(\mathrm{Bun}_G)) \\ {\scriptstyle \mathrm{Id} \otimes g^*}\uparrow & & \uparrow{\scriptstyle g^*} \\ \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z}) & \xrightarrow{\mathbf{ind}_{\mathrm{Hecke},\mathcal{Z}}} & \mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)) \end{array} \tag{14.10}$$

commute.

Remark 14.3.5. The above observations are equally applicable when instead of $\mathrm{Shv}(-)$ we consider $\mathrm{D\text{-}mod}(-)$ (and f is a map $\mathcal{Z} \to \mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X)$ or $\mathrm{Shv}^{\mathrm{all}}(-)$ and (and f is a map $\mathcal{Z} \to \mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)$).

14.4. **Beilinson's projector and nilpotence of singular support.**

14.4.1. Consider the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$. It follows from Sect. 13.2.7 that the action of $\mathrm{Rep}(\check{G}_{\mathrm{Ran}})$ on $\mathrm{Shv}(\mathrm{Bun}_G)$ preserves the subcategory $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Thus, we can consider $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ as a $\mathrm{Rep}(\check{G}_{\mathrm{Ran}})$-module category.

14.4.2. Let $\mathcal{Z}$ be a prestack over e, and fix a map $f : \mathcal{Z} \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ as above. Consider the resulting category $\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))$, equipped with the pair of adjoint functors

$$\mathbf{ind}_{\mathrm{Hecke},\mathcal{Z}} : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z}) \rightleftarrows \mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) : \mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}.$$

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z}) & \xleftarrow{\mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}} & \mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \\ \downarrow & & \downarrow \\ \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z}) & \xleftarrow{\mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}} & \mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)), \end{array}$$

where the horizontal arrows are monadic, and the left vertical arrow intertwines the acton of the corresponding monads (both are given by $\mathsf{R}_{\mathcal{Z}}$).

14.4.3. From now on we will assume that $\mathcal{Z}$ is such that $\mathrm{QCoh}(\mathcal{Z})$ is dualizable. This implies that the functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z}) \xrightarrow{\iota \otimes \mathrm{Id}} \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z})$$

is fully faithful.

From here is follows that the functor

$$\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \to \mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)) \tag{14.11}$$

is fully faithful, whose essential image consists equals

$$\{\mathcal{F} \in \mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)) \,|\, \mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}(\mathcal{F}) \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z})\}.$$

We claim:

Proposition 14.4.4. *Let* $\mathcal{Z}$ *be such that* $\mathrm{QCoh}(\mathcal{Z})$ *is dualizable. The functor* (14.11) *is an equivalence.*

Proof. We have to show that the essential image of the forgetful functor

$$\mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}} : \mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z})$$

is contained in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z})$.

Since $\mathrm{QCoh}(\mathcal{Z})$ is dualizable, we have to show that if T is a continuous functor $\mathrm{QCoh}(\mathcal{Z}) \to \mathrm{Vect}_{\mathsf{e}}$, the essential image of the composition

$$\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z}) \overset{\mathrm{Id}_{\mathrm{Shv}(\mathrm{Bun}_G)} \otimes \mathsf{T}}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G)$$

is contained in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Let $\mathcal{F}$ be an object of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z})$ that can be upgraded to an object of the category $\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G))$. We have to show that

$$(\mathrm{Id}_{\mathrm{Shv}(\mathrm{Bun}_G)} \otimes \mathsf{T})(\mathcal{F}) \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

We will show that $(\mathrm{Id}_{\mathrm{Shv}(\mathrm{Bun}_G)} \otimes \mathsf{T})(\mathcal{F}) \in \mathrm{Shv}(\mathrm{Bun}_G)^{\text{Hecke-lisse}}$. This would imply the required assertion by Theorem 13.4.3.

By Sect. 14.2.2, for $V \in \mathrm{Rep}(\check{G})$, we have:

$$\mathrm{H}(V, \mathcal{F}) \simeq \mathcal{F} \underset{\mathcal{O}_{\mathcal{Z}}}{\boxtimes} \mathsf{F}(V)$$

as objects of

$$\mathrm{Shv}(\mathrm{Bun}_G \times X) \otimes \mathrm{QCoh}(\mathcal{Z}).$$

Hence,

$$\mathrm{H}\left(V, (\mathrm{Id}_{\mathrm{Shv}(\mathrm{Bun}_G)} \otimes \mathsf{T})(\mathcal{F})\right) \simeq (\mathrm{Id}_{\mathrm{Shv}(\mathrm{Bun}_G \times X)} \otimes \mathsf{T})(\mathcal{F} \underset{\mathcal{O}_{\mathcal{Z}}}{\boxtimes} \mathsf{F}(V))$$

as objects of $\mathrm{Shv}(\mathrm{Bun}_G \times X)$.

Now, the functor $\mathrm{Id}_{\mathrm{Shv}(\mathrm{Bun}_G \times X)} \otimes \mathsf{T}$ maps objects in the essential image of the functor

$$(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z})) \otimes (\mathrm{QCoh}(\mathcal{Z}) \otimes \mathrm{QLisse}(X)) \overset{\underset{\mathcal{O}_{\mathcal{Z}}}{\boxtimes}}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G \times X) \otimes \mathrm{QCoh}(\mathcal{Z})$$

to $\mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X)$. Hence, we conclude that

$$\mathrm{H}\left(V, (\mathrm{Id}_{\mathrm{Shv}(\mathrm{Bun}_G)} \otimes \mathsf{T})(\mathcal{F})\right) \in \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X),$$

as required.

□

14.4.5. Combining with Sects. 12.3.1 and Sect. 12.3.3 we obtain:

Corollary 14.4.6. *Let $\mathcal{Z}$ be such that $\mathrm{QCoh}(\mathcal{Z})$ is dualizable.*
(a) *There exists a unique identification*

$$\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)) \simeq \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathcal{Z})$$

so that the forgetful functor

$$\mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}} : \mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z})$$

identifies with

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathcal{Z}) \simeq \tag{14.12}$$

$$\simeq (\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z})) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)) \to$$

$$\overset{\mathrm{Id} \otimes (\Delta_{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)})_*}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z}) \overset{\iota \otimes \mathrm{Id}}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z}).$$

(b) *Assume moreover that $\mathcal{O}_{\mathcal{Z}} \in \mathrm{QCoh}(\mathcal{Z})$ is compact. Then with respect to the identification of point (a), the forgetful functor*

$$\mathbf{oblv}_{\mathrm{Hecke}} : \mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}(\mathrm{Bun}_G)) \to \mathrm{Shv}(\mathrm{Bun}_G)$$

identifies with

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X))}{\otimes} \text{QCoh}(\mathcal{Z}) \overset{\text{Id}\otimes f_*}{\longrightarrow} \tag{14.13}$$

$$\to \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X))}{\otimes} \text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \simeq \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \overset{\iota}{\hookrightarrow} \text{Shv}(\text{Bun}_G).$$

In particular, we obtain:

Corollary 14.4.7. *Under the assumption of Corollary 14.4.6(b), the functor* (14.13) *admits a left adjoint, given by* $\mathsf{P}^{\text{enh}}_{\mathcal{Z}}$.

Remark 14.4.8. Note that the functor $\iota : \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \hookrightarrow \text{Shv}(\text{Bun}_G)$ itself does does *not* admit a left adjoint (see Sect. 16.1 for more details). This does not violate Corollary 14.4.7 since $\mathcal{O}_{\text{LocSys}^{\text{restr}}_{\check{G}}(X)}$ is not compact as an object of $\text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X))$.

14.4.9. Note that under the assumption of Corollary 14.4.6(a), we can regard $\mathsf{P}^{\text{enh}}_{\mathcal{Z}}$ as a functor

$$\text{Shv}(\text{Bun}_G) \to \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X))}{\otimes} \text{QCoh}(\mathcal{Z}),$$

so that the restriction of $\mathsf{P}^{\text{enh}}_{\mathcal{Z}}$ to $\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \subset \text{Shv}(\text{Bun}_G)$ identifies with

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \simeq \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X))}{\otimes} \text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \overset{\text{Id}\otimes f^*}{\longrightarrow}$$

$$\to \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X))}{\otimes} \text{QCoh}(\mathcal{Z}).$$

In particular, the functor $\mathsf{P}_{\mathcal{Z}}$ takes values in

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \otimes \text{QCoh}(\mathcal{Z}) \subset \text{Shv}(\text{Bun}_G) \otimes \text{QCoh}(\mathcal{Z}).$$

Furthermore, under the assumption of Corollary 14.4.6(b), we have

$$(\text{Id}\otimes\Gamma(\mathcal{Z},-)) \circ \mathsf{P}_{\mathcal{Z}} \simeq \iota \circ (\text{Id}\otimes f_*) \circ \mathsf{P}^{\text{enh}}_{\mathcal{Z}}, \tag{14.14}$$

where $\text{Id}\otimes f_*$ is the functor

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X))}{\otimes} \text{QCoh}(\mathcal{Z}) \to$$

$$\to \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X))}{\otimes} \text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \simeq \text{Shv}_{\text{Nilp}}(\text{Bun}_G).$$

14.4.10. Let us place ourselves in the situation of Sect. 14.3.4. Assume that both $\text{QCoh}(\mathcal{Z})$ and $\text{QCoh}(\mathcal{Z}')$ are dualizable. From Sect. 12.3.2 we obtain that the functor

$$g^* : \text{Hecke}(\mathcal{Z}, \text{Shv}(\text{Bun}_G)) \to \text{Hecke}(\mathcal{Z}', \text{Shv}(\text{Bun}_G))$$

identifies canonically with

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X))}{\otimes} \text{QCoh}(\mathcal{Z}) \overset{\text{Id}\otimes g^*}{\to} \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X))}{\otimes} \text{QCoh}(\mathcal{Z}').$$

Remark 14.4.11. The material in this subsection applies when instead of $\text{Shv}(-)$ we consider $\text{D-mod}(-)$, but f is a map $f : \mathcal{Z} \to \text{LocSys}^{\text{restr}}_{\check{G}}(X)$. Here we use the fact that the inclusion (13.8) is an equality.

Remark 14.4.12. This is a preview of the material in Sect. 17:

The discussion in this subsection applies when instead of $\text{Shv}(-)$ we consider $\text{Shv}^{\text{all}}(-)$. In fact, here we have two variants:

We can consider maps $f : \mathcal{Z} \to \text{LocSys}^{\text{restr}}_{\check{G}}(X)$, in which case the statements of the assertions in this subsection hold with no modifications.

However, we can also consider maps $f : \mathcal{Z} \to \text{LocSys}^{\text{Betti}}_{\check{G}}(X)$. In this case, one has to replace $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ by $\text{Shv}^{\text{all}}_{\text{Nilp}}(\text{Bun}_G)$.

14.5. **The projector onto the subcategory with nilpotent singular support.** In this subsection we will use Beilinson's projector to construct a functor

$$\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G),$$

left *inverse* the the embedding ι, and study its properties.

14.5.1. Let us apply the observation in Sect. 14.4.9 to $\mathcal{Z} = \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ and f being the identity map. We obtain a functor

$$\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

The property of $\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}}$ specified in Sect. 14.4.9 implies:

Lemma 14.5.2. *The functor* $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ *provides a* left inverse *to the embedding* ι.

14.5.3. Note that by Sects. 14.3.4 and Sect. 14.4.10, for an arbitrary $\mathcal{Z} \xrightarrow{f} \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ (with $\mathrm{QCoh}(\mathcal{Z})$ dualizable), we have

$$\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}} \simeq (f^* \otimes \mathrm{Id}) \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}, \tag{14.15}$$

where $f^* \otimes \mathrm{Id}$ is the pullback functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to$$
$$\to \mathrm{QCoh}(\mathcal{Z}) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

14.5.4. Let us rewrite the functor $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ somewhat differently.

Note that the functor

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \overset{(\Delta_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)})_*}{\longrightarrow} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X) \times \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \simeq$$
$$\simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \overset{\Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X),-)\otimes \mathrm{Id}}{\longrightarrow} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$$

is isomorphic to the identity functor, where $\Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X),-)$ is as in Sect. 7.6.1.

Hence, we obtain that the functor $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ can be rewritten as

$$\mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathsf{P}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}}{\longrightarrow} \tag{14.16}$$
$$\to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \overset{\mathrm{Id} \otimes \Gamma_!(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X),-)}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G),$$

where $\mathsf{P}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ can be written down explicitly as in Sect. 14.3.3.

Further, using (7.28), the functor (14.16) can be rewritten as

$$\mathcal{F} \mapsto \underset{\mathcal{Z}}{\oplus} \underset{n}{\mathrm{colim}} \left(\mathrm{Id} \otimes \Gamma(Z_n, f_n^!(-))\right) \circ \mathsf{P}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}(\mathcal{F}), \tag{14.17}$$

where $\mathcal{Z}$ runs over the set of connected components of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ and $Z_n \xrightarrow{f_n} \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ are as in Sects. 7.9.6-7.9.8 (see also Sect. 15.1.2) below.

14.5.5. The expression for $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ given by (14.17), combined with the explicit expression for $\mathsf{P}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ in Sect. 14.3.3, implies that the endofunctor $\iota \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ of $\mathrm{Shv}(\mathrm{Bun}_G)$ is an *integral Hecke functor*, i.e., a colimit of functors, each of which is the composition of a Hecke functor

$$\mathrm{H}(V,-) : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times X^I), \quad V \in \mathrm{Rep}(\check{G})^{\otimes I}$$

and a functor $\mathrm{Shv}(\mathrm{Bun}_G \times X^I) \to \mathrm{Shv}(\mathrm{Bun}_G)$ given by

$$\mathcal{F} \mapsto (p_1)_*(\mathcal{F} \overset{!}{\otimes} p_2^!(\mathcal{S})), \quad \mathcal{S} \in \mathrm{Shv}(X^I). \tag{14.18}$$

Furthermore, in the colimit expression for $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$, the above objects $\mathcal{S}$ belong to

$$\mathrm{QLisse}(X)^{\otimes I} \subset \mathrm{Shv}(X^I).$$

15. Applications of Beilinson's spectral projector

In this section we will combine our Theorems 13.3.2 and 13.4.4 with Beilinson's spectral projector to prove some key theorems about $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

15.1. Compact generation of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

In this subsection we will use Beilinson's projector to prove the following assertion:

Theorem 15.1.1. *The category* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ *is compactly generated.*

15.1.2. By Sects. 7.9.6-7.9.8, for every connected component $\mathcal{Z}$ of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$, we can find a family of algebraic stacks mapping to $\mathcal{Z}$

$$f_n : Z_n \to \mathcal{Z}$$

with the following properties:

- Each Z_n is of the form S/H with S an affine scheme and H an algebraic group;
- Each f_n is affine (so that $(f_n)_*$ is continuous and $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$-linear), and $(f_n)_*$ admits a continuous right adjoint $(f_n)^!$, which is also $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$-linear.
- The essential images of the functors $(f_n)_*$ generate $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$.

Remark 15.1.3. In what follows, in order to unburden the notations, we will group the connected components of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ together, and consider the stacks Z_n as mapping directly to $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$. So in the sequel, in this context, the index n no longer refers to a natural number but rather to an element $\underset{\mathcal{Z}}{\cup} \mathbb{N}$.

15.1.4. The adjunction

$$(f_n)_* : \mathrm{QCoh}(Z_n) \rightleftarrows \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))) : (f_n)^!$$

as $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$-module categories induces an adjunction

$$\mathrm{Id} \otimes (f_n)_* : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(Z_n) \rightleftarrows \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) : \mathrm{Id} \otimes f_n^!.$$

In particular, the functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(Z_n) \overset{\mathrm{Id} \otimes (f_n)_*}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \tag{15.1}$$

preserves compactness. Furthermore, since the essential images of the functors $(f_n)_*$ generate $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$, the essential images of the functors (15.1) generate $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Hence, it is enough to show that each of the categories

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(Z_n)$$

is compactly generated.

15.1.5. Consider the functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(Z_n) \overset{(15.1)}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\iota}{\to} \mathrm{Shv}(\mathrm{Bun}_G). \tag{15.2}$$

Since $\mathrm{Shv}(\mathrm{Bun}_G)$ is compactly generated, it suffices to show that this functor is conservative and admits a left adjoint.

The existence of the left adjoint follows from Corollary 14.4.7 (the conditions of the corollary are guaranteed by the assumption that Z_n is of the form S/H).

15.1.6. To prove that (15.2) is conservative we argue as follows. It suffices to show that the functor (15.1) is conservative. The latter is equivalent to the fact that the essential image of the functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \overset{\mathrm{Id}\otimes(f_n)^*}{\longrightarrow}$$
$$\to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(Z_n)$$

generates $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(Z_n)$.

To prove this, it is sufficient to show that the essential image of the functor

$$f_n^* : \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \to \mathrm{QCoh}(Z_n)$$

generates $\mathrm{QCoh}(Z_n)$. This is equivalent to the functor $(f_n)_*$ being conservative. But this is indeed the case since f_n is affine.

□[Theorem 15.1.1]

15.1.7. Note that Theorem 15.1.1 admits the following corollary:

Corollary 15.1.8. *Let* $\mathcal{U} \overset{j}{\hookrightarrow} \mathrm{Bun}_G$ *be an open substack such that the functor* $j_!$ *(equivalently,* j_**) sends* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U})^{\mathrm{constr}}$ *to* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{constr}}$*. Then* $\mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U})$ *is compactly generated.*

Proof. Follows from the fact that the functor

$$j^* : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathcal{U})$$

admits a conservative right adjoint, given by j_* (see Remark 13.1.6).

□

15.2. **A set of generators for** $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$**.** In the previous subsection we showed that the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is compactly generated. In this subsection, we will sharpen this by writing down an explicit set of generators for $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

15.2.1. Let $\mathcal{Y}$ be an algebraic stack and $\mathcal{N} \subset T^*(\mathcal{Y})$ a half-dimensional conical subset. We claim that we can find a collection of k-points y_i on $\mathcal{Y}$ (finitely many in every quasi-compact open of $\mathcal{Y}$) such that for every $0 \neq \mathcal{F} \in \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$, the !-fiber of $\mathcal{F}$ at least one y_i will be non-zero.

With no restriction of generality, we can assume that $\mathcal{Y}$ is a (quasi-compact) smooth scheme. We can partition $\mathcal{Y}$ into smooth, connected locally closed subschemes $\mathcal{Y}_i$ such that the dimensions of the fibers of the map

$$\mathcal{N} \hookrightarrow T^*(\mathcal{Y}) \to \mathcal{Y} \tag{15.3}$$

are $\leq \mathrm{codim}(\mathcal{Y}_i, \mathcal{Y})$ over $\mathcal{Y}_i$. By refining the partition, we can assume that for every n, the union

$$\underset{i, \mathrm{codim}(\mathcal{Y}_i,\mathcal{Y}) \geq n}{\cup} \mathcal{Y}_i$$

is closed.

Choose a point $y_i \in \mathcal{Y}_i(k)$. We claim that these points will have the required property.

Proof. For a given $0 \neq \mathcal{F} \in \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$, let i be an index with a minimal $\mathrm{codim}(\mathcal{Y}_i, \mathcal{Y})$, such that $\mathcal{F}|_{\mathcal{Y}_i} \neq 0$. It suffices to show that $\mathcal{F}|_{\mathcal{Y}_i}$ is lisse. I.e., we want to show that

$$\mathrm{SingSupp}(\mathcal{F}|_{\mathcal{Y}_i}) \subset T^*(\mathcal{Y}_i)$$

is the zero section.

Removing the (closed) subscheme

$$\bigcup_{j, \mathrm{codim}(\mathcal{Y}_j, \mathcal{Y}) > \mathrm{codim}(\mathcal{Y}_i, \mathcal{Y})} \mathcal{Y}_j,$$

we can assume that $\mathcal{Y}_i$ is a closed subscheme of $\mathcal{Y}$, and $\mathcal{F}$ is supported on $\mathcal{Y}_i$. Denote by π_i the projection

$$T^*(\mathcal{Y})|_{\mathcal{Y}_i} \to T^*(\mathcal{Y}_i).$$

This is a smooth surjective map, with fibers of dimension $\mathrm{codim}(\mathcal{Y}_i, \mathcal{Y})$.

We have

$$\mathrm{SingSupp}(\mathcal{F}) = \pi_i^{-1}(\mathrm{SingSupp}(\mathcal{F}|_{\mathcal{Y}_i})).$$

Hence, the dimension of the fibers of the map

$$\mathrm{SingSupp}(\mathcal{F}) \hookrightarrow T^*(\mathcal{Y}) \to \mathcal{Y}$$

over $\mathcal{Y}_i$ equals $\mathrm{codim}(\mathcal{Y}_i, \mathcal{Y})$ plus the dimension of the fibers of the map

$$\mathrm{SingSupp}(\mathcal{F}|_{\mathcal{Y}_i}) \hookrightarrow T^*(\mathcal{Y}_i) \to \mathcal{Y}_i. \tag{15.4}$$

Hence, since $\mathrm{SingSupp}(\mathcal{F}) \subset \mathcal{N}$ and by the assumption on (15.3), we obtain that the fibers of the map (15.4) are zero-dimensional.

Since $\mathrm{SingSupp}(\mathcal{F}|_{\mathcal{Y}_i})$ is conical, we obtain that it is necessarily the zero section.

□

15.2.2. We are now going to exhibit a particular set of generators for the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Recall that $\mathrm{Nilp} \subset T^*(\mathrm{Bun}_G)$ is half-dimensional, see Sect. D. For every i, let $\delta_{y_i} \in \mathrm{Shv}(\mathrm{Bun}_G)$ be the corresponding !-delta function object, i.e.,

$$\delta_{y_i} = (\mathbf{i}_{y_i})_!(\mathsf{e}),$$

where

$$\mathrm{Spec}(k) \overset{\mathbf{i}_{y_i}}{\to} \mathrm{Bun}_G$$

is the morphism corresponding to y_i.

Let

$$f_n : Z_n \to \mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X)$$

be one of the substacks as in Sect. 15.1.2.

Consider the objects

$$\mathsf{P}_{Z_n}^{\mathrm{enh}}(\delta_{y_i}) \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X))}{\otimes} \mathrm{QCoh}(Z_n),$$

and

$$(\mathrm{Id} \otimes (f_n)_*)(\mathsf{P}_{Z_n}^{\mathrm{enh}}(\delta_{y_i})) \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G). \tag{15.5}$$

We claim that the objects (15.5) provide a set of compact generators for $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

15.2.3. Since the functors (15.1) preserve compactness and the union of their essential images generates $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, it suffices to show that for a fixed n, the objects $\mathsf{P}^{\mathrm{enh}}_{Z_n}(\delta_{y_i})$ generate

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(Z_n).$$

By adjunction, for $\mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(Z_n)$, we have

$$\mathcal{H}om_{\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(Z_n)}(\mathsf{P}^{\mathrm{enh}}_{Z_n}(\delta_{y_i}), \mathcal{F}) \simeq \mathcal{H}om_{\mathrm{Shv}(\mathrm{Bun}_G)}\left(\delta_{y_i}, \iota \circ (\mathrm{Id} \otimes (f_n)_*)(\mathcal{F})\right).$$

Since the functor

$$\mathrm{Id} \otimes (f_n)_* : \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(Z_n) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

is conservative, the latter assertion is equivalent to the statement that if $\mathcal{F}' \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is *non-zero*, then *not all*

$$\mathcal{H}om_{\mathrm{Shv}(\mathrm{Bun}_G)}(\delta_{y_i}, \mathcal{F}') \simeq (\mathbf{i}_{y_i})^!(\mathcal{F}')$$

are zero.

However, the latter was proved in Sect. 15.2.1.

15.3. **The tensor product property.**

15.3.1. Let Y_1 and Y_2 be a pair of quasi-compact schemes (or algebraic stacks) over the ground field k. Let $\mathrm{Shv}(-)$ be a constructible sheaf theory (i.e., in the case of D-modules, we will consider the subcategory of holonomic D-modules or regular holonomic D-modules).

Consider the external tensor product functor

$$\mathrm{Shv}(Y_1) \otimes \mathrm{Shv}(Y_2) \to \mathrm{Shv}(Y_1 \times Y_2). \tag{15.6}$$

It is fully faithful (see [GKRV, Lemma A.2.6]), but very rarely an equivalence. However, it is an equivalence, for example, if either Y_1 or Y_2 is an algebraic stack with a *finite number of isomorphism classes of k-points*[29].

In fact, there is a clear obstruction for an object of $\mathrm{Shv}(Y_1 \times Y_2)$ to belong to the essential image of (15.6). Namely all objects in the essential image have their singular support contained in a subset of $T^*(Y_1 \times Y_2) \simeq T^*(Y_1) \times T^*(Y_2)$ of the form

$$\mathcal{N}_1 \times \mathcal{N}_2, \quad \mathcal{N}_i \subset T^*(Y_i).$$

Thus, one can wonder whether, given $\mathcal{N}_1$ and $\mathcal{N}_2$ as above, the functor

$$\mathrm{Shv}_{\mathcal{N}_1}(Y_1) \otimes \mathrm{Shv}_{\mathcal{N}_2}(Y_2) \to \mathrm{Shv}_{\mathcal{N}_1 \times \mathcal{N}_2}(Y_1 \times Y_2) \tag{15.7}$$

is an equivalence.

Now, this happens to always be the case for constructible sheaves in the Betti context, at least after passing to the left completion of the left-hand side (assuming $\mathcal{N}_i$ are Lagrangian). However, this is *not* the case of ℓ-adic sheaves over a field of positive characteristic, and not for holonomic (but irregular) D-modules.

For example, taking $Y_1 = Y_2 = \mathbb{A}^1$ and $\mathcal{N}_1 = \mathcal{N}_2 = \{0\}$, the map

$$\mathrm{QLisse}(\mathbb{A}^1) \otimes \mathrm{QLisse}(\mathbb{A}^1) \to \mathrm{QLisse}(\mathbb{A}^1 \times \mathbb{A}^1)$$

is *not* an equivalence. Indeed, the object

$$\mathrm{mult}^*(\text{A-Sch}) \in \mathrm{QLisse}(\mathbb{A}^1 \times \mathbb{A}^1)$$

does not lie in the essential image, where

$$\mathrm{mult} : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$$

[29] Such as, for example, $B\backslash G/N$, or the stack Bun_G for X of genus 0.

is the product map and A-Sch $\in \text{Shv}(\mathbb{A}^1)$ is the Artin-Schreier local system.

That said, Theorems F.9.4, F.9.6 and Theorem F.9.8 all say that the functor (15.7) is an equivalence in the case when Y_1 is a proper scheme and $\mathcal{N}_1 = \{0\}$, either up to left completions or under an additional assumption on Y_1. However, an assertion of this sort would still fail even when Y_1 is proper for a more general $\mathcal{N}_1$.

15.3.2. The main result of the present subsection is the following:

Theorem 15.3.3. *Let G_1 and G_2 be a pair of reductive groups. Then the functor*

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_{G_1}) \otimes \text{Shv}_{\text{Nilp}}(\text{Bun}_{G_2}) \to \text{Shv}_{\text{Nilp}}(\text{Bun}_{G_1 \times G_2}) \tag{15.8}$$

is an equivalence.

The rest of this subsection is devoted to the proof of this theorem.

15.3.4. We first show that (15.8) is fully faithful.

Indeed, a standard colimit argument shows that the fully faithfulness of (15.6) implies the fully faithfulness of (15.7), whenever one of the categories $\text{Shv}_{\mathcal{N}_i}(Y_i)$ is dualizable.

The dualizability assumption holds for $\text{Shv}_{\text{Nilp}}(\text{Bun}_{G_i})$ by Theorem 15.1.1.

15.3.5. We now show that the essential image of (15.8) generates the target category.

Let $\{y_{i_1}\} \in \text{Bun}_{G_1}(k)$ and $\{y_{i_2}\} \in \text{Bun}_{G_2}(k)$ be collections of points chosen as in Sect. 15.2.1 with respect to the subsets $\text{Nilp} \subset T^*(\text{Bun}_{G_1})$ and $\text{Nilp} \subset T^*(\text{Bun}_{G_2})$, respectively. Then, by construction, the collection of points

$$\{y_{i_1} \times y_{i_2}\} \in \text{Bun}_{G_1}(k) \times \text{Bun}_{G_2}(k) \simeq \text{Bun}_{G_1 \times G_2}(k)$$

will have the corresponding property with respect to

$$\text{Nilp} \subset T^*(\text{Bun}_{G_1 \times G_2}).$$

Consider the corresponding stacks

$$Z_{n_1} \overset{f_{n_1}}{\to} \text{LocSys}^{\text{restr}}_{\check{G}_1}(X) \text{ and } Z_{n_2} \overset{f_{n_2}}{\to} \text{LocSys}^{\text{restr}}_{\check{G}_2}(X)$$

and

$$Z_{n_1} \times Z_{n_2} \overset{f_{n_1} \times f_{n_2}}{\to} \text{LocSys}^{\text{restr}}_{\check{G}_1 \times \check{G}_2}(X).$$

By Sect. 15.2.2, it suffices to show that for all quadruples i_1, i_2, n_1, n_2, the object

$$(\text{Id} \otimes (f_{n_1} \times f_{n_2})_*)(\mathsf{P}^{\text{enh}}_{Z_{n_1} \times Z_{n_2}}(\delta_{y_{i_1} \times y_{i_2}})) \in \text{Shv}_{\text{Nilp} \times \text{Nilp}}(\text{Bun}_{G_1 \times G_2}) \tag{15.9}$$

lies in the essential image of the functor (15.8).

15.3.6. We claim the object (15.9) equals the external tensor product

$$(\text{Id} \otimes (f_{n_1})_*)(\mathsf{P}^{\text{enh}}_{Z_{n_1}}(\delta_{y_{i_1}})) \boxtimes (\text{Id} \otimes (f_{n_2})_*)(\mathsf{P}^{\text{enh}}_{Z_{n_2}}(\delta_{y_{i_2}})).$$

Using (14.14), in order to prove the latter assertion, it suffices to show that the diagram

$$\begin{array}{ccc} \text{Shv}(\text{Bun}_{G_1}) \otimes \text{Shv}(\text{Bun}_{G_2}) & \xrightarrow{\mathsf{P}_{Z_1} \boxtimes \mathsf{P}_{Z_2}} & \text{Shv}(\text{Bun}_{G_1}) \otimes \text{Shv}(\text{Bun}_{G_2}) \otimes \text{QCoh}(Z_1) \otimes \text{QCoh}(Z_2) \\ \downarrow & & \downarrow \\ \text{Shv}(\text{Bun}_{G_1 \times G_2}) & \xrightarrow{\mathsf{P}_{Z_1 \times Z_2}} & \text{Shv}(\text{Bun}_{G_1 \times G_2}) \otimes \text{QCoh}(Z_1 \times Z_2) \end{array}$$

commutes.

However, this follows from Proposition 12.2.2.

□[Theorem 15.3.3]

15.4. **Some consequences pertaining to Conjecture 13.1.8.**

15.4.1. Consider the tautological embedding

$$\iota : \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \hookrightarrow \text{Shv}(\text{Bun}_G). \tag{15.10}$$

Now that we know that the category $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ is compactly generated, we can equivalently reformulate Conjecture 13.1.8 as follows:

(i) The functor ι preserves compactness;

(ii) The right adjoint of ι is continuous.

15.4.2. We claim:

Theorem 15.4.3. *Conjecture 13.1.8 holds in the de Rham context.*

Proof. By Corollary 13.5.5, the composite functor

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \overset{\iota}{\hookrightarrow} \text{Shv}(\text{Bun}_G) \hookrightarrow \text{D-mod}(\text{Bun}_G)$$

admits a continuous right adjoint, where $\text{Shv}(-)$ is the category of holonomic D-modules.

In particular, the above composite functor sends compact objects in $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ to objects that are compact in $\text{D-mod}(\text{Bun}_G)$. However, since the embedding

$$\text{Shv}(\text{Bun}_G) \hookrightarrow \text{D-mod}(\text{Bun}_G)$$

is fully faithful, we obtain that ι preserves compactness as well.

□

Remark 15.4.4. Recall the functor

$$\mathsf{P}^{\text{enh}}_{\text{LocSys}^{\text{restr}}_{\check{G}}(X)} : \text{Shv}(\text{Bun}_G) \to \text{Shv}_{\text{Nilp}}(\text{Bun}_G),$$

see Sect. 14.5. In Proposition 16.2.3 below we will show that $\mathsf{P}^{\text{enh}}_{\text{LocSys}^{\text{restr}}_{\check{G}}(X)}$ *would be* the right adjoint of ι, if we knew that that right adjoint was continuous.

15.4.5. Recall (see Sect. F.8.2) that

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{access}} \subset \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \tag{15.11}$$

denotes the full subcategory generated by the essential image of

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \cap \text{Shv}(\text{Bun}_G)^c \subset \text{Shv}_{\text{Nilp}}(\text{Bun}_G).$$

The above definition of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{access}}$ coincides with the one given in Sect. F.8.1. This is due to the combination of Theorem 13.1.5 and Corollary F.8.11.

Note that Conjecture 13.1.8 can be reformulated as the assertion that the inclusion

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{access}} \subset \text{Shv}_{\text{Nilp}}(\text{Bun}_G)$$

is an equality.

15.4.6. We now claim:

Proposition 15.4.7. *The following statements are equivalent:*

(a) *Conjecture 13.1.8 holds;*

(b) *The endofunctor*

$$\iota \circ (\text{Id}_{\text{Shv}_{\text{Nilp}}(\text{Bun}_G)} \otimes \Gamma(Z_n, -)) \circ \mathsf{P}_{Z_n} \simeq \mathbf{oblv}_{\text{Hecke}} \circ \mathsf{P}^{\text{enh}}_{Z_n}$$

of $\text{Shv}(\text{Bun}_G)$ *for* Z_n *being each of the stacks from Sect. 15.1.2, has the following properties:*

(bi) *It preserves compactness;*

(bii) *It sends compact objects to objects that are comologically bounded;*

(biii) *It sends compact objects to objects that are comologically bounded on the left.*

Proof. By Corollary 14.4.6(b), we can rewrite the functor $\mathbf{oblv}_{\mathrm{Hecke}} \circ \mathsf{P}^{\mathrm{enh}}_{Z_n}$ as

$$\mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathsf{P}^{\mathrm{enh}}_{Z_n}}{\to} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(Z_n) \overset{\mathrm{Id}\otimes(f_n)_*}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\iota}{\to} \to \mathrm{Shv}(\mathrm{Bun}_G). \tag{15.12}$$

Point (a) implies (bi) because the endofunctor in (b) is the composition of ι (which preserves compactness by the assumption in (a)), the functor $(f_n)_* \otimes \mathrm{Id}$ (which preserves compactness by the assumption on (Z_n, f_n)), and the functor $\mathsf{P}^{\mathrm{enh}}_{Z_n}$ (which preserves compactness, being a left adjoint).

Vice versa, point (bi) implies point (a), because the images of the compacts under

$$\mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathsf{P}^{\mathrm{enh}}_{Z_n}}{\to} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(Z_n) \overset{\mathrm{Id}\otimes(f_n)_*}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

generate $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, see Sects. 15.1.5-15.1.6.

We have the tautological implications (bi) $\Rightarrow$ (bii) $\Rightarrow$ (biii). Suppose that (biii) holds, and let us deduce (a).

The embedding

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{\mathrm{access}} \hookrightarrow \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \tag{15.13}$$

induces an equivalence on bounded below (eventually coconnective) subcategories (see Sect. F.8.2). Point (a) is equivalent to this embedding being an equivalence, see Sect. 15.4.5.

The assumption in (biii) implies that the (compact) generators of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ are bounded on the left (=eventually coconnective), when considered as objects of $\mathrm{Shv}(\mathrm{Bun}_G)$; hence they belong to the essential image of (15.13). This implies that (15.13) is an equivalence.

□

Remark 15.4.8. Note that although we cannot prove Conjecture 13.1.8 in general, and as result we do not know that the endofunctors (15.12) preserve compactness, we do know that these functors send compact objects to objects that are bounded above and such that all of their individual cohomologies are constructible. This is because the functor ι sends objects that are compact in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ to objects of $\mathrm{Shv}(\mathrm{Bun}_G)$ with these properties.

This finiteness property of individual cohomologies is non-obvious from the presentation of the functors P_{Z_n} as colimits, see Sect. 14.3.3.

15.4.9. We will now prove:

Theorem 15.4.10. *Conjecture 13.1.8 holds in the Betti context.*

Proof. Consider the endofunctor (15.12). By Proposition 15.4.7, it suffices to show that it sends objects that are compact in $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ (for $\mathrm{Shv}(-)$ being the constructible category in the classical topology) to objects that are cohomologically bounded.

We can assume that our field of coefficients e is $\mathbb{C}$, and let us apply the Riemann-Hilbert equivalence. I.e., we can replace the initial $\mathrm{Shv}(-)$ by the *regular* holonomic category $\mathrm{Shv}^{\mathrm{reg.hol}}(-)$.

We now embed $\mathrm{Shv}^{\mathrm{reg.hol}}(-)$ into the entire holonomic category $\mathrm{Shv}^{\mathrm{hol}}(-)$, i.e., without the regularity assumption. By Theorem 15.4.3, we know that the functor (15.12) sends compact objects in $\mathrm{Shv}^{\mathrm{hol}}(\mathrm{Bun}_G)$ to objects that are cohomologically bounded.

Hence, it suffices to show that the embedding

$$\mathrm{Shv}^{\mathrm{reg.hol}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}^{\mathrm{hol}}(\mathrm{Bun}_G)$$

preserves compactness. However, this is true for any algebraic stack (e.g., this follows from the description of compact generators on a stack in Sect. F.1.2).

□

15.5. **The de Rham context.** In the de Rham context, the category $\text{D-mod}(\text{Bun}_G)$ carries an action of $\text{QCoh}(\text{LocSys}_{\check{G}}(X))$.

In this subsection we will recast some of the results of the preceding subsections in terms of this action.

15.5.1. We will use the functors $\mathsf{P}_{\mathcal{Z}}$ and $\mathsf{P}^{\text{enh}}_{\mathcal{Z}}$ on $\text{D-mod}(\text{Bun}_G)$, for a map

$$f : \mathcal{Z} \to \text{LocSys}^{\text{dR}}_{\check{G}}(X),$$

see Remark 14.3.5.

Unwinding the definitions, as in Sect. 12.3.1, we obtain that there is a canonical identification

$$\text{Hecke}(\mathcal{Z}, \text{D-mod}(\text{Bun}_G)) \simeq \text{D-mod}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X))}{\otimes} \text{QCoh}(\mathcal{Z}),$$

so that the functor $\mathsf{P}^{\text{enh}}_{\mathcal{Z}}$ corresponds to the pullback functor

$$\text{D-mod}(\text{Bun}_G) \simeq \text{D-mod}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X))}{\otimes} \text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X)) \overset{\text{Id}\otimes f^*}{\longrightarrow}$$
$$\to \text{D-mod}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X))}{\otimes} \text{QCoh}(\mathcal{Z}).$$

15.5.2. Let us denote the action functor of $\text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X))$ on $\text{D-mod}(\text{Bun}_G)$ by

$$\mathcal{E} \in \text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X)),\ \mathcal{M} \in \text{D-mod}(\text{Bun}_G) \mapsto \mathcal{E} \star \mathcal{M}.$$

In particular, we obtain that if f is such that the functor

$$f_* : \text{QCoh}(\mathcal{Z}) \to \text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X))$$

is continuous (in which case it is automatically a map of $\text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X))$-module categories since $\text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X))$ is rigid), then the monad

$$\mathbf{oblv}_{\text{Hecke}} \circ \mathsf{P}^{\text{enh}}_{\mathcal{Z}}$$

of $\text{D-mod}(\text{Bun}_G)$ identifies with

$$\mathcal{F} \mapsto f_*(\mathcal{O}_{\mathcal{Z}}) \star \mathcal{F}.$$

15.5.3. Recall that we have an identification

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \simeq \text{D-mod}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X))}{\otimes} \text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X))_{\text{LocSys}^{\text{restr}}_{\check{G}}(X)}, \tag{15.14}$$

see Proposition 13.5.3.

In particular, taking $\mathcal{Z} = \text{LocSys}^{\text{restr}}_{\check{G}}(X)$, we obtain that the endofunctor

$$\mathbf{oblv}_{\text{Hecke}} \circ \mathsf{P}^{\text{enh}}_{\text{LocSys}^{\text{restr}}_{\check{G}}(X)},$$

of $\text{D-mod}(\text{Bun}_G)$ identifies with $\iota^R \circ \iota$, and is given by the action of the object

$$\iota \circ \iota^R(\mathcal{O}_{\text{LocSys}^{\text{dR}}_{\check{G}}(X)}) \in \text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X)),$$

where by a slight abuse of notation we denote by ι the embedding

$$\text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X))_{\text{LocSys}^{\text{restr}}_{\check{G}}(X)} \hookrightarrow \text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X)),$$

and ι^R by its right adjoint (i.e., the functor of "local sections with set-theoretic support on $\text{LocSys}^{\text{restr}}_{\check{G}}(X)$").

15.5.4. Finally, we claim that the assertion of Theorem 15.3.3 can also be easily obtained from this perspective. Indeed, since $\text{D-mod}(\text{Bun}_G)$ is compactly generated, and hence dualizable, the functor

$$\text{D-mod}(\text{Bun}_G) \otimes \text{D-mod}(\text{Bun}_G) \to \text{D-mod}(\text{Bun}_G \times \text{Bun}_G)$$

is an equivalence.

Now, the equivalence in Theorem 15.3.3 can be obtained by tensoring both sides over

$$\text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X)) \otimes \text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X)) \simeq \text{QCoh}(\text{LocSys}^{\text{dR}}_{\check{G}}(X) \times \text{LocSys}^{\text{dR}}_{\check{G}}(X))$$

with

$$\text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \otimes \text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \simeq \text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X) \times \text{LocSys}^{\text{restr}}_{\check{G}}(X)).$$

15.5.5. *The regular singularity property.* There is, however, one new property of the category $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ that one obtains by the methods of the spectral projector:

Main Corollary 15.5.6. *All compact objects of* $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ *have regular singularities.*

Proof. It suffices to show that all objects of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ lie in the ind-completion of the *regular* holonomic subcategory. For that it suffices to show that the compact generators of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ have this property.

However, this follows from the description of the generators of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ given in Sect. 15.2.2:

We have to show that the objects (15.5) have regular singularities. The objects δ_{y_i} have regular singularities, so it suffices to show that the endofunctors

$$\iota \circ (\text{Id} \otimes (f_n)_*) \circ \mathsf{P}^{\text{enh}}_{Z_n} \simeq (\text{Id} \otimes \Gamma(Z_n, -)) \circ \mathsf{P}_{Z_n}$$

of $\text{Shv}(\text{Bun}_G)$ preserve the ind-completion of the regular holonomic subcategory.

However, this follows from the description of the functors $\mathsf{P}_{\mathcal{Z}}$ in Sect. 14.3.3.

□

Combining with Corollary 13.4.10, we obtain:

Main Corollary 15.5.7. *All Hecke eigensheaves have regular singularities.*

The above corollary was suggested as a conjecture in [BD1, Sect. 5.2.7].

16. More on Conjecture 13.1.8

The material of this section will not be used in the rest of the paper. Here we record several more statements that are logically equivalent to Conjecture 13.1.8.

16.1. **The pro-left adjoint to the embedding.**

16.1.1. The embedding

$$\iota : \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \hookrightarrow \text{Shv}(\text{Bun}_G)$$

does *not* admit a left adjoint. However, it admits a left adjoint with values in the pro-category

$$\iota^L : \text{Shv}(\text{Bun}_G) \to \text{Pro}(\text{Shv}_{\text{Nilp}}(\text{Bun}_G)).$$

16.1.2. The functor ι^L is related to the functors $\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}}$ as follows:

Let $\mathcal{Z}$ be a prestack equipped with a map $f : \mathcal{Z} \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$. Assume that $\mathcal{O}_{\mathcal{Z}} \in \mathrm{QCoh}(\mathcal{Z})$ is compact. Then it follows from Corollary 14.4.6(b) and Sect. 14.5.3 that the composition

$$\mathrm{Shv}(\mathrm{Bun}_G) \overset{\iota^L}{\to} \mathrm{Pro}(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \overset{\mathrm{Pro}(\mathrm{Id}\otimes f^*)}{\longrightarrow} \mathrm{Pro}\left(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathcal{Z})\right)$$

takes values in

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathcal{Z}) \subset \mathrm{Pro}\left(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathcal{Z})\right)$$

and identifies with the functor $\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}}$.

16.1.3. Vice versa, we can express the functor ι^L in terms of the functors $\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}}$:

Let $Z_n \overset{f_n}{\to} \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ be as in Sect. 15.1.2. We have:

Lemma 16.1.4. *For a* compact $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$, *the object*

$$\iota^L(\mathcal{F}) \in \mathrm{Pro}(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))$$

identifies canonically with

$$\text{“}\lim_{n}\text{”}\, (\mathrm{Id}\otimes(f_n)_*) \circ \mathsf{P}^{\mathrm{enh}}_{Z_n}(\mathcal{F}).$$

Proof. Follows from the fact that for $\mathcal{F}' \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, we have a canonical isomorphism

$$\underset{n}{\mathrm{colim}}\, (\mathrm{Id}\otimes(f_n)_*) \circ (\mathrm{Id}\otimes(f_n)^!)(\mathcal{F}') \simeq \mathcal{F}',$$

where $(\mathrm{Id}\otimes(f_n)_*, \mathrm{Id}\otimes(f_n)^!)$ are the adjoint functors

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(Z_n) \rightleftarrows \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G).$$

□

16.2. **The right adjoint.**

16.2.1. Let us now consider the right adjoint ι^R of ι. Note, however, that since we do not know Conjecture 13.1.8, the functor ι^R is a priori discontinuous.

We claim that there exists a natural transformation

$$\iota^R \to \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \tag{16.1}$$

as functors $\mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Indeed, we start with the counit of the adjunction

$$\iota \circ \iota^R \to \mathrm{Id}_{\mathrm{Shv}(\mathrm{Bun}_G)},$$

apply to both sides the functor $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$, and use the fact that $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \circ \iota \simeq \mathrm{Id}_{\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)}$.

16.2.2. We now claim:

Proposition 16.2.3. *The following conditions are equivalent:*

(a) *Conjecture 13.1.8 holds;*

(b) *The functor* $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ *provides a right adjoint to* ι*;*

(c) *The natural transformation* (16.1) *is an isomorphism.*

Proof. We have (c) ⇒ (b) tautologically. Also, (b) implies that ι^R is continuous, which implies (a). Let us show that (a) implies (c).

Let $\iota : \mathbf{C}' \to \mathbf{C}$ be a fully faithful functor of DG categories, and let P be its left inverse. In this case, as in (16.1), we construct a natural transformation

$$\iota^R \to \mathsf{P}.$$

We claim that this natural transformation is an isomorphism if and only if P annihilates the subcategory $(\mathbf{C}')^\perp \subset \mathbf{C}$. Indeed, this follows by looking at the fiber sequence

$$\iota \circ \iota^R(\mathbf{c}) \to \mathbf{c} \to \mathbf{c}'', \quad \mathbf{c} \in \mathbf{C},$$

where $\mathbf{c}'' \in (\mathbf{C}')^\perp$.

Note also that above, "annihilates" is equivalent to "preserves" since $\mathsf{P}(\mathbf{C}) \subset \mathbf{C}'$, while $\mathbf{C}' \cap (\mathbf{C}')^\perp = 0$.

Hence, we need to show that if (a) holds, then the functor $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ preserves

$$(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))^\perp \subset \mathrm{Shv}(\mathrm{Bun}_G).$$

We will now use the fact that $\iota \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ is an integral Hecke functor, see Sect. 14.5.5. We claim that assumption (a) implies that *any* integral Hecke functor preserves $(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))^\perp$.

Indeed, assumption (a) implies that $(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))^\perp$ is closed under colimits. Therefore, it is sufficient to show that $(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))^\perp$ is preserved by functors of the form

$$\mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathrm{H}(V,-)}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G \times X^I) \overset{(14.18)}{\longrightarrow} \mathrm{Shv}(\mathrm{Bun}_G), \quad V \in (\mathrm{Rep}(\check{G})^{\otimes I})^c, \quad \mathcal{S} \in \mathrm{Shv}(X^I)^c. \tag{16.2}$$

The functor (16.2) admits a left adjoint, which is again a functor of the same form with V replaced by its monoidal dual and $\mathcal{S}$ replaced by its Verdier dual.

Hence, we obtain that it is enough to show that functors of the form (16.2) preserve the subcategory $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$, which was already observed in Sect. 14.4.1.

□

16.3. **Conjecture 13.1.8 and cohomological amplitudes.**

16.3.1. Recall again the stacks Z_n from Sect. 15.1.2. Note, however, that by Sects. 7.9.6-7.9.8, we can assume that these substacks are obtained by base change of affine schemes S_n equipped with regular embeddings g_n into the coarse moduli spaces of connected components of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$.

For the duration of this subsection we will assume that Z_n has this form.

16.3.2. First, we claim:

Lemma 16.3.3. *For (Z_n, f_n) as above, the composite functor*

$$\mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathsf{P}^{\mathrm{enh}}_{Z_n}}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(Z_n) \overset{\mathrm{Id}\otimes(f_n)_*}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

is right t-exact.

Proof. By Sect. 16.1.2, it suffices to show that the endofunctor of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ given by

$$(\mathrm{Id}\otimes(f_n)_*) \circ (\mathrm{Id}\otimes(f_n)^*)$$

is right t-exact.

This endofunctor is given by the action on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ of the object

$$(f_n)_*(\mathcal{O}_{Z_n}) \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)).$$

However, by the construction of $\mathcal{O}_{Z_n}$, it admits a left resolution with terms being direct sums of copies of $\mathcal{O}_{\mathcal{Z}}$, where $\mathcal{Z}$ is a connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$. So, it suffices to show that the endofunctor of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ given by the action of $\mathcal{O}_{\mathcal{Z}}$ is right t-exact.

However, the latter functor is t-exact, being a direct summand of the identity functor.

□

16.3.4. By assumption, the maps g_n from the affine schemes S_n to coarse moduli spaces of connected components of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ are regular closed embeddings. For each n, let m_n denote the length of the corresponding regular sequence. Note that m_n only depends on the choice of a connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$.

We claim:

Proposition 16.3.5. *The collection of integers m_n is bounded.*

Proof. As in Sect. 9.7.2, there exists a curve X' over $\mathbb{C}$, such that $\mathcal{Z}^{\mathrm{coarse}}$ is isomorphic to its counterpart in the (restricted) Betti context. Hence, we can assume that we are in the Betti context. In this case, by Sect. 6.1, our $\mathcal{Z}^{\mathrm{coarse}}$ can be realized as the completion of $\mathrm{LocSys}^{\mathrm{Betti,coarse}}_{\check{G}}(X')$ at one of its closed points.

This implies that we can take $m = \dim(\mathrm{LocSys}^{\mathrm{Betti,coarse}}_{\check{G}}(X'))$: the schemes S_n can be taken to be fat points around the corresponding closed point on $\mathrm{LocSys}^{\mathrm{Betti,coarse}}_{\check{G}}(X')$, cut out by regular sequences on $\mathrm{LocSys}^{\mathrm{Betti,coarse}}_{\check{G}}(X')$.

□

Remark 16.3.6. The version of Proposition 16.3.5 holds in the general context of Theorem 5.4.2, i.e., for any target gentle Tannakian category $\mathbf{H}$:

Let $\mathcal{Z}$ be a connected component of $\mathbf{Maps}(\mathrm{Rep}(\mathsf{G}), \mathbf{H}))$, and let $\mathcal{Z}^{\mathrm{coarse}}$ be the corresponding coarse moduli space. Let σ be the unique closed point of $\mathcal{Z}$, and let

$$m = \dim(\mathsf{G}) \cdot \dim(H^0(T_\sigma(\mathcal{Z}))).$$

We claim that we can find a map $\mathcal{Z}^{\mathrm{coarse}} \to \mathbb{A}^m$ mapping the unique closed point to $0 \in \mathbb{A}^m$ such that the preimage of $0 \in \mathbb{A}^m$ is a scheme.

Indeed, let n be such that the pro-algebraic group $\mathsf{H}_{\phi\text{-isotyp}}$ is topologically generated by n elements. Then, according to the proof of Theorem 6.5.7, there exists a map

$$\mathcal{Z}^{\mathrm{coarse}} \to \mathsf{G}^n /\!/ \mathrm{Ad}(\mathsf{G}),$$

such that the preimage of the point in $\mathsf{G}^n /\!/ \mathrm{Ad}(\mathsf{G})$ equal to the image of $\mathbf{r}(\sigma)$ is a scheme. Hence, it suffices to show that we can choose $n \leq \dim(H^0(T_\sigma(\mathcal{Z})))$.

However, according the proof of Theorem 6.7.8, we can choose n be exactly $\dim(H^0(T_\sigma(\mathcal{Z})))$.

Let m denote the bound from Proposition 16.3.5.

16.3.7. We claim:

Proposition 16.3.8. *The functor $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ has a cohomological amplitude bounded on the right by M.*

Proof. We will show that the terms of the colimit (14.17) have cohomological amplitudes bounded on the right by m.

First, since the map f_n is a regular embedding, we have

$$f_n^! \simeq f_n^*[-m_n].$$

Hence, it is enough to show that the functor

$$\mathcal{F} \mapsto (\mathrm{Id} \otimes \Gamma(Z_n, f_n^*(-))) \circ \mathsf{P}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}(\mathcal{F})$$

is right t-exact.

Note, however, that the latter functor identifies with $(\mathrm{Id} \otimes (f_n)_*) \circ \mathsf{P}^{\mathrm{enh}}_{Z_n}$. Hence, it is right t-exact by Lemma 16.3.3.

□

16.3.9. We now claim:

Proposition 16.3.10. *The following statements are equivalent:*

(a) *Conjecture 13.1.8 holds;*

(b) *The endofunctor* (15.12) *has a cohomological amplitude bounded on the left;*

(b') *The endofunctor* (15.12) *has a cohomological amplitude bounded on the left by M;*

(c) *The functor $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ has a cohomological amplitude bounded on the left;*

(c') *The functor $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ is left t-exact;*

(d) *The functor ι^R has a cohomological amplitude bounded on the right;*

(d') *The functor ι^R has a cohomological amplitude bounded on the right by M.*

Proof. We have the tautological implications (b') ⇒ (b), (c') ⇒ (c), (d') ⇒ (d) The implication (b) ⇒ (a) was proved in Proposition 15.4.7. The implication (a) ⇒ (c') follows from Proposition 16.2.3. The implication (a) ⇒ (d') follows by combining Propositions 16.2.3 and 16.3.8.

It remains to show the implications (c') ⇒ (b'), (c) ⇒ (b), (d) ⇒ (a).

For (c') ⇒ (b') and (c) ⇒ (b), we note that the functor

$$\mathrm{Shv}(\mathrm{Bun}_G) \overset{\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}}}{\to} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(Z_n) \overset{\mathrm{Id} \otimes (f_n)_*}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

is the composite of $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$, and the endofunctor of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, given by the action of

$$(f_n)_*(\mathcal{O}_{Z_n}) \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)).$$

Now, $(f_n)_*(\mathcal{O}_{Z_n})$ admits a left resolution by copies of $\mathcal{O}_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$ of length m_n, which is $\leq M$.

Finally, the implication (d) ⇒ (a) holds for any *renormalization-adapted* pair of an algebraic stack $\mathcal{Y}$ and $\mathcal{N} \subset T^*(\mathcal{Y})$, see Sect. F.8.4. Indeed, suppose that the right adjoint to the embedding

$$\iota : \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \hookrightarrow \mathrm{Shv}(\mathcal{Y})$$

has a cohomological amplitude bounded on the right by some M'.

We need to show that for a collection of objects $\mathcal{F}_\alpha \in \mathrm{Shv}(\mathcal{Y})$, the map

$$\oplus\, \iota^R(\mathcal{F}_\alpha) \to \iota^R(\oplus\, \mathcal{F}_\alpha)$$

is an isomorphism. Since the t-structure on $\operatorname{Shv}_{\mathcal{N}}(\mathcal{Y})$ is separated, it suffices to show that for every cohomological degree n, the map

$$\oplus\, H^n\left(\iota^R(\mathcal{F}_\alpha)\right) \to H^n\left(\iota^R(\oplus\, \mathcal{F}_\alpha)\right)$$

is an isomorphism. By the assumption on the cohomological amplitude of ι^R, we can replace $\mathcal{F}_\alpha$ by $\mathcal{F}'_\alpha := \tau^{\geq n-M'}(\mathcal{F}_\alpha)$, so it is enough to check that the map

$$\oplus\, \iota^R(\mathcal{F}'_\alpha) \to \iota^R(\oplus\, \mathcal{F}'_\alpha)$$

is an isomorphism. However, the latter map takes place in the category

$$(\operatorname{Shv}_{\mathcal{N}}(\mathcal{Y}))^{\geq n-M'} \subset (\operatorname{Shv}_{\mathcal{N}}(\mathcal{Y}))^{>-\infty} \simeq (\operatorname{Shv}_{\mathcal{N}}(\mathcal{Y})^{\text{access}})^{>-\infty}.$$

Hence, we can test isomorphisms by mapping out of the compact generators of $\operatorname{Shv}_{\mathcal{N}}(\mathcal{Y})^{\text{access}}$. Now, the assertion follows from the fact that the functor ι, restricted to $\operatorname{Shv}_{\mathcal{N}}(\mathcal{Y})^{\text{access}}$ preserves compactness, by the assumption on $(\mathcal{Y}, \mathcal{N})$.

□

17. Spectral decomposition in the Betti context

In this section we will work over the ground field $k = \mathbb{C}$, and we will consider the sheaf-theoretic context of *all sheaves* in the classical topology, denoted $\operatorname{Shv}^{\text{all}}(-)$.

We will establish analogs of the results proved in the preceding sections in this context.

17.1. **Sheaves locally constant for the Hecke action.**

17.1.1. Consider the Hecke action in the Betti context, which is a compatible collection of functors

$$\mathrm{H}(-,-) : \operatorname{Rep}(\check{G})^{\otimes I} \otimes \operatorname{Shv}^{\text{all}}(\operatorname{Bun}_G) \to \operatorname{Shv}^{\text{all}}(\operatorname{Bun}_G \times X^I). \tag{17.1}$$

Let

$$\operatorname{Shv}^{\text{all}}(\operatorname{Bun}_G)^{\text{Hecke-loc.const.}} \subset \operatorname{Shv}^{\text{all}}(\operatorname{Bun}_G)$$

be the full subcategory, consisting of objects $\mathcal{F}$ for which for all $V \in \operatorname{Rep}(\check{G})$, we have

$$\mathrm{H}(V, F) \in \operatorname{Shv}^{\text{all}}(\operatorname{Bun}_G) \otimes \operatorname{Shv}^{\text{all}}_{\text{loc.const}}(X) \subset \operatorname{Shv}^{\text{all}}(\operatorname{Bun}_G \times X).$$

It is easy to see (see, e.g., [GKRV, Proposition C.2.5]) that the functors (17.1) send

$$\operatorname{Rep}(\check{G})^{\otimes I} \otimes \operatorname{Shv}^{\text{all}}(\operatorname{Bun}_G)^{\text{Hecke-loc.const.}} \to \operatorname{Shv}^{\text{all}}(\operatorname{Bun}_G)^{\text{Hecke-loc.const.}} \otimes \operatorname{Shv}^{\text{all}}_{\text{loc.const}}(X)^{\otimes I}. \tag{17.2}$$

Hence, by Theorem 9.1.2, we obtain that the category $\operatorname{Shv}^{\text{all}}(\operatorname{Bun}_G)^{\text{Hecke-loc.const.}}$ carries a monoidal action of $\operatorname{QCoh}(\operatorname{LocSys}^{\text{Betti}}_{\check{G}}(X))$.

17.1.2. Recall, following [NY1] (see also [GKRV, Theorem B.5.2]), that the Hecke functor

$$\mathrm{H}(-,-) : \operatorname{Rep}(\check{G}) \otimes \operatorname{Shv}^{\text{all}}(\operatorname{Bun}_G) \to \operatorname{Shv}^{\text{all}}(\operatorname{Bun}_G \times X)$$

sends

$$\operatorname{Shv}^{\text{all}}_{\text{Nilp}}(\operatorname{Bun}_G) \subset \operatorname{Shv}^{\text{all}}(\operatorname{Bun}_G)$$

to

$$\operatorname{Shv}^{\text{all}}_{\text{Nilp}\times\{0\}}(\operatorname{Bun}_G \times X) \subset \operatorname{Shv}^{\text{all}}(\operatorname{Bun}_G \times X).$$

By [GKRV, Theorem A.3.8, case (a)], the external tensor product functor

$$\operatorname{Shv}^{\text{all}}_{\text{Nilp}}(\operatorname{Bun}_G) \otimes \operatorname{Shv}_{\text{loc.const.}}(X) \to \operatorname{Shv}^{\text{all}}_{\text{Nilp}\times\{0\}}(\operatorname{Bun}_G \times X)$$

is an equivalence.

In particular, we obtain that

$$\operatorname{Shv}^{\text{all}}_{\text{Nilp}}(\operatorname{Bun}_G) \subset \operatorname{Shv}^{\text{all}}(\operatorname{Bun}_G)^{\text{Hecke-loc.const.}} \tag{17.3}$$

17.1.3. The functors (17.1) give rise to a system of functors

$$\mathrm{H}(-,-) : \mathrm{Rep}(\check{G})^{\otimes I} \otimes \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const}}(X)^{\otimes I}. \tag{17.4}$$

and the action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))$ on $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$ preserves the subcategory (17.3). In particular, we reproduce the following result of [NY1] (see also [GKRV, Corollary 5.4.5]):

Theorem 17.1.4. *The functors* (17.4) *combine to a monoidal action of* $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))$ *on* $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

17.1.5. The following assertion, which is the main result of this section, should be regarded as an analog of Theorem 13.4.3 for $\mathrm{Shv}^{\mathrm{all}}(-)$:

Theorem 17.1.6. *The inclusion* (17.3) *is an equality.*

17.2. **The left adjoint to the embedding.** In this subsection we will assume Theorem 17.1.6 and deduce some corollaries.

17.2.1. Consider the category $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)$ and its full subcategory

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\iota^{\mathrm{all}}}{\hookrightarrow} \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G). \tag{17.5}$$

According to Corollary G.7.6, the embedding (17.5) admits a left adjoint. We will now describe this left adjoint in terms of the Hecke action.

17.2.2. Let $\widetilde{\iota}^{\mathrm{all}}$ denote the embedding

$$\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \hookrightarrow \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G).$$

First, note that, parallel to Sects. 14.1-14.4, given a prestack $\mathcal{Z}$ over e and a map $f : \mathcal{Z} \to \mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)$, we can consider the category

$$\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)),$$

equipped with a pair of adjoint functors

$$\mathbf{ind}_{\mathrm{Hecke},\mathcal{Z}} : \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z}) \rightleftarrows \mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)) : \mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}.$$

Remark 17.2.3. We emphasize that unlike $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$, the category $\mathrm{Rep}(\check{G})^{\mathrm{Betti}}_{\mathrm{Ran}}$ is *not* rigid. So, we are using Sect. 11.8.2 to establish the existence of the above adjoint pair.

17.2.4. Denote by $\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}}$ the composition

$$\mathbf{ind}_{\mathrm{Hecke},\mathcal{Z}} \circ (- \otimes \mathcal{O}_{\mathcal{Z}}) : \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \to \mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G))$$

and by $\mathsf{P}_{\mathcal{Z}}$ the composition

$$\mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}} \circ \mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}} : \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \to \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z}).$$

When $\mathcal{Z}$ is such that $\mathcal{O}_{\mathcal{Z}} \in \mathrm{QCoh}(\mathcal{Z})$ is compact, the functor $\mathsf{P}^{\mathrm{enh}}_{\mathcal{Z}}$ is the left adjoint of

$$\mathbf{oblv}_{\mathrm{Hecke}} := (\mathrm{Id} \otimes \Gamma(\mathcal{Z}, -)) \circ \mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}.$$

17.2.5. Furthermore, assuming that $\mathrm{QCoh}(\mathcal{Z})$ is dualizable, the essential image of $\mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}$ lies in

$$\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \otimes \mathrm{QCoh}(\mathcal{Z}) \subset \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z})$$

and we have a canonical identification

$$\mathrm{Hecke}(\mathcal{Z}, \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)) \simeq \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathcal{Z}),$$

so that the functor $\mathbf{oblv}_{\mathrm{Hecke},\mathcal{Z}}$ identifies with the composition

$$\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathcal{Z}) \overset{\mathrm{Id} \otimes (\Delta_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)})_*}{\longrightarrow}$$

$$\to \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \otimes \mathrm{QCoh}(\mathcal{Z}) \overset{\widetilde{\iota}^{\mathrm{all}}}{\hookrightarrow} \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathcal{Z}).$$

In particular, the functor $\mathsf{P}_{\mathcal{Z}}$ maps

$$\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \to \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \otimes \mathrm{QCoh}(\mathcal{Z}).$$

If $\mathcal{O}_{\mathcal{Z}} \in \mathrm{QCoh}(\mathcal{Z})$ is compact (from which it formally follows that the functor f_* is continuous), we can rewrite the functor $\mathbf{oblv}_{\mathrm{Hecke}}$ as

$$\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathcal{Z}) \overset{\mathrm{Id} \otimes f_*}{\longrightarrow}$$
$$\to \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)) \simeq$$
$$\simeq \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \overset{\widetilde{\iota}^{\mathrm{all}}}{\hookrightarrow} \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G).$$

17.2.6. Applying this to $\mathcal{Z} = \mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)$ and f being the identity map[30], we obtain that the embedding $\widetilde{\iota}^{\mathrm{all}}$ admits a left adjoint, given by

$$\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)},$$

where

$$\widetilde{\iota}^{\mathrm{all}} \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)} \simeq (\mathrm{Id} \otimes \Gamma(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X), -)) \circ \mathsf{P}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}.$$

Remark 17.2.7. For future use we note that the functor $\widetilde{\iota}^{\mathrm{all}}$ realizes $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$ as a retract of $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)$; in particular, $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$ is dualizable.

17.2.8. Combining with Theorem 17.1.6, we obtain:

Corollary 17.2.9.
(a) *The functor* $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}$ *takes values in*

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$$

and identifies canonically with $(\iota^{\mathrm{all}})^L$.
(b) *The monad* $\iota^{\mathrm{all}} \circ (\iota^{\mathrm{all}})^L$ *acting on* $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)$ *identifies canonically with*

$$(\mathrm{Id} \otimes \Gamma(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X), -)) \circ \mathsf{P}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}.$$

17.2.10. Note that Corollary 17.2.9 contains the following assertion as a particular case:

Let y be a $\mathbb{C}$-point of Bun_G, and consider the corresponding object

$$\delta_y \in \mathrm{Shv}(\mathrm{Bun}_G) \subset \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G).$$

Theorem 17.2.11. *The object*

$$\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}(\delta_y) \in \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$$

belongs to $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Remark 17.2.12. In Sect. 17.4, we will show that the assertion of Theorem 17.2.11 actually implies that of Theorem 17.1.6. This is how Theorem 17.1.6 will be proved: we will prove Theorem 17.2.11 directly in Sect. 19.8, thereby establishing Theorem 17.1.6.

17.2.13. Finally, as in Sect. 15.2, we obtain:

Corollary 17.2.14. *Let* $y_i \in \mathrm{Bun}_G(\mathbb{C})$ *be points chosen as in Sect. 15.2.1. Then the objects* $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}(\delta_{y_i})$ *generate* $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

17.3. **Comparing** $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ **and** $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$**.** In this subsection we continue to assume the validity of Theorem 17.1.6, and we will deduce some further consequences.

[30]Note that, unlike the restricted situation, $\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)$ is a quasi-compact algebraic stack with an affine diagonal, and hence $\mathcal{O}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))$ is compact.

17.3.1. First, we claim:

Proposition 17.3.2. *The functor*

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \tag{17.6}$$

preserves compactness and is fully faithful.

Proof. By Conjecture 13.1.8, which holds in the Betti context (Theorem 15.4.10), the category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is generated by objects that are compact in $\mathrm{Shv}(\mathrm{Bun}_G)$. Applying Proposition G.7.10, we obtain that the functor (17.6) preserves compactness.

Given this, in order to prove that (17.6) is fully faithful, it suffices to show that it is fully faithful when restricted to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^c$. But this follows from the fact that (for any $\mathcal{Y}$) the functor

$$\mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}} \to \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y})$$

is fully faithful.

□

17.3.3. We will now explain how to single out *ind-constructible* sheaves with nilpotent singular support among *all* sheaves with nilpotent singular support in terms of the Hecke action.

Let

$$(\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G))^{\text{Hecke-fin.mon.}} \subset \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \tag{17.7}$$

be the full subcategory consisting of objects $\mathcal{F}$ such that for all $V \in \mathrm{Rep}(\check{G})$ we have

$$\mathrm{H}(V, \mathcal{F}) \in \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X) \subset \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}_{\mathrm{loc.const.}}(X),$$

cf. Sect. 9.8.1.

As in [GKRV, Proposition C.2.5], one easily shows that $(\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G))^{\text{Hecke-fin.mon.}}$ is stable under the Hecke action.

17.3.4. Note that by Proposition 9.8.3, the subcategory (17.7) equals

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \subset$$
$$\subset \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)) = \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G),$$

where we view

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)}$$

as a co-localization of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))$.

17.3.5. Note that the essential image of the functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \to \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

is contained in $(\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G))^{\text{Hecke-fin.mon.}}$, see, e.g., (13.3).

We claim:

Theorem 17.3.6. *The inclusion*

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow (\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G))^{\text{Hecke-fin.mon.}}$$

is an equality.

Proof. By Sect. 17.3.4, we have to show that the essential image of

$$\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \hookrightarrow$$
$$\to \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)) \simeq$$
$$\simeq \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$$

equals

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}.$$

We will do so by exhibiting a set of (compact) generators of

$$\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))_{\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)} \tag{17.8}$$

and show that they belong to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Namely, let y_i be as in Sect. 15.2.1. By Corollary 17.2.14, the objects $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}(\delta_{y_i})$ generate $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

Let

$$f_n : Z_n \to \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$$

be as in Sect. 15.1.2. Let $\widetilde{f}_n$ denote the composite map

$$Z_n \overset{f_n}{\to} \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X) \to \mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X).$$

We obtain that the objects

$$\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}(\delta_{y_i}) \otimes (\widetilde{f}_n)_*(\mathcal{O}_{Z_n})$$

generate (17.8).

However, diagram chase shows that these objects are isomorphic to the objects (15.5), and so they indeed belong to $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

□

17.4. **Proof of Theorem 17.1.6.** The rest of this section is devoted to the proof of Theorem 17.1.6. We will deduce it from Theorem 17.2.11. In its turn, Theorem 17.2.11 will be proved independently in Sect. 19.8.

17.4.1. Recall that ι^{all} denotes the embedding

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G),$$

and $\widetilde{\iota}^{\mathrm{all}}$ denotes the embedding

$$\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \hookrightarrow \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G).$$

Let $'\iota^{\mathrm{all}}$ denote the embedding

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}},$$

so that

$$\widetilde{\iota}^{\mathrm{all}} \circ {'\iota}^{\mathrm{all}} \simeq \iota^{\mathrm{all}}.$$

Recall that ι^{all} admits a left adjoint (by Corollary G.7.6), and $\widetilde{\iota}^{\mathrm{all}}$ admits a left adjoint, namely, $\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}$. Restricting the functor $(\iota^{\mathrm{all}})^L$ to $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$, we obtain a left adjoint to $'\iota^{\mathrm{all}}$, to be denoted $('\iota^{\mathrm{all}})^L$, so that

$$(\iota^{\mathrm{all}})^L \simeq ('\iota^{\mathrm{all}})^L \circ (\widetilde{\iota}^{\mathrm{all}})^L.$$

We wish to show that $('\iota^{\mathrm{all}})^L$ is conservative.

17.4.2. Recall (see Corollary G.9.3) that the category $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is naturally self-dual, so that with respect to the canonicaly self-duality of $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)$ (see Sect. G.9.1), we have

$$(\iota^{\mathrm{all}})^L \simeq (\iota^{\mathrm{all}})^\vee.$$

We claim now that the category $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$ is also canonically self-dual, so that

$$(\widetilde{\iota}^{\mathrm{all}})^L \simeq (\iota^{\mathrm{all}})^\vee. \tag{17.9}$$

Assuming this for a moment, let us prove that the functor $('\iota^{\mathrm{all}})^L$ is conservative.

17.4.3. It follows formally from the above properties that with respect to the above self-dualities of $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ and $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$, respectively, we have

$$('\iota^{\mathrm{all}})^L \simeq ('\iota^{\mathrm{all}})^\vee.$$

So, it suffices to show that the functor $('\iota^{\mathrm{all}})^\vee$ is conservative. Let $\mathcal{F}$ be a non-zero object of $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$. Let $y \in \mathrm{Bun}_G(\mathbb{C})$ be a point such that

$$\mathbf{i}_y^*(\widetilde{\iota}^{\mathrm{all}}(\mathcal{F})) \neq 0,$$

where $\mathbf{i}_y$ denotes the morphism $\mathrm{pt} \to \mathrm{Bun}_G$ corresponding to y.

17.4.4. Note that

$$\mathbf{i}_y^*(\widetilde{\iota}^{\mathrm{all}}(\mathcal{F})) \simeq \mathrm{counit}_{\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)}\langle \widetilde{\iota}^{\mathrm{all}}(\mathcal{F}), \delta_y\rangle,$$

which we rewrite as

$$\mathrm{counit}_{\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}}\langle \mathcal{F}, (\widetilde{\iota}^{\mathrm{all}})^\vee(\delta_y)\rangle,$$

and further as

$$\mathrm{counit}_{\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}}\langle \mathcal{F}, (\widetilde{\iota}^{\mathrm{all}})^L(\delta_y)\rangle \simeq \mathrm{counit}_{\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}}\langle \mathcal{F}, \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}(\delta_y)\rangle.$$

17.4.5. Now, by Theorem 17.2.11,

$$\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}(\delta_y) \in \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G),$$

hence, we have

$$\mathrm{counit}_{\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}}\langle \mathcal{F}, \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}(\delta_y)\rangle \simeq$$
$$\simeq \mathrm{counit}_{\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)}\langle ('\iota^{\mathrm{all}})^\vee(\mathcal{F}), \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}(\delta_y)\rangle.$$

Hence, we obtain

$$('\iota^{\mathrm{all}})^\vee(\mathcal{F}) \neq 0,$$

as desired.

$\square$[Theorem 17.1.6]

17.5. **Self-duality on** $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$**.** In this subsection we will construct a self-duality on $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$ with the properties specified in Sect. 17.4.2.

17.5.1. We let the counit on $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$ be induced by the counit on $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)$, i.e.,

$$\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \otimes \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \overset{\widetilde{\iota}^{\mathrm{all}} \otimes \widetilde{\iota}^{\mathrm{all}}}{\longrightarrow}$$
$$\to \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \overset{\mathrm{C}^\cdot_c(\mathrm{Bun}_G, - \overset{*}{\otimes} -)}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}}.$$

Recall that the unit for the self-duality on $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)$ is given by

$$(\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G}) \in \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G \times \mathrm{Bun}_G) \simeq \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G),$$

see Sect. G.9.1.

Proposition 17.5.2. *The unit maps*

$$((\widetilde{\iota}^{\mathrm{all}} \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}) \otimes \mathrm{Id})((\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G})) \to$$
$$\to ((\widetilde{\iota}^{\mathrm{all}} \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}) \otimes (\widetilde{\iota}^{\mathrm{all}} \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}))((\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G}))$$

and

$$(\mathrm{Id} \otimes (\widetilde{\iota}^{\mathrm{all}} \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}))((\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G})) \to$$
$$\to ((\widetilde{\iota}^{\mathrm{all}} \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}) \otimes (\widetilde{\iota}^{\mathrm{all}} \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}))((\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G}))$$

are isomorphisms.

Assuming the proposition, we obtain that the object

$$(\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)} \otimes \mathrm{Id})((\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G})) \in \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \otimes \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)$$

in fact belongs to

$$\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \otimes \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \subset$$
$$\subset \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \otimes \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G), \tag{17.10}$$

and the object

$$(\mathrm{Id} \otimes \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)})((\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G})) \in \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$$

belongs to

$$\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \otimes \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \subset$$
$$\subset \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}, \tag{17.11}$$

and, moreover, the above two objects of $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \otimes \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$ are isomorphic.

(Note the functors in (17.10) and (17.11) are indeed inclusions of full subcategories, since $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$ is dualizable, see Remark 17.2.7.)

This implies that the above object of $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}} \otimes \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$ defines a unit for a self-duality of $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$ so that (17.9) holds.

17.5.3. *Proof of Proposition 17.5.2.* We will prove the first isomorphism; the second one will follow by symmetry. We need to show that the unit of the adjunction

$$((\widetilde{\iota}^{\mathrm{all}} \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}) \otimes \mathrm{Id})((\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G})) \to$$
$$\to (\mathrm{Id} \otimes (\widetilde{\iota}^{\mathrm{all}} \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)})) \circ ((\widetilde{\iota}^{\mathrm{all}} \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}) \otimes \mathrm{Id})((\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G}))$$

is an isomorphism.

We will show that the object

$$((\widetilde{\iota}^{\mathrm{all}} \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}) \otimes \mathrm{Id})((\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G})) \in \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)$$

already belongs to the essential image of

$$\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}},$$

and hence the unit for the $(\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}, \widetilde{\iota}^{\mathrm{all}})$-adjunction on it is an isomorphism.

In fact, we will show that

$$((\widetilde{\iota}^{\mathrm{all}} \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}) \otimes \mathrm{Id})((\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G})) \simeq (\mathrm{Id} \otimes (\widetilde{\iota}^{\mathrm{all}} \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}))((\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G})) \tag{17.12}$$

as objects of $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)$.

17.5.4. Let τ denote the Cartan involution on $\check{G}$. The Hecke functors (17.1) have the basic property that for $V \in \mathrm{Rep}(\check{G})^{\otimes I}$,

$$(\mathrm{H}(V, -) \otimes \mathrm{Id})((\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G})) \simeq (\mathrm{Id} \otimes \mathrm{H}(V^{\tau}, -))((\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G}))$$

as objects of $\mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$, functorially in V and $I \in \mathrm{fSet}$.

This implies that for $\mathcal{V} \in \mathrm{Rep}(\check{G})^{\mathrm{Betti}}_{\mathrm{Ran}}$, we have

$$((\mathcal{V} \star -) \otimes \mathrm{Id})((\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G})) \simeq (\mathrm{Id} \otimes (\mathcal{V}^{\tau} \star -))((\Delta_{\mathrm{Bun}_G})_!(\underline{\mathsf{e}}_{\mathrm{Bun}_G})).$$

17.5.5. Recall that the functor $\widetilde{\iota}^{\mathrm{all}} \circ \mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}$ identifies with

$$(\mathrm{Id} \otimes \Gamma(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X), -)) \circ \mathsf{P}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)},$$

while $\mathsf{P}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}$ is the functor

$$\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \to \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)),$$

given by

$$(\mathsf{R}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)} \star -) \circ (\mathrm{Id} \otimes \mathcal{O}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)})$$

for the object

$$\mathsf{R}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)} \in \mathrm{Rep}(\check{G})^{\mathrm{Betti}}_{\mathrm{Ran}} \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)),$$

see Remark 12.1.12.

Hence, in order to prove (17.12), it suffices to show that

$$(\tau \otimes \mathrm{Id})(\mathsf{R}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}) \simeq (\mathrm{Id} \otimes \tau^*)(\mathsf{R}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}), \tag{17.13}$$

where in the right-hand side, τ denotes the involution of $\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)$, induced by τ.

However, (17.13) follows from the canonicity of the assignment

$$\mathcal{C} \mapsto \mathsf{R}_{\mathcal{C}}$$

with respect to $\mathcal{C}$.

□[Proposition 17.5.2]

18. Preservation of nilpotence of singular support

In this section we will prove Theorem 13.1.5. Let us indicate the main idea.

Let us ask the general question: how can we control the singular support of $f_*(\mathcal{F})$ for a morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ and $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y}_1)$ in terms of the singular support of $\mathcal{F}$? One situation in which we can do it is when f is proper. Namely, in this case, $\mathrm{SingSupp}(f_*(\mathcal{F}))$ is contained in the pull-push of $\mathrm{SingSupp}(\mathcal{F})$ along the diagram

$$T^*(\mathcal{Y}_2) \leftarrow \mathcal{Y}_1 \underset{\mathcal{Y}_2}{\times} T^*(\mathcal{Y}_2) \to T^*(\mathcal{Y}_1).$$

However, there is one more situation when this is possible: when f is the open embedding of stacks of the form

$$\mathbb{P}(E) \to E/\mathbb{G}_m,$$

where E is a vector bundle (over some base) and $\mathbb{P}(E)$ is its projectivization. In fact, this situation can be essentially reduced to one of a proper map, see Sect. 18.4. We call an open embedding of this form *contractive*.

The idea of the proof of Theorem 13.1.5, borrowed from [DrGa2], is to find open substacks $\mathcal{U}_i$ so that we can can calculate the singular supports of *- (or !-) extensions by reducing to the contractive situation.

18.1. **Statement of the result.** In this subsection we will give a more precise version of Theorem 13.1.5, in which we will specify what the open substacks $\mathcal{U}_i$ are.

18.1.1. Denote $\Lambda^{\mathbb{Q}} := \Lambda \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. We denote by $\leq$ the partial order relation on $\Lambda^{\mathbb{Q}}$

$$\lambda_1 \leq \lambda_2 \Leftrightarrow \lambda_2 - \lambda_1 \in \{\text{Positive integral span of simple coroots}\}.$$

Let

$$\Lambda^{\mathbb{Q},+} \subset \Lambda^{\mathbb{Q}}$$

be the cone of dominant coweights.

Recall (see e.g., [DrGa2, Theorem 7.4.3]) that the stack Bun_G admits a decomposition into locally closed substacks (known as the Harder-Narasimhan stratification)

$$\mathrm{Bun}_G = \underset{\theta \in \Lambda^{\mathbb{Q},+}}{\cup} \mathrm{Bun}_G^{(\theta)},$$

where each $\mathrm{Bun}_G^{(\theta)}$ is quasi-compact.

Moreover if a subset $S \subset \Lambda^+$ satisfies

$$\theta \in S,\ \theta' \leq \theta \Rightarrow \theta' \in S,$$

then

$$\underset{\theta \in S}{\cup} \mathrm{Bun}_G^{(\theta)}$$

is open in Bun_G.

18.1.2. For a fixed θ, let

$$\mathrm{Bun}_G^{(\leq \theta)} \overset{j^\theta}{\hookrightarrow} \mathrm{Bun}_G$$

denote the embedding of the open substack corresponding to $\underset{\theta' \leq \theta}{\cup} \mathrm{Bun}_G^{(\theta')}$.

The goal of this section is to prove the following theorem:

Theorem 18.1.3. *There exists an integer c (depending on G, $\mathrm{char}(k)$)[31], such that for θ satisfying*

$$\langle \theta, \check{\alpha}_i \rangle \geq (2g-2) + c, \quad \forall\, i \in I, \tag{18.1}$$

the functor

$$j_*^\theta : \mathrm{Shv}(\mathrm{Bun}_G^{(\leq \theta)})^{\mathrm{constr}} \to \mathrm{Shv}(\mathrm{Bun}_G)^{\mathrm{constr}}$$

preserves the condition of having nilpotent singular support.

18.1.4. *Example.* Let X have genus 1 and $\mathrm{char}(k) = 0$, so $\theta = 0$ satisfies (18.1). Note that

$$\mathrm{Bun}_G^{\leq 0} := \mathrm{Bun}_G^{\mathrm{ss}}$$

is the semi-stable locus.

Objects of $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G^{\mathrm{ss}})$ are known as character sheaves. So, in this case, Theorem 18.1.3 says that the functor of $*$-extension from the semi-stable locus sends character sheaves to sheaves with nilpotent singular support.

Remark 18.1.5. Since the functors

$$j_*^\theta : \mathrm{Shv}(\mathrm{Bun}_G^{(\leq \theta)})^{\mathrm{constr}} \to \mathrm{Shv}(\mathrm{Bun}_G)^{\mathrm{constr}}$$

and

$$j_!^\theta : \mathrm{Shv}(\mathrm{Bun}_G^{(\leq \theta)})^{\mathrm{constr}} \to \mathrm{Shv}(\mathrm{Bun}_G)^{\mathrm{constr}}$$

are related by Verdier duality, we obtain that the assertion of Theorem 18.1.3 automatically applies to the functor $j_!^\theta$ as well (Verdier duality preserves singular support).

In particular, it also applies to the functor

$$j_{!*}^\theta : \mathrm{Shv}(\mathrm{Bun}_G^{(\leq \theta)})^\heartsuit \to \mathrm{Shv}(\mathrm{Bun}_G)^\heartsuit.$$

18.2. **Set-up for the proof.** In this subsection we will explain how the calculation of extensions from the open substacks specified in Theorem 18.1.3 can be reduced to a contractive situation.

18.2.1. Let P be a parabolic in G with Levi quotient M and unipotent radical N. Let is call an open substack $\mathcal{U} \subset \mathrm{Bun}_M$ *good* if for $\mathcal{P}_M \in U$, we have

$$H^1(X, V^1_{\mathcal{P}_M}) = 0 \text{ and } H^0(X, V^2_{\mathcal{P}_M}) = 0$$

for any irreducible M-representation V^1 that appears as a subquotient of $\mathfrak{g}/\mathfrak{p}$ and an irreducible representation V^2 that appears as a subquotient of $\mathfrak{n}$.

Note that the above conditions guarantee that the map

$$\mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P \hookrightarrow \mathrm{Bun}_P \xrightarrow{\mathsf{p}} \mathrm{Bun}_G \tag{18.2}$$

is smooth and

$$\mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P \hookrightarrow \mathrm{Bun}_P \xrightarrow{\mathsf{q}} \mathrm{Bun}_M$$

is schematic, affine and smooth (see [DrGa2, Proposition 11.1.4]). In particular, the canonical map

$$\mathrm{Bun}_M \to \mathrm{Bun}_P$$

induces a closed embedding

$$\mathcal{U} \to \mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P\,. \tag{18.3}$$

[31] For $\mathrm{char} k = 0$ one can take $c = 0$.

18.2.2. We will use the following fact established in [DrGa2, Proposition 9.2.2 and Sect. 9.3]:

Theorem 18.2.3. *There exists an integer c such that for θ satisfying* (18.1)*, the closed substack $\mathrm{Bun}_G - \mathrm{Bun}_G^{(\leq\theta)}$ can be decomposed into a (locally finite) union of locally closed substacks $\mathcal{Y}$ of the following form:*

There exists a parabolic P with Levi quotient M and a good *open substack $\mathcal{U} \subset \mathrm{Bun}_M$ such that the image $\mathcal{V}$ of the map* (18.2) *contains $\mathcal{Y}$ as a closed substack, and the (closed) substack*

$$(\mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P) \underset{\mathrm{Bun}_G}{\times} \mathcal{Y} \subset \mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P$$

equals the (closed) substack

$$\mathcal{U} \subset \mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P$$

of (18.3).

18.2.4. Using a simple inductive argument, we obtain that in order to prove Theorem 18.1.3, it suffices to prove the following:

Let $\mathcal{Y} \subset \mathcal{V}$ be as in Theorem 18.2.3; in particular $\mathcal{Y}$ is closed in $\mathcal{V}$. Let j denote the open embedding

$$\mathcal{V} - \mathcal{Y} \overset{j}{\hookrightarrow} \mathcal{V}.$$

Let $\mathcal{F}$ be an object of $\mathrm{Shv}(\mathcal{V} - \mathcal{Y})^{\mathrm{constr}}$ with nilpotent singular support. Then $j_*(\mathcal{F}) \in \mathrm{Shv}(\mathcal{V})^{\mathrm{constr}}$ also has nilpotent singular support.

18.3. **What do we need to show?** Let us put ourselves in the situation of Sect. 18.2.4.

In this subsection we will formulate a general statement that estimates from the above the singular support of objects $j_*(\mathcal{F})$ in terms of the singular support of $\mathcal{F}$, see Sect. 18.3.3. We will show how this estimate implies the preservation of nilpotence of singular support.

The statement from Sect. 18.3.3 will be proved in Sect. 18.4.

18.3.1. Let $\widetilde{j}$ denote the embedding

$$\mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P - \mathcal{U} \hookrightarrow \mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P,$$

and let $\widetilde{\mathcal{F}}$ denote the pullback of $\mathcal{F}$ along the (smooth) projection

$$\mathcal{U} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P - \mathcal{U} \to \mathcal{V} - \mathcal{Y}.$$

Let $\mathcal{P}_M$ be a point of $\mathcal{U} \subset \mathrm{Bun}_M \subset \mathrm{Bun}_P$. Note that we have a canonical identification

$$T^*_{\mathcal{P}_M}(\mathrm{Bun}_P) \simeq \Gamma(X, \mathfrak{p}^*_{\mathcal{P}_M} \otimes \omega) \simeq \Gamma(X, \mathfrak{m}^*_{\mathcal{P}_M} \otimes \omega) \oplus \Gamma(X, \mathfrak{n}^*_{\mathcal{P}_M} \otimes \omega).$$

For $\widetilde{A} \in T^*_{\mathcal{P}_M}(\mathrm{Bun}_P)$, let A^0 and A^- denote its components in $\Gamma(X, \mathfrak{m}^*_{\mathcal{P}_M} \otimes \omega)$ and $\Gamma(X, \mathfrak{n}^*_{\mathcal{P}_M} \otimes \omega)$, respectively.

18.3.2. Let $\mathcal{P}_M$ be a point of $\mathcal{U}$, and let

$$A \in \Gamma(X, \mathfrak{g}^*_{\mathcal{P}_M} \otimes \omega) \simeq T^*_{\mathcal{P}_M}(\mathrm{Bun}_G)$$

be an element contained in $\mathrm{SingSupp}(j_*(\mathcal{F}))$. We wish to show that A is nilpotent.

Consider the map

$$\Gamma(X, \mathfrak{g}^*_{\mathcal{P}_M} \otimes \omega) \to \Gamma(X, \mathfrak{p}^*_{\mathcal{P}_M} \otimes \omega). \tag{18.4}$$

Let $\widetilde{A} \in \Gamma(X, \mathfrak{p}^*_{\mathcal{P}_M} \otimes \omega)$ denote the image of A under the map (18.4). Note that A is nilpotent if and only if the component

$$A^0 \in \Gamma(X, \mathfrak{m}^*_{\mathcal{P}_M} \otimes \omega)$$

of $\widetilde{A}$ is nilpotent.

Indeed, identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using an invariant bilinear form. Write

$$\Gamma(X, \mathfrak{g}^*_{\mathcal{P}_M} \otimes \omega) \simeq \Gamma(X, \mathfrak{g}_{\mathcal{P}_M} \otimes \omega) \simeq \Gamma(X, \mathfrak{n}_{\mathcal{P}_M} \otimes \omega) \oplus \Gamma(X, \mathfrak{m}_{\mathcal{P}_M} \otimes \omega) \oplus \Gamma(X, \mathfrak{n}^-_{\mathcal{P}_M} \otimes \omega).$$

The projection (18.4) corresponds to the projection on the last two factors. At the same time, the assumption on $\mathcal{U}$ implies that the first factor vanishes. So, we can think of A as an element of

$$\Gamma(X, \mathfrak{m}_{\mathcal{P}_M} \otimes \omega) \oplus \Gamma(X, \mathfrak{n}^-_{\mathcal{P}_M} \otimes \omega) \simeq \Gamma(X, \mathfrak{p}^-_{\mathcal{P}_M} \otimes \omega),$$

and it is nilpotent if and only if its Levi component is such.

18.3.3. We claim that it is enough to show the following:

Let $\widetilde{\mathcal{F}}$ be *an arbitrary* object of the category $\operatorname{Shv}(\mathcal{U} \underset{\operatorname{Bun}_M}{\times} \operatorname{Bun}_P - \mathcal{U})^{\operatorname{constr}}$, and let $\widetilde{A} \in T^*_{\mathcal{P}_M}(\operatorname{Bun}_P)$ belong to $\operatorname{SingSupp}(\widetilde{j}_*(\widetilde{\mathcal{F}}))$. Let A^0 be as in Sect. 18.3.1. Then there exists a point

$$\mathcal{P}'_P \in \{\mathcal{P}_M\} \underset{\operatorname{Bun}_M}{\times} \operatorname{Bun}_P - \{\mathcal{P}_M\}$$

such that the image, denoted $\widetilde{A}'$, of A^0 along

$$\Gamma(X, \mathfrak{m}^*_{\mathcal{P}_M} \otimes \omega) \simeq \Gamma(X, \mathfrak{m}^*_{\mathcal{P}'_M} \otimes \omega) \hookrightarrow \Gamma(X, \mathfrak{p}^*_{\mathcal{P}'_M} \otimes \omega) \simeq T^*_{\mathcal{P}'_P}(\operatorname{Bun}_P)$$

belongs to $\operatorname{SingSupp}(\widetilde{\mathcal{F}})$.

18.3.4. Let us show how the claim in Sect. 18.3.3 implies the needed property in Sect. 18.2.4.

Let $\widetilde{\mathcal{F}}$ be as in Sect. 18.3.1, and let A and $\widetilde{A}$ be as in Sect. 18.3.2. Since the projection

$$\mathcal{U} \underset{\operatorname{Bun}_M}{\times} \operatorname{Bun}_P \to \mathcal{V} \tag{18.5}$$

is smooth, the element

$$\widetilde{A} \in T^*_{\mathcal{P}_M}(\operatorname{Bun}_P)$$

belongs to the singular support of $\operatorname{SingSupp}(\widetilde{j}_*(\widetilde{\mathcal{F}}))$.

Let $\mathcal{P}'_P$ be as in Sect. 18.3.3. Using again the fact that (18.5) is smooth, we obtain that there exists

$$A' \in T^*_{\mathcal{P}'_P}(\operatorname{Bun}_G) \simeq \Gamma(X, \mathfrak{g}^*_{\mathcal{P}'_P} \otimes \omega)$$

that belongs to $\operatorname{SingSupp}(\mathcal{F})$, and whose image along

$$\Gamma(X, \mathfrak{g}^*_{\mathcal{P}'_P} \otimes \omega) \to \Gamma(X, \mathfrak{p}^*_{\mathcal{P}'_P} \otimes \omega)$$

is contained in

$$\Gamma(X, \mathfrak{m}^*_{\mathcal{P}'_P} \otimes \omega) \subset \Gamma(X, \mathfrak{p}^*_{\mathcal{P}'_P} \otimes \omega)$$

and equals the image of A^0 under the identification

$$\Gamma(X, \mathfrak{m}^*_{\mathcal{P}'_P} \otimes \omega) \simeq \Gamma(X, \mathfrak{m}^*_{\mathcal{P}'_M} \otimes \omega) \simeq \Gamma(X, \mathfrak{m}^*_{\mathcal{P}_M} \otimes \omega).$$

By assumption, A' is nilpotent, and is contained in

$$\Gamma(X, (\mathfrak{g}/\mathfrak{n})^*_{\mathcal{P}'_P} \otimes \omega) \subset \Gamma(X, \mathfrak{g}^*_{\mathcal{P}'_P} \otimes \omega).$$

Hence, its projection along

$$\Gamma(X, (\mathfrak{g}/\mathfrak{n})^*_{\mathcal{P}'_P} \otimes \omega) \to \Gamma(X, \mathfrak{m}^*_{\mathcal{P}'_P} \otimes \omega)$$

is nilpotent as well, while the latter identifies with A^0.

18.4. **Singular support in a contractive situation.** In this subsection we will provide a general context for the proof of the claim in Sect. 18.3.3.

18.4.1. Let us be given a schematic affine map of stacks $\pi : \mathcal{W} \to \mathcal{U}$, equipped with a section $s : \mathcal{U} \to \mathcal{W}$. Assume that $\mathcal{W}$, viewed as a stack over $\mathcal{U}$, is equipped with an action of the monoid $\mathbb{A}^1$ (with respect to multiplication), such that the action of $0 \in \mathbb{A}^1$ on $\mathcal{W}$ equals

$$\mathcal{W} \xrightarrow{\pi} \mathcal{U} \xrightarrow{s} \mathcal{W}.$$

Denote by j the open embedding $\mathcal{W} - \mathcal{U} \hookrightarrow \mathcal{W}$. Let $\mathcal{F}$ be an object of $\text{Shv}(\mathcal{W} - \mathcal{U})$. Assume that $\mathcal{F}$ is equivariant with respect to $\mathbb{G}_m \subset \mathbb{A}^1$, which acts on $\mathcal{W} - \mathcal{U}$.

Let u be a point of $\mathcal{U}$ and let ξ be an element of

$$T^*_u(\mathcal{U}) \oplus T^*_u(\{u\} \underset{\mathcal{U}}{\times} \mathcal{W}) \simeq T^*_u(\mathcal{W}).$$

Write ξ^0 and ξ^- for its $T^*_u(\mathcal{U})$ and $T^*_u(\{u\} \underset{\mathcal{U}}{\times} \mathcal{W})$ components, respectively.

We will prove:

Proposition 18.4.2. *Suppose that ξ belongs to* $\text{SingSupp}(j_*(\mathcal{F}))$. *Then there exists a point*

$$w \in \{u\} \underset{\mathcal{U}}{\times} \mathcal{W} - \{u\}$$

and an element $\xi' \in T^*_w(\mathcal{W})$ that belongs to $\text{SingSupp}(\mathcal{F})$ *and such that ξ' equals the image of ξ^0 under the codifferential map*

$$T^*_u(\mathcal{U}) \to T^*_w(\mathcal{W}).$$

By [DrGa2, Sect. 11.2], the set-up in Sect. 18.3.3 is a particular case of the situation in Sect. 18.4.1. Hence, the claim in Sect. 18.3.3 follows from Proposition 18.4.2.

The rest of this subsection is devoted to the proof of Proposition 18.4.2.

18.4.3. *Reduction steps.* First, by performing a smooth base change along $\mathcal{U}$, we can assume that $\mathcal{U}$ is an affine scheme.

Second, choosing homogeneous generators (for the given $\mathbb{G}_m$-action) of the ring of functions on $\mathcal{W}$, we can assume that $\mathcal{W}$ has the form $\mathcal{U} \times \text{Tot}(E)$, where E is a vector space, on which $\mathbb{G}_m$ acts via a collection of characters, which we regard as a string of positive integers denoted $(d_1, ..., d_n) = \underline{d}$.

18.4.4. We will consider a stacky blow-up, denoted $\widetilde{\text{Tot}}(E)_{\underline{d}}$ of $\text{Tot}(E)$ (it will be the usual blow up E at the origin for $(d_1, ..., d_n) = (1, ..., 1)$).

Namely, set:

$$\widetilde{\text{Tot}}(E)_{\underline{d}} := (\mathbb{A}^1 \times (\text{Tot}(E) - 0))/\mathbb{G}_m,$$

with respect to the anti-diagonal action (where the action on $\text{Tot}(E) - 0$ is given by the specified set of characters).

We have a naturally defined map

$$p : \widetilde{\text{Tot}}(E)_{\underline{d}} \to \text{Tot}(E),$$

given by the action of the monoid $\mathbb{A}^1$ on $\text{Tot}(E)$.

Denote also

$$\mathbb{P}(E)_{\underline{d}} := (\text{Tot}(E) - 0)/\mathbb{G}_m.$$

Note that $\mathbb{P}(E)_{\underline{d}}$ identifies with a closed substack of $\widetilde{\text{Tot}}(E)_{\underline{d}}$ corresponding to $0 \in \mathbb{A}^1$. Denote the embedding of the complement by $\widetilde{j}$.

Note that map p induces an *isomorphism*

$$(\widetilde{\text{Tot}}(E)_{\underline{d}} - \mathbb{P}(E)_{\underline{d}}) \to (\text{Tot}(E) - 0). \tag{18.6}$$

Let q denote the projection

$$\widetilde{\text{Tot}}(E)_{\underline{d}} \to \mathbb{P}(E)_{\underline{d}}.$$

Note that q realizes $\widetilde{\mathrm{Tot}}(E)_{\underline{d}}$ as a line bundle over $\mathbb{P}(E)_{\underline{d}}$, so that the embedding $\mathbb{P}(E)_{\underline{d}} \to \widetilde{\mathrm{Tot}}(E)_{\underline{d}}$ is the zero section.

By a slight abuse of notation, we will denote by the same characters $(p, \widetilde{j}, q)$ the corresponding morphisms after applying $\mathcal{U}\times$.

Let π denote the projection $\mathcal{U} \times \mathrm{Tot}(E) \to \mathcal{U}$, and let $\widetilde{\pi}$ denote the projection $\mathcal{U} \times \widetilde{\mathrm{Tot}}(E)_{\underline{d}} \to \mathcal{U}$, so that

$$\widetilde{\pi} = \pi \circ p.$$

Let $\overline{\pi}$ denote the projection $\mathcal{U} \times \mathbb{P}(E)_{\underline{d}} \to \mathcal{U}$, so that

$$\widetilde{\pi} = \overline{\pi} \circ q.$$

18.4.5. We claim:

Lemma 18.4.6. *The maps*

$$\widetilde{\mathrm{Tot}}(E)_{\underline{d}} \overset{p}{\to} \mathrm{Tot}(E) \text{ and } \mathbb{P}(E)_{\underline{d}} \overset{\overline{\pi}}{\to} \mathrm{pt}$$

are proper[32].

The lemma will be proved in Sect. 18.5.

18.4.7. Let $\widetilde{\mathcal{F}}$ denote the pullback of $\mathcal{F}$ along the isomorphism (18.6). We have

$$j_*(\mathcal{F}) \simeq p_*(\widetilde{j}_*(\widetilde{\mathcal{F}})).$$

We record the following lemma:

Lemma 18.4.8. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a proper map between algebraic stacks. Let $\mathcal{F}_1 \in \mathrm{Shv}(\mathcal{Y}_1)$ and denote $\mathcal{F}_2 := f_*(\mathcal{F}_1)$. Let $y_2 \in \mathcal{Y}_2$ be a point and $\xi_2 \in T^*_{y_2}(\mathcal{Y}_2)$ an element contained in $\mathrm{SingSupp}(\mathcal{F}_2)$. Then there exists $y_1 \in f^{-1}(y_2)$ such that*

$$df^*(\xi_2) =: \xi_1 \in T^*_{y_1}(Y_1)$$

belongs to $\mathrm{SingSupp}(\mathcal{F}_1)$.

Proof. It follows from the definition of singular support in [Be2] that the assertion of the lemma holds for any separated morphism f, for which the canonical natural transformation $f_! \to f_*$ is an isomorphism.

The required property for proper maps follows from [Ols, Theorem 1.1] (see, however Remark 18.5.4 for an alternative argument in our specific case).

□

18.4.9. We proceed with the proof of Proposition 18.4.2. Let $\xi = (\xi^0, \xi^-)$ be as in the statement of the proposition.

We claim that we can assume that $\xi^- = 0$.

Indeed, the action of $\mathbb{A}^1$ (viewed as a monoid with respect to *multiplication*) on $\mathcal{U} \times \mathrm{Tot}(E)$ induces an action of $\mathbb{A}^1$ on $T^*_{(u,0)}(\mathcal{U} \times \mathrm{Tot}(E))$. Since $\mathcal{F}$ is $\mathbb{G}_m$-equivariant, the subset

$$\mathrm{SingSupp}(j_*(\mathcal{F})) \cap T^*_{(u,0)}(\mathcal{U} \times \mathrm{Tot}(E))$$

is $\mathbb{G}_m$-invariant. Hence, it is invariant with respect to all of $\mathbb{A}^1$, and in particular, with respect to the action of $0 \in \mathbb{A}^1$. However, the action of 0 sends the pair

$$(\xi^0, \xi^-) \in T^*_u(\mathcal{U}) \oplus T^*_0(\mathrm{Tot}(E))$$

to $(\xi^0, 0)$.

Hence,

$$(\xi^0, 0) \in \mathrm{SingSupp}(j_*(\mathcal{F})) \cap T^*_{(u,0)}(\mathcal{U} \times \mathrm{Tot}(E)).$$

[32] Note that the notion of properness is applied here to maps of algebraic stacks that are not necessarily schematic.

18.4.10. Thus, let $\xi^0 \in T^*_u(\mathcal{U})$ be an element so that

$$d\pi^*(\xi^0) = (\xi^0, 0) \in T^*_{(u,0)}(\mathcal{U} \times \operatorname{Tot}(E))$$

belongs to $\operatorname{SingSupp}(j_*(\mathcal{F}))$.

By Lemmas 18.4.6 and 18.4.8, we can find a point $\overline{e} \in \mathbb{P}(E)_{\underline{d}} \subset \widetilde{\operatorname{Tot}}(E)_{\underline{d}}$ such that the element

$$dp^* \circ d\pi^*(\xi^0) \in T^*_{(u,\overline{e})}(\mathcal{U} \times \widetilde{\operatorname{Tot}}(E)_{\underline{d}}) \tag{18.7}$$

belongs to $\operatorname{SingSupp}(\widetilde{j}_*(\widetilde{\mathcal{F}}))$.

Let e be a point of $\operatorname{Tot}(E) - 0$ that projects to $\overline{e}$. Set

$$\xi' := d\pi^*(\xi^0) \in T^*_{(u,e)}(\mathcal{U} \times \operatorname{Tot}(E)).$$

We will show that $\xi' \in \operatorname{SingSupp}(\mathcal{F})$, which will prove Proposition 18.4.2.

18.4.11. Since $\widetilde{\mathcal{F}}$ is $\mathbb{G}_m$-equivariant, it is of the form

$$q^*(\overline{\mathcal{F}})$$

for a canonically defined $\overline{\mathcal{F}} \in \operatorname{Shv}(\mathcal{U} \times \mathbb{P}(E)_{\underline{d}})$.

As was mentioned above, $\mathcal{U} \times \widetilde{\operatorname{Tot}}(E)_{\underline{d}}$ is a total space of a line bundle over $\mathcal{U} \times \mathbb{P}(E)_{\underline{d}}$ by means of the projection q. We identify

$$T^*_{(u,\overline{e})}(\mathcal{U} \times \widetilde{\operatorname{Tot}}(E)_{\underline{d}}) \simeq T^*_{(u,\overline{e})}(\mathcal{U} \times \mathbb{P}(E)_{\underline{d}}) \oplus k,$$

where the first summand is the image of dq^*.

It is easy to see that

$$\operatorname{SingSupp}(\widetilde{j}_*(\widetilde{\mathcal{F}})) \cap T^*_{(u,\overline{e})}(\mathcal{U} \times \mathbb{P}(E)_{\underline{d}}) \subset T^*_{(u,\overline{e})}(\mathcal{U} \times \widetilde{\operatorname{Tot}}(E)_{\underline{d}})$$

equals the image of

$$\operatorname{SingSupp}(\overline{\mathcal{F}}) \cap T^*_{(u,\overline{e})}(\mathcal{U} \times \mathbb{P}(E)_{\underline{d}})$$

along dq^*.

The condition in (18.7) reads as

$$d\widetilde{\pi}^*(\xi^0) \in \operatorname{SingSupp}(\widetilde{j}_*(\widetilde{\mathcal{F}})),$$

where we also note that

$$d\widetilde{\pi}^* = dq^* \circ d\overline{\pi}^*.$$

Hence,

$$d\overline{\pi}^*(\xi^0) \in \operatorname{SingSupp}(\overline{\mathcal{F}}) \cap T^*_{(u,\overline{e})}(\mathcal{U} \times \mathbb{P}(E)_{\underline{d}}). \tag{18.8}$$

Using the isomorphism (18.6), q restricts to a map

$$\mathcal{U} \times (\operatorname{Tot}(E) - 0) \to \mathcal{U} \times \mathbb{P}(E)_{\underline{d}}.$$

In terms of this map,

$$\operatorname{SingSupp}(\mathcal{F}) \cap T^*_{(u,e)}(\mathcal{U} \times \operatorname{Tot}(E))$$

equals the image of

$$\operatorname{SingSupp}(\overline{\mathcal{F}}) \cap T^*_{(u,\overline{e})}(\mathcal{U} \times \mathbb{P}(E)_{\underline{d}})$$

along dq^*, where we also note that

$$d\pi^* = dq^* \circ d\overline{\pi}^*.$$

Hence, from (18.8), we obtain

$$d\pi^*(\xi^0) \in \operatorname{SingSupp}(\mathcal{F}),$$

as desired.

□[Proposition 18.4.2]

18.5. **The stacky weighted projective space.** In this subsection we will prove Lemma 18.4.6.

18.5.1. First, we observe that the morphism $p : \widetilde{\mathrm{Tot}}(E)_{\underline{d}} \to \mathrm{Tot}(E)$ factors as

$$\widetilde{\mathrm{Tot}}(E)_{\underline{d}} \to \mathbb{P}(E)_{\underline{d}} \times \mathrm{Tot}(E) \to \mathrm{Tot}(E),$$

where the first arrow is a finite morphism[33].

Hence, it is enough to prove the assertion of the lemma that concerns $\mathbb{P}(E)_{\underline{d}}$.

18.5.2. We first consider the case when the action of $\mathbb{G}_m$ on E is given by the d-th power of the standard character, i.e., $\underline{d} = (d, ..., d)$. We will denote the resulting stack $\mathbb{P}(E)_{\underline{d}}$ simply by $\mathbb{P}(E)_d$.

In this case, the map

$$\mathbb{P}(E)_d \to \mathbb{P}(E)$$

(where $\mathbb{P}(E)$ is the usual projectivization of E) is a Zariski locally trivial fibration with fiber pt/μ_d, where μ_d is the (finite) group-scheme of d-th roots of unity.

18.5.3. We now consider the case of a general $\underline{d}$. It suffices to find another vector space E' and an integer d' so that we have a $\mathbb{G}_m$-equivariant finite morphism

$$E \to E',$$

where $\mathbb{G}_m$ acts on E' by the d'-th power of the standard character.

Write $E = (\mathbb{A}^1)^n$, where $\mathbb{G}_m$ acts on the i-th copy by the d_i-th power of the standard character. Set $d' := \mathrm{lcm}(d_i)$ and $E' := (\mathbb{A}^1)^n$.

The morphism $E \to E'$ is given by raising to the power $\frac{d'}{d_i}$ along the i-th coordinate.

□[Lemma 18.4.6]

Remark 18.5.4. The proof of Proposition 18.4.2 used Lemma 18.4.8, in whose proof one of the ingredients was Olsson's theorem, which implies that for proper map between algebraic stacks, the natural transformation $f_! \to f_*$ is an isomorphism.

In our case, the morphism in question is

$$p : \mathcal{U}' \times \widetilde{\mathrm{Tot}}(E)_{\underline{d}} \to \mathcal{U}' \times \mathrm{Tot}(E),$$

(for some base $\mathcal{U}'$), and we claim that this corresponding property can be established directly.

Indeed, tracing through the above proof of Lemma 18.5, and using the fact that the direct image along a finite map is a conservative functor, we obtain that it is sufficient to establish the corresponding properties for the morphism

$$\mathcal{U}' \times \mathbb{P}(E)_d \to \mathcal{U}'.$$

However, this follows from the corresponding property for the morphisms

$$\mathcal{U}' \times \mathbb{P}(E)_d \to \mathcal{U}' \times \mathbb{P}(E),$$

(which is easy) and

$$\mathcal{U}' \times \mathbb{P}(E) \to \mathcal{U}'$$

(which follows from the fact that $\mathbb{P}(E)$ is a proper *scheme*).

19. Proof of Theorem 13.4.4

In this section we will prove Theorem 13.4.4.

We first consider the case of $G = GL_2$, which explains the main idea of the argument. We then implement this idea in a slightly more involved case of $G = GL_n$ (where it is sufficient consider the minuscule Hecke functors).

Finally, we treat the case of an arbitrary G; the proof reduces to the analysis of the local Hitchin map and affine Springer fibers.

[33]Note, however, that unlike the usual blowup, this map is not necessarily a closed embedding.

19.1. **Estimating singular support from below.** In this subsection we will state a general result that allows us to guarantee that a certain cotangent vector does belong to the singular support of a sheaf obtained as a direct image.

19.1.1. Let $\mathcal{Y}$ be an algebraic stack. In this section we will be operating with the notion of singular support of objects of $\mathrm{Shv}(\mathcal{Y})$ that do not necessarily belong to $\mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}$.

By definition, for $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$, its singular support $\mathrm{SingSupp}(\mathcal{F})$ is the subset of $T^*(\mathcal{Y})$ equal to the set-theoretic union of singular supports of constructible subsheaves of each of its perverse cohomologies.

We refer the reader to Sect. E.6 for the explanation of what we mean by $T^*(\mathcal{Y})$ when $\mathcal{Y}$ is a *not necessarily* smooth *scheme* and [GKRV, Sect. A.3.6] for how one passes from schemes to algebraic stacks; the upshot is that in practice we can always assume that $\mathcal{Y}$ is a smooth scheme.

We emphasize that with this definition, $\mathrm{SingSupp}(\mathcal{F})$ is *not necessarily closed* as a subset of $T^*(\mathcal{Y})$.

That said, for a closed subset $\mathcal{N} \subset T^*(\mathcal{Y})$, we have

$$\mathrm{SingSupp}(\mathcal{F}) \subset \mathcal{N} \Leftrightarrow \mathcal{F} \in \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}).$$

In particular, if $\mathcal{Y}$ is smooth,

$$\mathrm{SingSupp}(\mathcal{F}) \subset \{0\} \Leftrightarrow \mathcal{F} \in \mathrm{QLisse}(\mathcal{Y}).$$

19.1.2. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a schematic morphism between algebraic stacks with $\mathcal{Y}_2$ smooth. We denote by df^* the codifferential map

$$T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to T^*(\mathcal{Y}_1).$$

Theorem 19.1.3. *Let $\mathcal{F}_1$ be an object of* $\mathrm{Shv}(\mathcal{Y}_1)$ *and let $\xi_2 \neq 0$ be an element of $T^*_{y_2}(\mathcal{Y}_2)$ for some $y_2 \in \mathcal{Y}_2$. Assume there exists a point $y_1 \in f^{-1}(y_2) \subset \mathcal{Y}_1$ such that the following conditions hold:*
(i) *The point*

$$(\xi_2, y_1) \in T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$$

satisfies

$$df^*(\xi_2) \in T^*_{y_1}(\mathcal{Y}_1) \cap \mathrm{SingSupp}(\mathcal{F}_1),$$

i.e., (ξ_2, y_1) belongs to the intersection

$$(df^*)^{-1}(\mathrm{SingSupp}(\mathcal{F}_1)) \cap (\{\xi_2\} \times f^{-1}(y_2)) \subset T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1. \tag{19.1}$$

(ii) *For every cohomological degree m, for every constructible sub-object $\mathcal{F}'_1$ of $H^m(\mathcal{F}_1)$ and for every irreducible component $\mathcal{N}_1$ of* $\mathrm{SingSupp}(\mathcal{F}'_1)$*, if*

$$(\xi_2, y_1) \in (df^*)^{-1}(\mathcal{N}_1),$$

then the following conditions are satisfied:

(iia) *The composite map*

$$(df^*)^{-1}(\mathcal{N}_1) \hookrightarrow T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to T^*(\mathcal{Y}_2)$$

is quasi-finite on a neighborhood of the point (ξ_2, y_1), or equivalently, the point (ξ_2, y_1) is isolated in the intersection

$$(df^*)^{-1}(\mathcal{N}_1) \cap (\xi_2 \times f^{-1}(y_2));$$

(iib) *The subscheme $(df^*)^{-1}(\mathcal{N}_1)$ has dimension*[34] *$\leq \dim(\mathcal{Y}_2)$ at the point (ξ_2, y_1).*

Finally, assume:

- *Our sheaf-theoretic context is étale, Betti or ind-*regular *holonomic.*

Then ξ_2 belongs to $\mathrm{SingSupp}(f_*(\mathcal{F}_1))$.

The proof will be given in Sect. H. Several remarks are in order:

[34] This inequality is automatically an equality: we can assume that $\mathcal{Y}_1$ is smooth; then by [Be2], every $\mathcal{N}_1$ has dimension $\dim(\mathcal{Y}_1)$, and hence $(df^*)^{-1}(\mathcal{N}_1)$, if non-empty, has dimension $\geq \dim(\mathcal{Y}_2)$.

Remark 19.1.4. The statement of Theorem 19.1.3 appeals to the notion of dimension of a Zariski-closed subset in

$$T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1. \tag{19.2}$$

Note that, although for a non-smooth scheme/stack $\mathcal{Y}$, its cotangent bundle $T^*(\mathcal{Y})$ is defined only up to a unipotent gerbe (see Sect. E.6), so one cannot a priori talk unambiguously about the dimension of its closed subsets, this difficulty is not present for (19.2), since $\mathcal{Y}_2$ is smooth.

Remark 19.1.5. When $\text{char}(k) = 0$ and we work either with holonomic D-modules, or when $k = \mathbb{C}$ and we work with constructible sheaves in the classical topology, it is known that $\text{SingSupp}(\mathcal{F}_1)$ is Lagrangian, and hence $f((df^*)^{-1}(\text{SingSupp}(\mathcal{F}_1)))$ is isotropic.

This implies that, given (iia), condition (iib) is automatic in this case.

Remark 19.1.6. One can ask whether a statement analogous to Theorem 19.1.3 with condition (iib) omitted holds when instead of $\text{Shv}(-)$ we work with entire category of D-modules (not necessarily holonomic ones).

We believe that the answer is yes. In fact, when the object $\mathcal{F}_1 \in \text{D-mod}(\mathcal{Y}_1)$ is coherent, the proof was explained to us by P. Schapira.

Remark 19.1.7. Since the statement of Theorem 19.1.3 excludes the de Rham context (i.e., all D-modules or even holonomic ones), we will not be able to apply it directly to prove Theorem 13.4.3 in this case.

Instead, we will deduce Theorem 13.4.3 in the de Rham context as follows:

The validity of Theorem 19.1.3 for $\text{Shv}(-)$ in the Betti context implies, by Lefschetz principle and Riemann-Hilbert, its validity in the context of *regular holonomic* D-modules.

We will then formally deduce the assertion of Theorem 13.4.3 the entire category of D-modules from the regular holonomic case, see Sect. 19.7. The validity of Theorem 13.4.4 for holonomic D-modules would then follow from Remark 13.4.5.

(Recall, however, that we believe that (the stronger) Theorem 13.4.4 holds for the entire category $\text{D-mod}(-)$, see Remark 13.4.7.)

Remark 19.1.8. We do not know[35] a viable analog of Theorem 19.1.3 for the category $\text{Shv}^{\text{all}}(-)$. This is why our method of proof of Theorem 17.1.6 is indirect.

19.2. **The case of $G = GL_2$.** In this section we will assume that $\text{char}(k) > 2$.

19.2.1. Take $G = GL_2$. To shorten the notation, we will write Bun_2 instead of Bun_{GL_2}. Let $\mathcal{F}$ be an object in $\text{Shv}(\text{Bun}_2)^{\text{Hecke-lisse}}$. We will show that the singular support of $\mathcal{F}$ is contained in the nilpotent cone.

Let

$$\text{H} : \text{Shv}(\text{Bun}_2) \to \text{Shv}(\text{Bun}_2 \times X)$$

be the basic Hecke functor, i.e., pull-push along the diagram

$$\text{Bun}_2 \xleftarrow{\overleftarrow{h}} \mathcal{H}_2 \xrightarrow{\overrightarrow{h} \times s} \text{Bun}_2 \times X,$$

where $\mathcal{H}_2$ is the moduli space of triples

$$\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}', \tag{19.3}$$

where $\mathcal{M}$ and $\mathcal{M}'$ are vector bundles on X and $\mathcal{M}'/\mathcal{M}$ is a torsion sheaf of length 1 on X. The maps $\overleftarrow{h}$ and $\overrightarrow{h}$ send the triple $\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'$ to $\mathcal{M}$ and $\mathcal{M}'$, respectively, and s sends it to the support of $\text{coker}(\alpha)$.

[35]This was explained to us by P. Schapira.

19.2.2. We will argue by contradiction, so assume that $\mathrm{SingSupp}(\mathcal{F})$ is not contained in the nilpotent cone.

Let

$$\xi \in T^*_{\mathcal{M}}(\mathrm{Bun}_2), \quad \mathcal{M} \in \mathrm{Bun}_2$$

be an element contained in $\mathrm{SingSupp}(\mathcal{F})$. Recall that the cotangent space $T^*_{\mathcal{M}}(\mathrm{Bun}_2)$ identifies with the space of

$$A \in \mathrm{Hom}(\mathcal{M}, \mathcal{M} \otimes \omega).$$

We wish to show that if A corresponds to ξ, then A is nilpotent.

19.2.3. First, we claim that $\mathrm{Tr}(A) = 0$ as an element of $\Gamma(X, \omega)$. Indeed, consider the action

$$\mathrm{Pic} \times \mathrm{Bun}_2 \to \mathrm{Bun}_2, \quad \mathcal{L}, \mathcal{M} \mapsto \mathcal{L} \otimes \mathcal{M}. \tag{19.4}$$

As in Sect. 13.4.11, it is easy to see that the pullback of $\mathcal{F}$ along (19.4) belongs to

$$\mathrm{Shv}_{\{0\} \times T^*(\mathrm{Bun}_2)}(\mathrm{Pic} \times \mathrm{Bun}_2) \subset \mathrm{Shv}(\mathrm{Pic} \times \mathrm{Bun}_2).$$

Hence, A lies in the subspace of $T^*_{\mathcal{M}}(\mathrm{Bun}_2)$ perpendicular to

$$\mathrm{Im}(T_{\mathbf{1}}(\mathrm{Pic}) \overset{\mathcal{L} \mapsto \mathcal{L} \otimes \mathcal{M}}{\longrightarrow} T_{\mathcal{M}}(\mathrm{Bun}_2)) \subset T_{\mathcal{M}}(\mathrm{Bun}_2),$$

and this subspace exactly consists of those A that have trace 0.

19.2.4. Assume now that A is non-nilpotent. This means that $\det(A) \neq 0$ as an element of $\Gamma(X, \omega^{\otimes 2})$. The conditions

$$\mathrm{Tr}(A) = 0 \ \text{ and } \ \det(A) \neq 0 \tag{19.5}$$

(plus the assumption that $\mathrm{char}(k) > 2$) imply that at the generic point of X, the operator A is regular semi-simple.

Let

$$\widetilde{X} \subset T^*(X)$$

be the spectral curve corresponding to A. The fact that A is generically regular semi-simple implies that over the generic point of X, the projection

$$\widetilde{X} \to X$$

is generically étale.

Let $x \in X$ be a point which has two distinct preimages in $\widetilde{X}$. Let $\widetilde{x}$ be one of them. We can think of $\widetilde{x}$ as an element $T^*_x(X)$, which we will denote by ξ_x.

We will construct a point $\mathcal{M}' \in \mathrm{Bun}_2$ and $A' \in T^*_{\mathcal{M}'}(\mathrm{Bun}_2)$, such that the element

$$(A', \xi_x) \in T^*_{\mathcal{M}',x}(\mathrm{Bun}_2 \times X)$$

belongs to $\mathrm{SingSupp}(\mathrm{H}(\mathcal{F}))$.

19.2.5. For a point (19.3) of $\mathcal{H}_2$, the intersection of

$$(d\overleftarrow{h}^*)(T^*_{\mathcal{M}}(\mathrm{Bun}_2)) \cap (d(\overrightarrow{h} \times s)^*)(T^*_{\mathcal{M}',x}(\mathrm{Bun}_2 \times X)) \subset T^*_{(\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}')}(\mathcal{H}_2)$$

consists of commutative diagrams

$$\begin{array}{ccc} \mathcal{M}' & \xrightarrow{A'} & \mathcal{M}' \otimes \omega \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\alpha \otimes \mathrm{id}} \\ \mathcal{M} & \xrightarrow{A} & \mathcal{M} \otimes \omega, \end{array} \tag{19.6}$$

where the corresponding element of $T^*_x(X)$ is given by the induced map

$$\mathcal{M}'/\mathcal{M} \to (\mathcal{M}'/\mathcal{M}) \otimes \omega.$$

19.2.6. We can think of $\mathcal{M}$ as a torsion-free sheaf $\mathcal{L}$ on $\widetilde{X}$, which is generically a line bundle. The possible diagrams (19.6) correspond to upper modifications of

$$\mathcal{L} \hookrightarrow \mathcal{L}', \quad \operatorname{supp}_X(\mathcal{L}'/\mathcal{L}) \subset \{x\} \underset{X}{\times} \widetilde{X}$$

as coherent sheaves on $\widetilde{X}$.

By the assumption on x, there are exactly two such modifications, corresponding to the two preimages of x in $\widetilde{X}$. We let $(\mathcal{M}', A')$ be the modification corresponding to the chosen point $\widetilde{x}$, so $A' \in T^*_{\mathcal{M}'}(\mathrm{Bun}_2)$.

19.2.7. We claim that $(A', \xi_x) \in T^*_{\mathcal{M}',x}(\mathrm{Bun}_2 \times X)$ indeed belongs to $\mathrm{SingSupp}(\mathrm{H}(X))$. We will do so by applying Theorem 19.1.3 to

$$\mathcal{Y}_1 = \mathcal{H}_2,\ \mathcal{Y}_2 = \mathrm{Bun}_2 \times X,\ f = (\overrightarrow{h} \times s),\ \mathcal{F}_1 = \overleftarrow{h}^*(\mathcal{F}),$$
$$y_1 = (x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'),\ y_2 = (\mathcal{M}', x),\ \xi_2 = (A', \xi_x).$$

Note that since $\overleftarrow{h}$ is smooth,

$$\mathrm{SingSupp}(\overleftarrow{h}^*(\mathcal{F})) \subset T^*(\mathcal{H}_2)$$

equals the image of

$$\mathrm{SingSupp}(\mathcal{F}) \underset{\mathrm{Bun}_2, \overleftarrow{h}}{\times} \mathcal{H}_2$$

along the codifferential of $\overleftarrow{h}$

$$\mathrm{SingSupp}(\mathcal{F}) \underset{\mathrm{Bun}_2, \overleftarrow{h}}{\times} \mathcal{H}_2 \subset T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_2, \overleftarrow{h}}{\times} \mathcal{H}_2 \to T^*(\mathcal{H}_2).$$

19.2.8. We first verify condition (i) of Theorem 19.1.3. The fact that the point $((A', \xi_x), (\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'))$ belongs to

$$\mathrm{SingSupp}(\overleftarrow{h}^*(\mathcal{F})) \cap \left((A', \xi_x) \times (\overrightarrow{h} \times s)^{-1}(\mathcal{M}', x)\right) \subset T^*(\mathrm{Bun}_2 \times X) \underset{\mathrm{Bun}_2 \times X, (\overrightarrow{h} \times s)}{\times} \mathcal{H}_2 \tag{19.7}$$

follows from the construction.

19.2.9. We now verify condition (iia). We have to show that the point $((A', \xi_x), (\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'))$ is isolated in the intersection (19.7). For that end, suffices to show that the intersection

$$\left(T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_2, \overleftarrow{h}}{\times} \mathcal{H}_2\right) \cap \left((A', \xi_x) \times (\overrightarrow{h} \times s)^{-1}(\mathcal{M}', x)\right) \subset T^*(\mathcal{H}_2)$$

is finite.

We will establish a slightly stronger assertion, namely that the intersection

$$\left(T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_2, \overleftarrow{h}_x}{\times} \mathcal{H}_{2,x}\right) \cap \left(A' \times (\overrightarrow{h}_x)^{-1}(\mathcal{M}')\right) \subset T^*(\mathcal{H}_{2,x}) \tag{19.8}$$

is finite, where

$$\mathrm{Bun}_2 \xleftarrow{\overleftarrow{h}_x} \mathcal{H}_{2,x} \xrightarrow{\overrightarrow{h}_x} \mathrm{Bun}_2$$

is the fiber of

$$\mathrm{Bun}_2 \xleftarrow{\overleftarrow{h}} \mathcal{H}_2 \xrightarrow{\overrightarrow{h}} \mathrm{Bun}_2$$

over $x \in X$.

The intersection (19.8) consists of diagrams (19.6) with fixed $(\mathcal{M}', A', x)$. By Sect. 19.2.6, such diagrams are in bijection with lower modifications of $\mathcal{L}'$ as a coherent sheaf on $\widetilde{X}$ supported at x, and there are exactly two of those.

Remark 19.2.10. Note that most of the above argument would apply to Bun_n for $n \geq 2$, except for the last finiteness assertion. The latter used the fact that A is generically semi-simple, which in the case $n = 2$ is guaranteed by the conditions (19.5).

19.2.11. We now verify condition (iib) of Theorem 19.1.3. Note that [Be2], for every cohomological degree m and every constructible sub-object $\mathcal{F}'$ of $H^m(\mathcal{F})$, all irreducible components of $\mathrm{SingSupp}(\mathcal{F}')$ have dimension equal to $\dim(\mathrm{Bun}_G)$.

Hence, it suffices to show that for every $\mathcal{F}'$ as above, the fibers of the composite map

$$\mathrm{SingSupp}(\overleftarrow{h}^*(\mathcal{F}')) \underset{T^*(\mathcal{H}_2)}{\times} \left(T^*(\mathrm{Bun}_2 \times X) \underset{\mathrm{Bun}_2 \times X, (\overrightarrow{h} \times s)}{\times} \mathcal{H}_2\right) \to \mathrm{SingSupp}(\overleftarrow{h}^*(\mathcal{F}')) \overset{\overleftarrow{h}}{\to} \mathrm{SingSupp}(\mathcal{F}')$$

have dimension ≤ 1 near $((A', \xi_x), (\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'))$.

We will show that the fibers of the map

$$\left(T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_2, \overleftarrow{h}}{\times} \mathcal{H}_2\right) \underset{T^*(\mathcal{H}_2)}{\times} \left(T^*(\mathrm{Bun}_2 \times X) \underset{\mathrm{Bun}_2 \times X, (\overrightarrow{h} \times s)}{\times} \mathcal{H}_2\right) \to$$
$$\to T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_2, \overleftarrow{h}}{\times} \mathcal{H}_2 \to T^*(\mathrm{Bun}_2)$$

have dimension ≤ 1 near $((A', \xi_x), (\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'))$. It suffices to show that the map

$$\left(T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_2, \overleftarrow{h}_x}{\times} \mathcal{H}_{2,x}\right) \underset{T^*(\mathcal{H}_2)}{\times} \left(T^*(\mathrm{Bun}_2 \times X) \underset{\mathrm{Bun}_2 \times X, \overrightarrow{h}}{\times} \mathcal{H}_2\right) \simeq \tag{19.9}$$
$$\simeq \left(T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_2, \overleftarrow{h}_x}{\times} \mathcal{H}_{2,x}\right) \underset{T^*(\mathcal{H}_{2,x})}{\times} \left(T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_2, \overrightarrow{h}_x}{\times} \mathcal{H}_{2,x}\right) \to$$
$$\to T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_2, \overleftarrow{h}_x}{\times} \mathcal{H}_{2,x} \to T^*(\mathrm{Bun}_2)$$

is finite near $(A', (\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'))$.

Since the map (19.9) is proper, it suffices to show that the point $(A', (\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'))$ is isolated in its fiber with respect to (19.9).

However, this is a similar finiteness assertion to what we proved in Sect. 19.2.9.

19.3. **The case of $G = GL_n$.**

In this section we will assume that $\mathrm{char}(k) > n$.

We will essentially follow the same argument as in the case of $n = 2$, with the difference that we will have to use all minuscule Hecke functors, and not just the basic one.

19.3.1. Let $G = GL_n$, and we will write Bun_n instead of Bun_{GL_n}. Let $\mathcal{F}$ be an object in $\mathrm{Shv}(\mathrm{Bun}_n)^{\text{Hecke-lisse}}$. We will show that the singular support of $\mathcal{F}$ is contained in the nilpotent cone.

For an integer $1 \leq i \leq n$, let

$$\mathrm{H}^i : \mathrm{Shv}(\mathrm{Bun}_n) \to \mathrm{Shv}(\mathrm{Bun}_n \times X)$$

denote the i-th Hecke functor, i.e., pull-push along the diagram

$$\mathrm{Bun}_n \overset{\overleftarrow{h}}{\longleftarrow} \mathcal{H}^i_n \overset{\overrightarrow{h} \times s}{\longrightarrow} \mathrm{Bun}_n \times X,$$

where $\mathcal{H}^i_n$ is the moduli space of quadruples $(x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}')$, where:

- x is a point of X;
- $\mathcal{M}$ and $\mathcal{M}'$ are rank n bundles on X;

- α is an injection of coherent sheaves

$$\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}', \tag{19.10}$$

  such that coker(α) has length i and is *scheme-theoretically* supported at x.

For future use, let

$$\mathrm{Bun}_n \xleftarrow{\overleftarrow{h}_x} \mathcal{H}^i_{n,x} \xrightarrow{\overrightarrow{h}_x} \mathrm{Bun}_n$$

denote the fiber of the above picture over a given $x \in X$.

19.3.2. We will argue by contradiction, so assume that SingSupp($\mathcal{F}$) is not contained in the nilpotent cone.

Let

$$\xi_1 \in T^*_{\mathcal{M}}(\mathrm{Bun}_n), \quad \mathcal{M} \in \mathrm{Bun}_n$$

be an element contained in SingSupp($\mathcal{F}$). Thus ξ_1 corresponds to an element

$$A \in \mathrm{Hom}(\mathcal{M}, \mathcal{M} \otimes \omega),$$

and assume that A is non-nilpotent. Let $x \in X$ be a point such that

$$A_x \in \mathrm{Hom}(\mathcal{M}_x, \mathcal{M}_x \otimes T^*_x(X))$$

has a non-zero eigenvalue, to be denoted $\xi_x \in T^*_x(X)$. Let i denote its multiplicity (as a generalized eigenvalue). We will construct a point $\mathcal{M}' \in \mathrm{Bun}_n$ and $\xi_2 \in T^*_{\mathcal{M}'}(\mathrm{Bun}_n)$, such that the element

$$(\xi_2, i \cdot \xi_x) \in T^*_{\mathcal{M}',x}(\mathrm{Bun}_n \times X)$$

belongs to SingSupp($\mathrm{H}^i(X)$) (it is here that we use the assumption that char(k) $> n$, namely that the integer i is non-zero in k).

19.3.3. For a point $(x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}')$ of $\mathcal{H}^i_n$, the intersection

$$(d\overleftarrow{h}^*)(T^*_{\mathcal{M}}(\mathrm{Bun}_n)) \cap (d(\overrightarrow{h} \times s)^*)(T^*_{\mathcal{M}',x}(\mathrm{Bun}_n)) \subset T^*_{(x,\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}')}(\mathcal{H}^i_n)$$

consists of commutative diagrams

$$\begin{array}{ccc} \mathcal{M}' & \xrightarrow{A'} & \mathcal{M}' \otimes \omega \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\alpha\otimes\mathrm{id}} \\ \mathcal{M} & \xrightarrow{A} & \mathcal{M} \otimes \omega, \end{array} \tag{19.11}$$

where the corresponding element of $T^*_x(X)$ is given by the *trace* of the induced map

$$\mathcal{M}'/\mathcal{M} \to (\mathcal{M}'/\mathcal{M}) \otimes \omega.$$

19.3.4. Let $\widetilde{X} \subset T^*(X)$ be the spectral curve corresponding to A. We can think of $\mathcal{M}$ as a torsion-free sheaf $\mathcal{L}$ on $\widetilde{X}$. Its modifications

$$\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'$$

that fit into (19.11) are in bijection with modifications

$$\mathcal{L} \overset{\widetilde{\alpha}}{\hookrightarrow} \mathcal{L}' \tag{19.12}$$

as torsion-free coherent sheaves on $\widetilde{X}$.

19.3.5. Let $\mathcal{D}_x$ be the formal disc around x, and set

$$\widetilde{\mathcal{D}}_x := \mathcal{D}_x \underset{X}{\times} \widetilde{X}.$$

Modifications as in (19.12) are in bijection with similar modifications of $\mathcal{L}|_{\widetilde{\mathcal{D}}_x}$.

The multi-disc $\widetilde{\mathcal{D}}_x$ can be written as

$$\widetilde{\mathcal{D}}_x := \widetilde{\mathcal{D}}_x^1 \sqcup \widetilde{\mathcal{D}}_x^2,$$

where $\widetilde{\mathcal{D}}_x^1$ is the connected component containing the element $\xi_x \in T_x^*(X) \subset T^*(X)$. By assumption,

$$\widetilde{\mathcal{D}}_x^1 \to \mathcal{D}_x \tag{19.13}$$

is a finite flat ramified cover, such that the preimage of $x \in \mathcal{D}_x$ is a "fat point" of length i. Hence, the rank of (19.13) is i.

In particular, we obtain that $\mathcal{L}|_{\widetilde{\mathcal{D}}_x^1}$, viewed as a coherent sheaf on $\mathcal{D}_x$ via the pushforward along (19.13), is a vector bundle of rank equal to i. (Note, however, that it is not in general true that $\mathcal{L}|_{\widetilde{\mathcal{D}}_x^1}$ itself is a line bundle on $\widetilde{\mathcal{D}}_x^1$; that only be the case if ξ_x is a *regular* eigenvalue, i.e., if the dimension of the actual eigenspace with eigenvalue ξ_x is 1.)

We let the sought-for modification of $\mathcal{L}_{\mathcal{D}_x}$ be given by

$$\mathcal{L}'_{\mathcal{D}_x}|_{\widetilde{\mathcal{D}}_x^1} = \mathcal{L}'_{\mathcal{D}_x}(x)|_{\widetilde{\mathcal{D}}_x^1} \text{ and } \mathcal{L}'_{\mathcal{D}_x}|_{\widetilde{\mathcal{D}}_x^2} = \mathcal{L}'_{\mathcal{D}_x}|_{\widetilde{\mathcal{D}}_x^2},$$

i.e., we leave $\mathcal{L}$ intact on $\widetilde{\mathcal{D}}_x^2$, and twist by the divisor equal to the preimage of x on $\widetilde{\mathcal{D}}_x^1$.

19.3.6. In order to show that the pair $(\xi_2, i \cdot \xi_x)$ indeed belongs to $\mathrm{SingSupp}(\mathrm{H}^i(X))$, we will apply Theorem 19.1.3 to

$$\mathcal{Y}_1 = \mathcal{H}_n^i,\ \mathcal{Y}_2 = \mathrm{Bun}_n \times X,\ f = (\overrightarrow{h} \times s),\ \mathcal{F}_1 = \overleftarrow{h}^*(\mathcal{F}),$$
$$y_1 = (x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'),\ y_2 = (\mathcal{M}', x),\ \xi_2 = (A', i \cdot \xi_x).$$

Let us verify conditions (i) and (ii) of Theorem 19.1.3. We start with condition (i).

The point

$$((A', i \cdot \xi_x), (x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}')) \in T^*_{(x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}')}(\mathcal{H}_n^i)$$

belongs to $\mathrm{SingSupp}(\overleftarrow{h}^*(\mathcal{F}))$ by assumption.

19.3.7. Next we verify condition (iia). As in Sect. 19.2.9, it suffices to show that the intersection

$$\left(T^*(\mathrm{Bun}_2) \underset{\mathrm{Bun}_n, \overleftarrow{h}_x}{\times} \mathcal{H}_{n,x}^i\right) \cap \left(A' \times (\overrightarrow{h}_x)^{-1}(\mathcal{M}')\right) \subset T^*(\mathcal{H}_{n,x}^i) \tag{19.14}$$

is finite.

We interpret the pair $(\mathcal{M}', A')$ as a torsion-free sheaf $\mathcal{L}'$ on $\widetilde{X}$, and the intersection (19.14) consists of its lower modifications (19.12), such that the quotient $\mathcal{L}'/\mathcal{L}$, viewed as a coherent sheaf on X, is scheme-theoretically supported at x and has length i.

Lower modifications of $\mathcal{L}'$ on $\widetilde{X}$ over $x \in X$ are in bijection with lower modifications of $\mathcal{L}'|_{\widetilde{\mathcal{D}}_x}$. Those split into connected components indexed by the length of the quotient $\mathcal{L}'/\mathcal{L}$ on *each* connected component of $\widetilde{\mathcal{D}}_x$.

Take the connected component, where the length of the modification is i over $\widetilde{\mathcal{D}}_x^1$, and 0 on all other components. We claim that this connected component consists of a single point, which corresponds to our $(x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}')$.

Indeed, the condition on the scheme-theoretic support of $\mathcal{L}'/\mathcal{L}$ implies that

$$\mathcal{L}'(-x) \subset \mathcal{L},$$

while the requirement on the length implies that the above inclusion is an equality.

19.3.8. Let us verify condition (iib) in Theorem 19.1.3. As in Sect. 19.2.11, it suffices to show that the map

$$\left(T^*(\mathrm{Bun}_n) \underset{\mathrm{Bun}_n, \overleftarrow{h}}{\times} \mathcal{H}^i_n\right) \underset{T^*(\mathcal{H}^i_n)}{\times} \left(T^*(\mathrm{Bun}_n \times X) \underset{\mathrm{Bun}_n \times X, \overrightarrow{h} \times s}{\times} \mathcal{H}^i_n\right) \to$$
$$\to \left(T^*(\mathrm{Bun}_n) \underset{\mathrm{Bun}_n, \overleftarrow{h}}{\times} \mathcal{H}^i_n\right) \to T^*(\mathrm{Bun}_n)$$

has fibers of dimension ≤ 1 near $((A', i \cdot \xi_x), (x, \mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'))$.

Furthermore, it suffices to show that the map

$$\left(T^*(\mathrm{Bun}_n) \underset{\mathrm{Bun}_n, \overleftarrow{h}_x}{\times} \mathcal{H}^i_{n,x}\right) \underset{T^*(\mathcal{H}^i_n)}{\times} \left(T^*(\mathrm{Bun}_n \times X) \underset{\mathrm{Bun}_n \times X, \overrightarrow{h}}{\times} \mathcal{H}^i_n\right) \simeq \tag{19.15}$$
$$\simeq \left(T^*(\mathrm{Bun}_n) \underset{\mathrm{Bun}_n, \overleftarrow{h}_x}{\times} \mathcal{H}^i_{n,x}\right) \underset{T^*(\mathcal{H}^i_{n,x})}{\times} \left(T^*(\mathrm{Bun}_n) \underset{\mathrm{Bun}_n, \overrightarrow{h}_x}{\times} \mathcal{H}^i_{n,x}\right) \to$$
$$\to \left(T^*(\mathrm{Bun}_n) \underset{\mathrm{Bun}_n, \overleftarrow{h}_x}{\times} \mathcal{H}^i_{n,x}\right) \to T^*(\mathrm{Bun}_n)$$

is finite near $(A', (\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}'))$.

Since the map (19.15) is proper, it suffices to show that the point $(A', (\mathcal{M} \overset{\alpha}{\hookrightarrow} \mathcal{M}')$ is isolated in its fiber with respect to (19.15). The latter is proved by the same consideration as in Sect. 19.3.7.

19.4. **The case of an arbitrary reductive group** G**.** The proof in the case of an arbitrary G will follow the same idea as in the case of GL_n. What will be different is the local analysis:

In the case of GL_n, to a cotangent vector to Bun_G (a.k.a. Higgs field), we attached its spectral curve $\widetilde{X}$, and proved the theorem by analyzing the behavior of modifications of sheaves on it.

For an arbitrary G, there is no spectral curve. Instead, our local analysis will amount to studying the fibers of the affine (parabolic) Springer map.

19.4.1. Recall that the first assumption on $\mathrm{char}(k)$ in Sect. 13.4.1 says that there exists a non-degenerate G-equivariant pairing

$$\mathfrak{g} \otimes \mathfrak{g} \to k, \tag{19.16}$$

whose restriction to the center of any Levi subalgebra remains non-degenerate.

We will use the pairing (19.16) to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ as G-modules, and also $\mathfrak{m}^*$ with $\mathfrak{m}$ for any Levi subgroup $M \subset G$.

19.4.2. Let $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$ be an object with non-nilpotent singular support. We will find an irreducible representation $V^\lambda \in \mathrm{Rep}(\check{G})$, such that the corresponding Hecke functor

$$\mathrm{H}(V^\lambda, -) : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times X),$$

sends $\mathcal{F}$ to an object of $\mathrm{Shv}(\mathrm{Bun}_G \times X)$ whose singular support is *not* contained in

$$T^*(\mathrm{Bun}_G) \times \{\text{zero-section}\} \subset T^*(\mathrm{Bun}_G) \times T^*(X) = T^*(\mathrm{Bun}_G \times X).$$

19.4.3. Using the pairing (19.16), we can think of points of $T^*(\mathrm{Bun}_G)$ as pairs $(\mathcal{P}_G, A)$, where $\mathcal{P}_G$ is a G-bundle on X and A is an element of

$$\Gamma(X, \mathfrak{g}_{\mathcal{P}_G} \otimes \omega).$$

The Chevaley map attaches to A above a global section $\mathrm{ch}(A)$ of $\mathfrak{a}_\omega$, where the latter is the ω-twist of

$$\mathfrak{a} := \mathfrak{g}/\!/G \simeq \mathfrak{t}/\!/W,$$

where $\mathfrak{t}$ is the Cartan subalgebra and W is the Weyl group. Note that the latter isomorphism uses the second assumption on $\mathrm{char}(k)$, namely, that it does not divide $|W|$.

By assumption, $\mathrm{SingSupp}(\mathcal{F})$ contains a point $(\mathcal{P}_G, A)$ for which A is non-nilpotent, i.e., $\mathrm{ch}(A) \neq 0$. Let $x \in X$ be a point for which the value

$$\mathrm{ch}(A)_x \in \mathfrak{a}_{\omega_x} \simeq (\mathfrak{a} \times (T^*_x(X) - 0)) / \mathbb{G}_m$$

of $\mathrm{ch}(A)$ at x is non-zero.

Choose a preimage $t_x \in \mathfrak{t} \otimes T^*_x(X)$ of $\mathrm{ch}(A)_x$ along the projection

$$\mathfrak{t} \otimes T^*_x(X) \to \mathfrak{a}_{\omega_x}.$$

Let M be the Levi subgroup of G equal to the centralizer of t_x. (Thus, if $\mathrm{ch}(A)_x$ were zero, we would get $M = G$, and if A_x was regular semi-simple, we would get $M = T$, the Cartan subgroup.)

Let λ be a coweight of $Z(M)$ that is G-dominant and (G, M)-regular (the latter means that the centralizer of λ in G is contained in M). By the non-degeneracy assumption on (19.16), we can choose λ so that the value of the pairing (19.16) on the pair (t_x, λ) is non-zero.

We claim that with this choice of λ, the singular support of the object

$$\mathrm{H}(V^\lambda, \mathcal{F}) \in \mathrm{Shv}(\mathrm{Bun}_G \times X)$$

at the point $(\mathcal{P}'_G, x) \in \mathrm{Bun}_G \times X$ will contain an element (A', ξ_x), where $\mathcal{P}'_G$ is the Hecke modification of $\mathcal{P}_G$ at x of type λ specified in Sect. 19.4.4 below, and

$$0 \neq \xi_x \in T^*_x(X).$$

The element A' will also be specified in Sect. 19.4.4 below.

19.4.4. By the choice of M, the fiber $(\mathcal{P}_{G,x}, A_x)$ of $(\mathcal{P}_G, A)$ at x admits a reduction $(\mathcal{P}_{M,x}, A_x)$ to M, so that

$$A_x \in \mathfrak{m}_{\mathcal{P}_{M,x}} \otimes T^*_x(X)$$

is such that its semi-simple part lies in

$$Z(\mathfrak{m}_{\mathcal{P}_{M,x}}) \otimes T^*_x(X) \subset \mathfrak{m}_{\mathcal{P}_{M,x}} \otimes T^*_x(X)$$

and is (G, M)-regular.

Note now that the map of the stack-theoretic quotients

$$\mathfrak{m}/\mathrm{Ad}(M) \to \mathfrak{g}/\mathrm{Ad}(G)$$

is étale at points of $\mathfrak{m}$, whose centralizer in G is contained in M (see Lemma 19.6.3 below). This implies that the restriction $\mathcal{P}^{\mathrm{loc}}_G := \mathcal{P}_G|_{\mathcal{D}_x}$ admits a *unique* reduction to M, to be denoted $\mathcal{P}^{\mathrm{loc}}_M$, such that:

- The value of $\mathcal{P}^{\mathrm{loc}}_M$ at x is $\mathcal{P}_{M,x}$;
- $A^{\mathrm{loc}} := A|_{\mathcal{D}_x}$ lies in $\Gamma(\mathcal{D}_x, \mathfrak{m}_{\mathcal{P}^{\mathrm{loc}}_M} \otimes \omega)$;
- $A^{\mathrm{loc}}_x = A_x$ as elements of $\mathfrak{m}_{\mathcal{P}_{M,x}} \otimes T^*_x(X)$.

Being a cocharacter of $Z(M)$, the element λ defines a distinguished modification $\mathcal{P}'^{\mathrm{loc}}_M$ of $\mathcal{P}^{\mathrm{loc}}_M$. We let $\mathcal{P}'^{\mathrm{loc}}_G$ be the induced modification of $\mathcal{P}^{\mathrm{loc}}_G := \mathcal{P}_G|_{\mathcal{D}_x}$, and we let $\mathcal{P}'_G$ denote the resulting modification of $\mathcal{P}_G$, i.e.,

$$\begin{cases} \mathcal{P}'_G|_{\mathcal{D}_x} = \mathcal{P}'^{\mathrm{loc}}_G, \\ \mathcal{P}'_G|_{X-x} = \mathcal{P}_G|_{X-x}. \end{cases}$$

The centrality of λ implies that we have a natural identification

$$\mathfrak{m}_{\mathcal{P}'^{\mathrm{loc}}_M} \simeq \mathfrak{m}_{\mathcal{P}^{\mathrm{loc}}_M},$$

and hence A^{loc} gives rise to a section

$$A'^{\mathrm{loc}} \in \Gamma(\mathcal{D}_x, \mathfrak{m}_{\mathcal{P}'^{\mathrm{loc}}_M} \otimes \omega).$$

By a slight abuse of notation we will denote by the same symbol A'^{loc} its image along

$$\Gamma(\mathcal{D}_x, \mathfrak{m}_{\mathcal{P}'^{\mathrm{loc}}_M} \otimes \omega) \to \Gamma(\mathcal{D}_x, \mathfrak{g}_{\mathcal{P}'^{\mathrm{loc}}_M} \otimes \omega) = \Gamma(\mathcal{D}_x, \mathfrak{g}_{\mathcal{P}'^{\mathrm{loc}}_G} \otimes \omega) = \Gamma(\mathcal{D}_x, \mathfrak{g}_{\mathcal{P}'_G} \otimes \omega).$$

Let

$$A' \in \Gamma(X, \mathfrak{g}_{\mathcal{P}'_G} \otimes \omega)$$

denote the element such that

$$\begin{cases} A'|_{\mathcal{D}_x} = A'^{\mathrm{loc}}, \\ A'|_{X-x} = A|_{X-x}. \end{cases}$$

19.4.5. Consider the Hecke stack

$$\mathrm{Bun}_G \xleftarrow{\overleftarrow{h}} \mathcal{H}_G \xrightarrow{\overrightarrow{h}\times s} \mathrm{Bun}_G \times X.$$

For future use, denote by

$$\mathrm{Bun}_G \xleftarrow{\overleftarrow{h}_x} \mathcal{H}_{G,x} \xrightarrow{\overrightarrow{h}_x} \mathrm{Bun}_G$$

the fiber of this picture over a given $x \in X$.

We will apply Theorem 19.1.3 to

$$\mathcal{Y}_1 = \mathcal{H}_G,\ \mathcal{Y}_2 = \mathrm{Bun}_G \times X,\ f = \overrightarrow{h} \times s,\ \mathcal{F}_1 = \overleftarrow{h}^*(\mathcal{F}) \otimes \tau^*(\mathcal{V}^\lambda),$$

where:

- $\tau : \mathcal{H}_G \to \mathcal{H}_G^{\mathrm{loc}}$ is the projection on the local Hecke stack (see [GKRV, Sect.B.3.2]);
- $\mathcal{V}^\lambda \in \mathrm{Shv}(\mathcal{H}_G^{\mathrm{loc}})$ corresponds to $V^\lambda \in \mathrm{Rep}(\check{G})$ by geometric Satake.

We take $y_2 = (\mathcal{P}'_G, x)$ and y_1 corresponding to the modification $\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G$.

Remark 19.4.6. In what follows, we will appeal to the cotangent bundle of $\mathcal{H}_G$ and related geometric objects, and to the notion of singular support of sheaves on them. The apotropaic definitions that justify these manipulations are spelled out in Sect. E.6.

19.4.7. We will show the following:

(a) There exists *some* $\xi_x \in T^*_x(X)$ such that

$$((A', \xi_x), (x, \mathcal{P}_G \rightsquigarrow \mathcal{P}'_G)) \in \left(T^*(\mathrm{Bun}_G \times X) \underset{\mathrm{Bun}_G \times X, \overrightarrow{h}\times s}{\times} \overline{\mathcal{H}}^\lambda_G \right) \cap \mathrm{SingSupp}\left(\overleftarrow{h}^*(\mathcal{F}) \otimes \tau^*(\mathcal{V}^\lambda) \right);$$

(b) ξ_x is the value of the pairing (19.16) on the pair (A_x, λ), or equivalently, (t_x, λ), and hence, is non-zero by the choice of λ;

(c) The point $((A', \xi_x), (x, \mathcal{P}_G \rightsquigarrow \mathcal{P}'_G))$ is isolated in the intersection

$$\left(T^*(\mathrm{Bun}_G \times \mathcal{H}_G^{\mathrm{loc}}) \underset{\mathrm{Bun}_G \times \mathcal{H}_G^{\mathrm{loc}}, \overleftarrow{h}\times \tau}{\times} \overline{\mathcal{H}}^\lambda_G \right) \cap \left((A', \xi_x) \times (\overrightarrow{h} \times s)^{-1}(\mathcal{P}'_G, x) \right) \subset T^*(\mathcal{H}_G),$$

where $\overline{\mathcal{H}}^\lambda_G$ is the closure of $\mathcal{H}^\lambda_G \subset \mathcal{H}_G$, the latter being the locus of modifications of type λ.

(d) The point $(A', (\mathcal{P}_G \rightsquigarrow \mathcal{P}'_G))$ is isolated in its fiber along the map

$$\left(T^*(\mathrm{Bun}_G \times \mathcal{H}^{\mathrm{loc}}_{G,x}) \underset{\mathrm{Bun}_G \times \mathcal{H}^{\mathrm{loc}}_{G,x}, \overleftarrow{h}_x \times \tau_x}{\times} \overline{\mathcal{H}}^\lambda_{G,x}\right) \underset{T^*(\overline{\mathcal{H}}^\lambda_{G,x})}{\times} \left(T^*(\mathrm{Bun}_G) \underset{\mathrm{Bun}_G, \overrightarrow{h}_x}{\times} \overline{\mathcal{H}}^\lambda_{G,x}\right) \to$$
$$\to T^*(\mathrm{Bun}_G \times \mathcal{H}^{\mathrm{loc}}_{G,x}) \underset{\mathrm{Bun}_G \times \mathcal{H}^{\mathrm{loc}}_{G,x}, \overleftarrow{h}_x \times \tau_x}{\times} \overline{\mathcal{H}}^\lambda_{G,x} \to T^*(\mathrm{Bun}_G),$$

where

$$\tau_x : \mathcal{H}_{G,x} \to \mathcal{H}^{\mathrm{loc}}_{G,x}, \quad \mathcal{V}^\lambda_x \in \mathrm{Shv}(\mathcal{H}^{\mathrm{loc}}_{G,x})$$

are the counterparts of $(\tau, \mathcal{V}^\lambda)$ at x.

Arguing as in Sects. 19.2 and 19.3, and using the fact that the map $\tau : \mathcal{H}_G \to \mathcal{H}^{\mathrm{loc}}_G$ is pro-smooth, once we establish properties (a)-(d), the assertion of Theorem 13.4.4 will follow by applying Theorem 19.1.3.

19.4.8. Since the map

$$\overleftarrow{h} \times s : \mathcal{H}_G \to \mathrm{Bun}_G \times X,$$

locally in the smooth topology, it can be isomorphed to the product situation

$$\mathrm{Bun}_G \times X \times \mathrm{Gr}_G \to \mathrm{Bun}_G \times X,$$

so that $\overleftarrow{h}^*(\mathcal{F}) \otimes \tau^*(\mathcal{V}) \in \mathrm{Shv}(\mathcal{H}_G)$ identifies with

$$\mathcal{F} \boxtimes \underline{\mathsf{e}}_X \boxtimes \mathcal{V}' \in \mathrm{Shv}(\mathrm{Bun}_G \times X \times \mathrm{Gr}_G), \quad \mathcal{V}' \in \mathrm{Shv}(\mathrm{Gr}_G),$$

in order to prove point (a) it is sufficient (in fact, equivalent) to show:

(a')

$$(A', (\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G)) \in \left(T^*(\mathrm{Bun}_G) \underset{\mathrm{Bun}_G, \overrightarrow{h}_x}{\times} \overline{\mathcal{H}}^\lambda_{G,x}\right) \cap \mathrm{SingSupp}(\overleftarrow{h}^*_x(\mathcal{F}) \otimes \tau^*_x(\mathcal{V}^\lambda_x)).$$

19.4.9. Recall (see, for example, [GKRV, Formula (B.23)]) that for a point

$$\mathcal{P}^{\mathrm{loc}}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'^{\mathrm{loc}}_G$$

of $\mathcal{H}^{\mathrm{loc}}_{G,x}$, the cotangent space

$$T^*_{\mathcal{P}^{\mathrm{loc}}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'^{\mathrm{loc}}_G}(\mathcal{H}^{\mathrm{loc}}_{G,x})$$

identifies with the set of pairs

$$A^{\mathrm{loc}} \in \Gamma(\mathcal{D}_x, \mathfrak{g}_{\mathcal{P}^{\mathrm{loc}}_G} \otimes \omega), \ A'^{\mathrm{loc}} \in \Gamma(\mathcal{D}_x, \mathfrak{g}_{\mathcal{P}'^{\mathrm{loc}}_G} \otimes \omega), \tag{19.17}$$

such that

$$\alpha(A^{\mathrm{loc}}) = A'^{\mathrm{loc}}$$

as elements of $\Gamma(\overset{\circ}{\mathcal{D}}_x, \mathfrak{g}_{\mathcal{P}'^{\mathrm{loc}}_G} \otimes \omega_X)$.

Furthermore, given

$$A \in T^*_{\mathcal{P}_G}(\mathrm{Bun}_G) \simeq \Gamma(X, \mathfrak{g}_{\mathcal{P}_G} \otimes \omega), \ A' \in T^*_{\mathcal{P}'_G}(\mathrm{Bun}_G) \simeq \Gamma(X, \mathfrak{g}_{\mathcal{P}'_G} \otimes \omega)$$

their images in $T^*_{\mathcal{P}_G \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G}(\mathcal{H}_{G,x})$ differ by the image of an element in $T^*_{\mathcal{P}_G|_{\mathcal{D}_x} \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G|_{\mathcal{D}_x}}(\mathcal{H}^{\mathrm{loc}}_{G,x})$ if and only if

$$\alpha(A|_{X-x}) = A'|_{X-x},$$

and in this case the corresponding element of $T^*_{\mathcal{P}_G|_{\mathcal{D}_x} \overset{\alpha}{\rightsquigarrow} \mathcal{P}'_G|_{\mathcal{D}_x}}(\mathcal{H}^{\mathrm{loc}}_{G,x})$ is given in terms of (19.17) by

$$A^{\mathrm{loc}} := A|_{\mathcal{D}_x}, \ A'^{\mathrm{loc}} = A'|_{\mathcal{D}_x}.$$

19.4.10. Hence, in order to prove (a'), it suffices to show that for a point

$$(\mathcal{P}_G^{\text{loc}} \overset{\alpha}{\rightsquigarrow} \mathcal{P}'^{\text{loc}}_G) \in \mathcal{H}_{G,x}^{\text{loc}}$$

induced by a point

$$(\mathcal{P}_M^{\text{loc}} \overset{\alpha}{\rightsquigarrow} \mathcal{P}'^{\text{loc}}_M) \in \mathcal{H}_{M,x}^{\text{loc}},$$

corresponding to λ (see Sect. 19.4.4), *any* pair

$$(A^{\text{loc}}, A'^{\text{loc}}) \in T^*_{\mathcal{P}_G^{\text{loc}} \overset{\alpha}{\rightsquigarrow} \mathcal{P}'^{\text{loc}}_G}(\mathcal{H}_{G,x}^{\text{loc}})$$

belongs to $\text{SingSupp}(\mathcal{V}_x^\lambda)$.

We identify

$$\mathcal{H}_{G,x}^{\text{loc}} = G[\![t]\!]\backslash G(\!(t)\!)/G[\![t]\!]$$

so that the point $\mathcal{P}_G^{\text{loc}} \overset{\alpha}{\rightsquigarrow} \mathcal{P}'^{\text{loc}}_G$ corresponds to t^λ.

Recall that $\mathcal{V}_x^\lambda$ is the IC-sheaf on the closure of the double coset of

$$t^\lambda \in G[\![t]\!]\backslash G(\!(t)\!)/G[\![t]\!].$$

Hence, the fiber of $\text{SingSupp}(\mathcal{V}_x^\lambda)$ at t^λ is the conormal to this double coset, and hence equals the entire cotangent space at this point.

19.4.11. To prove point (b), we mimic the argument of [GKRV, Sect. B.6.7]. We consider $\mathcal{H}_G^{\text{loc}}$, equipped with its natural crystal structure along X, and the corresponding splitting of the short exact sequence

$$0 \to T_x^*(X) \to T^*_{\mathcal{P}_G^{\text{loc}} \overset{\alpha}{\rightsquigarrow} \mathcal{P}'^{\text{loc}}_G}(\mathcal{H}_G^{\text{loc}}) \to T^*_{\mathcal{P}_G^{\text{loc}} \overset{\alpha}{\rightsquigarrow} \mathcal{P}'^{\text{loc}}_G}(\mathcal{H}_{G,x}^{\text{loc}}) \to 0,$$

i.e.,

$$T^*_{\mathcal{P}_G^{\text{loc}} \overset{\alpha}{\rightsquigarrow} \mathcal{P}'^{\text{loc}}_G}(\mathcal{H}_G^{\text{loc}}) \simeq T_x^*(X) \oplus T^*_{\mathcal{P}_G^{\text{loc}} \overset{\alpha}{\rightsquigarrow} \mathcal{P}'^{\text{loc}}_G}(\mathcal{H}_{G,x}^{\text{loc}}).$$

It suffices to show that, in terms of this identification, for an element

$$(\xi_x, (A^{\text{loc}}, A'^{\text{loc}})) \in T^*_{\mathcal{P}_G^{\text{loc}} \overset{\alpha}{\rightsquigarrow} \mathcal{P}'^{\text{loc}}_G}(\mathcal{H}_G^{\text{loc}})$$

that belongs to $\text{SingSupp}(\mathcal{V}^\lambda)$, we have

$$\xi_x := \langle A_x^{\text{loc}}, \lambda \rangle, \tag{19.18}$$

where A_x^{loc} is the value of A^{loc} at x.

The assertion is local, so we can assume that X is $\mathbb{A}^1$, with coordinate t. This allows us to trivialize the line $T_x^*(X)$. Further, we can assume that $\mathcal{P}_G^{\text{loc}}$ is trivial. Then we can think of

$$A^{\text{loc}} \in \Gamma(\mathcal{D}_x, \mathfrak{g}_{\mathcal{P}_G^{\text{loc}}} \otimes \omega)$$

as an element of $\mathfrak{g}[\![t]\!]$.

By [GKRV, Formula (B.33)], the element ξ_x equals

$$\text{Res}_x(A^{\text{loc}}, \lambda \cdot \frac{dt}{t}),$$

whence (19.18).

19.5. **Proof of point (c) and affine Springer fibers.** To prove point (c), it suffices to show:

(c') The point $(A', (\mathcal{P}_G \rightsquigarrow \mathcal{P}'_G))$ is isolated in the intersection

$$\left(T^*(\text{Bun}_G \times \mathcal{H}_{G,x}^{\text{loc}}) \underset{\text{Bun}_G \times \mathcal{H}_{G,x}^{\text{loc}}, \overleftarrow{h}_x \times \tau_x}{\times} \overline{\mathcal{H}}_{G,x}^\lambda \right) \cap \left(A' \times (\overrightarrow{h}_x)^{-1}(\mathcal{P}'_G) \right) \subset T^*(\mathcal{H}_{G,x}). \tag{19.19}$$

Point (d) in Sect. 19.4.5 is proved similarly.

19.5.1. Consider first the larger intersection

$$\left(T^*(\mathrm{Bun}_G \times \mathcal{H}^{\mathrm{loc}}_{G,x}) \underset{\mathrm{Bun}_G \times \mathcal{H}^{\mathrm{loc}}_{G,x}, \overleftarrow{h}_x \times \tau_x}{\times} \mathcal{H}_{G,x}\right) \cap \left(A' \times (\overrightarrow{h}_x)^{-1}(\mathcal{P}'_G)\right) \subset T^*(\mathcal{H}_{G,x}). \tag{19.20}$$

By Sect. 19.4.9, the scheme in (19.20) is the space of modifications of $\mathcal{P}'_G|_{\mathcal{D}_x} \rightsquigarrow \mathcal{P}^{\mathrm{loc}}_G$, for which the element

$$A'|_{\mathcal{D}_x} \in \Gamma(\mathcal{D}_x, \mathfrak{g}_{\mathcal{P}'_G} \otimes \omega) \subset \Gamma(\overset{\circ}{\mathcal{D}}_x, \mathfrak{g}_{\mathcal{P}'_G} \otimes \omega) \simeq \Gamma(\overset{\circ}{\mathcal{D}}_x, \mathfrak{g}_{\mathcal{P}^{\mathrm{loc}}_G} \otimes \omega)$$

belongs to

$$\Gamma(\mathcal{D}_x, \mathfrak{g}_{\mathcal{P}^{\mathrm{loc}}_G} \otimes \omega) \subset \Gamma(\overset{\circ}{\mathcal{D}}_x, \mathfrak{g}_{\mathcal{P}^{\mathrm{loc}}_G} \otimes \omega).$$

Denote this space by $\mathrm{Spr}_{G,A'}$: it is isomorphic to a parahoric affine Springer fiber over the element A'. Denote the intersection (19.19) by $\mathrm{Spr}^{\leq\lambda}_{G,A'}$.

19.5.2. If we trivialize $\mathcal{P}'^{\mathrm{loc}}_G := \mathcal{P}'_G|_{\mathcal{D}_x}$, we can think of $\mathrm{Spr}_{G,A'}$ as a (closed) subscheme in Gr_G, and we have

$$\mathrm{Spr}^{\leq\lambda}_{G,A'} = \mathrm{Spr}_{G,A'} \cap \overline{\mathrm{Gr}}^{\lambda}_G.$$

We need to show that our particular point

$$\mathcal{P}'_G|_{\mathcal{D}_x} \rightsquigarrow \mathcal{P}_G|_{\mathcal{D}_x} \tag{19.21}$$

is isolated in $\mathrm{Spr}^{\leq\lambda}_{G,A'}$.

19.5.3. By Sect. 19.4.4, the G-bundle $\mathcal{P}'^{\mathrm{loc}}_G$ on $\mathcal{D}_x$ is equipped with a reduction to M, denoted $\mathcal{P}'^{\mathrm{loc}}_M$ and $A'|_{\mathcal{D}_x}$ belongs to

$$\Gamma(\mathcal{D}_x, \mathfrak{m}_{\mathcal{P}'^{\mathrm{loc}}_M} \otimes \omega).$$

So along with $\mathrm{Spr}_{G,A'}$, we can consider its variant for M, to be denoted $\mathrm{Spr}_{M,A'}$. Since

$$\mathrm{Gr}_M \to \mathrm{Gr}_G$$

is a closed embedding, so is the embedding $\mathrm{Spr}_{M,A'} \hookrightarrow \mathrm{Spr}_{G,A'}$.

We claim:

Proposition 19.5.4. *The inclusion* $\mathrm{Spr}_{M,A'} \hookrightarrow \mathrm{Spr}_{G,A'}$ *is an equality.*

Remark 19.5.5. For our purposes, which is proving that (19.21) is isolated in $\mathrm{Spr}^{\leq\lambda}_{G,A'}$, we only need the assertion Proposition 19.5.4 at the level of sets of k-points.

19.5.6. Let us show how Proposition 19.5.4 implies that (19.21) is isolated in $\mathrm{Spr}^{\leq\lambda}_{G,A'}$.

By Proposition 19.5.4, it suffices to show that the point t^λ is isolated in

$$\overline{\mathrm{Gr}}^{\lambda}_G \cap \mathrm{Gr}_M. \tag{19.22}$$

Since t^λ belongs to

$$\mathrm{Gr}^{\lambda}_G \cap \mathrm{Gr}_M, \tag{19.23}$$

it suffices to show that it is isolated in (19.23).

Note, however, that the intersection $\mathrm{Gr}^{\lambda}_G \cap \mathrm{Gr}_M$ is the union of $M[\![t]\!]$-orbits Gr^{μ}_M over M-dominant coweights μ for which there exists $w \in W$ such that

$$\mu = w(\lambda).$$

Note that the point t^λ equals Gr^{λ}_M, because λ is a coweight of $Z(M)$. The assertion follows now from the regularity assumption on λ: the orbit Gr^{λ}_M belongs to a different connected component of Gr_M than the other Gr^{μ}_M with $\mu = w(\lambda)$.

19.6. **Proof of Proposition 19.5.4.**

19.6.1. Fix a Cartan subgroup $T \subset G$, and a Levi subgroup $T \subset M \subset G$. We consider the affine schemes

$$\mathfrak{a} := \mathfrak{g}/\!/ \operatorname{Ad}(G) \simeq \mathfrak{t}/\!/W \text{ and } \mathfrak{a}_M := \mathfrak{m}/\!/ \operatorname{Ad}(M) \simeq \mathfrak{t}/\!/W_M,$$

and a natural map between them.

Let $\overset{\circ}{\mathfrak{t}} \subset \mathfrak{t}$ be the open subset consisting of elements $\mathfrak{t} \in \mathfrak{t}$ for which $\check{\alpha}(\mathfrak{t}) \neq 0$ for all roots $\check{\alpha}$ that are *not* roots of M. Since this subset is W_M-invariant, it corresponds to an open subset

$$\overset{\circ}{\mathfrak{a}}_M \subset \mathfrak{a}_M,$$

so that we have a Cartesian diagram

$$\begin{array}{ccc} \overset{\circ}{\mathfrak{t}} & \longrightarrow & \mathfrak{t} \\ \downarrow & & \downarrow \\ \overset{\circ}{\mathfrak{a}}_M & \longrightarrow & \mathfrak{a}_M. \end{array} \tag{19.24}$$

We will refer to $\overset{\circ}{\mathfrak{a}}_M$ as the (G, M)-*regular* locus of $\mathfrak{a}_M$.

We observe:

Lemma 19.6.2. *For an element $A \in \mathfrak{m}$ the following conditions on are equivalent:*

(i) $Z_{\mathfrak{g}}(A) \subset \mathfrak{m}$*;*

(i') *The adjoint action of A on $\mathfrak{g}/\mathfrak{m}$ is invertible;*

(ii) $Z_{\mathfrak{g}}(A^{\mathrm{ss}}) \subset \mathfrak{m}$*, where A^{ss} is the semi-simple part of A;*

(ii') *The adjoint action of A^{ss} on $\mathfrak{g}/\mathfrak{m}$ is invertible;*

(iii) *The image of A in $\mathfrak{a}_M$ belongs to $\overset{\circ}{\mathfrak{a}}_M$.*

Proof. Clearly (i) $\Leftrightarrow$ (i') and (ii) $\Leftrightarrow$ (ii'). However, it is also clear that (i') $\Leftrightarrow$ (ii'). The equivalence (iii) $\Leftrightarrow$ (ii') is the fact that the diagram (19.24) is Cartesian. □

Let us say that an element $A \in \mathfrak{m}$ is (G, M)-*regular* if it satisfies the equivalent conditions of Lemma 19.6.2. Elements of $\mathfrak{m}$ that are (G, M)-regular form a Zariski-open subset to be denoted $\overset{\circ}{\mathfrak{m}}$. We have a Cartesian diagram

$$\begin{array}{ccc} \overset{\circ}{\mathfrak{m}} & \longrightarrow & \mathfrak{m} \\ \downarrow & & \downarrow \\ \overset{\circ}{\mathfrak{a}}_M & \longrightarrow & \mathfrak{a}_M. \end{array} \tag{19.25}$$

We now claim:

Lemma 19.6.3.

(a) *The open subset $\overset{\circ}{\mathfrak{a}}_M \subset \mathfrak{a}_M$ is the locus of etaleness of the map*

$$\mathfrak{a}_M \to \mathfrak{a}.$$

(b) *The open subset $\overset{\circ}{\mathfrak{m}} \subset \mathfrak{m}$ is the locus of etaleness of the map*

$$\mathfrak{m}/\operatorname{Ad}(M) \to \mathfrak{g}/\operatorname{Ad}(G).$$

(c) *The diagram*

$$\begin{array}{ccc} \overset{\circ}{\mathfrak{m}}/\operatorname{Ad}(M) & \longrightarrow & \mathfrak{g}/\operatorname{Ad}(G) \\ \downarrow & & \downarrow \\ \overset{\circ}{\mathfrak{a}}_M & \longrightarrow & \mathfrak{a} \end{array}$$

is Cartesian.

Proof. Point (a) follows from the third assumption on char(k) in Sect. 13.4.1: an element $\mathfrak{t} \in \mathfrak{t}$ belongs to $\overset{\circ}{\mathfrak{m}}$ if and only if its stabilizer in W is contained in W_M.

Point (b) is a straightforward tangent space calculation.

For point (c), we note that by points (a) and (b), the map

$$\overset{\circ}{\mathfrak{m}}/\operatorname{Ad}(M) \to \mathfrak{a}_M \underset{\mathfrak{a}}{\times} \mathfrak{g}/\operatorname{Ad}(G)$$

is étale. So, it is sufficient to check that it is bijective at the level of field-valued points, which follows from Jordan decomposition and Lemma 19.6.2.

□

19.6.4. For a prestack $\mathcal{Y}$ denote by

$$\mathbf{Maps}(\mathcal{D}_x, \mathcal{Y}) \text{ and } \mathbf{Maps}(\overset{\circ}{\mathcal{D}}_x, \mathcal{Y})$$

the corresponding prestacks of arcs and loops into $\mathcal{Y}$, respectively:

$$\operatorname{Maps}(\operatorname{Spec}(R), \mathbf{Maps}(\mathcal{D}_x, \mathcal{Y})) = \operatorname{Maps}(\operatorname{Spec}(R[\![t]\!]), \mathcal{Y})$$

and

$$\operatorname{Maps}(\operatorname{Spec}(R), \mathbf{Maps}(\overset{\circ}{\mathcal{D}}_x, \mathcal{Y})) = \operatorname{Maps}(\operatorname{Spec}(R(\!(t)\!)), \mathcal{Y}).$$

19.6.5. We now return to Proposition 19.5.4. Choose a trivialization of $\omega|_{\mathcal{D}_x}$. Thus, we are given a map

$$\mathcal{D}_x \to \mathfrak{m}/\operatorname{Ad}(M), \tag{19.26}$$

which is (G, M)-regular; that is, it is in fact a map to $\overset{\circ}{\mathfrak{m}}/\operatorname{Ad}(M)$. The map between the Springer fibers can be written explicitly as

$$\mathbf{Maps}(\mathcal{D}_x, \mathfrak{m}/\operatorname{Ad}(M)) \underset{\mathbf{Maps}(\overset{\circ}{\mathcal{D}}_x, \mathfrak{m}/\operatorname{Ad}(M))}{\times} \{*\} \to \mathbf{Maps}(\mathcal{D}_x, \mathfrak{g}/\operatorname{Ad}(G)) \underset{\mathbf{Maps}(\overset{\circ}{\mathcal{D}}_x, \mathfrak{g}/\operatorname{Ad}(G))}{\times} \{*\};$$

we need to show that the map is an isomorphism. Here the map $\{*\} \to \mathbf{Maps}(\overset{\circ}{\mathcal{D}}_x, \mathfrak{m}/\operatorname{Ad}(M))$ is given by the restriction of (19.26) to $\overset{\circ}{\mathcal{D}}_x$.

19.6.6. Notice that the map

$$\mathbf{Maps}(\mathcal{D}_x, \overset{\circ}{\mathfrak{m}}/\operatorname{Ad}(M)) \underset{\mathbf{Maps}(\overset{\circ}{\mathcal{D}}_x, \overset{\circ}{\mathfrak{m}}/\operatorname{Ad}(M))}{\times} \{*\} \to \mathbf{Maps}(\mathcal{D}_x, \mathfrak{m}/\operatorname{Ad}(M)) \underset{\mathbf{Maps}(\overset{\circ}{\mathcal{D}}_x, \mathfrak{m}/\operatorname{Ad}(M))}{\times} \{*\},$$

is an isomorphism. Indeed, using (19.25), it is obtained by base change from the map

$$\mathbf{Maps}(\mathcal{D}_x, \overset{\circ}{\mathfrak{a}}_M) \underset{\mathbf{Maps}(\overset{\circ}{\mathcal{D}}_x, \overset{\circ}{\mathfrak{a}}_M)}{\times} \{*\} \to \mathbf{Maps}(\mathcal{D}_x, \mathfrak{a}_M) \underset{\mathbf{Maps}(\overset{\circ}{\mathcal{D}}_x, \mathfrak{a}_M)}{\times} \{*\}, \tag{19.27}$$

which is an isomorphism.

19.6.7. It remains to show that the map

$$\mathbf{Maps}(\mathcal{D}_x, \overset{\circ}{\mathfrak{m}}/\operatorname{Ad}(M)) \underset{\mathbf{Maps}(\overset{\circ}{\mathcal{D}}_x, \overset{\circ}{\mathfrak{m}}/\operatorname{Ad}(M))}{\times} \{*\} \to \mathbf{Maps}(\mathcal{D}_x, \mathfrak{g}/\operatorname{Ad}(G)) \underset{\mathbf{Maps}(\overset{\circ}{\mathcal{D}}_x, \mathfrak{g}/\operatorname{Ad}(G))}{\times} \{*\}$$

is an isomorphism. By Lemma 19.6.3(c), the map is obtained by base change from the map

$$\mathbf{Maps}(\mathcal{D}_x, \overset{\circ}{\mathfrak{a}}_M) \underset{\mathbf{Maps}(\overset{\circ}{\mathcal{D}}_x, \overset{\circ}{\mathfrak{a}}_M)}{\times} \{*\} \to \mathbf{Maps}(\mathcal{D}_x, \mathfrak{a}) \underset{\mathbf{Maps}(\overset{\circ}{\mathcal{D}}_x, \mathfrak{a})}{\times} \{*\}. \tag{19.28}$$

Therefore, it suffices to prove that (19.28) is an isomorphism. This is clear at the classical level: as both $\mathfrak{a}$ and $\overset{\circ}{\mathfrak{a}}_M$ are separated, for a classical affine scheme S, the sets of S-points in both the source and the target of (19.28) are singleton sets. (As was mentioned above, Proposition 19.5.4 on the level of sets of k-points actually suffices for our purposes.)

To complete the proof, we notice that (19.28) is formally étale, because $\overset{\circ}{\mathfrak{a}}_M \to \mathfrak{a}$ is étale.

19.7. **Proof of Theorem 13.4.3 for non-holonomic D-modules.** We will deduce the assertion of Theorem 13.4.3 for D-mod$(-)$ from its validity for the subcategory Shv$(-)$ consisting of objects with regular holonomic cohomologies.

The proof is based on considering field extensions of the initial ground field k (cf. the proof of Observation 20.4.4).

19.7.1. By Proposition 9.8.6, it suffices to show that the inclusion

$$\operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G) \simeq \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G) \underset{\operatorname{QCoh}(\operatorname{LocSys}^{\operatorname{restr}}_{\check{G}}(X))}{\otimes} \operatorname{QCoh}(\operatorname{LocSys}^{\operatorname{restr}}_{\check{G}}(X)) \simeq$$

$$\simeq \operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G) \underset{\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}}(X))}{\otimes} \operatorname{QCoh}(\operatorname{LocSys}^{\operatorname{restr}}_{\check{G}}(X)) \hookrightarrow$$

$$\hookrightarrow \operatorname{D-mod}(\operatorname{Bun}_G) \underset{\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}}(X))}{\otimes} \operatorname{QCoh}(\operatorname{LocSys}^{\operatorname{restr}}_{\check{G}}(X))$$

is an equality.

Let S be an affine scheme mapping to $\operatorname{LocSys}^{\operatorname{restr}}_{\check{G}}(X)$. It suffices to show that the inclusion

$$\operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G) \underset{\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}}(X))}{\otimes} \operatorname{QCoh}(S) \hookrightarrow \operatorname{D-mod}(\operatorname{Bun}_G) \underset{\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}}(X))}{\otimes} \operatorname{QCoh}(S) \tag{19.29}$$

is an equality[36].

19.7.2. Let $k \subset k'$ be a field extension. Let X' (resp., S', Bun'_G) be the base change of X (resp., S, Bun_G) from k to k'. Note that for any prestack $\mathcal{Y}$ over k and its base change $\mathcal{Y}'$ to k', we have

$$\operatorname{D-mod}(\mathcal{Y}') \simeq \operatorname{D-mod}(\mathcal{Y}) \underset{\operatorname{Vect}_k}{\otimes} \operatorname{Vect}_{k'} \tag{19.30}$$

and

$$\operatorname{LocSys}_{\check{G}}(X') \simeq \operatorname{LocSys}_{\check{G}}(X) \underset{\operatorname{Spec}(k)}{\times} \operatorname{Spec}(k'). \tag{19.31}$$

For a fixed $\mathcal{N} \subset T^*(\mathcal{Y})$, we have a fully faithful embedding

$$\operatorname{D-mod}_{\mathcal{N}}(\mathcal{Y}) \underset{\operatorname{Vect}_k}{\otimes} \operatorname{Vect}_{k'} \hookrightarrow \operatorname{D-mod}_{\mathcal{N}'}(\mathcal{Y}'), \tag{19.32}$$

but which is no longer an equivalence (indeed, for example, for $\mathcal{N} = \{0\}$, there are more local systems over k' than over k).

[36]Indeed, $\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}}(X))$ is rigid and $\operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$ and $\operatorname{D-mod}(\operatorname{Bun}_G)$ are dualizable, so the operations $\operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G) \underset{\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}}(X))}{\otimes} -$ and $\operatorname{D-mod}(\operatorname{Bun}_G) \underset{\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}}(X))}{\otimes} -$ commute with limits.

From (19.30) and (19.31) we obtain an equivalence

$$\left(\text{D-mod}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}_{\check{G}}(X))}{\otimes} \text{QCoh}(S)\right) \underset{\text{Vect}_k}{\otimes} \text{Vect}_{k'} \simeq \text{D-mod}(\text{Bun}'_G) \underset{\text{QCoh}(\text{LocSys}_{\check{G}}(X'))}{\otimes} \text{QCoh}(S')$$

and from (19.32) a fully faithful embedding

$$\left(\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}_{\check{G}}(X))}{\otimes} \text{QCoh}(S)\right) \underset{\text{Vect}_k}{\otimes} \text{Vect}_{k'} \hookrightarrow \to \text{Shv}_{\text{Nilp}}(\text{Bun}'_G) \underset{\text{QCoh}(\text{LocSys}_{\check{G}}(X'))}{\otimes} \text{QCoh}(S'). \tag{19.33}$$

However, we claim:

Lemma 19.7.3. *The fully faithful embedding* (19.33) *is an equivalence.*

Proof. We will show that the image of the functor (19.33) contains the generators of the target category.

Indeed, let $y_i \in \text{Bun}_G$ be as Sect. 15.2.2. Let y'_i be the corresponding k'-points of Bun'_G. Then the generators of

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}_{\check{G}}(X))}{\otimes} \text{QCoh}(S)$$

are given by $\mathsf{P}^{\text{enh}}_S(\delta_{y_i})$, and the generators of

$$\text{Shv}_{\text{Nilp}}(\text{Bun}'_G) \underset{\text{QCoh}(\text{LocSys}_{\check{G}}(X'))}{\otimes} \text{QCoh}(S')$$

are given by $\mathsf{P}^{\text{enh}}_{S'}(\delta_{y'_i})$. Now these generators are sent to one another by the functor (19.33). □

19.7.4. We are now ready to prove that (19.29) is an equality. Let $\mathcal{F}$ be an object in the right-hand side, which is right-orthogonal to the left-hand side. By Lemma 19.7.3 for any $k \subset k'$, the pullback $\mathcal{F}'$ to Bun'_G will have the same property.

We now clam that for any $\mathcal{F}$ as above, its image in $\text{D-mod}(\text{Bun}_G)$ is right-orthogonal to $\text{Shv}(\text{Bun}_G)$. Indeed, for any $\mathcal{F}_1 \in \text{Shv}(\text{Bun}_G)$, we have

$$\mathcal{H}om_{\text{D-mod}(\text{Bun}_G)}(\mathcal{F}_1, \mathcal{F}) \simeq \mathcal{H}om_{\text{D-mod}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}_{\check{G}}(X))}{\otimes} \text{QCoh}(S)}(\mathsf{P}^{\text{enh}}_S(\mathcal{F}_1), \mathcal{F}),$$

while

$$\mathsf{P}^{\text{enh}}_S(\mathcal{F}_1) \in \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \underset{\text{QCoh}(\text{LocSys}_{\check{G}}(X))}{\otimes} \text{QCoh}(S).$$

Hence, we obtain that for $\mathcal{F}$ as above and any $k \subset k'$, the corresponding object $\mathcal{F}' \in \text{D-mod}(\text{Bun}'_G)$ is right-orthogonal to $\text{Shv}(\text{Bun}'_G)$.

We wish to show that $\mathcal{F} = 0$. It suffices to show that the image of $\mathcal{F}$ in $\text{D-mod}(\text{Bun}_G)$ is zero. This follows from the next assertion:

Lemma 19.7.5. *Let* $\mathcal{F} \in \text{D-mod}(\mathcal{Y})$ *be such that for any* $k \subset k'$*, the corresponding object* $\mathcal{F}' \in \text{D-mod}(\mathcal{Y}')$ *is right-orthogonal to* $\text{Shv}(\mathcal{Y}')$*. Then* $\mathcal{F} = 0$.

Proof. Let $\mathcal{F} \neq 0$. Consider the underlying object $\mathbf{oblv}_{\text{D-mod}}(\mathcal{F}) \in \text{QCoh}(\mathcal{Y})$. Then we can find a geometric point

$$\text{Spec}(k') \xrightarrow{\mathbf{i}_y} \mathcal{Y},$$

so that $\mathbf{i}^*_y(\mathbf{oblv}_{\text{D-mod}}(\mathcal{F})) \neq 0$.

Let $\mathbf{i}_{y'}$ denote the resulting geometric point of $\mathcal{Y}'$. Then $\mathbf{i}^*_{y'}(\mathbf{oblv}_{\text{D-mod}}(\mathcal{F}')) \neq 0$. However, the latter means that

$$\mathcal{H}om_{\text{D-mod}(\mathcal{Y}')}(\delta_{y'}, \mathcal{F}') \neq 0.$$

□

19.8. **Proof of Theorem 17.2.11.** We retain the notations of Sect. 17.2.

19.8.1. We need to show that

$$\mathsf{P}^{\mathrm{enh}}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}(\delta_y) \in \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)^{\text{Hecke-loc.const.}}$$

belongs to $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. This is equivalent to showing that the object

$$\mathsf{P}_{\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)}(\delta_y) \in \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))$$

belongs to

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)) \subset \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)).$$

Furthermore, the latter is equivalent to showing that the object

$$\mathsf{P}_{\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\check{G}}(X)}(\delta_y) \in \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\check{G}}(X))$$

belongs to

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\check{G}}(X)) \subset \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\check{G}}(X)).$$

19.8.2. Since $\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\check{G}}(X)$ is an eventually coconnective affine scheme, its QCoh is generated under colimits by objects of the form

$$\widetilde{\mathbf{i}}_*(\widetilde{\mathsf{e}}'),$$

where

$$\widetilde{\mathbf{i}} : \mathrm{Spec}(\widetilde{\mathsf{e}}') \to \mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\check{G}}(X)$$

and $\widetilde{\mathsf{e}}' \supset \mathsf{e}$ are the residue fields of scheme-theoretic points of $\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\check{G}}(X)$.

In particular, $\mathcal{O}_{\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\check{G}}(X)}$ can be expressed as a colimit of such objects.

Hence, it suffices to show that all

$$\mathsf{P}_{\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\check{G}}(X)}(\delta_y) \otimes \widetilde{\mathbf{i}}_*(\widetilde{\mathsf{e}}') \in \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\check{G}}(X))$$

belong to

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\check{G}}(X)).$$

Note, however, that

$$\mathsf{P}_{\mathrm{LocSys}^{\mathrm{Betti,rigid}_x}_{\check{G}}(X)}(\delta_y) \otimes \widetilde{\mathbf{i}}_*(\widetilde{\mathsf{e}}') \simeq (\mathrm{Id} \otimes \widetilde{\mathbf{i}}_*)(\mathsf{P}_{\mathrm{Spec}(\widetilde{\mathsf{e}}')}(\delta_y)).$$

Hence, it suffices to show that the objects

$$\mathsf{P}_{\mathrm{Spec}(\widetilde{\mathsf{e}}')}(\delta_y) \in \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{Vect}_{\widetilde{\mathsf{e}}'}$$

belong to

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Vect}_{\widetilde{\mathsf{e}}'} \subset \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{Vect}_{\widetilde{\mathsf{e}}'}.$$

19.8.3. Let $\mathsf{e}' \supset \widetilde{\mathsf{e}}'$ be the algebraic closure of $\widetilde{\mathsf{e}}'$. It is easy to see that it is sufficient to show that the objects

$$\mathsf{P}_{\mathrm{Spec}(\mathsf{e}')}(\delta_y) \in \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{Vect}_{\mathsf{e}'}$$

belong to

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{Vect}_{\mathsf{e}'} \subset \mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G) \otimes \mathrm{Vect}_{\mathsf{e}'}.$$

19.8.4. Note, however, that we have canonical identifications

$$\mathrm{Shv}^{\mathsf{e},\mathrm{all}}(\mathrm{Bun}_G)\otimes\mathrm{Vect}_{\mathsf{e}'}\simeq\mathrm{Shv}^{\mathsf{e}',\mathrm{all}}(\mathrm{Bun}_G),\quad \mathrm{Shv}^{\mathsf{e},\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\otimes\mathrm{Vect}_{\mathsf{e}'}\simeq\mathrm{Shv}^{\mathsf{e}',\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \tag{19.34}$$

and

$$\mathrm{LocSys}^{\mathrm{Betti},\mathsf{e}}_{\check{G}}(X)\underset{\mathrm{Spec}(\mathsf{e})}{\times}\mathrm{Spec}(\mathsf{e}')\simeq\mathrm{LocSys}^{\mathrm{Betti},\mathsf{e}'}_{\check{G}}(X), \tag{19.35}$$

where the superscripts e and e' indicate the fields of coefficients of our sheaves.

We can regard

$$\mathbf{i}:\mathrm{Spec}(\mathsf{e}')\to\mathrm{LocSys}^{\mathrm{Betti},\mathsf{e}}_{\check{G}}(X)$$

as an e'-point

$$\mathbf{i}':\mathrm{Spec}(\mathsf{e}')\to\mathrm{LocSys}^{\mathrm{Betti},\mathsf{e}'}_{\check{G}}(X).$$

Under these identifications, the functor

$$\mathsf{P}_{\mathrm{Spec}(\mathsf{e}')}:\mathrm{Shv}^{\mathsf{e},\mathrm{all}}(\mathrm{Bun}_G)\to\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)\otimes\mathrm{Vect}_{\mathsf{e}'}$$

identifies with

$$\mathrm{Shv}^{\mathsf{e},\mathrm{all}}(\mathrm{Bun}_G)\overset{-\otimes\mathsf{e}'}{\longrightarrow}\mathrm{Shv}^{\mathsf{e}',\mathrm{all}}(\mathrm{Bun}_G)\overset{\mathsf{P}_{\mathrm{Spec}(\mathsf{e}')}}{\longrightarrow}\mathrm{Shv}^{\mathsf{e}',\mathrm{all}}(\mathrm{Bun}_G),$$

where the second arrow is Beilinson's projector corresponding to the point $\mathbf{i}'$.

19.8.5. However, $\mathbf{i}'$, being a rational point, factors as

$$\mathrm{Spec}(\mathsf{e}')\to\mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x,\mathsf{e}'}_{\check{G}}(X)\to\mathrm{LocSys}^{\mathrm{Betti},\mathsf{e}'}_{\check{G}}(X).$$

Hence,

$$\mathsf{P}_{\mathrm{Spec}(\mathsf{e}')}(\delta_y)\in\mathrm{Shv}^{\mathsf{e}'}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\subset\mathrm{Shv}^{\mathsf{e}',\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G),$$

see Sect. 14.4.9.

Remark 19.8.6. Note, however, that although we have an equivalence (19.34), the functor

$$\mathrm{Shv}^{\mathsf{e}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)\otimes\mathrm{Vect}_{\mathsf{e}'}\subset\mathrm{Shv}^{\mathsf{e}'}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

is a *proper containment.*

Similarly, although we have an isomorphism (19.35), the map

$$\mathrm{LocSys}^{\mathrm{restr},\mathsf{e}}_{\check{G}}(X)\underset{\mathrm{Spec}(\mathsf{e})}{\times}\mathrm{Spec}(\mathsf{e}')\to\mathrm{LocSys}^{\mathrm{restr},\mathsf{e}'}_{\check{G}}(X),$$

is an embedding of a union of connected components, but *not* an isomorphism.

Part IV: Langlands theory with nilpotent singular support

Let us make a brief overview of the contents of this Part.

In Sect. 20 we state the (categorical) Geometric Langlands conjecture with restricted variation:

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)),$$

and compare it to the de Rham and Betti versions of the GLC. A priori the restricted version follows from these other two. However, we show that the restricted version is actually equivalent to the full de Rham version (assuming Hypothesis 20.4.2).

In Sect. 21 we formulate one of the key points of this paper, the Trace Conjecture. We start by reviewing the *local term* map

$$\mathrm{LT} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q)),$$

where $\mathcal{Y}$ is an algebraic stack defined over $\mathbb{F}_q$, but considered over $\overline{\mathbb{F}}_q$. The Trace Conjecture says that the composition

$$\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \to \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}(\mathrm{Bun}_G)) \overset{\mathrm{LT}}{\to} \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)) =: \mathrm{Autom}$$

is an isomorphism. We then discuss a generalization of the Trace Conjecture that recovers cohomology of shtukas also as traces of functors acting on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$.

In Sect. 22 we make a digression and proof a version of the Trace Conjecture for the category of lisse sheaves on an abelian variety.

In Sect. 23 we explain how the Trace Conjecture allows us to recover V. Lafforgue's spectral decomposition of Autom with respect to (the coarse moduli space of) Langlands parameters.

We start by defining the (prestack) $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ as Frobenius-fixed points on $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$); in Theorem 23.1.4 we prove that $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ is actually an algebraic stack.

We view $(\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)), \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))$ as a pair of a monoidal category with its module category, equipped with endofunctors (both given by Frobenius). In this case we can consider

$$\mathrm{cl}(\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G), (\mathrm{Frob}_{\mathrm{Bun}_G})_*) \in \mathrm{HH}_\bullet(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)), \mathrm{Frob}^*)$$

attached to this data (see [GKRV, Sect. 3.8.1]). We identify

$$\mathrm{HH}_\bullet(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)), \mathrm{Frob}^*) \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$$

(see [GKRV, Example 3.7.3]), and denote the resulting object

$$\mathrm{Drinf} \in \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)).$$

By design (see [GKRV, Theorem 3.8.5]), we have

$$\Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathrm{Drinf}) \simeq \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)),$$

where the right-hand side is naturally acted on by

$$\mathcal{E}xc := \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathcal{O}_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}).$$

Combining with the Trace Conjecture, we obtain an action of $\mathcal{E}xc$ on Autom, i.e., a spectral decomposition of Autom with respect to the coarse moduli space of $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$.

Finally, assuming the Geometric Langlands Conjecture (plus a more elementary Conjecture 23.6.9), we deduce an equivalence

$$\mathrm{Drinf} \simeq \omega_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)},$$

as objects of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$. Combining with the Trace Conjecture, we obtain a conjectural identification

$$\mathrm{Autom} \simeq \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \omega_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}),$$

i.e., a description of the space of (unramified) automorphic functions purely in terms of the stack of Langlands parameters.

In Sect. 24 we prove Theorem 23.1.4. The key tools for the proof are the properties of the map r from Theorem 5.4.2, combined with results from [De] and [LLaf].

20. Geometric Langlands Conjecture with nilpotent singular support

In this section we formulate a version of the Geometric Langlands Conjecture that involves $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. Its main feature is that it makes sense for any sheaf theory from our list.

We then explain the relationship between this version of the Geometric Langlands Conjecture and the de Rham and Betti versions. We will show that both these versions imply the one with nilpotent singular support.

Vice versa, we show (under a certain plausible assumption, see Hypthesis 20.4.2) that the restricted version actually implies the full de Rham version.

20.1. **Digression: coherent singular support.** In this subsection we will show how to adapt the theory of singular support, developed in [AG] for quasi-smooth *schemes*, to the case of quasi-smooth *formal schemes*. We will assume that the reader is familiar with the main tenets of the paper [AG].

20.1.1. Let $\mathcal{Y}$ be a formal affine scheme (see Remark 1.4.7), locally almost of finite type as a prestack.

We shall say that $\mathcal{Y}$ is *quasi-smooth* if for every e-point y of $\mathcal{F}$, the cotangent space $T^*_y(\mathcal{Y})$ is acyclic off degrees 0 and -1.

Equivalently, $\mathcal{Y}$ is quasi-smooth if

$$T^*(\mathcal{Y})|_{{}^{\mathrm{red}}\mathcal{Y}} \in \mathrm{Coh}({}^{\mathrm{red}}\mathcal{Y})$$

can be locally written as a 2-step complex of vector bundles $\mathcal{E}_{-1} \to \mathcal{E}_0$.

20.1.2. We will denote by

$$T(\mathcal{Y})|_{{}^{\mathrm{red}}\mathcal{Y}} \in \mathrm{Coh}({}^{\mathrm{red}}\mathcal{Y})^{\leq 1}$$

the *naive* dual of $T^*(\mathcal{Y})|_{{}^{\mathrm{red}}\mathcal{Y}}$, i.e.,

$$T(\mathcal{Y})|_{{}^{\mathrm{red}}\mathcal{Y}} = \underline{\mathrm{Hom}}(T^*(\mathcal{Y})|_{{}^{\mathrm{red}}\mathcal{Y}}, \mathcal{O}_{{}^{\mathrm{red}}\mathcal{Y}}).$$

We define the (reduced) scheme $\mathrm{Sing}(\mathcal{Y})$ to be the reduced scheme underlying

$$\mathrm{Spec}_{{}^{\mathrm{red}}\mathcal{Y}}(\mathrm{Sym}_{\mathcal{O}_{{}^{\mathrm{red}}\mathcal{Y}}}(T(\mathcal{Y})|_{{}^{\mathrm{red}}\mathcal{Y}}[1])).$$

20.1.3. Let $\mathcal{Y}$ be quasi-smooth. It follows from Theorem 3.1.4 that we can write $\mathcal{Y}$ as a filtered colimit

$$\mathcal{Y} = \underset{i}{\mathrm{colim}}\, Y_n, \tag{20.1}$$

where:

- Y_n are quasi-smooth affine schemes;
- The maps $Y_{n_1} \to Y_{n_2}$ are closed embeddings that induce isomorphisms ${}^{\mathrm{red}}Y_{n_1} \to {}^{\mathrm{red}}Y_{n_2}$;
- For every n, the map $Y_n \to \mathcal{Y}$ is a closed embedding such that the induced map

$$\mathrm{Sing}(\mathcal{Y}) \underset{\mathcal{Y}}{\times} Y_n \to \mathrm{Sing}(Y_n) \tag{20.2}$$

  is a closed embedding.

Indeed, in the notations of Theorem 3.1.4, let

$$\mathcal{Y}_n := \mathcal{Y} \underset{\mathbb{A}^m}{\times} \{0\},$$

where the map $\mathcal{Y} \to \mathbb{A}^m$ is

$$\mathcal{Y} \to \mathrm{Spec}(R) \overset{\{f^n_1,\dots,f^n_m\}}{\longrightarrow} \mathbb{A}^m.$$

Then, on the one hand, $\mathcal{Y}_n$ is a quasi-smooth formal affine scheme (since $\mathcal{Y}$ is such), and

$$\mathrm{Sing}(\mathcal{Y}) \underset{\mathcal{Y}}{\times} \mathcal{Y}_n \to \mathrm{Sing}(\mathcal{Y}_n)$$

is a closed embedding.

On the other hand, $\mathcal{Y}_n$ is actually an affine scheme isomorphic to $\operatorname{Spec}(R_n)$. So we can take $Y_n := \mathcal{Y}_n$.

20.1.4. Recall that for a prestack $\mathcal{Y}$ locally almost of finite type it makes sense to talk about the category $\operatorname{IndCoh}(\mathcal{Y})$, which is a module category over the (symmetric) monoidal category $\operatorname{QCoh}(\mathcal{Y})$.

If $\mathcal{Y}$ is an ind-scheme, we have a well-defined (small) subcategory

$$\operatorname{Coh}(\mathcal{Y}) \subset \operatorname{IndCoh}(\mathcal{Y})^c,$$

so that $\operatorname{IndCoh}(\mathcal{Y})$ identifies with the ind-completion of $\operatorname{Coh}(\mathcal{Y})$, see [GR3, Sect. 2.4.3].

For $\mathcal{Y}$ written as (20.1), we have

$$\operatorname{IndCoh}(\mathcal{Y}) \simeq \lim_n \operatorname{IndCoh}(Y_n), \tag{20.3}$$

where the limit is formed using the functors $i^!_{n_1,n_2}$ for $Y_{n_1} \overset{i_{n_1,n_2}}{\longrightarrow} Y_{n_2}$, and also

$$\operatorname{IndCoh}(\mathcal{Y}) \simeq \operatorname*{colim}_n \operatorname{IndCoh}(Y_n),$$

where the colimit is formed inside DGCat using the functors $(i_{n_1,n_2})^{\operatorname{IndCoh}}_*$, see [GR3, Sect. 2.4.2].

In terms of this identification, we have

$$\operatorname{Coh}(\mathcal{Y}) \simeq \operatorname*{colim}_n \operatorname{Coh}(Y_n), \tag{20.4}$$

where the colimit is formed using the *-pullback functors, but inside the ∞-category of *not-necessarily cocomplete* DG categories.

20.1.5. The theory of singular support for quasi-smooth *schemes* developed in [AG] applies "as-is" in the case of formal affine schemes that are quasi-smooth.

In particular, to an object

$$\mathcal{M} \in \operatorname{Coh}(\mathcal{Y}),$$

one can attach its singular support $\operatorname{SingSupp}(\mathcal{M})$, which is a conical Zariski-closed subset in $\operatorname{Sing}(\mathcal{Y})$.

Explicitly, for a given $\mathcal{M} \in \operatorname{Coh}(\mathcal{Y})$, the fiber of $\operatorname{SingSupp}(\mathcal{M})$ over a given $\operatorname{Spec}(\mathsf{e}) \overset{\mathbf{i}_y}{\longrightarrow} \mathcal{Y}$ is the support of

$$\underset{n}{\oplus} H^n(\mathbf{i}_y^!(\mathcal{M})),$$

viewed as a module over the algebra

$$\operatorname{Sym}^n(H^1(T_y(\mathcal{Y}))),$$

where the action is defined as in [AG, Sect. 6.1.1].

20.1.6. For a given conical Zariski-closed subset $\mathcal{N} \subset \operatorname{Sing}(\mathcal{Y})$, we can talk about a full subcategory

$$\operatorname{Coh}_{\mathcal{N}}(\mathcal{Y}) \subset \operatorname{Coh}(\mathcal{Y}),$$

consisting of objects whose singular support is contained in $\mathcal{N}$. We denote by $\operatorname{IndCoh}_{\mathcal{N}}(\mathcal{Y})$ its ind-completion, which is a full subcategory in $\operatorname{IndCoh}(\mathcal{Y})$.

One can describe the category $\operatorname{IndCoh}_{\mathcal{N}}(\mathcal{Y})$ in terms of (20.3) as follows: Given $\mathcal{N} \subset \operatorname{Sing}(\mathcal{Y})$, let $\mathcal{N}_n \subset \operatorname{Sing}(Y_n)$ be the image of

$$\mathcal{N} \underset{\mathcal{Y}}{\times} Y_n \subset \operatorname{Sing}(\mathcal{Y}) \underset{\mathcal{Y}}{\times} Y_n$$

under the map (20.2). Then for $Y_{n_1} \to Y_{n_2}$, the pullback functor

$$\operatorname{IndCoh}(Y_{n_2}) \to \operatorname{IndCoh}(Y_{n_1})$$

sends

$$\operatorname{IndCoh}_{\mathcal{N}_{n_2}}(Y_{n_2}) \to \operatorname{IndCoh}_{\mathcal{N}_{n_1}}(Y_{n_1})$$

(see [AG, Proposition 7.1.3.(a)]) and we have

$$\operatorname{IndCoh}_{\mathcal{N}}(\mathcal{Y}) \simeq \lim_n \operatorname{IndCoh}_{\mathcal{N}_n}(Y_n), \tag{20.5}$$

as subcategories in the two sides of (20.3).

20.1.7. Finally, one checks, using [AG, Proposition 7.1.3.(b)] and base change, that for a pair of indices n_1, n_2 the composite functor

$$\text{IndCoh}(Y_{n_1}) \overset{\text{*-pushforward}}{\longrightarrow} \text{IndCoh}(\mathcal{Y}) \overset{\text{!-pullback}}{\longrightarrow} \text{IndCoh}(Y_{n_2})$$

sends

$$\text{IndCoh}_{\mathcal{N}_{n_1}}(Y_{n_1}) \to \text{IndCoh}_{\mathcal{N}_{n_2}}(Y_{n_2}).$$

This implies that the *-pushforward functors $\text{IndCoh}(Y_i) \to \text{IndCoh}(\mathcal{Y})$ send

$$\text{IndCoh}_{\mathcal{N}_{n_1}}(Y_{n_1}) \to \text{IndCoh}_{\mathcal{N}}(\mathcal{Y}). \tag{20.6}$$

This shows that the category $\text{IndCoh}_{\mathcal{N}}(\mathcal{Y})$ is compactly generated, namely, by the essential images of the functors (20.6).

Remark 20.1.8. Note, however, that the individual functors

$$(i_{n_1,n_2})^{\text{IndCoh}}_* : \text{IndCoh}(Y_{n_1}) \to \text{IndCoh}(Y_{n_2})$$

do *not* necessarily send $\text{IndCoh}_{\mathcal{N}_{n_1}}(Y_{n_1})$ to $\text{IndCoh}_{\mathcal{N}_{n_1}}(Y_{n_2})$.

20.1.9. The action of $\text{QCoh}(\mathcal{Y})$ on $\text{IndCoh}(\mathcal{Y})$ preserves the subcategory $\text{IndCoh}_{\mathcal{N}}(\mathcal{Y})$.

It follows formally from (20.5) the action on the object $\omega_{\mathcal{Y}} \in \text{IndCoh}(\mathcal{Y})$ defines an equivalence from $\text{QCoh}(\mathcal{Y})$ onto the full subcategory

$$\text{IndCoh}_{\{0\}}(\mathcal{Y}) \subset \text{IndCoh}(\mathcal{Y}),$$

where $\{0\} \subset \text{Sing}(\mathcal{Y})$ is the zero-section.

20.1.10. Suppose for a moment that we are given a quasi-smooth algebraic stack $\mathcal{Y}'$ and a map $\mathcal{Y} \to \mathcal{Y}'$, which is an ind-closed embedding and a formal isomorphism (see Remark 4.2.5).

Let $\mathcal{N}' \subset \text{Sing}(\mathcal{Y}')$, and set $\mathcal{N} := \mathcal{N}'|_{\mathcal{Y}}$. Then it is easy to see that the full subcategories

$$\text{IndCoh}_{\mathcal{N}}(\mathcal{Y}) \subset \text{IndCoh}(\mathcal{Y})$$

and

$$\text{IndCoh}_{\mathcal{N}'}(\mathcal{Y}') \cap \text{IndCoh}(\mathcal{Y}')_{\mathcal{Y}} \subset \text{IndCoh}(\mathcal{Y}')_{\mathcal{Y}}$$

coincide under the identification

$$\text{IndCoh}(\mathcal{Y}) \simeq \text{IndCoh}(\mathcal{Y}')_{\mathcal{Y}}.$$

20.2. Geometric Langlands Conjecture for $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$.

20.2.1. Note that the identification of cotangent spaces of $\text{LocSys}^{\text{restr}}_{\check{G}}(X)$ given by Proposition 2.2.2(b) and (1.19) implies that for a e-point of σ of $\text{LocSys}^{\text{restr}}_{\check{G}}(X)$, we have

$$T^*_\sigma(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \simeq \text{C}^\cdot_c(X, \check{\mathfrak{g}}^\vee_\sigma \overset{*}{\otimes} \omega_X)[-1] \simeq \text{C}^\cdot(X, \check{\mathfrak{g}}^\vee_\sigma)[1], \tag{20.7}$$

where the last isomorphism uses the fact that X is a proper smooth curve.

In the above formula $\check{\mathfrak{g}}^\vee_\sigma$ is the local system on X associated to σ and $\check{\mathfrak{g}}^\vee \in \text{Rep}(\check{G})$.

20.2.2. In particular, we obtain that the cotangent spaces to $\text{LocSys}^{\text{restr}}_{\check{G}}(X)$ live in cohomological degrees $[-1, 1]$.

This implies that the cotangent spaces to $\text{LocSys}^{\text{restr,rigid}_x}_{\check{G}}(X)$ live in cohomological degrees $[-1, 0]$. I.e., $\text{LocSys}^{\text{restr,rigid}_x}_{\check{G}}(X)$ is a union of quasi-smooth formal affine schemes.

This allows us to talk about

$$\text{Sing}(\text{LocSys}^{\text{restr,rigid}_x}_{\check{G}}(X)),$$

which is a (reduced) algebraic stack over ${}^{\text{red}}\text{LocSys}^{\text{restr,rigid}_x}_{\check{G}}(X)$.

Thus, to a given conical Zariski-closed $\mathcal{N}' \subset \text{Sing}(\text{LocSys}^{\text{restr,rigid}_x}_{\check{G}}(X))$ we can attach a full subcategory

$$\text{IndCoh}_{\mathcal{N}'}(\text{LocSys}^{\text{restr,rigid}_x}_{\check{G}}(X)) \subset \text{IndCoh}(\text{LocSys}^{\text{restr,rigid}_x}_{\check{G}}(X)).$$

20.2.3. The $\check{G}$-action on $\operatorname{LocSys}_{\check{G}}^{\operatorname{restr},\operatorname{rigid}_x}(X)$ naturally extends to $\operatorname{Sing}(\operatorname{LocSys}_{\check{G}}^{\operatorname{restr},\operatorname{rigid}_x}(X))$.

We define

$$\operatorname{Sing}(\operatorname{LocSys}_{\check{G}}^{\operatorname{restr}}(X)) := \operatorname{Sing}(\operatorname{LocSys}_{\check{G}}^{\operatorname{restr},\operatorname{rigid}_x}(X))/\check{G},$$

which is a (reduced) algebraic stack over ${}^{\operatorname{red}}\operatorname{LocSys}_{\check{G}}^{\operatorname{restr}}(X)$.

Given a conical Zariski-closed

$$\mathcal{N} \subset \operatorname{Sing}(\operatorname{LocSys}_{\check{G}}^{\operatorname{restr}}(X)),$$

let $\mathcal{N}'$ be its preimage in $\operatorname{Sing}(\operatorname{LocSys}_{\check{G}}^{\operatorname{restr},\operatorname{rigid}_x}(X))$. Define

$$\operatorname{IndCoh}_{\mathcal{N}}(\operatorname{LocSys}_{\check{G}}^{\operatorname{restr}}(X))$$

as the full subcategory of $\operatorname{IndCoh}(\operatorname{LocSys}_{\check{G}}^{\operatorname{restr}}(X))$ consisting of objects whose *- (or, equivalently, -!) pullback along the (smooth) projection

$$\operatorname{LocSys}_{\check{G}}^{\operatorname{restr}}(X) \simeq \operatorname{LocSys}_{\check{G}}^{\operatorname{restr},\operatorname{rigid}_x}(X)/\check{G}$$

belongs to the full subcategory $\operatorname{IndCoh}_{\mathcal{N}'}(\operatorname{LocSys}_{\check{G}}^{\operatorname{restr},\operatorname{rigid}_x}(X))$ of $\operatorname{IndCoh}(\operatorname{LocSys}_{\check{G}}^{\operatorname{restr},\operatorname{rigid}_x}(X))$.

20.2.4. Note that the identification of cotangent spaces of $\operatorname{LocSys}_{\check{G}}^{\operatorname{restr}}(X)$ given by (20.7) allows us to identify e-points of

$$\operatorname{Arth}_{\check{G}}(X) := \operatorname{Sing}(\operatorname{LocSys}_{\check{G}}^{\operatorname{restr}}(X))$$

with pairs

$$(\sigma, A),$$

where:

- σ is a $\check{G}$-point of $\operatorname{LocSys}_{\check{G}}^{\operatorname{restr}}(X)$;
- A is an element in $H^0(X, \check{\mathfrak{g}}_\sigma^\vee)$.

20.2.5. Let

$$\operatorname{Nilp} \subset \operatorname{Arth}_{\check{G}}(X)$$

be the closed subset whose e-points consist of pairs (σ, A) for which A is nilpotent. Thus, we can consider the fullcategory

$$\operatorname{IndCoh}_{\operatorname{Nilp}}(\operatorname{LocSys}_{\check{G}}^{\operatorname{restr}}(X)) \subset \operatorname{IndCoh}(\operatorname{LocSys}_{\check{G}}^{\operatorname{restr}}(X)).$$

20.2.6. We propose the following "restricted" version of the Geometric Langlands Conjecture:

Main Conjecture 20.2.7. *There exists a canonical equivalence*

$$\operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G) \simeq \operatorname{IndCoh}_{\operatorname{Nilp}}(\operatorname{LocSys}_{\check{G}}^{\operatorname{restr}}(X)),$$

compatible with the action of $\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}}^{\operatorname{restr}}(X))$ *on both sides.*

20.2.8. *Example.* Let X have genus zero. Then the inclusion

$$\operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G) \subset \operatorname{Shv}(\operatorname{Bun}_G),$$

is an equality, and $\operatorname{LocSys}_{\check{G}}^{\operatorname{restr}}(X)$ is actually an algebraic stack isomorphic to

$$(\operatorname{pt} \underset{\check{G}}{\times} \operatorname{pt})/\check{G}.$$

In this case, the assertion of Conjecture 20.2.7 is known: it follows from the (derived) geometric Satake.

20.2.9. *Example.* Let us see what Conjecture 20.2.7 says for $G = \mathbb{G}_m$. Note that in this case

$$\mathrm{Nilp} \subset \mathrm{Arth}_{\check{G}}(X)$$

is the 0-section, so

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) = \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_m}(X)).$$

Given a $\mathbb{G}_m$-local system σ, let $\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_m}(X)_\sigma$ be the corresponding component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_m}(X)$. Let

$$\mathrm{QLisse}(\mathrm{Pic})_\sigma \subset \mathrm{QLisse}(\mathrm{Pic})$$

be the corresponding direct factor of QLisse(Pic), see Sect. 13.3.6.

Thus, Conjecture 20.2.7 says that we have an equivalence

$$\mathrm{QLisse}(\mathrm{Pic})_\sigma \simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_m}(X)_\sigma). \tag{20.8}$$

Let us show how establish the isomorphism (20.8) directcly.

Up to translation by σ on $\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_m}(X)$ (resp., tensor product by E_σ on Pic), we can assume that σ is trivial, so E_σ is the constant sheaf $\underline{\mathsf{e}}_{\mathrm{Pic}}$. The corresponding equivalence (20.8) is then the following statement:

Pick a point $x \in X$. Write

$$\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_m}(X)_\sigma \simeq \mathrm{pt}/\mathbb{G}_m \times \mathrm{Tot}(H^1(X, \underline{\mathsf{e}}_X))^\wedge_{\{0\}} \times \mathrm{Tot}(\mathsf{e}[-1]),$$

see Sect. 1.5.1, and

$$\mathrm{Pic} \simeq \mathbb{Z} \times \mathrm{Jac}(X) \times \mathrm{pt}/\mathbb{G}_m.$$

So

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathbb{G}_m}(X)_\sigma) \simeq \mathrm{Vect}^{\mathbb{Z}}_{\mathsf{e}} \otimes \big(\mathrm{Sym}(H^1(X, \underline{\mathsf{e}}_X)[-1])\text{-mod}\big) \otimes (\mathsf{e}[\xi]\text{-mod}), \quad \deg(\xi) = -1,$$

and

$$\mathrm{QLisse}(\mathrm{Pic})_\sigma \simeq \mathrm{Vect}^{\mathbb{Z}}_{\mathsf{e}} \otimes (\mathrm{Shv}(\mathrm{Jac}(X))_0) \otimes (\mathrm{Shv}(\mathrm{pt}/\mathbb{G}_m)),$$

where $\mathrm{Jac}(X)$ is the Jacobian *variety* of X, and $\mathrm{Shv}(\mathrm{Jac}(X))_0 \subset \mathrm{Shv}(\mathrm{Jac}(X))$ is the full subcategory generated by the constant sheaf.

Now the result follows from the canonical identifications

$$\mathrm{Shv}(\mathrm{pt}/\mathbb{G}_m) \simeq \mathrm{C}_\cdot(\mathbb{G}_m)\text{-mod} \simeq \mathsf{e}[\xi]\text{-mod}, \quad \deg(\xi) = -1,$$

and

$$\mathrm{Shv}(\mathrm{Jac}(X))_0 \simeq \mathcal{E}nd(\underline{\mathsf{e}}_{\mathrm{Jac}(X)})\text{-mod} = \mathrm{Sym}(H^1(X, \underline{\mathsf{e}}_X)[-1])\text{-mod},$$

see Sect. 22.3.2 for the latter isomorphism.

20.3. **Comparison to other forms of the Geometric Langlands Conjecture.**

20.3.1. Let us specialize to the de Rham context. Recall that in this case, we have a version of the geometric Langlands conjecture from [AG, Conjecture 11.2.2], which predicts the existence of a canonical equivalence

$$\mathrm{D\text{-}mod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X)), \tag{20.9}$$

compatible with the actions of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))$ on both sides.

We note that Conjecture 20.2.7 (in the de Rham context) is a formal corollary of this statement. Namely, tensoring both sides of (20.9) with $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ over $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))$, we obtain:

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_G) & \longrightarrow & \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \\ \downarrow \sim & & \downarrow \sim \\ \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X)) & \longrightarrow & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \end{array}$$

where the top horizontal arrow comes from Proposition 13.5.3, and the bottom horizontal arrow is an equivalence by Sect. 20.1.10.

20.3.2. Let us now specialize to the Betti context. In this case, we have a version of the geometric Langlands conjecture, proposed in [BN, Conjecture 1.5], which says that there is an equivalence

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}(X)), \tag{20.10}$$

compatible with the actions of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$ on both sides.

We note that Conjecture 20.2.7 (in the Betti context) is a formal corollary of this statement. Namely, tensoring both sides of (20.10) with $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ over $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))$, we obtain:

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))}{\otimes} \mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) & \longrightarrow & \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \\ \downarrow \sim & & \downarrow \sim \\ \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X))}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X)) & \longrightarrow & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \end{array}$$

where the top horizontal arrow comes from Theorem 17.3.6, and the bottom horizontal arrow is an equivalence by Sect. 20.1.10.

20.4. **A converse implication.** Above we have seen that the full de Rham version of the Geometric Langlands Conjecture implies the restricted version. Here we will show that the converse implication also takes place, under a plausible hypothesis about the de Rham version.

20.4.1. We place ourselves into the de Rham context of the Geometric Langlands Conjecture. Let us assume the following:

Hypothesis 20.4.2. *There exists a functor*

$$\mathbb{L} : \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X)) \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

that preserves compactness and is compatible with the actions of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))$ *on both sides.*

This hypothesis would be a theorem if one accepted Quasi-Theorems 6.7.2 and 9.5.3 from [Ga7].

20.4.3. We now claim:

Observation 20.4.4. *Assume that the functor* $\mathbb{L}$ *from Hypothesis 20.4.2 induces an equivalence*

$$\begin{aligned} &\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \simeq \\ &\simeq \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X)) \overset{\mathrm{Id}\otimes\mathbb{L}}{\to} \\ &\to \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)) \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \simeq \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G). \end{aligned} \tag{20.11}$$

Then the functor $\mathbb{L}$ *itself is an equivalence.*

The rest of this subsection is devoted to the proof this Observation.

20.4.5. Since the functor $\mathbb{L}$ preserves compactness, it admits a continuous right adjoint. Since $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))$ is rigid, this right adjoint is compatible with the action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))$.

Consider the adjunction maps

$$\mathrm{Id} \to \mathbb{L}^R \circ \mathbb{L} \text{ and } \mathbb{L} \circ \mathbb{L}^R \to \mathrm{Id}\,. \tag{20.12}$$

We wish to show that they are isomorphisms.

We have the following general assertion:

Lemma 20.4.6. *Let $\mathbf{C}$ be a category acted on by $\mathrm{QCoh}(\mathcal{Y})$, where $\mathcal{Y}$ is a quasi-compact eventually coconnective algebraic stack almost of finite type with affine diagonal. Then an object $\mathbf{c} \in \mathbf{C}$ is zero if and only if for every geometric point*

$$\mathbf{i} : \mathrm{Spec}(k') \to \mathcal{Y},$$

the image of $\mathbf{c}$ under

$$\mathbf{C} \simeq \mathrm{QCoh}(\mathcal{Y}) \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C} \overset{\mathbf{i}^* \otimes \mathrm{Id}}{\longrightarrow} \mathrm{Vect}_{k'} \underset{\mathrm{QCoh}(\mathcal{Y})}{\otimes} \mathbf{C}$$

vanishes.

The proof of the lemma is given below. Let us accept it temporarily.

20.4.7. Applying the lemma, we obtain that in order to prove that the maps (20.12) are isomorphisms, it suffices to show that the functor

(20.13)
$$\mathrm{Vect}_{k'} \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X)) \overset{\mathrm{Id} \otimes \mathbb{L}}{\longrightarrow} \mathrm{Vect}_{k'} \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

is an equivalence for all

$$\mathbf{i} : \mathrm{Spec}(k') \to \mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X).$$

With no restriction of generality we can assume that k' is algebraically closed.

Let X', G' denote the base change of X, G along $k \rightsquigarrow k'$. Let Bun'_G denote the corresponding algebraic stack over k'. We have

$$\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}'}(X') \simeq \mathrm{Spec}(k') \underset{\mathrm{Spec}(k)}{\times} \mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X) \text{ and } \mathrm{Bun}'_G \simeq \mathrm{Spec}(k') \underset{\mathrm{Spec}(k)}{\times} \mathrm{Bun}_G,$$

and hence

$$\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}'}(X')) \simeq \mathrm{Vect}_{k'} \underset{\mathrm{Vect}_k}{\otimes} \mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X)),$$

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}'}(X')) \simeq \mathrm{Vect}_{k'} \underset{\mathrm{Vect}_k}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}}(X))$$

and

$$\mathrm{D\text{-}mod}(\mathrm{Bun}'_G) \simeq \mathrm{Vect}_{k'} \underset{\mathrm{Vect}_k}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}_G).$$

Hence, we can rewrite the map in (20.13) as

$$\mathrm{Vect}_{k'} \underset{\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{dR}}_{\check{G}'}(X'))}{\otimes} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}'}(X')) \to \mathrm{Vect}_{k'} \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}'}(X'))}{\otimes} \mathrm{D\text{-}mod}(\mathrm{Bun}'_G).$$

Thus, we have reduced the verification of the isomorphism (20.13) to the case when $k' = k$.

20.4.8. Note now that (for $k' = k$), the map $\mathbf{i} : \operatorname{Spec}(k) \to \operatorname{LocSys}^{\mathrm{dR}}_{\check{G}}(X)$ factors as

$$\operatorname{Spec}(k) \to \operatorname{LocSys}^{\mathrm{restr}}_{\check{G}}(X) \to \operatorname{LocSys}^{\mathrm{dR}}_{\check{G}}(X).$$

Hence, the map (20.13) is obtained by

$$\operatorname{Vect}_k \underset{\operatorname{QCoh}(\operatorname{LocSys}^{\mathrm{restr}}_{\check{G}}(X))}{\otimes} -$$

from (20.11), and hence is an equivalence.

□[Observation 20.4.4]

Remark 20.4.9. Note that whereas

$$\operatorname{LocSys}_{\check{G}'}(X') \simeq \operatorname{Spec}(k') \underset{\operatorname{Spec}(k)}{\times} \operatorname{LocSys}_{\check{G}}(X),$$

the same *no longer* holds for $\operatorname{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$. Similarly, while

$$\operatorname{D-mod}(\operatorname{Bun}'_G) \simeq \operatorname{Vect}_{k'} \underset{\operatorname{Vect}_k}{\otimes} \operatorname{D-mod}(\operatorname{Bun}_G),$$

the same is no longer true for $\operatorname{Shv}_{\mathrm{Nilp}}(\operatorname{Bun}_G)$.

Remark 20.4.10. A counterpart of Observation 20.4.4 would apply also in the Betti version, using the identifications (19.34) and (19.35) (cf. Sect. 19.8). We are just less confident of the status of the analog of Hypothesis 20.4.2 in this case.

20.4.11. *Proof of Lemma 20.4.6.* First, by [Ga2, Theorem 2.2.6], $\mathcal{Y}$ is 1-affine[37], hence we can replace $\mathcal{Y}$ by an affine scheme $S = \operatorname{Spec}(A)$.

Since $\mathcal{Y}$ was assumed eventually coconnective, S has the same property. Hence, $\operatorname{QCoh}(S)$ is generated under colimits by objects of the form $\widetilde{\mathbf{i}}_*(\widetilde{k}')$, for

$$\widetilde{\mathbf{i}} : \operatorname{Spec}(\widetilde{k}') \to S,$$

$\widetilde{k}'$ are residue fields of scheme-theoretic points of S, see Lemma 9.2.8. In particular $\mathcal{O}_S$ can be expressed as a colimit of objects of this form.

Hence, $\mathbf{c}$ can be expressed as a colimit of objects of the form

$$\widetilde{\mathbf{i}}_*(\widetilde{k}') \otimes \mathbf{c} \simeq (\widetilde{\mathbf{i}}_* \otimes \operatorname{Id}) \circ (\widetilde{\mathbf{i}}^* \otimes \operatorname{Id})(\mathbf{c}).$$

Hence, if all $(\widetilde{\mathbf{i}}^* \otimes \operatorname{Id})(\mathbf{c})$ vanish, then $\mathbf{c}$ vanishes.

Let k' be the algebraic closure of $\widetilde{k}'$. It is easy to see that for any $\operatorname{Vect}_{\widetilde{k}'}$-linear category $\widetilde{\mathbf{C}}$, the pullback functor

$$\widetilde{\mathbf{C}} \to \operatorname{Vect}_{k'} \underset{\operatorname{Vect}_{\widetilde{k}'}}{\otimes} \widetilde{\mathbf{C}}$$

is conservative.

Hence, for $\mathbf{i}$ equal to

$$\operatorname{Spec}(k') \to \operatorname{Spec}(\widetilde{k}') \overset{\widetilde{\mathbf{u}}}{\to} S,$$

if $(\mathbf{i}^* \otimes \operatorname{Id})(\mathbf{c})$ vanishes, then so does $(\widetilde{\mathbf{i}}^* \otimes \operatorname{Id})(\mathbf{c})$.

□[Lemma 20.4.6]

[37]For our applications, we only need in the case when $\mathcal{Y}$ is of the form S/H, where S is an affine scheme and H is an algebraic group, in which case the assertion of [Ga2, Theorem 2.2.6] easily follows from the case of $\operatorname{pt}/\mathsf{H}$.

21. The trace conjecture

Throughout this section we will be working with schemes/algebraic stacks of finite type over $\overline{\mathbb{F}}_q$, that are defined over $\mathbb{F}_q$, so that they carry the geometric Frobenius endomorphism.

Our sheaf-theoretic context will (by necessity) be that of ℓ-adic sheaves, so $\mathsf{e} = \overline{\mathbb{Q}}_\ell$.

This section contains what is the main point of this paper. We propose a conjecture that expresses the space of automorphic functions as the categorical trace of Frobenius acting on the category $\operatorname{Shv}_{\operatorname{Nilp}}(\operatorname{Bun}_G)$.

21.1. **The categorical trace of Frobenius.**

21.1.1. Let $\mathcal{Y}$ be an algebraic stack. While discussing general algebraic stacks in this and the next subsection, we will assume that $\mathcal{Y}$ is locally a quotient (of a scheme by an algebraic group); this is an assumption under which the results of [GaVa] are established.

We consider the Frobenius endomorphism $\operatorname{Frob}_\mathcal{Y}$ of $\mathcal{Y}$. One word of warning is that when $\mathcal{Y}$ is an Artin stack, $\operatorname{Frob}_\mathcal{Y}$ is not necessarily schematic. However, it is surjective and radicial, in the sense that it becomes an isomorphism after we apply sheafificaton in the topology generated by surjective radicial maps.

Hence, the action of $\operatorname{Frob}_\mathcal{Y}$ on the category $\operatorname{Shv}(\mathcal{Y})$ has properties of a surjective radicial map. In particular, the functor

$$(\operatorname{Frob}_\mathcal{Y})^* : \operatorname{Shv}(\mathcal{Y}) \to \operatorname{Shv}(\mathcal{Y})$$

is an equivalence (and hence its right adjoint $(\operatorname{Frob}_\mathcal{Y})_*$ is also an equivalence).

Furthermore, the natural transformation

$$(\operatorname{Frob}_\mathcal{Y})_! \to (\operatorname{Frob}_\mathcal{Y})_* \tag{21.1}$$

(which is a priori well-defined due to the fact that $\operatorname{Frob}_\mathcal{Y}$ is separated) is an isomorphism, and both functors are equivalences.

From here it follows that left adjoint of $(\operatorname{Frob}_\mathcal{Y})_*$ is isomorophic to the right adjoint of $(\operatorname{Frob}_\mathcal{Y})_!$, i.e.,

$$(\operatorname{Frob}_\mathcal{Y})^* \simeq (\operatorname{Frob}_\mathcal{Y})^!.$$

21.1.2. Assume first that $\mathcal{Y}$ is quasi-compact.

The category $\operatorname{Shv}(\mathcal{Y})$ is compactly generated, and hence dualizable. Hence, we can consider the categorical trace of $(\operatorname{Frob}_\mathcal{Y})_*$:

$$\operatorname{Tr}((\operatorname{Frob}_\mathcal{Y})_*, \operatorname{Shv}(\mathcal{Y})) \in \operatorname{Vect}_\mathsf{e}.$$

To $\mathcal{F} \in \operatorname{Shv}(\mathcal{Y})^c$ equipped with a map

$$\mathcal{F} \overset{\alpha}{\to} (\operatorname{Frob}_\mathcal{Y})_*(\mathcal{F}), \tag{21.2}$$

we can attach its class

$$\operatorname{cl}(\mathcal{F}, \alpha) \in \operatorname{Tr}((\operatorname{Frob}_\mathcal{Y})_*, \operatorname{Shv}(\mathcal{Y})),$$

see [GKRV, Sect. 3.4.3].

We will refer to the data of α as a *lax Weil structure* on $\mathcal{F}$, and to the pair $(\mathcal{F}, \alpha)$ as a *lax Weil sheaf* on $\mathcal{Y}$.

21.1.3. We claim that there is a canonically defined map, called the *Local Term*,

$$\operatorname{LT} : \operatorname{Tr}((\operatorname{Frob}_\mathcal{Y})_*, \operatorname{Shv}(\mathcal{Y})) \to \operatorname{Funct}(\mathcal{Y}(\mathbb{F}_q)), \tag{21.3}$$

where $\operatorname{Funct}(-)$ stands for the (classical) vector space of e-valued functions on the set of isomorphism classes of a given groupoid.

In fact, there are two such maps, denoted $\operatorname{LT}^{\text{naive}}$ and $\operatorname{LT}^{\text{true}}$.

21.1.4. The map $\mathrm{LT}^{\mathrm{naive}}$ is designed so that for a lax Weil sheaf $(\mathcal{F}, \alpha)$, we have

$$\mathrm{LT}^{\mathrm{naive}}(\mathrm{cl}(\mathcal{F}, \alpha)) = \mathrm{funct}(\mathcal{F}, \alpha),$$

where $\mathrm{funct}(\mathcal{F}, \alpha)$ is the usual function on $\mathcal{Y}(\mathbb{F}_q)$ attached to $(\mathcal{F}, \alpha)$ obtained by taking traces of the Frobenius on $\mathbb{F}_q$-points:

By adjunction, the datum of α is equivalent to the datum of a map

$$\alpha^L : (\mathrm{Frob}_{\mathcal{Y}})^*(\mathcal{F}) \to \mathcal{F}.$$

Now the value of $\mathrm{funct}(\mathcal{F}, \alpha)$ on a given $y \in \mathcal{Y}(\mathbb{F}_q)$,

$$\mathrm{pt} \overset{\mathbf{i}_y}{\to} \mathcal{Y}$$

equals the trace of

$$\mathbf{i}_y^*(\mathcal{F}) \overset{y \text{ is Frob-invariant}}{\longrightarrow} (\mathrm{Frob}_{\mathcal{Y}} \circ \mathbf{i}_y)^*(\mathcal{F}) \simeq \mathbf{i}_y^* \circ \mathrm{Frob}_{\mathcal{Y}}^*(\mathcal{F}) \overset{\alpha^L}{\to} \mathbf{i}_y^*(\mathcal{F}).$$

21.1.5. The actual definition of $\mathrm{LT}^{\mathrm{naive}}$ proceeds as follows. Every y as above defines a functor

$$\mathbf{i}_y^* : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}},$$

which admits a continuous right adjoint (namely, $(\mathbf{i}_y)_*$), and is equipped with a morphism (in fact, an isomorphism)

$$\mathbf{i}_y^* \circ (\mathrm{Frob}_{\mathcal{Y}})_* \to \mathbf{i}_y^*. \tag{21.4}$$

Hence, by [GKRV, Sect. 3.4.1], it defines a map

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Tr}(\mathrm{Id}, \mathrm{Vect}_{\mathsf{e}}) \simeq \mathsf{e}.$$

This map is, by definition, the composition of $\mathrm{LT}^{\mathrm{naive}}$ with the evaluation map

$$\mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)) \overset{\mathrm{ev}_y}{\to} \mathsf{e}.$$

The map $\mathrm{LT}^{\mathrm{naive}}$ has the following features.

21.1.6. For a lax Weil sheaf $(\mathcal{F}_0, \alpha_0)$ on $\mathrm{Shv}(\mathcal{Y})$, consider the functor

$$\mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y}), \quad \mathcal{F} \mapsto \mathcal{F}_0 \overset{*}{\otimes} \mathcal{F}.$$

This functor is endowed with a natural transformation

$$\mathcal{F}_0 \overset{*}{\otimes} (\mathrm{Frob}_{\mathcal{Y}})_*(\mathcal{F}) \to (\mathrm{Frob}_{\mathcal{Y}})_*(\mathcal{F}_0 \overset{*}{\otimes} \mathcal{F}), \tag{21.5}$$

and it admits a (continuous) right adjoint, given by

$$\mathcal{F} \mapsto \mathbb{D}(\mathcal{F}_0) \overset{!}{\otimes} \mathcal{F}.$$

Hence, by [GKRV, Sect. 3.4.1], it defines a map

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})). \tag{21.6}$$

We claim that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) & \xrightarrow{\mathrm{LT}^{\mathrm{naive}}} & \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)) \\ {\scriptstyle (21.6)}\downarrow & & \downarrow{\scriptstyle \mathrm{funct}(\mathcal{F}_0, \alpha_0) \cdot -} \\ \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) & \xrightarrow{\mathrm{LT}^{\mathrm{naive}}} & \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)). \end{array}$$

Indeed, this follows from the fact that for a given $y \in \mathcal{Y}(\mathbb{F}_q)$, we have a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{Shv}(\mathcal{Y}) & \xrightarrow{\mathbf{i}_y^*} & \mathrm{Vect}_{\mathsf{e}} \\ {\scriptstyle \mathcal{F}_0 \overset{*}{\otimes}}\downarrow & & \downarrow{\scriptstyle \mathbf{i}_y^*(\mathcal{F}_0)\otimes -} \\ \mathrm{Shv}(\mathcal{Y}) & \xrightarrow{\mathbf{i}_y^*} & \mathrm{Vect}_{\mathsf{e}} \end{array}$$

compatible with the natural transformations (21.4) and (21.5) via the endomorphism on $\mathbf{i}_y^*(\mathcal{F}_0)$ given by α_0^L. This implies that the resulting map

$$\mathsf{e} \simeq \mathrm{Tr}(\mathrm{Id}, \mathrm{Vect}_{\mathsf{e}}) \overset{(\mathbf{i}_y^*(\mathcal{F}_0), \alpha_0^L)}{\longrightarrow} \mathrm{Tr}(\mathrm{Id}, \mathrm{Vect}_{\mathsf{e}}) \simeq \mathsf{e}$$

is given by multiplication by

$$\mathrm{Tr}(\alpha_0^L, \mathbf{i}_y^*(\mathcal{F}_0)) = \mathrm{funct}(\mathcal{F}_0, \alpha_0)(y),$$

as desired.

21.1.7. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map. Consider the functor

$$f^* : \mathrm{Shv}(\mathcal{Y}_2) \to \mathrm{Shv}(\mathcal{Y}_1).$$

This functor is endowed with a natural transformation (in fact, an isomorphism)

$$f^* \circ (\mathrm{Frob}_{\mathcal{Y}_2})_* \to (\mathrm{Frob}_{\mathcal{Y}_1})_* \circ f^*. \tag{21.7}$$

The right adjoint of f^* is the usual direct image functor f_*. However, for morphisms between stacks, the functor f_* is not necessarily continuous. Therefore, in what follows we will assume that f is *safe* in the sense of [DrGa1, Definition 10.2.2]. Concretely, this condition means that for any geometric point of any geometric fiber of f, the neutral connected component of its group of automorphisms is unipotent. In particular, any schematic map between algebraic stacks is safe.

Assuming that f_* is continuous, by [GKRV, Sect. 3.4.1], the functor f^* defines a map

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_2})_*, \mathrm{Shv}(\mathcal{Y}_2)) \to \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_1})_*, \mathrm{Shv}(\mathcal{Y}_1)). \tag{21.8}$$

We claim that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_2})_*, \mathrm{Shv}(\mathcal{Y}_2)) & \xrightarrow{\mathrm{LT}^{\mathrm{naive}}} & \mathrm{Funct}(\mathcal{Y}_2(\mathbb{F}_q)) \\ {\scriptstyle (21.8)}\downarrow & & \downarrow{\scriptstyle \text{pull back}} \\ \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_1})_*, \mathrm{Shv}(\mathcal{Y}_1)) & \xrightarrow{\mathrm{LT}^{\mathrm{naive}}} & \mathrm{Funct}(\mathcal{Y}_1(\mathbb{F}_q)), \end{array}$$

where the right vertical arrow is given by pullback of functions along the induced map

$$\mathcal{Y}_1(\mathbb{F}_q) \to \mathcal{Y}_2(\mathbb{F}_q).$$

This follows just from the fact that the *-pullback functor is compatible with compositions.

21.1.8. Finally, let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be as above. Consider the functor

$$f_! : \mathrm{Shv}(\mathcal{Y}_1) \to \mathrm{Shv}(\mathcal{Y}_2).$$

This functor is endowed with a natural transformation (in fact, an isomorphism)

$$f_! \circ (\mathrm{Frob}_{\mathcal{Y}_1})_* \to (\mathrm{Frob}_{\mathcal{Y}_2})_* \circ f_!, \tag{21.9}$$

(coming from (21.1)), and it admits a (continuous) right adjoint, given by $f^!$.

Hence, by [GKRV, Sect. 3.4.1], it defines a map

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_1})_*, \mathrm{Shv}(\mathcal{Y}_1)) \to \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_2})_*, \mathrm{Shv}(\mathcal{Y}_2)). \tag{21.10}$$

Theorem 21.1.9. *We have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_1})_*, \mathrm{Shv}(\mathcal{Y}_1)) & \xrightarrow{\mathrm{LT}^{\mathrm{naive}}} & \mathrm{Funct}(\mathcal{Y}_1(\mathbb{F}_q)) \\ {\scriptstyle (21.10)}\downarrow & & \downarrow{\scriptstyle \text{push forward}} \\ \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_2})_*, \mathrm{Shv}(\mathcal{Y}_2)) & \xrightarrow{\mathrm{LT}^{\mathrm{naive}}} & \mathrm{Funct}(\mathcal{Y}_2(\mathbb{F}_q)), \end{array}$$

where the right vertical arrow is given by (weighted)[38] *summation along the fiber of the induced map*

$$\mathcal{Y}_1(\mathbb{F}_q) \to \mathcal{Y}_2(\mathbb{F}_q).$$

This theorem is a version of the Grothendieck-Lefschetz trace formula. The proof is given in [GaVa].

Remark 21.1.10. It is easy to prove Theorem 21.1.9 when f is a locally closed embedding. And this is the only case we will need in order to formulate Conjecture 21.3.7.

21.1.11. Let now $\mathcal{Y}$ be an algebraic stack that is not necessarily quasi-compact. We write

$$\mathcal{Y} := \underset{\mathcal{U}}{\cup} \mathcal{U}, \tag{21.11}$$

where $\mathcal{U} \overset{j}{\hookrightarrow} \mathcal{Y}$ is a filtered collection of quasi-compact open prestacks, so that

$$\mathrm{Shv}(\mathcal{Y}) \simeq \underset{\mathcal{U}}{\lim}\, \mathrm{Shv}(\mathcal{U}),$$

with respect to the restriction maps and also

$$\mathrm{Shv}(\mathcal{Y}) \simeq \underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Shv}(\mathcal{U}),$$

with respect to !-pushforwards, see [DrGa2, Proposition 1.7.5].

We claim that the functors $j_! : \mathrm{Shv}(\mathcal{U}) \to \mathrm{Shv}(\mathcal{Y})$ induce an isomorphism

$$\underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Tr}((\mathrm{Frob}_{\mathcal{U}})_*, \mathrm{Shv}(\mathcal{U})) \to \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})). \tag{21.12}$$

Indeed, we have

$$\mathrm{Id}_{\mathrm{Shv}(\mathcal{Y})} \simeq \underset{\mathcal{U}}{\mathrm{colim}}\, j_! \circ j^*,$$

and hence

$$\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \simeq \underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_* \circ j_! \circ j^*, \mathrm{Shv}(\mathcal{Y})) \simeq$$
$$\simeq \underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Tr}(j_! \circ (\mathrm{Frob}_{\mathcal{U}})_* \circ j^*, \mathrm{Shv}(\mathcal{Y})) \overset{\text{cyclicity of trace}}{\simeq} \underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Tr}((\mathrm{Frob}_{\mathcal{U}})_* \circ j^* \circ j_!, \mathrm{Shv}(\mathcal{U})) \simeq$$
$$\simeq \underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Tr}((\mathrm{Frob}_{\mathcal{U}})_*, \mathrm{Shv}(\mathcal{U})),$$

as desired.

From here, using Theorem 21.1.9 for open embeddings, we obtain that the maps $\mathrm{LT}^{\mathrm{naive}}$ for $\mathcal{U}$ give rise to a map

$$\mathrm{LT}^{\mathrm{naive}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q)), \tag{21.13}$$

where $\mathrm{Funct}_c(-)$ stands for "functions with finite support", so

$$\mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q)) \simeq \underset{\mathcal{U}}{\mathrm{colim}}\, \mathrm{Funct}_c(\mathcal{U}(\mathbb{F}_q)).$$

[38] We weigh each point by $\frac{1}{|\text{order of its group of automorphisms}|}$.

21.2. **The true local term.** We now proceed to the definition of the map

$$\mathrm{LT}^{\mathrm{true}}: \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)). \tag{21.14}$$

As in the previous subsection, on the first pass we will assume that $\mathcal{Y}$ is quasi-compact[39]. We will also assume that $\mathcal{Y}$ is duality-adapted, see Sect. F.2.6 for what this means. (According to Conjecture F.2.7, all quasi-compact algebraic stacks with an affine diagonal have this property; in Theorem F.2.8 it is shown that algebraic stacks that can locally be written as quotients are such.)

21.2.1. We recall that the algebraic stack $\mathcal{Y}^{\mathrm{Frob}}$ is *discrete*, i.e., has the form

$$\sqcup\, (\mathrm{pt}\,/\Gamma), \quad \Gamma \in \text{Finite Groups},$$

so we can identify

$$\mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)) \simeq \mathrm{C}^\cdot(\mathcal{Y}^{\mathrm{Frob}}, \omega_{\mathcal{Y}^{\mathrm{Frob}}}).$$

Let $\mathbf{i}_{\mathcal{Y}}$ denote the forgetful map

$$\mathcal{Y}^{\mathrm{Frob}} \to \mathcal{Y}.$$

Let us rewrite

$$\mathrm{C}^\cdot(\mathcal{Y}^{\mathrm{Frob}}, \omega_{\mathcal{Y}^{\mathrm{Frob}}}) \simeq \mathrm{C}^\cdot(\mathcal{Y}, (\mathbf{i}_{\mathcal{Y}})_*(\omega_{\mathcal{Y}^{\mathrm{Frob}}})).$$

Using base change along

$$\begin{array}{ccc} \mathcal{Y}^{\mathrm{Frob}} & \xrightarrow{\mathbf{i}_{\mathcal{Y}}} & \mathcal{Y} \\ {\scriptstyle \mathbf{i}_{\mathcal{Y}}}\downarrow & & \downarrow{\scriptstyle (\mathrm{Frob}_{\mathcal{Y}}, \mathrm{id}_{\mathcal{Y}})} \\ \mathcal{Y} & \xrightarrow{\Delta_{\mathcal{Y}}} & \mathcal{Y} \times \mathcal{Y}, \end{array}$$

we can rewrite

$$(\mathbf{i}_{\mathcal{Y}})_*(\omega_{\mathcal{Y}^{\mathrm{Frob}}}) \simeq \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}).$$

To summarize, we have

$$\mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)) \simeq \mathrm{C}^\cdot(\mathcal{Y}, \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})).$$

21.2.2. In order to compute $\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y}))$, we identify $\mathrm{Shv}(\mathcal{Y})$ with its own dual, see Sect. F.4.1. We recall that the corresponding pairing

$$\mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}}$$

is given by

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathrm{C}^\cdot_{\blacktriangle}(\mathcal{Y}, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2) \simeq \mathrm{C}^\cdot_{\blacktriangle}(\mathcal{Y}, \Delta^!_{\mathcal{Y}}(\mathcal{F}_1 \boxtimes \mathcal{F}_2)),$$

where the notation $\mathrm{C}^\cdot_{\blacktriangle}$ is as in Sect. F.4.2.

Let $\mathrm{u}_{\mathrm{Shv}(\mathcal{Y})} \in \mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y})$ be the unit of the self-duality on $\mathrm{Shv}(\mathcal{Y})$.

We obtain that $\mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y}))$ is given by

$$\mathrm{C}^\cdot_{\blacktriangle}(\mathcal{Y}, \Delta^!_{\mathcal{Y}} \circ \boxtimes \circ ((\mathrm{Frob}_{\mathcal{Y}})_* \otimes \mathrm{Id}_{\mathrm{Shv}(\mathcal{Y})})(\mathrm{u}_{\mathrm{Shv}(\mathcal{Y})})),$$

where $\boxtimes$ denotes the external tensor product functor

$$\mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}).$$

[39]This is essential for the construction because Verdier duality a priori works only for quasi-compact stacks.

21.2.3. We note that

$$(\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ \boxtimes \simeq \boxtimes \circ ((\mathrm{Frob}_{\mathcal{Y}})_* \otimes \mathrm{Id}_{\mathrm{Shv}(\mathcal{Y})}).$$

Hence, in order to construct the map (21.14), it suffices to construct a map

$$\boxtimes(\mathrm{u}_{\mathrm{Shv}(\mathcal{Y})}) \to (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}), \tag{21.15}$$

and a map

$$\mathrm{C}^{\cdot}_{\blacktriangle}(\mathcal{Y}, \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})) \to \mathrm{C}^{\cdot}(\mathcal{Y}, \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})) \simeq \\ \simeq \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)). \tag{21.16}$$

The first arrow in (21.16) is the map (F.6). In our case, it is in fact an isomorphism, because the object

$$\Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}) \simeq (\mathbf{i}_{\mathcal{Y}})_*(\omega_{\mathcal{Y}^{\mathrm{Frob}}}) \in \mathrm{Shv}(\mathcal{Y})$$

is compact, since $\mathcal{Y}$ is duality-adapted (by Theorem F.2.8).

We proceed to the construction of the map (21.15), cf. Remark 11.6.3.

21.2.4. Let $\boxtimes^R$ denote the right adjoint of the functor

$$\boxtimes : \mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y}).$$

We claim that we have a canonical isomorphism

$$\mathrm{u}_{\mathrm{Shv}(\mathcal{Y})} \simeq \boxtimes^R((\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})), \tag{21.17}$$

which would then give rise to the desired map (21.15) by adjunction.

To establish (21.17) we note that for $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Shv}(\mathcal{Y})^c$, we have

$$\mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})\otimes\mathrm{Shv}(\mathcal{Y})}(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathrm{u}_{\mathrm{Shv}(\mathcal{Y})}) \simeq \mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})}(\mathcal{F}_1, \mathbb{D}(\mathcal{F}_2)),$$

by the definition of the self-duality on $\mathrm{Shv}(\mathcal{Y})$ (here $\mathbb{D}$ is Verdier duality), while

$$\mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})\otimes\mathrm{Shv}(\mathcal{Y})}(\mathcal{F}_1 \otimes \mathcal{F}_2, \boxtimes^R((\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}))) \simeq \mathcal{H}om_{\mathrm{Shv}(\mathcal{Y}\times\mathcal{Y})}(\mathcal{F}_1 \boxtimes \mathcal{F}_2, (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})) \simeq \\ \simeq \mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})}(\mathcal{F}_1, \mathbb{D}(\mathcal{F}_2))$$

as well.

21.2.5. *Example.* Let $(\mathcal{F}, \alpha)$ be a lax Weil sheaf on $\mathcal{Y}$. Unwinding the construction, we obtain that the image of

$$\mathrm{cl}(\mathcal{F}, \alpha) \in \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y}))$$

along the map $\mathrm{LT}^{\mathrm{true}}$, thought of as an element of

$$\mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q)) \simeq \mathrm{C}^{\cdot}(\mathcal{Y}, \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})) \simeq \\ \simeq \mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})}(\underline{\mathsf{e}}_{\mathcal{Y}}, \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}})),$$

equals

$$\underline{\mathsf{e}}_{\mathcal{Y}} \to \mathcal{F} \overset{!}{\otimes} \mathbb{D}(\mathcal{F}) \simeq \Delta^!_{\mathcal{Y}}(\mathcal{F} \boxtimes \mathbb{D}(\mathcal{F})) \overset{\alpha\boxtimes\mathrm{id}}{\longrightarrow} \Delta^!_{\mathcal{Y}}((\mathrm{Frob}_{\mathcal{Y}})_*(\mathcal{F}) \boxtimes \mathbb{D}(\mathcal{F})) \simeq \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_*(\mathcal{F} \boxtimes \mathbb{D}(\mathcal{F})) \to \\ \to \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_* \circ \Delta^*_{\mathcal{Y}}(\mathcal{F} \boxtimes \mathbb{D}(\mathcal{F})) \simeq \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\mathcal{F} \overset{*}{\otimes} \mathbb{D}(\mathcal{F})) \to \\ \to \Delta^!_{\mathcal{Y}} \circ (\mathrm{Frob}_{\mathcal{Y}} \times \mathrm{id}_{\mathcal{Y}})_* \circ (\Delta_{\mathcal{Y}})_*(\omega_{\mathcal{Y}}).$$

Remark 21.2.6. The map (21.15) constructed above is, in general, *not* an isomorphism. In fact it is an isomorphism *if and only if* the functor

$$\mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y}) \overset{\boxtimes}{\to} \mathrm{Shv}(\mathcal{Y} \times \mathcal{Y})$$

is an equivalence, see Sect. 15.3.1.

The fact that the map (21.15) is not in general an isomorphism prevents the map (21.14) from being an isomorphism.

However (as was remarked in Sect. 15.3.1), we obtain that (21.15) is an isomorphism for algebraic stacks that have finitely many isomorphism classes of $\overline{\mathbb{F}}_q$-points, e.g., for $N\backslash G/B$, or a quasi-compact substack of Bun_G for a curve X of genus 0. Hence, (21.14) is an isomorphism in these cases as well.

21.2.7. We claim:

Theorem 21.2.8. *The maps*

$$\mathrm{LT}^{\mathrm{naive}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))$$

and

$$\mathrm{LT}^{\mathrm{true}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))$$

are canonically homotopic.

The proof is given in [GaVa]. From now on, we will just use the symbol

$$\mathrm{LT} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))$$

for the local term map.

Remark 21.2.9. Theorem 21.2.8 implies that for a lax Weil sheaf $(\mathcal{F}, \alpha)$, the images of $\mathrm{cl}(\mathcal{F}, \alpha) \in \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y}))$ in $\mathrm{Funct}(\mathcal{Y}(\mathbb{F}_q))$ under the above two maps coincide.

Interpreting the image of $\mathrm{cl}(\mathcal{F}, \alpha)$ along $\mathrm{LT}^{\mathrm{true}}$ as in Sect. 21.2.5, the latter assertion becomes equivalent to one in [Var2, Theorem 2.1.3] (when $\mathcal{Y}$ is a scheme). The proof of Theorem 21.2.8 is an elaboration of the ideas from *loc. cit.*

Corollary 21.2.10. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a map between stacks.*

(a) *If f is safe, then the diagram*

$$\begin{array}{ccc} \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_2})_*, \mathrm{Shv}(\mathcal{Y}_2)) & \xrightarrow{\mathrm{LT}^{\mathrm{true}}} & \mathrm{Funct}(\mathcal{Y}_2(\mathbb{F}_q)) \\ {\scriptstyle (21.8)}\big\downarrow & & \big\downarrow{\scriptstyle \textit{pull back}} \\ \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_1})_*, \mathrm{Shv}(\mathcal{Y}_1)) & \xrightarrow{\mathrm{LT}^{\mathrm{true}}} & \mathrm{Funct}(\mathcal{Y}_1(\mathbb{F}_q)), \end{array}$$

commutes.

(b) *The diagram*

$$\begin{array}{ccc} \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_1})_*, \mathrm{Shv}(\mathcal{Y}_1)) & \xrightarrow{\mathrm{LT}^{\mathrm{naive}}} & \mathrm{Funct}(\mathcal{Y}_1(\mathbb{F}_q)) \\ {\scriptstyle (21.10)}\big\downarrow & & \big\downarrow{\scriptstyle \text{push forward}} \\ \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}_2})_*, \mathrm{Shv}(\mathcal{Y}_2)) & \xrightarrow{\mathrm{LT}^{\mathrm{naive}}} & \mathrm{Funct}(\mathcal{Y}_2(\mathbb{F}_q)), \end{array}$$

commutes.

21.2.11. We now let $\mathcal{Y}$ be an arbitrary algebraic stack locally of finite type (i.e., not necessarily quasi-compact), written as (21.11)

Using (21.12) and Corollary 21.2.8(a), we combine the maps $\mathrm{LT}^{\mathrm{true}}$ for the quasi-compact open substacks $\mathcal{U} \subset \mathcal{Y}$ to a map

$$\mathrm{LT}^{\mathrm{true}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q)).$$

Furthermore, Theorem 21.2.8 implies that the above map equals the map

$$\mathrm{LT}^{\mathrm{naive}} : \mathrm{Tr}((\mathrm{Frob}_{\mathcal{Y}})_*, \mathrm{Shv}(\mathcal{Y})) \to \mathrm{Funct}_c(\mathcal{Y}(\mathbb{F}_q))$$

of (21.13).

21.3. **From geometric to classical: the Trace Conjecture.**

21.3.1. We start with the following observation. Let $\mathcal{Y}$ be an algebraic stack over $\overline{\mathbb{F}}_q$, but defined over $\mathbb{F}_q$. Consider the diagram

$$\begin{array}{ccccc} \mathcal{Y} & \xrightarrow{\text{Frob}^{\text{arithm}}_{\mathcal{Y}}} & \mathcal{Y} & \xrightarrow{\text{Frob}_{\mathcal{Y}}} & \mathcal{Y} \\ \downarrow & & \downarrow & & \\ \text{Spec}(\overline{\mathbb{F}}_q) & \xrightarrow{\text{Frob}^{\text{arithm}}_{\mathcal{Y}}} & \text{Spec}(\overline{\mathbb{F}}_q), & & \end{array}$$

where:

- The bottom horizontal square is the Frobenius automorphism of $\text{Spec}(\mathbb{F}_q)$;
- The square is Cartesian;
- The composite top horizontal arrow is the absolute Frobenius on $\mathcal{Y}$.

For $\mathcal{N} \subset T^*(\mathcal{Y})$, let $\mathcal{N}' \subset T^*(\mathcal{Y})$ denote the base-change of $\mathcal{N}$ along $\text{Frob}^{\text{arithm}}_{\mathcal{Y}}$.

We claim:

Lemma 21.3.2. *The functor* $(\text{Frob}_{\mathcal{Y}})_* : \text{Shv}(\mathcal{Y}) \to \text{Shv}(\mathcal{Y})$ *sends* $\text{Shv}_{\mathcal{N}}(\mathcal{Y}) \to \text{Shv}_{\mathcal{N}'}(\mathcal{Y})$.

Proof. The functor $(\text{Frob}_{\mathcal{Y}})_*$ is the inverse of the pullback functor $(\text{Frob}_{\mathcal{Y}})^*$, and the latter is the inverse of $(\text{Frob}^{\text{arithm}}_{\mathcal{Y}})^*$, since pullback by the absolute Frobenius acts as identity.

Hence, the functor $(\text{Frob}_{\mathcal{Y}})_*$ identifies with $(\text{Frob}^{\text{arithm}}_{\mathcal{Y}})^*$. Therefore, it is sufficient to show that $(\text{Frob}^{\text{arithm}}_{\mathcal{Y}})^*$ sends $\text{Shv}_{\mathcal{N}}(\mathcal{Y}) \to \text{Shv}_{\mathcal{N}'}(\mathcal{Y})$.

Now, for any map of fields $k \to k'$, the pullback functor along

$$\mathcal{Y}' := \text{Spec}(k') \underset{\text{Spec}(k)}{\times} \mathcal{Y} \to \mathcal{Y}$$

sends $\text{Shv}_{\mathcal{N}}(\mathcal{Y}) \subset \text{Shv}(\mathcal{Y})$ to $\text{Shv}_{\mathcal{N}'}(\mathcal{Y}') \subset \text{Shv}(\mathcal{Y}')$.

□

21.3.3. We now come to one of the central ideas of this paper.

Recall that the category $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ is compactly generated, and in particular, dualizable. By Lemma 21.3.2, the action of $(\text{Frob}_{\text{Bun}_G})_*$ on $\text{Shv}(\text{Bun}_G)$ preserves the subcategory

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G) \overset{\iota}{\hookrightarrow} \text{Shv}(\text{Bun}_G); \tag{21.18}$$

in particular, $(\text{Frob}_{\text{Bun}_G})_*$ restricts to an endofunctor of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$.

Hence, we can form the object

$$\text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \in \text{Vect}_{\mathsf{e}}\,.$$

21.3.4. Recall also the subcategory

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{access}} \subset \text{Shv}_{\text{Nilp}}(\text{Bun}_G), \tag{21.19}$$

see Sect. 15.4.5[40].

By the combination of Theorem 13.1.5 and Corollary F.8.11, it is generated by objects compact in $\text{Shv}(\text{Bun}_G)$; in particular it is compactly generated. The endofunctor $(\text{Frob}_{\text{Bun}_G})_*$ of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ preserves the subcategory $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{access}}$ (indeed, its compact generators are those objects of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$ that are compact in $\text{Shv}(\text{Bun}_G)$).

Hence, we can form also the object

$$\text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{access}}) \in \text{Vect}_{\mathsf{e}}\,.$$

[40]Note, however, that according to (the very plausible) Conjecture 13.1.8, the inclusion (21.19) is an equivalence.

21.3.5. The composite functor

$$\text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{access}} \hookrightarrow \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \overset{\iota}{\hookrightarrow} \text{Shv}(\text{Bun}_G) \tag{21.20}$$

preserves compactness. In particular, the functor (21.19) also preserves compactness, and hence both functors (21.19) and (21.20) admit continuous right adjoints.

Since the above functors commute with the action of $(\text{Frob}_{\text{Bun}_G})_*$, by [GKRV, Sect. 3.4.1], we obtain maps

$$\text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{access}}) \to \text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G))$$

and

$$\text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{access}}) \to \text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}(\text{Bun}_G)).$$

21.3.6. We propose the following:

Main Conjecture 21.3.7.

(a) *There is a canonical isomorphism*

$$\text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \simeq \text{Funct}_c(\text{Bun}_G(\mathbb{F}_q)). \tag{21.21}$$

(b) *The diagram*

$$\begin{array}{ccc} \text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{access}}) & \longrightarrow & \text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \\ \downarrow & & \downarrow{\scriptstyle (21.21)} \\ \text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) & \xrightarrow{\text{LT}} & \text{Funct}_c(\text{Bun}_G(\mathbb{F}_q)) \end{array}$$

commutes.

In what follows we will use the notation

$$\text{Autom} := \text{Funct}_c(\text{Bun}_G(\mathbb{F}_q))$$

and

$$\widetilde{\text{Autom}} := \text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)).$$

So, the statement of Conjecture 21.3.7(a) is that we have a canonical isomorphism

$$\widetilde{\text{Autom}} \simeq \text{Autom}\,.$$

21.3.8. Let us explain a concrete meaning of point (b) of Conjecture 21.3.7.

Let $\mathcal{F}$ be a compact object of $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)$, which is bounded below as an object of $\text{Shv}(\text{Bun}_G)$. In particular, $\mathcal{F}$ belongs to $\text{Shv}_{\text{Nilp}}(\text{Bun}_G)^{\text{access}}$ and is compact there, so it is compact also as an object of $\text{Shv}(\text{Bun}_G)$, and, in particular, constructible.

Assume that $\mathcal{F}$ is equipped with a lax Weil structure as in (21.2). Then we can consider the elements

$$\text{cl}(\mathcal{F}, \alpha) \in \text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G))$$

and

$$\text{funct}(\mathcal{F}, \alpha) \in \text{Funct}_c(\text{Bun}_G(\mathbb{F}_q)).$$

Now point (b) of Conjecture 21.3.7 says that these two elements match under the isomorphism (21.21).

Remark 21.3.9. Note that if we knew Conjecture 13.1.8, we could formulate Conjecture 21.3.7 just as saying that the composite map

$$\text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}_{\text{Nilp}}(\text{Bun}_G)) \to \text{Tr}((\text{Frob}_{\text{Bun}_G})_*, \text{Shv}(\text{Bun}_G)) \overset{\text{LT}}{\to} \text{Funct}_c(\text{Bun}_G(\mathbb{F}_q))$$

is an isomorphism.

Remark 21.3.10. Note that Conjecture 21.3.7 defines a direct bridge from the geometric Langlands theory to the classical one, since it implies that the space of automorphic functions with compact support, can be expressed as the categorical trace of the Frobenius endofunctor acting on the category of sheaves on Bun_G with nilpotent singular support.

Such a bridge allows us to transport structural assertions about $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ as a category, to assertions about Autom as a vector space. We will see some examples of this in Sect. 23.

21.3.11. There are several pieces of evidence towards the validity of Conjecture 21.3.7.

(I) It is true when G is a torus. We will analyze this case in the next subsection.

(II) It is true when X is of genus 0. Indeed, in this case the inclusion

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \subset \mathrm{Shv}(\mathrm{Bun}_G)$$

is an equality, and the map LT is an isomorphism because

$$\mathrm{Shv}(\mathrm{Bun}_G) \otimes \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G) \tag{21.22}$$

is an equivalence, see Remark 21.2.6.

(III) We have seen in Remark 21.2.6 that the failure of the map LT originates in the failure of the functor (21.22) to be equivalence. Now, this obstruction goes away for $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ thanks to Theorem 15.3.3.

(IV) The lisseness property of the cohomology of shtukas, recently established by C. Xue in [Xue2], see Remark 21.5.2.

21.4. **The case of $G = \mathbb{G}_m$.** In this subsection we will verify by hand the assertion of Conjecture 21.3.7(a) for $G = \mathbb{G}_m$. In Sect. 22 we will reprove it by a different method.

To simplify the notation, we will work not with the entire $\mathrm{Bun}_{\mathbb{G}_m} \simeq \mathrm{Pic}$, but with its neutral connected component Pic_0.

21.4.1. Recall that according to Sect. 20.2.9, the category

$$\mathrm{Shv}_{\{0\}}(\mathrm{Pic}_0) = \mathrm{QLisse}(\mathrm{Pic}_0)$$

is the direct sum over isomorphism classes of $\mathbb{G}_m$-local systems σ of copies of

$$\big(\mathrm{Sym}(H^1(X, \underline{\mathsf{e}}_X)[-1])\text{-mod}\big) \otimes (\mathrm{C}_\cdot(\mathbb{G}_m)\text{-mod}), \tag{21.23}$$

where for every σ, we send the module

$$\big(\mathrm{Sym}(H^1(X, \underline{\mathsf{e}}_X)[-1])\text{-mod}\big) \otimes \mathsf{e} \in \big(\mathrm{Sym}(H^1(X, \underline{\mathsf{e}}_X)[-1])\text{-mod}\big) \otimes (\mathrm{C}_\cdot(\mathbb{G}_m)\text{-mod})$$

(where e denotes the augmentation module over $\mathrm{C}_\cdot(\mathbb{G}_m)$) to the irreducible Hecke eigensheaf

$$E_\sigma \in \mathrm{QLisse}(\mathrm{Pic}_0)$$

corresponding to σ.

21.4.2. When we compute $\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Pic}_0})_*, -)$ on this category, only the direct summands, for which

$$\mathrm{Frob}_X^*(\sigma) \simeq \sigma \tag{21.24}$$

can contribute.

For each such σ choose an isomorphism in (21.24). This choice defines a Weil sheaf structure on the corresponding E_σ. Further, this choice identifies the action of $\mathrm{Tr}((\mathrm{Frob}_{\mathrm{Pic}_0})_*, -)$ on the direct summand (21.23) with the action induced by the Frobenius automorphism of the algebra

$$A := \mathrm{Sym}(H^1(X, \underline{\mathsf{e}}_X)[-1]) \otimes \mathrm{C}_\cdot(\mathbb{G}_m).$$

We will show that

$$\mathrm{Tr}(\mathrm{Frob}, A\text{-mod}) \simeq \mathsf{e}, \tag{21.25}$$

and that the induced map

$$\mathsf{e} \mapsto \mathrm{Tr}(\mathrm{Frob}, A\text{-mod}) \to \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Pic}_0})_*, \mathrm{QLisse}(\mathrm{Pic}_0)) \to \\ \to \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Pic}_0})_*, \mathrm{Shv}(\mathrm{Pic}_0)) \overset{\mathrm{LT}}{\to} \mathrm{Funct}(\mathrm{Pic}_0(\mathbb{F}_q)) \tag{21.26}$$

sends $1 \in \mathsf{e}$ to $\mathrm{funct}(E_\sigma) \cdot (1-q)$.

This will prove the required assertion since the functions $\mathrm{funct}(E_\sigma)$ form a basis of $\mathrm{Funct}(\mathrm{Pic}_0(\mathbb{F}_q))$, by Class Field Theory.

21.4.3. We have

$$A = A_1 \otimes A_2, \quad A_1 = \mathrm{Sym}(H^1(X, \underline{\mathsf{e}}_X)[-1]),\ A_2 = \mathrm{C}_\cdot(\mathbb{G}_m).$$

This corresponds to writing

$$\mathrm{Pic}_0 \simeq \mathrm{Jac}(X) \times \mathrm{pt}/\mathbb{G}_m.$$

We will perform the calculation for each factor separately.

21.4.4. Note that if A' is a polynomial algebra

$$A' \simeq \mathrm{Sym}(V),$$

where V is equipped with an endomorphism F with no eigenvalue 1, then the functor

$$\mathrm{Vect}_{\mathsf{e}} \to A'\text{-mod}, \quad \mathsf{e} \mapsto A'$$

defines an isomorphism

$$\mathsf{e} \simeq \mathrm{Tr}(\mathrm{Id}, \mathrm{Vect}_{\mathsf{e}}) \to \mathrm{Tr}(F, A'\text{-mod}).$$

Applying this to $A' = A_1$ and $A' = A_2$, we obtain the desired identifications

$$\mathrm{Tr}(\mathrm{Frob}, A_1\text{-mod}) \simeq \mathsf{e} \text{ and } \mathrm{Tr}(\mathrm{Frob}, A_2\text{-mod}) \simeq \mathsf{e},$$

as required in (21.25).

21.4.5. To prove (21.26) for A_1, we consider the composite functor

$$\mathrm{Vect}_{\mathsf{e}} \overset{\mathsf{e} \mapsto A_1}{\longrightarrow} A_1\text{-mod} \to \mathrm{QLisse}(\mathrm{Pic}_0) \to \mathrm{Shv}(\mathrm{Pic}_0),$$

equipped with its datum of compatibility with the Frobenius.

It sends

$$\mathsf{e} \mapsto E_\sigma,$$

equipped with its Weil structure, to be denoted α.

Hence, the corresponding map

$$\mathsf{e} \simeq \mathrm{Tr}(\mathrm{Id}, \mathrm{Vect}_{\mathsf{e}}) \to \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Pic}_0})_*, \mathrm{Shv}(\mathrm{Pic}_0))$$

sends

$$1 \in \mathsf{e} \ \mapsto \ \mathrm{cl}(E_\sigma, \alpha) \in \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Pic}_0})_*, \mathrm{Shv}(\mathrm{Pic}_0)).$$

Hence, its image under LT is $\mathrm{funct}(E_\sigma)$.

21.4.6. We now consider A_2. The composite functor

$$\mathrm{Vect}_{\mathsf{e}} \overset{\mathsf{e} \mapsto A_2}{\longrightarrow} A_2\text{-mod} \to \mathrm{Shv}(\mathrm{pt}/\mathbb{G}_m) \overset{\mathrm{pullback}}{\to} \mathrm{Shv}(\mathrm{pt}) = \mathrm{Vect}_{\mathsf{e}}$$

sends

$$\mathsf{e} \mapsto \mathrm{C}_\cdot(\mathbb{G}_m),$$

equipped with the natural datum of compatibility with the Frobenius.

Hence the resulting map

$$\mathsf{e} \simeq \mathrm{Tr}(\mathrm{Id}, \mathrm{Vect}_{\mathsf{e}}) \to \mathrm{Tr}((\mathrm{Frob}_{\mathrm{pt}/\mathbb{G}_m})_*, \mathrm{Shv}(\mathrm{pt}/\mathbb{G}_m)) \to \mathrm{Tr}((\mathrm{Frob}_{\mathrm{pt}})_*, \mathrm{Shv}(\mathrm{pt})) = \mathrm{Tr}(\mathrm{Id}, \mathrm{Vect}_{\mathsf{e}}) \simeq \mathsf{e}$$

sends

$$1 \in \mathsf{e} \ \mapsto \mathrm{Tr}(\mathrm{Frob}, \mathrm{C}_\cdot(\mathbb{G}_m)) = 1 - q \in \mathsf{e}.$$

21.5. **A generalization: cohomologies of shtukas.** In this subsection we will formulate a generalization of the Trace Conjecture, which gives a trace interpretation to cohomologies of shtukas.

21.5.1. Let us recall the construction of cohomologies of shtukas, following [VLaf1] and [Var1].

Let I be a finite set and V an object of $\mathrm{Rep}(\check{G})^{\otimes I}$. To this data we attach an object

$$\mathrm{Sht}_{I,V} \in \mathrm{Shv}(X^I)$$

as follows.

We consider the I-legged Hecke stack

$$\begin{array}{ccccc} \mathrm{Bun}_G & \xleftarrow{\overleftarrow{h}} & \mathrm{Hecke}_{X^I} & \xrightarrow{\overrightarrow{h}} & \mathrm{Bun}_G \\ & & {\scriptstyle\pi}\big\downarrow & & \\ & & X^I. & & \end{array}$$

The I-legged shtuka space is defined as the fiber product

$$\begin{array}{ccc} \mathrm{Sht}_I & \longrightarrow & \mathrm{Hecke}_{X^I} \\ \big\downarrow & & \big\downarrow{\scriptstyle (\overleftarrow{h},\overrightarrow{h})} \\ \mathrm{Bun}_G & \xrightarrow{(\mathrm{Frob}_{\mathrm{Bun}_G},\mathrm{Id})} & \mathrm{Bun}_G \times \mathrm{Bun}_G\,. \end{array}$$

Let π' denote the composite map

$$\mathrm{Sht}_I \to \mathrm{Hecke}_{X^I} \xrightarrow{\pi} X^I.$$

Recall that (the classical[41]) geometric Satake attaches to $V \in \mathrm{Rep}(\check{G})^{\otimes I}$ an object

$$\mathcal{S}_V \in \mathrm{Shv}(\mathrm{Hecke}_{X^I}).$$

Let $\mathcal{S}'_V \in \mathrm{Shv}(\mathrm{Sht}_I)$ denote its *-restriction to $\mathrm{Shv}(\mathrm{Sht}_I)$. Finally, we set

$$\mathrm{Sht}_{I,V} := \pi'_!(\mathcal{S}'_V) \in \mathrm{Shv}(X^I).$$

Remark 21.5.2. A recent result of [Xue2] says that the objects $\mathrm{Sht}_{I,V}$ actually belong to

$$\mathrm{QLisse}(X^I) \subset \mathrm{Shv}(X^I).$$

21.5.3. *Example.* Take $I = \emptyset$ and V to be e. Then

$$\mathrm{Sht}_\emptyset \simeq (\mathrm{Bun}_G)^{\mathrm{Frob}} \simeq \mathrm{Bun}_G(\mathbb{F}_q).$$

We obtain that

$$\mathrm{Sht}_{\emptyset,\mathsf{e}} = \mathrm{C}^\cdot_c(\mathrm{Bun}_G(\mathbb{F}_q), \underline{\mathsf{e}}_{\mathrm{Bun}_G(\mathbb{F}_q)}) \simeq \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)) = \mathrm{Autom}\,. \tag{21.27}$$

21.5.4. We will now construct a different system of objects

$$\widetilde{\mathrm{Sht}}_{I,V} \in \mathrm{QLisse}(X^I).$$

21.5.5. Note that the categorical trace construction has the following variant. Let $\mathbf{C}$ be a dualizable DG category and let

$$F : \mathbf{C} \to \mathbf{C} \otimes \mathbf{D},$$

where $\mathbf{D}$ is some other DG category.

Then we can consider an object

$$\mathrm{Tr}(F, \mathbf{C}) \in \mathbf{D}.$$

Namely, $\mathrm{Tr}(F, \mathbf{C})$ is the composition

$$\mathrm{Vect}_{\mathsf{e}} \xrightarrow{\text{unit}} \mathbf{C}^\vee \otimes \mathbf{C} \xrightarrow{\mathrm{Id}\otimes F} \mathbf{C}^\vee \otimes \mathbf{C} \otimes \mathbf{D} \xrightarrow{\text{counit}\otimes\mathrm{Id}_{\mathbf{D}}} \mathbf{D}.$$

(The usual trace construction is when $\mathbf{D} = \mathrm{Vect}_{\mathsf{e}}$, so $\mathrm{Tr}(F, \mathbf{C}) \in \mathrm{Vect}_{\mathsf{e}}$.)

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[41]As opposed to derived.

21.5.6. We apply this to $\mathbf{C} := \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, $\mathbf{D} = \mathrm{QLisse}(X^I)$ and F being the functor

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{(\mathrm{Frob}_{\mathrm{Bun}_G})_*}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \overset{\mathrm{H}(V,-)}{\longrightarrow} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X^I).$$

We set

$$\widetilde{\mathrm{Sht}}_{I,V} := \mathrm{Tr}(\mathrm{H}(V,-) \circ (\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) \in \mathrm{QLisse}(X^I).$$

We propose:

Main Conjecture 21.5.7. *The objects* $\mathrm{Sht}_{I,V}$ *and* $\widetilde{\mathrm{Sht}}_{I,V}$ *are canonically isomorphic.*

21.5.8. Consider the case of $I = \emptyset$. As we have seen in Sect. 21.5.3,

$$\mathrm{Sht}_{\emptyset,\mathsf{e}} = \mathrm{C}^{\cdot}_c(\mathrm{Bun}_G(\mathbb{F}_q), \underline{\mathsf{e}}_{\mathrm{Bun}_G(\mathbb{F}_q)}) \simeq \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q)) = \mathrm{Autom}\,.$$

This is while,

$$\widetilde{\mathrm{Sht}}_{\emptyset,\mathsf{e}} = \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)) =: \widetilde{\mathrm{Autom}},$$

which, according to Conjecture 21.3.7(a), is isomorphic to Autom.

So, Conjecture 21.3.7(a) is a special case of Conjecture 21.5.7.

Remark 21.5.9. A crucial piece of evidence for the validity of Conjecture 21.5.7 is provided by the result of [Xue2] mentioned in Remark 21.5.2.

21.5.10. *Partial Frobeniuses.* Recall (see [VLaf1, Sect. 3]) that the objects $\mathrm{Sht}_{I,V}$ carry an additional structure, namely, equivariance with respect to the *partial Frobenius maps*.

The construction from [GKRV, Sect. 5.3] endows the objects $\widetilde{\mathrm{Sht}}_{I,V}$ with a similar structure. (See also Remark 23.4.7 for a conceptual explanation of this structure.)

The statement of Conjecture 21.5.7 should be strengthened as follows: the isomorphism

$$\mathrm{Sht}_{I,V} \simeq \widetilde{\mathrm{Sht}}_{I,V}$$

is compatible with the structure of equivariance with respect to the partial Frobenius maps.

22. The trace conjecture for abelian varieties

In this section we will prove a statement parallel to Conjecture 21.3.7 for the category of lisse sheaves on an abelian variety.

The material of this section is not logically related to the contents of the rest of the paper.

22.1. **Statement of the result.**

22.1.1. Let A be an abelian variety. Consider the subcategory

$$\mathrm{QLisse}(A) \overset{\iota_A}{\hookrightarrow} \mathrm{Shv}(A). \tag{22.1}$$

The category QLisse(A) is compactly generated by the character sheaves, to be denoted E_σ (see Sect. 22.3.2 below). From here it follows that the pair $(A, \{0\})$ is constraccessible (see Definition F.7.5). In particular, the embedding ι_A admits a continuous right adjoint.

22.1.2. We now take our ground field k to be $\overline{\mathbb{F}}_q$, but we assume that A is defined over $\mathbb{F}_q$, so it carries an action of the geometric Frobenius endomorphism Frob_A.

Consider the composition

$$\mathrm{Tr}((\mathrm{Frob}_A)_*, \mathrm{QLisse}(A)) \to \mathrm{Tr}((\mathrm{Frob}_A)_*, \mathrm{Shv}(A)) \overset{\mathrm{LT}}{\longrightarrow} \mathrm{Funct}(A(\mathbb{F}_q)). \tag{22.2}$$

The main result of this section reads:

Theorem 22.1.3. *The map* (22.2) *is an isomorphism.*

The rest of this section is devoted to the proof of this theorem.

22.2. **The right adjoint to the embedding of the lisse subcategory.** In this subsection we will study the right adjoint of the functor ι_A of (22.1).

22.2.1. Deviating from the notations of the rest of the paper, for the duration of this section, we let X denote a smooth scheme (not necessarily a curve). In the applications to the proof of Theorem 22.1.3, we will take X to be the abelian variety A.

Consider the embedding

$$\text{QLisse}(X) \overset{\iota_X}{\hookrightarrow} \text{Shv}(X). \tag{22.3}$$

We will assume that the pair $(X, \{0\})$ is constraccessible. I.e., QLisse(X) is generated by objects that are compact in Shv(X). In particular, the above functor ι_X admits a continuous right adjoint[42].

We will now give an explicit formula for this right adjoint.

22.2.2. For a pair of objects $E_1, E_2 \in \text{Lisse}(X) = \text{QLisse}(X)^c$, consider the functor

$$\mathsf{P}_{E_1,E_2} : \text{Shv}(X) \to \text{QLisse}(X), \quad \mathcal{F} \mapsto E_1 \otimes \mathcal{H}om_{\text{Shv}(X)}(E_2, \mathcal{F}).$$

As E_1, E_2 vary we obtain a functor

$$\text{Lisse}(X) \times \text{Lisse}(X)^{\text{op}} \to \text{Funct}(\text{Shv}(X), \text{QLisse}(X)). \tag{22.4}$$

Let $\mathsf{P}_X \in \text{Funct}(\text{Shv}(X), \text{QLisse}(X))$ be the coEnd of the functor (22.4), i.e., the colimit of (22.4) over the index category TwArr(QLisse$(X)^c$).

The following is standard:

Lemma 22.2.3. *The functor P_X identifies canonically with ι_X^R.*

22.2.4. Let Y_1 and Y_2 be a pair of quasi-compact schemes. For an object $\mathcal{Q} \in \text{Shv}(Y_1 \times Y_2)$ we will denote by $\mathsf{K}_\mathcal{Q}$ the functor

$$\text{Shv}(Y_1) \to \text{Shv}(Y_2), \quad \mathcal{F} \mapsto (p_2)_*(p_1^!(\mathcal{F}) \overset{!}{\otimes} \mathcal{Q}).$$

Let Z be yet another scheme. We will denote by $\mathsf{K}_\mathcal{Q} \boxtimes \text{Id}_Z$ the functor

$$\text{Shv}(Y_1 \times Z) \to \text{Shv}(Y_2 \times Z)$$

equal to $\mathsf{K}_{\mathcal{Q} \boxtimes (\Delta_Z)_*(\omega_Z)}$, where

$$\mathcal{Q} \boxtimes (\Delta_Z)_*(\omega_Z) \in \text{Shv}(Y_1 \times Y_2 \times Z \times Z) \simeq \text{Shv}((Y_1 \times Z) \times (Y_2 \times Z)).$$

22.2.5. For $E_1, E_2 \in \text{Lisse}(X)^c$, set

$$\mathcal{Q}_{E_1,E_2} := \mathbb{D}(E_2) \boxtimes E_1 \in \text{Shv}(X \times X),$$

so that the functor P_{E_1,E_2} above identifies with $\mathsf{K}_{\mathcal{Q}_{E_1,E_2}}$.

The assignment

$$E_1, E_2 \mapsto \mathcal{Q}_{E_1,E_2}$$

is a functor

$$\text{Lisse}(X) \times \text{Lisse}(X)^{\text{op}} \to \text{Shv}(X \times X),$$

and let $\mathcal{Q}_{\text{QLisse}(X)}$ be its coEnd. By construction,

$$\mathsf{K}_{\mathcal{Q}_{\text{QLisse}(X)}} \simeq \iota_X \circ \mathsf{P}_X.$$

[42]Even if $(X, \{0\})$ is not constraccessible, the discussion below applies if we replace the entire QLisse(X) by an arbitrary full subcategory of QLisse(X) which is generated by objects that are compact in Shv(X).

22.2.6. Note that for a scheme Z, the essential image of the endofunctor $\mathsf{K}_{\mathcal{Q}_{\mathrm{QLisse}(X)}} \boxtimes \mathrm{Id}_Z$ of $\mathrm{Shv}(X \times Z)$ is contained in the full subcategory

$$\mathrm{QLisse}(X) \otimes \mathrm{Shv}(Z) \hookrightarrow \mathrm{Shv}(X \times Z).$$

We claim:

Proposition 22.2.7. *For a scheme Z, the endofunctor $\mathsf{K}_{\mathcal{Q}_{\mathrm{QLisse}(X)}} \boxtimes \mathrm{Id}_Z$ of $\mathrm{Shv}(X \times Z)$ identifies with the (pre)composition of the fully faithful embedding*

$$\mathrm{QLisse}(X) \otimes \mathrm{Shv}(Z) \hookrightarrow \mathrm{Shv}(X \times Z)$$

and its right adjoint.

Proof. We need to establish a functorial isomorphism

$$\mathcal{H}om_{\mathrm{QLisse}(X)\otimes\mathrm{Shv}(Z)}(E \boxtimes \mathcal{F}_Z, \mathsf{K}_{\mathcal{Q}_{\mathrm{QLisse}(X)}}(\mathcal{F})) \simeq \mathcal{H}om_{\mathrm{Shv}(X\times Z)}(E \boxtimes \mathcal{F}_Z, \mathcal{F}) \tag{22.5}$$

for $E \in \mathrm{Lisse}(X),\ \mathcal{F}_Z \in \mathrm{Shv}(Z)^c,\ \mathcal{F} \in \mathrm{Shv}(X \times Z)$.

Set

$$\mathcal{F}_X := (p_X)_*(\mathcal{F} \overset{!}{\otimes} p_Z^!(\mathbb{D}(\mathcal{F}_Z))).$$

Then we can rewrite the left-hand side in (22.5) as

$$\mathcal{H}om_{\mathrm{QLisse}(X)}(E, \mathsf{P}_X(\mathcal{F}_X)),$$

and the right-hand side as

$$\mathcal{H}om_{\mathrm{Shv}(X)}(E, \mathcal{F}_X).$$

So the assertion of the proposition follows from Lemma 22.2.3.

□

22.3. **Lisse sheaves an abelian variety.** Let us now specialize to the case when $X = A$ is an abelian variety.

22.3.1. Let σ denote an isomorphism class of 1-dimensional local systems on A, and let

$$E_\sigma \in \mathrm{QLisse}(X)$$

be an object in the given isomorphism, canonically fixed by the requirement that its *-fiber at $1 \in A$ is identified with e.

The following are standard facts about local systems on an abelian variety:

- Each E_σ has a (unique) structure of character sheaf, i.e., it is equipped with an isomorphism

$$\mathrm{mult}_A^*(E_\sigma) \simeq E_\sigma \boxtimes E_\sigma,$$

  normalized so that its fiber at $1 \times 1 \in A \times A$ is the identity map (it then automatically satisfies the associativity requirement);
- Each irreducible object in $\mathrm{Lisse}(A)^\heartsuit$ is isomorphic to some E_σ;
- $\mathcal{H}om(E_{\sigma_1}, E_{\sigma_2} = 0$ if $\sigma_1 \neq \sigma_2$;
- $\mathcal{H}om(E_\sigma, E_\sigma) \simeq \mathrm{C}^\cdot(A, \mathsf{e}) \simeq \mathrm{Sym}(H^1(A, \mathsf{e})[-1])$, as associative algebras.

The second and the third points above imply that the category $\mathrm{IndLisse}(A)$ splits as a direct sum

$$\underset{\sigma}{\oplus} \mathrm{IndLisse}(A)_\sigma,$$

where each $\mathrm{IndLisse}(A)_\sigma$ is compactly generated by σ.

22.3.2. We now claim that the embedding

$$\mathrm{IndLisse}(A) \hookrightarrow \mathrm{QLisse}(A)$$

is an equivalence, i.e., the objects E_σ (compactly) generate all of QLisse(X).

For that it suffices to show that each IndLisse$(A)_\sigma$ is left-complete in the t-structure (induced by the usual t-structure on QLisse(A).

However, this t-structure, when viewed as a t-structure on

$$\mathrm{IndLisse}(A)_\sigma \simeq \mathrm{Sym}(H^1(A,\mathsf{e}))\text{-mod}$$

translates via the Koszul duality

$$\mathrm{Sym}(H^1(A,\mathsf{e})[-1])\text{-mod} \simeq \mathrm{Sym}(H^1(A,\mathsf{e})^\vee)\text{-mod}_{\{0\}}$$

to the t-structure on $\mathrm{Sym}(H^1(A,\mathsf{e})^\vee)\text{-mod}_{\{0\}}$ compatible with the forgetful functor

$$\mathrm{Sym}(H^1(A,\mathsf{e})^\vee)\text{-mod}_{\{0\}} \hookrightarrow \mathrm{Sym}(H^1(A,\mathsf{e})^\vee)\text{-mod} \to \mathrm{Vect}_{\mathsf{e}},$$

and the latter is manifestly left-complete.

22.3.3. The above description of QLisse(A) implies that we can rewrite the object $\mathsf{K}_{\mathcal{Q}_{\mathrm{QLisse}(A)}}$ more concisely. Namely, we have

$$\mathcal{Q}_{\mathrm{QLisse}(A)} \simeq \underset{\sigma}{\oplus} \mathbb{D}(E_\sigma) \underset{\mathrm{C}^\cdot(A)}{\otimes} E_\sigma.$$

In particular, the functor ι_A^R acts as

$$\mathcal{F} \mapsto \underset{\sigma}{\oplus} \left(E_\sigma \underset{\mathrm{C}^\cdot(A)}{\otimes} \mathcal{H}om_{\mathrm{Shv}(A)}(E_\sigma, \mathcal{F}) \right). \tag{22.6}$$

22.3.4. The operation of convolution defines on Shv(A) a structure of symmetric monoidal category. We will denote the corresponding binary operation by $\star$ (in order to distinguish it from the pointwise symmetric monoidal structure, which is denote by $\overset{!}{\otimes}$).

The full subcategory QLisse(A) is preserved by $\star$, and is in fact a monoidal ideal. Hence, the functor ι_A^R acquires a right-lax symmetric monoidal structure.

However, formula (22.6) implies:

Lemma 22.3.5. *The right-lax symmetric monoidal structure on ι_A^R is strict.*

22.3.6. Thus, we obtain that ι_A is a symmetric monoidal co-localization. Denote by

$$\delta_{1,\mathrm{QLisse}} \in \mathrm{QLisse}(A)$$

the object

$$\iota_A^R(\delta_1).$$

It follows formally that the functor ι_A^R is given by convolution with $\delta_{1,\mathrm{QLisse}}$:

$$\iota_A^R(\mathcal{F}) \simeq \mathcal{F} \star \delta_{1,\mathrm{QLisse}}.$$

Similarly, we obtain that the object

$$\mathcal{Q}_{\mathrm{QLisse}(A)} \in \mathrm{Shv}(A \times A)$$

is obtained by convolving $(\Delta_A)_*(\omega_A)$ with $\delta_{1,\mathrm{QLisse}}$ along the second factor.

Remark 22.3.7. By formula (22.6), the object $\delta_{1,\mathrm{QLisse}}$ is a direct sum

$$\underset{\sigma}{\oplus} \delta_{1,\sigma}, \quad \delta_{1,\sigma} \simeq E_\sigma \underset{\mathrm{C}^\cdot(A)}{\otimes} \mathsf{e},$$

where e is the augmentation module for $\mathrm{C}^\cdot(A)$, corresponding to the point $1 \in A$.

So, each $\delta_{1,\sigma}$ is an infinite Jordan block. It is naturally filtered and the associated graded identifies with

$$E_\sigma \otimes \mathrm{Sym}(H^1(A,\mathsf{e})).$$

22.4. **Calculating the trace of Frobenius.** We return to the general set-up of Sect. 22.2.1, where we now assume that the ground field is $\overline{\mathbb{F}}_q$, but the scheme X is defined over $\mathbb{F}_q$, so we can talk about

$$\operatorname{Tr}((\operatorname{Frob}_X)_*, \operatorname{QLisse}(X)) \in \operatorname{Vect}_{\mathsf{e}}.$$

In this subsection we will produce an explicit formula for this trace.

22.4.1. Let $\mathcal{Q} \in \operatorname{Shv}(Y \times Y)$ and $\mathsf{K}_{\mathcal{Q}}$ be as in Sect. 22.2.4. We introduce the following notation

$$\operatorname{Tr}_{\operatorname{geom}}(\mathcal{Q}, Y) := \Gamma(Y, \Delta_Y^!(\mathcal{Q})).$$

Note that we have a canonically defined map

$$\operatorname{Tr}(\mathsf{K}_{\mathcal{Q}}, \operatorname{Shv}(Y)) \to \operatorname{Tr}_{\operatorname{geom}}(\mathcal{Q}, Y), \tag{22.7}$$

which is functorial in $\mathcal{Q}$.

Indeed, this map comes from the map (21.15).

22.4.2. Note that for

$$\mathcal{Q} = \mathcal{Q}_{\operatorname{Frob}_Y} := (\operatorname{Id} \times \operatorname{Frob}_Y)_*((\Delta_Y)_*(\omega_Y)) \simeq (\operatorname{Frob}_Y \times \operatorname{Id})^!((\Delta_Y)_*(\omega_Y)),$$

so that

$$\mathsf{K}_{\mathcal{Q}_{\operatorname{Frob}_Y}} = (\operatorname{Frob}_Y)_*,$$

we have a canonical identification

$$\operatorname{Tr}_{\operatorname{geom}}(\mathcal{Q}_{\operatorname{Frob}_Y}, Y) \simeq \operatorname{Funct}(Y(\mathbb{F}_q)).$$

By construction, the map $\operatorname{LT}^{\operatorname{true}}$ is the map (22.7) for $\mathcal{Q}_{\operatorname{Frob}_Y}$.

22.4.3. We take $Y = X$ and

$$\mathcal{Q}_1 = \mathcal{Q}_{\operatorname{Frob}_X}$$

and

$$\mathcal{Q}_2 = \mathcal{Q}_{\operatorname{Frob}_X, \operatorname{QLisse}} := (\operatorname{Id}_X \boxtimes \mathsf{K}_{\operatorname{QLisse}(X)})(\mathcal{Q}_1),$$

so that

$$\mathsf{K}_{\mathcal{Q}_2} = \mathsf{K}_{\operatorname{QLisse}(X)} \circ (\operatorname{Frob}_X)_*.$$

The counit of the adjunction of Proposition 22.2.7 defines a map

$$\mathcal{Q}_2 \to \mathcal{Q}_1.$$

In particular, we obtain a commutative diagram

$$\begin{array}{ccc}
\operatorname{Tr}(\mathsf{K}_{\operatorname{QLisse}(X)} \circ (\operatorname{Frob}_X)_*, \operatorname{QLisse}(X)) & \longrightarrow & \operatorname{Tr}((\operatorname{Frob}_X)_*, \operatorname{QLisse}(X)) \\
\downarrow & & \downarrow \\
\operatorname{Tr}(\mathsf{K}_{\operatorname{QLisse}(X)} \circ (\operatorname{Frob}_X)_*, \operatorname{Shv}(X)) & \longrightarrow & \operatorname{Tr}((\operatorname{Frob}_X)_*, \operatorname{Shv}(X)) \\
{\scriptstyle (22.7)}\downarrow & & \downarrow{\scriptstyle (22.7)} \\
\operatorname{Tr}_{\operatorname{geom}}(\mathcal{Q}_{\operatorname{Frob}_X, \operatorname{QLisse}}, X) & \longrightarrow & \operatorname{Tr}_{\operatorname{geom}}(\mathcal{Q}_{\operatorname{Frob}_X}, X) \\
& & \downarrow\sim \\
& & \operatorname{Funct}(X(\mathbb{F}_q))
\end{array} \tag{22.8}$$

Note that the right vertical composition in (22.8) is that map

$$\operatorname{Tr}((\operatorname{Frob}_X)_*, \operatorname{QLisse}(X)) \to \operatorname{Tr}((\operatorname{Frob}_X)_*, \operatorname{Shv}(X)) \xrightarrow{\operatorname{LT}} \operatorname{Funct}(X(\mathbb{F}_q))$$

of (22.2) when $X = A$.

22.4.4. The proof of Theorem 22.1.3 will amount to showing that:

- The top horizontal arrow in (22.8) is an isomorphism (this is true for any X);
- The upper left vertical arrow in (22.8) is an isomorphism (this is true for any X);
- The lower left vertical arrow in (22.8) is an isomorphism (this is true for any X);
- The bottom horizontal arrow in (22.8) is an isomorphism when $X = A$ is an abelian variety.

Note that the combination of the first three isomorphisms implies that we have a canonical isomorphism

$$\mathrm{Tr}((\mathrm{Frob}_X)_*, \mathrm{QLisse}(X)) \simeq \mathrm{Tr}_{\mathrm{geom}}(\mathcal{Q}_{\mathrm{Frob}_X,\mathrm{QLisse}}, X).$$

22.4.5. The fact that the top horizontal arrow in (22.8) is an isomorphism is immediate: by Proposition 22.2.7, the functor $\mathsf{K}_{\mathrm{QLisse}(X)}$ acts as identity on $\mathrm{QLisse}(X)$.

Similarly, the fact that the upper left vertical arrow in (22.8) is an isomorphism follows from the fact that the functor $\mathsf{K}_{\mathrm{QLisse}(X)}$ identifies with the composition $\iota_X \circ \iota_X^R$.

22.4.6. To prove that the lower left vertical arrow in (22.8) is an isomorphism, it suffices to show that the map

$$(\mathrm{Id}_X \boxtimes \mathsf{K}_{\mathrm{QLisse}(X)})(\boxtimes(\mathrm{u}_{\mathrm{Shv}(X)})) \to (\mathrm{Id}_X \boxtimes \mathsf{K}_{\mathrm{QLisse}(X)})((\Delta_X)_*(\omega_X)), \tag{22.9}$$

induced by (21.15), is an isomorphism.

Let us denote by Φ the $\boxtimes$ functor

$$\mathrm{Shv}(X) \otimes \mathrm{Shv}(X) \to \mathrm{Shv}(X \times X).$$

Then by Proposition 22.2.7, the map (22.9) identifies with the value of the natural transformation

$$(\Phi \circ (\mathrm{id} \otimes \iota_X)) \circ (\Phi \circ (\mathrm{id} \otimes \iota_X))^R \circ \Phi \circ \Phi^R \to (\Phi \circ (\mathrm{id} \otimes \iota_X)) \circ (\Phi \circ (\mathrm{id} \otimes \iota_X))^R \tag{22.10}$$

on $(\Delta_X)_*(\omega_X)$.

We claim that the natural transformation (22.10) itself is an isomorphism. By definition, (22.10) is the map

$$\Phi \circ (\mathrm{id} \otimes \iota_X) \circ (\mathrm{id} \otimes \iota_X)^R \circ \Phi^R \circ \Phi \circ \Phi^R \to \Phi \circ (\mathrm{id} \otimes \iota_X) \circ (\mathrm{id} \otimes \iota_X)^R \circ \Phi^R,$$

given by the counit $\Phi \circ \Phi^R \to \mathrm{Id}$. We precompose this map with

$$\Phi \circ (\mathrm{id} \otimes \iota_X) \circ (\mathrm{id} \otimes \iota_X)^R \circ \Phi^R \to \Phi \circ (\mathrm{id} \otimes \iota_X) \circ (\mathrm{id} \otimes \iota_X)^R \circ \Phi^R \circ \Phi \circ \Phi^R,$$

given by the unit $\mathrm{Id} \to \Phi \circ \Phi^R$, which is an isomorphism since Φ is fully faithful. Hence, it suffices to show that the composition

$$\Phi \circ (\mathrm{id} \otimes \iota_X) \circ (\mathrm{id} \otimes \iota_X)^R \circ \Phi^R \to$$
$$\to \Phi \circ (\mathrm{id} \otimes \iota_X) \circ (\mathrm{id} \otimes \iota_X)^R \circ \Phi^R \circ \Phi \circ \Phi^R \to \Phi \circ (\mathrm{id} \otimes \iota_X) \circ (\mathrm{id} \otimes \iota_X)^R \circ \Phi^R$$

is an isomorphism. However, this composition is the identity map, by the adjunction axioms.

22.5. **A calculation using Lang's isogeny.** In this subsection we will prove that the bottom horizontal arrow in (22.8) is an isomorphism.

22.5.1. Let $\mathcal{F}$ be an arbitrary object of $\mathrm{Shv}(A)$. For any scheme Z, let $- \star \mathcal{F}$ denote the endofunctor of $\mathrm{Shv}(Z \times A)$ obtained by convolving with $\mathcal{F}$ along the A factor.

The following results from a diagram chase:

Lemma 22.5.2. *There is a canonical isomorphism*

$$\mathrm{Tr}_{\mathrm{geom}}(\mathcal{Q}_{\mathrm{Frob}_A} \star \mathcal{F}, A) \simeq \mathrm{C}^{\cdot}(A, L^!(\mathcal{F})),$$

where $L : A \to A$ is the Lang isogeny.

22.5.3. Let us observe that the object $\mathcal{Q}_{\mathrm{Frob}_X,\mathrm{QLisse}}$ identifies with $\mathcal{Q}_{\mathrm{Frob}_X} \star \delta_{1,\mathrm{QLisse}}$ (see Sect. 22.3.6). Under this identification, the map

$$\mathcal{Q}_{\mathrm{Frob}_X,\mathrm{QLisse}} \to \mathcal{Q}_{\mathrm{Frob}_X}$$

is obtained from counit of the adjunction.

$$\delta_{1,\mathrm{QLisse}} \to \delta_1.$$

Hence, by Lemma 22.5.2, in order to show that the bottom horizontal arrow in (22.8) is an isomorphism, we have to show that the map

$$\mathrm{C}^{\cdot}(A, L^!(\delta_{1,\mathrm{QLisse}})) \to \mathrm{C}^{\cdot}(A, L^!(\delta_1)) \tag{22.11}$$

is an isomorphism.

22.5.4. We rewrite the map (22.11) as

$$\mathcal{H}om_{\mathrm{Shv}(A)}(\underline{\mathsf{e}}_A, L^!(\delta_{1,\mathrm{QLisse}})) \to \mathcal{H}om_{\mathrm{Shv}(A)}(\underline{\mathsf{e}}_A, L^!(\delta_1)),$$

and further by adjunction as

$$\mathcal{H}om_{\mathrm{Shv}(A)}(L_!(\underline{\mathsf{e}}_A), \delta_{1,\mathrm{QLisse}}) \to \mathcal{H}om_{\mathrm{Shv}(A)}(L_!(\underline{\mathsf{e}}_A), \delta_1).$$

Now, the latter map is indeed an isomorphism because

$$L_!(\underline{\mathsf{e}}_A) \in \mathrm{QLisse}(A),$$

since L is a finite étale map.

□[Theorem 22.1.3]

23. Localization of the space of automorphic functions

In this section we will introduce the space (in fact, a quasi-compact algebraic stack) of *arithmetic* Langlands parameters, denoted $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$.

We will see how our Trace Conjecture leads to a *localization* of the space of automorphic forms onto $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$.

23.1. The arithmetic $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}$.

23.1.1. Consider the automorphism of the symmetric monoidal category $\mathrm{QLisse}(X)$, given by pullback with respect to the Frobenius endomorphism of X:

$$\mathrm{Frob}_X^* : \mathrm{QLisse}(X) \to \mathrm{QLisse}(X).$$

By transport of structure, the prestack $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ acquires an automorphism, which we will denote simply by Frob.

23.1.2. Consider the prestack

$$(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))^{\mathrm{Frob}}$$

of Frob-fixed points of $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$.

Note that e-points of $(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))^{\mathrm{Frob}}$ are Weil $\check{G}$-local systems on X, i.e., $\check{G}$-local systems on X equipped with a Weil structure.

23.1.3. In Sect. 24 we will prove:

Theorem 23.1.4. *The fixed-point locus* $(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))^{\mathrm{Frob}}$ *is a quasi-compact, mock-affine*[43] *algebraic stack, locally almost of finite type.*

[43] See Sect. 5.3.1 for what this means.

23.1.5. In the same Sect. 24 we will also prove:

Theorem 23.1.6. *Assume that $\check{G}$ is semi-simple. Let σ be an* e*-point of* $(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))^{\mathrm{Frob}}$*, which is irreducible as a Weil $\check{G}$-local system. Then the group of its automorphisms is finite, and the resulting map*

$$\mathrm{pt}/\mathrm{Aut}(\sigma) \to (\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))^{\mathrm{Frob}}$$

is the embedding of a connected component.

Combining with the quasi-compactness assertion from Theorem 23.1.4, we obtain:

Corollary 23.1.7. *Let $\check{G}$ be semi-simple. Then there is only a finite number of irreducible Weil $\check{G}$-local systems on X.*

23.1.8. We will think of $(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))^{\mathrm{Frob}}$ as the stack parameterizing $\check{G}$-local systems on X equipped with a Weil structure, and henceforth denote it by

$$\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X).$$

Remark 23.1.9. We propose $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ as a candidate for the stack $\mathcal{S}$, alluded to in [VLaf2, Remark 8.5].

Recently, P. Scholze (unpublished) and X. Zhu (in [Zhu]) proposed two more definitions of the stack of Weil $\check{G}$-local systems on X. Their definitions are different from each other, and are of completely different flavor from ours. It is likely, however, that the resulting three versions of $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ are actually equivalent.

Remark 23.1.10. As we shall see in Sect. 23.5.3, the stack $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ is *non-classical*, i.e., its structure sheaf has non-trivial negative cohomology.

23.2. **The excursion algebra.**

23.2.1. Denote

$$\mathcal{E}xc := \Gamma(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \mathcal{O}_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}).$$

This is a commutative algebra object in $\mathrm{Vect}_{\mathsf{e}}$. Since $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$ is mock-affine, the algebra $\mathcal{E}xc$ is connective.

Set

$$\mathrm{LocSys}^{\mathrm{arithm,coarse}}_{\check{G}}(X) := \mathrm{Spec}(\mathcal{E}xc). \tag{23.1}$$

This is the coarse moduli space of arithmetic Langlands parameters. By construction, it is a *derived* affine scheme.

23.2.2. The algebra $\mathcal{E}xc$ is related to V. Lafforgue's algebra of excursion operators as follows.

Let $\mathrm{Weil}(X,x)^{\mathrm{discr}}$ be the Weil group of X (for some choice of a base point $x \in X$), considered as a discrete group. Set

$$\mathcal{X}^{\mathrm{discr}} := B(\mathrm{Weil}(X,x)^{\mathrm{discr}}) \in \mathrm{Spc}.$$

Consider the (mock-affine) algebraic stack

$$\mathrm{LocSys}_{\check{G}}(\mathcal{X}^{\mathrm{discr}}) \simeq \mathrm{LocSys}^{\mathrm{rigid}_x}_{\check{G}}(\mathcal{X}^{\mathrm{discr}})/\check{G},$$

and set

$$\mathcal{E}xc^{\mathrm{discr}} := \Gamma(\mathrm{LocSys}_{\check{G}}(\mathcal{X}^{\mathrm{discr}}), \mathcal{O}_{\mathrm{LocSys}_{\check{G}}(\mathcal{X}^{\mathrm{discr}})}),$$

$$\mathrm{LocSys}^{\mathrm{arithm,coarse,discr}}_{\check{G}}(X) := \mathrm{Spec}(\mathcal{E}xc^{\mathrm{discr}}).$$

23.2.3. The algebra $\mathcal{E}xc^{\mathrm{discr}}$ is the algebra of excursion operators attached to the group $\mathrm{Weil}(X,x)^{\mathrm{discr}}$, see [GKRV, Sect. 2.7] (it is denoted there by $\mathcal{E}nd_{\mathcal{A}\otimes Y}(\mathbf{1}_{\mathcal{A}\otimes Y})$; in our case $Y=\mathfrak{X}^{\mathrm{discr}}$ and $\mathcal{A}=\mathrm{Rep}(\check{G})$).

The classical commutative algebra $H^0(\mathcal{E}xc^{\mathrm{discr}})$ is the algebra of excursion operators in [VLaf1], so

$$ {}^{\mathrm{cl}}\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm,coarse,discr}}(X) $$

is the classical coarse moduli space of representations of $\mathrm{Weil}(X,x)^{\mathrm{discr}}$, considered as a discrete group.

23.2.4. We have a naturally defined closed embedding

$$ \mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X)\hookrightarrow \mathrm{LocSys}_{\check{G}}(\mathfrak{X}^{\mathrm{discr}}), $$

which induces a map

$$ \mathcal{E}xc^{\mathrm{discr}}\to \mathcal{E}xc, $$

surjective on H^0, and hence a closed embedding

$$ \mathrm{LocSys}_{\check{G}}^{\mathrm{arithm,coarse}}(X)\to \mathrm{LocSys}_{\check{G}}^{\mathrm{arithm,coarse,discr}}(X). $$

23.3. **Enhanced trace and Drinfeld's object.** In this subsection we will explain how the procedure of *2-categorical trace* produces from $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, equipped with the Frobenius endofunctor, an object of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X))$, to be denoted Drinf.

23.3.1. Recall the set-up of Sect. 7.10.1.

Thus, we take $\mathbf{A}$ to be the (symmetric) monoidal category

$$ \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X)), $$

and we take $F_{\mathbf{A}}$ to be given by Frob^*, where Frob is as in Sect. 23.1.1.

By Sect. 7.10.4, we have a canonical identification

$$ \mathrm{HH}_\bullet(\mathrm{Frob}^*,\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X)))\simeq \mathrm{QCoh}((\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X))^{\mathrm{Frob}})=:\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X)) $$

as (symmetric) monoidal categories.

We take the module category $\mathbf{M}$ to be $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, and $F_{\mathbf{M}}$ to be $(\mathrm{Frob}_{\mathrm{Bun}_G})_*$, which is equipped with a natural structure of compatibility with Frob^*. Since $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is dualizable as a DG category (thanks to Theorem 15.1.1), we can consider the objects

$$ \mathrm{Tr}((\mathrm{Frob}_{\mathrm{Bun}_G})_*,\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))=:\widetilde{\mathrm{Autom}}\in\mathrm{Vect}_{\mathsf{e}} $$

and

$$ \mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}(X))}((\mathrm{Frob}_{\mathrm{Bun}_G})_*,\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))=:\mathrm{Drinf}\in\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X)). $$

In the next subsection we will explain that Drinf can be regarded as a "universal shtuka", see Proposition 23.4.5.

23.3.2. From Theorem 7.10.6, we obtain:

Corollary 23.3.3. *There exists a canonical isomorphism*

$$ \widetilde{\mathrm{Autom}}\simeq\Gamma(\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X),\mathrm{Drinf}). \tag{23.2} $$

Remark 23.3.4. Note that according to Theorem 7.10.6, a priori, in the right-hand side in (23.2), we had to consider the functor $\Gamma_!(\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X),\mathrm{Drinf})$. However, since $\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X)$ is actually an algebraic stack (thanks to Theorem 23.1.4), we have

$$ \Gamma_!(\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X),-)\simeq\Gamma(\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X),-). $$

23.3.5. In particular, from Corollary 23.3.3, we obtain an action of the algebra

$$ \mathcal{E}xc:=\Gamma(\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X),\mathcal{O}_{\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X)}) $$

on $\widetilde{\mathrm{Autom}}$.

23.3.6. Let us now combine Corollary 23.3.3 with Conjecture 21.3.7. We obtain:

Corollary-of-Conjecture 23.3.7. *There exists a canonical isomorphism of vector spaces*

$$\text{Autom} \simeq \Gamma(\text{LocSys}_{\check{G}}^{\text{arithm}}(X), \text{Drinf}). \tag{23.3}$$

As a consequence, we obtain:

Corollary-of-Conjecture 23.3.8. *There exists a canonically defined action of the algebra* $\mathcal{E}xc$ *on* Autom.

Combining with Sects. 23.2.2-23.2.4, we obtain an action of the algebra $\mathcal{E}xc^{\text{discr}}$ on Autom. Thus, we obtain a spectral decomposition of Autom over the affine scheme $\text{LocSys}_{\check{G}}^{\text{arithm,coarse,discr}}(X)$.

Remark 23.3.9. As we will see in Remark 23.4.7, if we furthermore input Conjecture 21.5.7 (along with its complement in Sect. 21.5.10), we will see that the resulting action of $\mathcal{E}xc^{\text{discr}}$ on Autom equals the action defined in [VLaf1] (with the extension by C. Xue in [Xue1]) by excursion operators.

23.3.10. We can view the conclusion of Corollary 23.3.7 as "localization" of the space Autom of automorphic functions onto the stack $\text{LocSys}_{\check{G}}^{\text{arithm}}(X)$ of arithmetic Langlands parameters, in the sense that we realize Autom as the space of sections of a quasi-coherent sheaf on this stack.

23.4. **Relation to shtukas.** In this subsection we will explain that the Shtuka Conjecture (i.e., Conjecture 21.5.7) implies that the object Drinf constructed above, encodes all the shtuka cohomology.

23.4.1. Recall the objects

$$\widetilde{\text{Sht}}_{I,V} \in \text{QLisse}(X),$$

see Sect. 21.5.6.

We will now show, following [GKRV, Sect. 5.2], how the object

$$\text{Drinf} \in \text{QCoh}(\text{LocSys}_{\check{G}}^{\text{arithm}}(X))$$

recovers these objects, and endows them with a structure of equivariance with respect to the partial Frobenius maps.

23.4.2. For$I \in \text{fSet}$ and $V \in \text{Rep}(\check{G})^{\otimes I}$, let $\mathcal{E}_V^I$ be the corresponding tautological object of

$$\text{QCoh}(\text{LocSys}_{\check{G}}^{\text{restr}}(X)) \otimes \text{QLisse}(X)^{\otimes I},$$

see (13.6).

Namely, for $S \to \text{LocSys}_{\check{G}}^{\text{restr}}(X)$, the value of $\mathcal{E}_V^I$ on S, viewed as an object of

$$\text{QCoh}(S) \otimes \text{QLisse}(X)^{\otimes I}$$

is the value on V of the symmetric monoidal functor

$$\text{Rep}(\check{G})^{\otimes I} \to \text{QCoh}(S)^{\otimes I} \otimes \text{QLisse}(X)^{\otimes I} \to \text{QCoh}(S) \otimes \text{QLisse}(X)^{\otimes I},$$

where:

- The first arrow is the I tensor power of the functor $\text{Rep}(\check{G}) \to \text{QCoh}(S) \otimes \text{QLisse}(X)$ defining the map $S \to \text{LocSys}_{\check{G}}^{\text{restr}}(X)$;
- The second arrow uses the I-fold tensor product functor $\text{QCoh}(S)^{\otimes I} \to \text{QCoh}(S)$.

In what follows, by a slight abuse of notation, we will denote by the same character $\mathcal{E}_V^I$ the image of $\mathcal{E}_V^I$ under the fully faithful functor[44]

$$\text{QCoh}(\text{LocSys}_{\check{G}}^{\text{restr}}(X)) \otimes \text{QLisse}(X)^{\otimes I} \to \text{QCoh}(\text{LocSys}_{\check{G}}^{\text{restr}}(X)) \otimes \text{QLisse}(X^I).$$

23.4.3. Let

$$\mathcal{E}_V^{I,\text{arithm}} \in \text{QCoh}(\text{LocSys}_{\check{G}}^{\text{arithm}}(X)) \otimes \text{QLisse}(X^I)$$

denote the restriction of $\mathcal{E}_V^I$ along

$$\text{LocSys}_{\check{G}}^{\text{arithm}}(X) \to \text{LocSys}_{\check{G}}^{\text{restr}}(X).$$

[44]This functor is fact an equivalence, by the combination of Corollary E.4.7 and Theorem E.9.9.

23.4.4. We claim:

Proposition 23.4.5. *There exists a canonical isomorphism in* $\mathrm{QLisse}(X^I)$

$$\widetilde{\mathrm{Sht}}_{I,V} \simeq \Big(\Gamma(\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X), -) \otimes \mathrm{Id}\Big)\,(\mathrm{Drinf} \otimes \mathcal{E}_V^{I,\mathrm{arithm}}). \tag{23.4}$$

Proof. This is a variant of [GKRV, Theorem 4.4.4], combined with Theorem 7.10.6. □

Remark 23.4.6. For I as above and $i \in I$, let Frob_{i,X^I} denote the Frobenius map along the i-th factor in X^I. By construction, the object $\mathcal{E}_V^{I,\mathrm{arithm}}$ carries a natural structure of equivariance with respect to these endomorphisms:

$$((\mathrm{Frob}_{X^I,i})^* \otimes \mathrm{Id})(\mathcal{E}_V^{I,\mathrm{arithm}}) \simeq \mathcal{E}_V^{I,\mathrm{arithm}}.$$

This structure endows the left-hand side in (23.4) with a similar structure. Thus, we obtain a structure of equivariance with respect to the partial Frobenius maps on the objects $\widetilde{\mathrm{Sht}}_{I,V}$.

It follows as in [GKRV, Proposition 5.3.3] that the resulting structure on $\widetilde{\mathrm{Sht}}_{I,V}$ can be described by explicit excursion operators.

Remark 23.4.7. It follows from Proposition 23.4.5 and [GKRV, Proposition 5.4.3] (combined with Theorem 7.10.6 and its complement in Remark 7.10.8) that one can describe the action of $\mathcal{E}xc^{\mathrm{discr}}$ on $\widetilde{\mathrm{Autom}}$ (see Corollary 23.3.3 and Sects. 23.2.2-23.2.4) by explicit excursion operators.

Thus, if we assume Conjecture 21.5.7 (along with its complement in Sect. 21.5.10), we obtain that the above action matches under the isomorphism

$$\widetilde{\mathrm{Autom}} \simeq \mathrm{Autom}$$

with the action of $\mathcal{E}xc^{\mathrm{discr}}$ on Autom, defined in V. Lafforgue's work (with the extension by C. Xue in [Xue1]).

Remark 23.4.8. As has been remarked above, we propose our $\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X)$ as a candidate for the stack sought-for in [VLaf2, Remark 8.5] and [LafZh, Sect. 6] (it was denoted $\mathcal{S}/\check{G}$ in both these papers).

The space $\mathcal{S}$ is supposed to be the affine scheme parameterizing homomorphisms from the Weil group of X (based at x) to $\check{G}$. So, our proposal for $\mathcal{S}$ itself is

$$\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X) \underset{\mathrm{pt}/\check{G}}{\times} \mathrm{pt}\,.$$

Although in *loc.cit.* the space $\mathcal{S}/\check{G}$ is only defined heuristically, it is designed so that it carries a collection of quasi-coherent sheaves $\mathcal{E}_V^{I,\mathrm{arithm},\mathcal{S}}$ for (I,V) as above.

The goal of *loc.cit.* was to define an object

$$\mathrm{Drinf}^{\mathcal{S}} \in \mathrm{QCoh}(\mathcal{S}/\check{G}),$$

so that

$$\big(\Gamma(\mathcal{S}/\check{G}, -) \otimes \mathrm{Id}\big)\,(\mathrm{Drinf}^{\mathcal{S}} \otimes \mathcal{E}_V^{I,\mathrm{arithm},\mathcal{S}}) \simeq \mathrm{Sht}_{I,V}$$

Thus, assuming Conjecture 21.5.7, our $\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X)$ with the object

$$\mathrm{Drinf} \in \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{\mathrm{arithm}}(X))$$

achieves this goal.

23.5. **Arithmetic Arthur parameters.**

23.5.1. Fix an e-point σ of $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{arithm}}(X)$. By (20.7) and Verdier duality, the tangent space $T_\sigma(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))$ identifies with

$$\mathrm{C}^{\cdot}(X, \check{\mathfrak{g}}_\sigma)[1].$$

Hence, the tangent space $T_\sigma(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{arithm}}(X))$ identifies with

$$\mathrm{Fib}\left(T_\sigma(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)) \overset{\mathrm{Frob}-\mathrm{id}}{\longrightarrow} T_\sigma(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))\right) \simeq \mathrm{Fib}\left(\mathrm{C}^{\cdot}(X, \check{\mathfrak{g}}_\sigma) \overset{\mathrm{Frob}-\mathrm{id}}{\longrightarrow} \mathrm{C}^{\cdot}(X, \check{\mathfrak{g}}_\sigma)\right)[1],$$

and thus is concentrated in the cohomological degrees $[-1, 2]$.

This implies that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{arithm}}(X)$ has a perfect cotangent complex and is *quasi-quasi-smooth*. The latter by definition means that it can be smoothly covered by an derived affine scheme, whose cotangent spaces are are concentrated in the cohomological degrees $[-2, 0]$.

23.5.2. We have

$$H^2(T_\sigma(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{arithm}}(X))) \simeq \mathrm{coker}\left(H^2(X, \check{\mathfrak{g}}_\sigma) \overset{\mathrm{Frob}-\mathrm{id}}{\longrightarrow} H^2(X, \check{\mathfrak{g}}_\sigma)\right).$$

Hence, by Verdier duality

$$H^{-2}(T^*_\sigma(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{arithm}}(X))) \simeq \left(H^0(X, \check{\mathfrak{g}}_\sigma(1))\right)^{\mathrm{Frob}},$$

where (1) means Tate twist, and where we have identified $\check{\mathfrak{g}}$ with its dual using an invariant form.

In other words, we can think of elements of $H^{-2}(T^*_\sigma(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{arithm}}(X)))$ as elements

$$A \in H^0(X, \check{\mathfrak{g}}_\sigma)$$

such that

$$\mathrm{Frob}(A) = q \cdot A.$$

We note that such elements A is necessarily nilpotent.

23.5.3. Note that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{arithm}}(X)$ does contain points σ which admit a non-zero element A as above. For example, take σ to be geometrically trivial and fix an arbitrary non-zero nilpotent element

$$A \in \check{\mathfrak{g}} \simeq H^0(X, \check{\mathfrak{g}}_\sigma).$$

Now let the Weil structure be given by the image of $q \in \mathbb{G}_m$ under

$$\mathbb{G}_m \to SL_2 \to \check{G},$$

where $SL_2 \to \check{G}$ is a Jacobson-Morozov map corresponding to A.

This implies that $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{arithm}}(X)$ is *non-classical* and not even eventually coconnective: for a quasi-quasi smooth scheme that is *not* just quasi-smooth, its structure sheaf necessarily lives in infinitely many cohomological degrees.

23.5.4. Let $\mathcal{Z}$ be a quasi-quasi-smooth algebraic stack. Following a suggestion of D. Beraldo, one can mimic the construction of [AG, Sect. 2.3.3] and produce a classical algebraic stack, denoted $\mathrm{Sing}_2(\mathcal{Z})$, whose e-points are pairs

$$(z, \xi), \quad z \in \mathcal{Z},\, \xi \in H^{-2}(T^*_z(\mathcal{Z})).$$

23.5.5. We will denote:

$$\mathrm{Arth}^{\mathrm{arithm}}(X) := \mathrm{Sing}_2(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{arithm}}(X)),$$

and refer to it as the stack of *arithmetic Arthur parameters*.

Thus, the stack $\mathrm{Arth}^{\mathrm{arithm}}(X)$ projects to $\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{arithm}}(X)$, and the fiber over a given σ is the vector space

$$A \in H^0(X, \check{\mathfrak{g}}_\sigma),\ \mathrm{Frob}(A) = q \cdot A.$$

Remark 23.5.6. The terminology "Arthur parameters" is justified as follows:

If σ is semi-simple (as a Weil local system), then using a Jacobson-Morozov argument, we can identify the *set*

$$\{A,\ \mathrm{Frob}(A) = q \cdot A\}/\,\mathrm{Ad}(\mathrm{Aut}(\sigma))$$

with the *set*

$$\{SL_2 \to \mathrm{Aut}(\sigma)\}/\,\mathrm{Ad}(\mathrm{Aut}(\sigma)).$$

(Note, however, nilpotent elements have more automorphisms than SL_2-triples.)

23.6. **A digression: categorical trace on** IndCoh **on stacks.** In order to formulate our conjecture that expresses the space of automorphic forms explicitly in terms of the spectral side, we will need to make a digression and discuss properties of the categorical trace construction applied to $\mathrm{IndCoh}(\mathcal{Y})$, where $\mathcal{Y}$ is a quasi-smooth stack.

The material in this subsection was obtained as a result of communications with D. Beraldo.

23.6.1. Let $\mathcal{Y}$ be a quasi-compact algebraic stack equipped with an endomorphism ϕ. Then according to [GKRV, Sect. 3.5.3], we have

$$\mathrm{Tr}(\phi^*, \mathrm{QCoh}(\mathcal{Y})) \simeq \Gamma(\mathcal{Y}^\phi, \mathcal{O}_{\mathcal{Y}^\phi}). \tag{23.5}$$

23.6.2. Assume now that $\mathcal{Y}$ locally almost of finite type. In this case, along with $\mathrm{QCoh}(\mathcal{Y})$, we can consider the category $\mathrm{IndCoh}(\mathcal{Y})$, and the functor

$$\Upsilon_\mathcal{Y} : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y}), \quad \mathcal{F} \mapsto \mathcal{F} \otimes \omega_\mathcal{Y}.$$

Recall that we have the canonical self-dualities

$$\mathrm{QCoh}(\mathcal{Y})^\vee \simeq \mathrm{QCoh}(\mathcal{Y}) \text{ and } \mathrm{IndCoh}(\mathcal{Y})^\vee \simeq \mathrm{IndCoh}(\mathcal{Y}),$$

with pairings given by

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \Gamma(\mathcal{Y}, \mathcal{F}_1 \otimes \mathcal{F}_2) \text{ and } \mathcal{F}_1, \mathcal{F}_2 \mapsto \Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2),$$

respectively.

With respect to these self-dualities, the functor $\Upsilon_\mathcal{Y}$ is the dual of the (tautological) functor

$$\text{un-ren}_\mathcal{Y} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y}).$$

Recall also that the functor $\text{un-ren}_\mathcal{Y}$ has the property that for a schematic map $f : \mathcal{Y}_1 \to \mathcal{Y}_2$, the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{Y}_1) & \xrightarrow{\text{un-ren}_{\mathcal{Y}_1}} & \mathrm{QCoh}(\mathcal{Y}_1) \\ {\scriptstyle f_*^{\mathrm{IndCoh}}}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{IndCoh}(\mathcal{Y}_2) & \xrightarrow{\text{un-ren}_{\mathcal{Y}_2}} & \mathrm{QCoh}(\mathcal{Y}_2) \end{array}$$

commutes.

In particular,

$$\Gamma^{\mathrm{IndCoh}}(\mathcal{Y}, -) \simeq \Gamma(\mathcal{Y}, -) \circ \text{un-ren}_\mathcal{Y}\,. \tag{23.6}$$

23.6.3. A parallel computation to (23.5) shows that

$$\mathrm{Tr}(\phi^!, \mathrm{IndCoh}(\mathcal{Y})) \simeq \Gamma^{\mathrm{IndCoh}}(\mathcal{Y}^\phi, \omega_{\mathcal{Y}^\phi}). \tag{23.7}$$

23.6.4. Furthermore, we can place ourselves in the paradigm of Sect. 23.3.1, and consider $\mathrm{QCoh}(\mathcal{Y})$ and $\mathrm{IndCoh}(\mathcal{Y})$ as module categories over $\mathrm{QCoh}(\mathcal{Y})$, equipped with compatible endofunctors.

Thus, we can consider the objects

$$\mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathcal{Y})}(\phi^*, \mathrm{QCoh}(\mathcal{Y})) \text{ and } \mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathcal{Y})}(\phi^!, \mathrm{IndCoh}(\mathcal{Y}))$$

in $\mathrm{QCoh}(Y^\phi)$.

A computation similar to [GKRV, Sect. 3.5.3] shows that

$$\mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathcal{Y})}(\phi^*, \mathrm{QCoh}(\mathcal{Y})) \simeq \mathcal{O}_{\mathcal{Y}^\phi} \text{ and } \mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathcal{Y})}(\phi^!, \mathrm{IndCoh}(\mathcal{Y})) \simeq \text{un-ren}(\omega_{\mathcal{Y}^\phi}), \tag{23.8}$$

as objects of $\mathrm{QCoh}(Y^\phi)$ (note that the latter isomorphism is compatible with (23.7) via (23.6)).

23.6.5. Assume now that $\mathcal{Y}$ is quasi-smooth. Let $\mathcal{N}$ be a conical Zariski-closed subset in $\mathrm{Sing}(\mathcal{Y})$. Assume that the codifferential map

$$\mathrm{Sing}(\phi) : \mathcal{Y} \underset{\phi, \mathcal{Y}}{\times} \mathrm{Sing}(\mathcal{Y}) \to \mathrm{Sing}(\mathcal{Y}),$$

sends $\mathcal{Y} \underset{\phi, \mathcal{Y}}{\times} \mathcal{N} \subset \mathcal{Y} \underset{\phi, \mathcal{Y}}{\times} \mathrm{Sing}(\mathcal{Y})$ to $\mathcal{N} \subset \mathrm{Sing}(\mathcal{Y})$, so that the functor $\phi^!$ sends

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}) \to \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}),$$

see [AG, Proposition 7.1.3(a)].

Then it makes sense to consider

$$\mathrm{Tr}(\phi^!, \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})) \in \mathrm{Vect}_{\mathsf{e}}\,. \tag{23.9}$$

Furthermore, we can regard $\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})$ as a module category over $\mathrm{QCoh}(\mathcal{Y})$ and consider the object

$$\mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathcal{Y})}(\phi^!, \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})) \in \mathrm{QCoh}(\mathcal{Y}^\phi),$$

so that by (7.29) we have

$$\mathrm{Tr}(\phi^!, \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})) \simeq \Gamma\left(\mathcal{Y}^\phi, \mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathcal{Y})}(\phi^!, \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}))\right).$$

Remark 23.6.6. Unfortunately, we do not have an explicit answer for what the above object $\mathrm{Tr}^{\mathrm{enh}}(\phi^!, \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}))$ is in general. We expect, however, that one can give such an answer in terms of the subset

$$\mathcal{N}^\phi \subset \mathrm{Sing}_2(\mathcal{Y}^\phi),$$

defined in (23.13) below.

Yet, we know some particular cases: by (23.8), we have

$$\mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathcal{Y})}(\phi^!, \mathrm{IndCoh}_{\{0\}}(\mathcal{Y})) \simeq \mathcal{O}_{\mathcal{Y}^\phi} \tag{23.10}$$

and

$$\mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathcal{Y})}(\phi^!, \mathrm{IndCoh}(\mathcal{Y})) \simeq \text{un-ren}_{\mathcal{Y}^\phi}(\omega_{\mathcal{Y}^\phi}), \tag{23.11}$$

23.6.7. Note now that for $\mathcal{Y}$ quasi-smooth, the stack $\mathcal{Y}^\phi$ is quasi-quasi-smooth and

$$\mathrm{Sing}_2(\mathcal{Y}^\phi) := \{y \in \mathcal{Y},\ \phi(y) \sim y,\ \xi \in H^{-1}(T^*_y(\mathcal{Y})),\ \mathrm{Sing}(\phi)(\xi) = \xi\}. \tag{23.12}$$

Let $\mathcal{N} \subset \mathrm{Sing}(\mathcal{Y})$ be as Sect. 23.6.5. Set

$$\mathcal{N}^\phi \subset \mathrm{Sing}_2(\mathcal{Y}^\phi) \tag{23.13}$$

be the subset that in terms of (23.12) corresponds to the condition that $\xi \in \mathcal{N} \underset{\mathcal{Y}}{\times} \{y\}$.

23.6.8. We propose:

Conjecture 23.6.9. *Suppose that for a pair of conical subsets* $\mathcal{N}_1 \subset \mathcal{N}_2$ *as above, the inclusion*

$$\mathcal{N}_1^\phi \subset \mathcal{N}_2^\phi$$

is an equality. Then the inclusion functor

$$\mathrm{IndCoh}_{\mathcal{N}_1}(\mathcal{Y}) \hookrightarrow \mathrm{IndCoh}_{\mathcal{N}_2}((\mathcal{Y})$$

defines an isomorphism

$$\mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathcal{Y})}(\phi^!, \mathrm{IndCoh}_{\mathcal{N}_1}(\mathcal{Y})) \simeq \mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathcal{Y})}(\phi^!, \mathrm{IndCoh}_{\mathcal{N}_2}(\mathcal{Y}))$$

in $\mathrm{QCoh}(\mathcal{Y}^\phi)$.

This conjecture is not far-fetched, and might have been already established in the works of D. Beraldo.

23.6.10. As a particular case, and combining with (23.11) we obtain:

Corollary-of-Conjecture 23.6.11. *Suppose that for* $\mathcal{N}$ *as above, the inclusion*

$$\mathcal{N}^\phi \subset \mathrm{Sing}_2(\mathcal{Y}^\phi)$$

is an equality. Then the inclusion functor

$$\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y}) \hookrightarrow \mathrm{IndCoh}(\mathcal{Y})$$

defines an isomorphism

$$\mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathcal{Y})}(\phi^!, \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})) \simeq \mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathcal{Y})}(\phi^!, \mathrm{IndCoh}(\mathcal{Y})) \simeq \text{un-ren}_{\mathcal{Y}^\phi}(\omega_{\mathcal{Y}^\phi})$$

in $\mathrm{QCoh}(\mathcal{Y}^\phi)$. *In particular,*

$$\mathrm{Tr}(\phi^!, \mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})) \simeq \Gamma^{\mathrm{IndCoh}}(\mathcal{Y}^\phi, \omega_{\mathcal{Y}^\phi}).$$

23.7. **A digression: categorical trace on** IndCoh **on *formal* stacks.** We now generalize the discussion in Sect. 23.6 to the case when instead of a quasi-compact algebraic stack $\mathcal{Y}$, we have a *formal algebraic stack* $\mathcal{Y}$ as in Sect. 7.9.1. I.e., $\mathcal{Y}$ is a disjoint of prestacks, each of which is the quotient of a formal affine scheme by an action of an algebraic group.

23.7.1. First we observe that by Propositions 7.5.4 and 7.6.4, we have a canonical identification

$$\mathrm{Tr}(\phi^*, \mathrm{QCoh}(\mathcal{Y})) \simeq \Gamma_!(\mathcal{Y}^\phi, \mathcal{O}_{\mathcal{Y}^\phi}). \tag{23.14}$$

(Note the difference with formula (23.14): for a formal stack we have $\Gamma_!(\mathcal{Y}^\phi, -)$ instead of $\Gamma_!(\mathcal{Y}^\phi, -)$.)

23.7.2. By contrast, the formula for $\mathrm{Tr}(\phi^*, \mathrm{IndCoh}(\mathcal{Y}))$ remains unchanged:

$$\mathrm{Tr}(\phi^!, \mathrm{IndCoh}(\mathcal{Y})) \simeq \Gamma^{\mathrm{IndCoh}}(\mathcal{Y}^\phi, \omega_{\mathcal{Y}^\phi}), \tag{23.15}$$

where Γ^{IndCoh} is (an equivariant version of) the functor in [GR2, Chapter 3, Sect. 1.4].

23.7.3. The functor

$$\Upsilon_{\mathcal{Y}} : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y})$$

is defined as before:

$$\mathcal{F} \mapsto \mathcal{F} \otimes \omega_{\mathcal{Y}}$$

(in fact, this functor makes sense for *any* prestack locally almost of finite type).

Let

$$\text{un-ren}_{\mathcal{Y}} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y})$$

be the functor dual to the functor $\Upsilon_{\mathcal{Y}}$.

This functor can be characterized by the property that the diagrams

$$\begin{array}{ccc} \mathrm{IndCoh}(S) & \xrightarrow{\text{un-ren}_S} & \mathrm{QCoh}(S) \\ {\scriptstyle f_*^{\mathrm{IndCoh}}}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{IndCoh}(\mathcal{Y}) & \xrightarrow{\text{un-ren}_{\mathcal{Y}}} & \mathrm{QCoh}(\mathcal{Y}) \end{array}$$

are commutative for all $S \xrightarrow{f} \mathcal{Y}$, where S is an affine scheme almost of finite type.

23.7.4. Then parallel to (23.8), we have:

$$\mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathcal{Y})}(\phi^*, \mathrm{QCoh}(\mathcal{Y})) \simeq \mathcal{O}_{\mathcal{Y}^\phi} \text{ and } \mathrm{Tr}^{\mathrm{enh}}_{\mathrm{QCoh}(\mathcal{Y})}(\phi^!, \mathrm{IndCoh}(\mathcal{Y})) \simeq \text{un-ren}_{\mathcal{Y}^\phi}(\omega_{\mathcal{Y}^\phi}), \tag{23.16}$$

23.7.5. Finally, we conjecture that a generalization of Conjecture 23.6.9, stated "as-is" holds in the case of formal stacks as well.

23.8. **Towards an explicit spectral description of the space of automorphic functions.** In this subsection we will assume two of our Main Conjectures, 21.3.7 and 20.2.7 and (try to) deduce consequences for Autom.

23.8.1. First, putting the above two conjectures together, we obtain:

Main Conjecture 23.8.2. *We have a canonical isomorphism*

$$\mathrm{Autom} \simeq \mathrm{Tr}(\mathrm{Frob}^*, \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))).$$

Since Frob is an automorphism of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, in the above conjecture we could replace the functor Frob^* by $\mathrm{Frob}^!$.

Thus, assuming the above conjecture, in order to describe Autom, we wish to have an explicit description of the object

$$\mathrm{Tr}(\mathrm{Frob}^!, \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))) \in \mathrm{Vect}_{\mathsf{e}}\,.$$

23.8.3. We apply the discussion in Sect. 23.7 to $\mathcal{Y} = \mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ with $\phi = \mathrm{Frob}$. We note that the inclusion

$$\mathrm{Nilp}^{\mathrm{Frob}} \hookrightarrow \mathrm{Sing}_2(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)) = \mathrm{Arth}^{\mathrm{arithm}}(X)$$

is indeed an equality.

Hence, combining Conjecture 20.2.7 with Corollary 23.6.11 (for formal stacks, see Sect. 23.7.5), we obtain:

Main Conjecture 23.8.4. *We have a canonical isomorphism in* $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$*:*

$$\mathrm{Drinf} \simeq \text{un-ren}_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}(\omega_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}).$$

23.8.5. Taking global sections over $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$, and taking into account Conjecture 21.3.7, we obtain:

Main Conjecture 23.8.6. *We have a canonical isomorphism*

$$\mathrm{Autom} \simeq \Gamma^{\mathrm{IndCoh}}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X), \omega_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}).$$

Note that Conjecture 23.8.6 provides an explicit description of the space of automorphic functions in terms of Galois representations.

Remark 23.8.7. Note also that the statement of Conjecture 23.8.6 is close to the most naive guess for the expression of Autom in terms of the moduli space of Galois representations: the latter would say that we should take sections of the structure sheaf on $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)$, whereas Conjecture 23.8.6 says that we should rather take sections of the dualizing complex.

Note, however, that the objects

$$\mathcal{O}_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)} \text{ and } \text{un-ren}_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)}(\omega_{\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X)})$$

of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$ are very far apart:

Since $\mathrm{LocSys}^{\mathrm{arithm}}_{\check{G}}(X))$ is not quasi-smooth, the structure sheaf goes infinitely off in the connective direction, while the dualizing complex goes infinitely off in the coconnective direction.

24. Proofs of Theorems 23.1.4 and 23.1.6

24.1. **Proof of Theorem 23.1.4.** In this subsection we will prove Theorem 23.1.4. We go back to the notations of Part I, and denote the reductive group for which we are considering local systems by G.

The key ingredient will be provided by Theorem 5.4.2, which gives us a handle on "how far is $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ from being an algebraic stack", combined with some fundamental facts from algebraic geometry pertaining to Weil sheaves on curves (specifically, Weil-II and L. Lafforgue's theorem, which says that every irreducible Weil local system is pure).

24.1.1. First off, the assertion that $(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))^{\mathrm{Frob}}$ is mock-affine and locally almost of finite type as a prestack follows from the corresponding property of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$.

To prove the remaining assertions of the theorem, by Proposition 3.3.8, it suffices to consider the case of $\mathsf{G} = GL_n$.

The assertion of the theorem can be broken into two parts:

(a) There are only finitely many connected components of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ that are invariant under Frob.

(b) For each connected component $\mathcal{Z}$ of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$, the fiber product

$$(\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X))^{\mathrm{Frob}} \underset{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)}{\times} \mathcal{Z}$$

is an algebraic stack (as opposed to an ind-algebraic stack, see Sect. 5.2).

24.1.2. We start by proving (a). Recall (see Proposition 3.7.2) that connected components of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ are in bijection with isomorphism classes of semi-simple local systems. To such a local system we can associate a partition

$$n = (n_1 + ... + n_1 + n_2 + ... + n_2 + ... + n_k + ... + n_k), \quad n_i \neq n_j$$

with n_i appearing m_i times, and a collection of irreducible local systems

$$(\sigma^1_{n_1}, ..., \sigma^{m_1}_{n_1}, \sigma^1_{n_2}, ..., \sigma^{m_2}_{n_2}, \sigma^1_{n_k}, ..., \sigma^{m_k}_{n_k}), \tag{24.1}$$

where each $\sigma^?_{n_i}$ has rank n_i.

We claim that there is only a finite number of possibilities for a string (24.1), provided that its isomorphism class is invariant under the Frobenius.

Indeed, the isomorphism class as above is invariant under the Frobenius if for every $j = 1, ..., k$ there exists an element $g_j \in \Sigma_{m_j}$ (the symmetric group on m_j letters) such that

$$(\mathrm{Frob}_X(\sigma^1_{n_j}), ..., \mathrm{Frob}_X(\sigma^{m_j}_{n_j})) = (\sigma^{g_j(1)}_{n_j}, ..., \sigma^{g_j(m_j)}_{n_j}).$$

For every j let $d_j := \mathrm{ord}(g_j)$. We obtain that all local systems $\sigma^?_{n_j}$ are invariant under $(\mathrm{Frob}_X)^{d_j}$. I.e., each such local system is an irreducible local system (over $\overline{\mathbb{F}}_q$) that can be equipped with a Weil structure (with respect to $\mathbb{F}_{q^{d_j}}$).

We claim that the number of isomorphism classes of such local systems is finite.

To prove this, it suffices to show that the number of irreducible Weil local systems of a given rank r and a fixed determinant is finite. The latter follows from L. Lafforgue's theorem ([LLaf]), which says that such Weil local systems are in bijection with unramified cuspidal automorphic representations of GL_r with a fixed central character. Now, the number of such automorphic representations (for a given function field) is finite.

24.1.3. We now start tackling point (b) from Sect. 24.1.1.

Let $\mathcal{Z}$ be a connected component of $\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)$ invariant under the Frobenius. Denote

$$\mathcal{Z}^{\mathrm{rigid}_x} := \mathcal{Z} \underset{\mathrm{LocSys}^{\mathrm{restr}}_{\mathsf{G}}(X)}{\times} \mathrm{LocSys}^{\mathrm{restr},\mathrm{rigid}_x}_{\mathsf{G}}(X).$$

It is enough to show that

$$(\mathcal{Z}^{\mathrm{Frob}})^{\mathrm{rigid}_x} := \mathcal{Z}^{\mathrm{Frob}} \underset{\mathcal{Z}}{\times} \mathcal{Z}^{\mathrm{rigid}_x} \tag{24.2}$$

is an affine scheme; a priori we know that it is an ind-affine ind-scheme.

It follows from Corollary 2.2.6(a) that $(\mathcal{Z}^{\mathrm{Frob}})^{\mathrm{rigid}_x}$ has a connective corepresentable deformation theory. Therefore, by [Lu3, Theorem 18.1.0.1], it suffices to show that ${}^{\mathrm{cl}}((\mathcal{Z}^{\mathrm{Frob}})^{\mathrm{rigid}_x})$ is a classical affine scheme. Equivalently, it suffices to show that the underlying classical prestack of $\mathcal{Z}^{\mathrm{Frob}}$ itself is a classical algebraic stack (as opposed to an ind-algebraic stack).

24.1.4. Set

$$\mathcal{W}^{\mathrm{rigid}_x} := \mathrm{pt} \underset{\mathcal{Z}^{\mathrm{coarse}}}{\times} \mathcal{Z}^{\mathrm{rigid}_x},$$

where $\mathcal{Z}^{\mathrm{coarse}}$ is as in Theorem 5.4.2, and set also

$$\mathcal{W} := \mathcal{W}^{\mathrm{rigid}_x}/\mathsf{G} \simeq \mathrm{pt} \underset{\mathcal{Z}^{\mathrm{coarse}}}{\times} \mathcal{Z}.$$

We will prove:

Proposition 24.1.5. *The map $\mathcal{W}^{\mathrm{Frob}} \to \mathcal{Z}^{\mathrm{Frob}}$ induces an isomorphism of the underlying classical prestacks.*

This proposition immediately implies that ${}^{\mathrm{cl}}(\mathcal{Z}^{\mathrm{Frob}})$ is an algebraic stack, since we know that $\mathcal{W}$ (and hence $\mathcal{W}^{\mathrm{Frob}}$) is an algebraic stack, by Corollary 5.4.5.

24.1.6. Note that on the level of the underlying classical prestacks, the map

$$\mathrm{pt} \to \mathcal{Z}^{\mathrm{coarse}}$$

is fully faithful (since $\mathcal{Z}^{\mathrm{coarse}}$ is a derived scheme).

Hence, the assertion of Proposition 24.1.5 is equivalent to the following one:

Proposition 24.1.7. *The composition*

$$\mathcal{Z}^{\mathrm{Frob}} \to \mathcal{Z} \xrightarrow{r} \mathcal{Z}^{\mathrm{coarse}} \tag{24.3}$$

factors as

$$\mathcal{Z}^{\mathrm{Frob}} \to \mathrm{pt} \to \mathcal{Z}^{\mathrm{coarse}} \tag{24.4}$$

at the level of the underlying classical prestacks.

24.2. **Proof of Proposition 24.1.7: the pure case.** Proposition 24.1.7 says that all global functions on $\mathcal{Z}$ become constant, when restricted to $\mathcal{Z}^{\mathrm{Frob}}$.

In this subsection we will prove this assertion on a neighborhood of a point of $\mathcal{Z}^{\mathrm{Frob}}$ that corresponds to a *pure* local system. The proof will use Weil-II.

24.2.1. Let

$$S \to \mathcal{Z}^{\mathrm{Frob}} \tag{24.5}$$

be a map, where $S = \mathrm{Spec}(A)$ with A classical Artinian.

It suffices to show that for any such map, the composition

$$S \to \mathcal{Z}^{\mathrm{Frob}} \to \mathcal{Z} \overset{r}{\to} \mathcal{Z}^{\mathrm{coarse}} \tag{24.6}$$

factors as

$$S \to \mathrm{pt} \to \mathcal{Z}^{\mathrm{coarse}}. \tag{24.7}$$

We can think of (24.5) as a local system E_A on X, endowed with a Weil structure, and equipped with an action of A, whose fiber at $x \in X$ is a (locally) free A-module of rank n.

24.2.2. Let E be the Weil local system corresponding to the composition

$$\mathrm{pt} \to S \to \mathcal{Z}^{\mathrm{Frob}}.$$

Let us first consider the case when E is *pure of weight* 0 (with respect to some identification $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$).

Let $\overline{E}$ denote the underlying local system, when we forget the Weil structure. Let $\mathrm{Aut}(\overline{E})$ denote the *classical* algebraic group of automorphisms of $\overline{E}$.

Varying the Weil structure on $\overline{E}$ defines a map

$$\mathrm{Aut}(\overline{E})/\,\mathrm{Ad}_{\mathrm{Frob}}(\mathrm{Aut}(\overline{E})) \to \mathcal{Z}^{\mathrm{Frob}}, \tag{24.8}$$

where $\mathrm{Ad}_{\mathrm{Frob}}(\mathrm{Aut}(\overline{E}))$ stands for the action of $\mathrm{Aut}(\overline{E})$ on itself given by

$$g(g_1) = \mathrm{Frob}(g) \cdot g_1 \cdot g^{-1},$$

and where Frob is the automorphism of $\mathrm{Aut}(\overline{E})$ induced by

$$\mathrm{Aut}(\overline{E}) \overset{\text{functoriality}}{\longrightarrow} \mathrm{Aut}(\mathrm{Frob}(\overline{E})) \overset{\text{Weil structure}}{\longrightarrow} \mathrm{Aut}(\overline{E}).$$

24.2.3. We claim that the map (24.8) defines a formal isomorphism at E. In order to prove this, it suffices to show that the map (24.8) induces an isomorphism at the level of tangent spaces.

We have:

$$T_1(\mathrm{Aut}(\overline{E})) \simeq H^0(X, \mathrm{End}(\overline{E})),$$

and

$$T_1(\mathrm{Aut}(\overline{E})/\,\mathrm{Ad}_{\mathrm{Frob}}(\mathrm{Aut}(\overline{E}))) \simeq \mathrm{coFib}\left(H^0(X, \mathrm{End}(\overline{E})) \overset{\mathrm{Frob}-\mathrm{Id}}{\longrightarrow} H^0(X, \mathrm{End}(\overline{E}))\right).$$

We also have

$$T_E(\mathcal{Z}^{\mathrm{Frob}}) \simeq \mathrm{Fib}\left(T_{\overline{E}}(\mathcal{Z}) \overset{\mathrm{Frob}-\mathrm{Id}}{\longrightarrow} T_{\overline{E}}(\mathcal{Z})\right),$$

where

$$T_{\overline{E}}(\mathcal{Z}) \simeq \mathrm{C}^{\cdot}(X, \mathrm{End}(\overline{E}))[1].$$

The map that (24.8) induces at the level of tangent spaces corresponds to canonical map

$$H^0(X, \mathrm{End}(\overline{E})) \to \mathrm{C}^{\cdot}(X, \mathrm{End}(\overline{E})).$$

Hence, in order to show that

$$T_1(\mathrm{Aut}(\overline{E})/\,\mathrm{Ad}_{\mathrm{Frob}}(\mathrm{Aut}(\overline{E}))) \to T_E(\mathcal{Z}^{\mathrm{Frob}})$$

is an isomorphism, it suffices to show that $\mathrm{Frob} - \mathrm{Id}$ induces an isomorphism on $H^1(X, \mathrm{End}(\overline{E}))$ and $H^2(X, \mathrm{End}(\overline{E}))$. In other words, we have to show that Frob does not have eigenvalue 1 on either $H^1(X, \mathrm{End}(\overline{E}))$ or $H^2(X, \mathrm{End}(\overline{E}))$.

24.2.4. We will now use the assumption that E is pure of weight 0.

This assumption implies that the induced Weil structure on $\mathrm{End}(\overline{E})$ is also pure of weight 0. Hence, the eigenvalues of Frob on $H^1(X, \mathrm{End}(\overline{E}))$ (resp., $H^2(X, \mathrm{End}(\overline{E}))$) are algebraic numbers that under any complex embedding have Archimedean absolute values $q^{\frac{1}{2}}$ (resp., q).

In particular, these eigenvalues are different from 1.

24.2.5. Thus, we have established that the map (24.8) is a formal isomorphism at E. Hence, by deformation theory, the initial map

$$S \to \mathcal{Z}^{\mathrm{Frob}}$$

of (24.5) can be lifted to a map

$$S \to \mathrm{Aut}(\overline{E}) / \mathrm{Ad}_{\mathrm{Frob}}(\mathrm{Aut}(\overline{E})).$$

However, the composite map

$$\mathrm{Aut}(\overline{E}) / \mathrm{Ad}_{\mathrm{Frob}}(\mathrm{Aut}(\overline{E})) \to \mathcal{Z}^{\mathrm{Frob}} \to \mathcal{Z}$$

by definition factors as

$$\mathrm{Aut}(\overline{E}) / \mathrm{Ad}_{\mathrm{Frob}}(\mathrm{Aut}(\overline{E})) \to \mathrm{pt} / \mathrm{Aut}(\overline{E}) \to \mathcal{Z},$$

while the composition

$$\mathrm{pt} / \mathrm{Aut}(\overline{E}) \to \mathcal{Z} \to \mathcal{Z}^{\mathrm{coarse}}$$

factors as

$$\mathrm{pt} / \mathrm{Aut}(\overline{E}) \to \mathrm{pt} \to \mathcal{Z}^{\mathrm{coarse}}.$$

This proves the required factorization of (24.6) as (24.7) (in the case when E was pure of weight 0).

24.3. **Reduction to the pure case.** Above we have established the factorization of (24.6) as (24.7) when the Weil local system E was pure of weight 0.

In this subsection we will perform the reduction to this case. The source of pure local systems will be provided by the theorem of L. Lafforgue, which says that every irreducible Weil local system is pure.

24.3.1. Let us choose an isomorphism

$$\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}, \tag{24.9}$$

so we can assign the Archimedean absolute value to elements of $\overline{\mathbb{Q}}_\ell$. With this choice, we claim that every Weil local system E' on X acquires a *canonical* (weight) filtration, indexed by real numbers

$$0 \subset ... \subset E'_{r_1} \subset E'_{r_2} \subset ... \subset E',$$

such that each subquotient

$$\mathrm{gr}_r(E')$$

is "pure of weight r" in the sense that it is of the form

$$E_0 \otimes \ell_r, \tag{24.10}$$

where:

- E_0 is pure of weight 0 (with respect to (24.9));
- ℓ_r is a character of $\mathbb{Z} = \mathrm{Weil}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, on which the generator $1 \in \mathbb{Z}$ acts by a scalar with Archimedean absolute value $q^{\frac{r}{2}}$.

Moreover, this filtration is functorial in E' and is compatible with tensor products.

The existence and properties of such a filtration follow from the combination of the following three facts:

- For two local systems $E_0^1 \otimes \ell_{r^1}$ and $E_0^2 \otimes \ell_{r^2}$ of the form (24.10),

$$r_1 \neq r_2 \;\Rightarrow\; \mathrm{Hom}(E_0^1 \otimes \ell_{r^1}, E_0^2 \otimes \ell_{r^2}) = 0.$$

- For two local systems $E_0^1 \otimes \ell_{r^1}$ and $E_0^2 \otimes \ell_{r^2}$ of the form (24.10),
$$r_1 < r_2 \Rightarrow \mathrm{Ext}^1(E_0^1 \otimes \ell_{r^1}, E_0^2 \otimes \ell_{r^2}) = 0.$$
This follows from [De].
- Every irreducible Weil local system on X is of the form (24.10). This is a theorem of L. Lafforgue, [LLaf].

24.3.2. Applying this construction to $E' = E_A$ (see Sect. 24.2.1), we obtain a filtration
$$0 = E_{A,0} \subset E_{A,1} \subset ... \subset E_{A,k} = E_A$$
by Weil local systems, stable under the action of A.

We claim that the fibers of $\mathrm{gr}_i(E_A)$ at any $x \in X$ are (locally) free over A. For that end, it suffices to show that the induced filtration on $\mathrm{ev}_x(E_A)$ canonically splits.

Indeed, let d be such that x is defined over $\overline{\mathbb{F}}_{q^d}$. Then Frob^d acts on $\mathrm{ev}_x(E_A)$, and its action on the different subquotients
$$\mathrm{gr}_i(E_A)$$
has distinct generalized eigenvalues.

24.3.3. Thus, we obtain that we can lift our initial S-point of $\mathcal{Z}^{\mathrm{Frob}}$ to a point of
$$(\mathrm{LocSys}_{\mathsf{P}}^{\mathrm{restr}}(X))^{\mathrm{Frob}},$$
where P is the parabolic corresponding to the ranks of $\mathrm{gr}_i(E_A)$.

Let $\mathcal{Z}_{\mathsf{P}}$ denote the corresponding connected component of $\mathrm{LocSys}_{\mathsf{P}}^{\mathrm{restr}}(X)$, i.e., we now have a map
$$S \to (\mathcal{Z}_{\mathsf{P}})^{\mathrm{Frob}}. \tag{24.11}$$

It suffices to show that the composition
$$S \overset{(24.11)}{\longrightarrow} (\mathcal{Z}_{\mathsf{P}})^{\mathrm{Frob}} \to \mathcal{Z}_{\mathsf{P}} \to \mathcal{Z} \to \mathcal{Z}^{\mathrm{coarse}} \tag{24.12}$$
factors as
$$S \to \mathrm{pt} \to \mathcal{Z}^{\mathrm{coarse}}. \tag{24.13}$$

24.3.4. In what follows we will want to consider the coarse moduli space corresponding to $\mathcal{Z}_{\mathsf{P}}$. The slight inconvenience is that $\mathcal{Z}_{\mathsf{P}}$ is *not* ind mock-affine (because P is not reductive). We will overcome this as follows.

Set
$$\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} := \mathcal{Z}_{\mathsf{P}} \underset{\mathrm{pt}/\mathsf{P}}{\times} \mathrm{pt}/\mathsf{M},$$
which we can also think of as
$$\mathcal{Z}_{\mathsf{P}}^{\mathrm{rigid}_x}/\mathsf{M}$$
for a choice of a Levi splitting $\mathsf{M} \to \mathsf{P}$.

The map
$$\mathrm{pt}/\mathsf{M} \to \mathrm{pt}/\mathsf{P}$$
is smooth, so the map
$$S \overset{(24.11)}{\longrightarrow} (\mathcal{Z}_{\mathsf{P}})^{\mathrm{Frob}}$$
can be lifted to a map
$$S \to (\mathcal{Z}_{\mathsf{P}})^{\mathrm{Frob}} \underset{\mathcal{Z}_{\mathsf{P}}}{\times} \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x}.$$

It suffices to show that the composition
$$S \to (\mathcal{Z}_{\mathsf{P}})^{\mathrm{Frob}} \underset{\mathcal{Z}_{\mathsf{P}}}{\times} \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{Z}_{\mathsf{P}} \to \mathcal{Z} \to \mathcal{Z}^{\mathrm{coarse}} \tag{24.14}$$

factors as

$$S \to \mathrm{pt} \to \mathcal{Z}^{\mathrm{coarse}}. \tag{24.15}$$

24.3.5. Since

$$\mathcal{Z}_{\mathsf{P}}^{\mathrm{rigid}_x} := \mathcal{Z}_{\mathsf{P}} \underset{\mathrm{pt}/\mathsf{P}}{\times} \mathrm{pt}$$

is ind-affine ind-scheme, and M is reductive, the ind-algebraic stack $\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x}$ is ind mock-affine. Hence, we have the well-defined ind-affine ind-scheme

$$\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x,\mathrm{coarse}},$$

and by construction, any map

$$\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{U},$$

where $\mathcal{U}$ is an ind-affine ind-scheme, factors as

$$\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x,\mathrm{coarse}} \to \mathcal{U}.$$

24.3.6. In particular, the map

$$\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{Z}_{\mathsf{P}} \to \mathcal{Z} \to \mathcal{Z}^{\mathrm{coarse}}$$

that appears in (24.14) factors as

$$\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x,\mathrm{coarse}} \to \mathcal{Z}^{\mathrm{coarse}}.$$

Hence, it suffices to show that the composition

$$S \to (\mathcal{Z}_{\mathsf{P}})^{\mathrm{Frob}} \underset{\mathcal{Z}_{\mathsf{P}}}{\times} \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x,\mathrm{coarse}} \tag{24.16}$$

factors as

$$S \to \mathrm{pt} \to \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x,\mathrm{coarse}}, \tag{24.17}$$

where

$$\mathrm{pt} \to \mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x,\mathrm{coarse}}$$

is the unique e-point of $\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x,\mathrm{coarse}}$, see isomorphism (24.18) below.

24.3.7. Let $\mathcal{Z}_{\mathsf{M}}$ be the connected component of $\mathrm{LocSys}_{\mathsf{M}}^{\mathrm{restr}}(X)$, corresponding to $\mathcal{Z}_{\mathsf{P}}$. By the argument in Sect. 5.1.11, the projection

$$\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x} \to \mathcal{Z}_{\mathsf{M}}$$

induces an isomorphism

$$\mathcal{Z}_{\mathsf{P}}^{\text{unip-rigid}_x,\mathrm{coarse}} \simeq \mathcal{Z}_{\mathsf{M}}^{\mathrm{coarse}}. \tag{24.18}$$

Hence, it suffices to show that the composition

$$S \overset{(24.11)}{\longrightarrow} (\mathcal{Z}_{\mathsf{P}})^{\mathrm{Frob}} \to (\mathcal{Z}_{\mathsf{M}})^{\mathrm{Frob}} \to \mathcal{Z}_{\mathsf{M}} \to \mathcal{Z}_{\mathsf{M}}^{\mathrm{coarse}}$$

factors as

$$S \to \mathrm{pt} \to \mathcal{Z}_{\mathsf{M}}^{\mathrm{coarse}}.$$

24.3.8. Write

$$\mathsf{M} = \prod_i GL_{n_i},$$

so it is enough to prove the corresponding factorization assertion for each of the GL_{n_i} factors separately.

However, by the assumption on $\mathrm{gr}_i(E_A)$, this reduces us to the pure of weight 0 case considered in Sect. 24.2. Indeed, the resulting local systems $\mathrm{gr}_i(E_A)$ are pure of weight 0 (up to a twist by a line).

□[Theorem 23.1.4]

24.4. **Proof of Theorem 23.1.6.**

24.4.1. To prove the theorem, it suffices to show that for an irreducible Weil local system σ, the tangent space

$$T_\sigma\left((\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))^{\mathrm{Frob}}\right)=0.$$

We have:

$$T_\sigma\left((\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X))^{\mathrm{Frob}}\right)\simeq \mathrm{Fib}(T_\sigma\left(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)\right)\overset{\mathrm{Frob}-\mathrm{Id}}{\longrightarrow} T_\sigma\left(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)\right),$$

while

$$T_\sigma\left(\mathrm{LocSys}_{\mathsf{G}}^{\mathrm{restr}}(X)\right)\simeq \mathrm{C}^\cdot(X,\mathsf{g}_\sigma)[1].$$

So, it is enough to show that Frob does not have fixed vectors when acting on $H^i(X,\mathsf{g}_\sigma)$, $i=0,1,2$.

24.4.2. We first consider the case of $i=0$.

Note that

$$(H^0(X,\mathsf{g}_\sigma))^{\mathrm{Frob}}$$

is the Lie algebra of the *classical* group of automorphisms of σ as a Weil local system.

If this group has a non-trivial connected component, a standard argument implies that σ can be reduced to a proper parabolic.

24.4.3. To treat the cases $i=1,2$, it suffices to prove the following:

Proposition 24.4.4. *Let G be a semi-simple group and let σ be an irreducible Weil G-local system. Then for any $V\in\mathrm{Rep}(\mathsf{G})^{c,\heartsuit}$, the associated Weil local system V_σ is pure of weight 0.*

This proposition is likely well-known. We will provide a proof for completeness.

24.4.5. First, we recall the following general construction:

Let

$$\mathrm{Fil}_{\mathbb{R}}(\mathrm{Vect}_{\mathsf{e}}^{c,\heartsuit})$$

be the abelian symmetric monoidal category consisting of finite-dimensional vector spaces, endowed with a filtration indexed by the real numbers.

Let us be given a G-torsor σ, thought of as a symmetric monoidal functor

$$\mathsf{F}_\sigma:\mathrm{Rep}(\mathsf{G})^{c,\heartsuit}\to\mathrm{Vect}_{\mathsf{e}}^{c,\heartsuit}.$$

Note that the datum of a lift of F to a symmetric monoidal functor

$$\mathsf{F}_\sigma^{\mathrm{Fil}_{\mathbb{R}}}:\mathrm{Rep}(\mathsf{G})^{c,\heartsuit}\to\mathrm{Fil}_{\mathbb{R}}(\mathrm{Vect}_{\mathsf{e}}^{c,\heartsuit})$$

is equivalent to the datum of a reduction σ_{P} of σ to a parabolic P (denote its Levi quotient by M) and an element $\lambda\in\pi_{1,\mathrm{alg}}(Z^0_{\mathsf{M}})\otimes\mathbb{R}$, which is dominant and (G,M)-regular (see Sect. 19.4.3 for what this means).

24.4.6. With respect to this bijection, for $V\in\mathrm{Rep}(\mathsf{G})^{c,\heartsuit}$, the filtration on $\mathsf{F}_\sigma(V)$ is recovered as follows:

The choice of P defines a filtration on V by P-subrepresentations, indexed by the poset of characters of Z^0_{M},

$$V_\chi\subset V,\quad \chi\in\mathrm{Hom}(Z^0_{\mathsf{M}},\mathbb{G}_m).$$

such that for a given character χ, the action of P on the subquotient $\mathrm{gr}_\chi(V)$ factors through M with Z^0_{M} acting by χ. Denote by

$$\mathsf{F}_{\sigma_{\mathsf{P}}}(V_\chi)\subset\mathsf{F}_\sigma(V)$$

the induced filtration on $\mathsf{F}_\sigma(V)$.

Now for $r\in\mathbb{R}$, the subspace

$$(\mathsf{F}_\sigma(V))_r\subset\mathsf{F}_\sigma(V)$$

is the sum of the subspaces

$$\mathsf{F}_{\sigma_{\mathsf{P}}}(V_\chi),\quad \langle\chi,\lambda\rangle\le r.$$

In particular, if G is semi-simple, then $\mathsf{P} = \mathsf{G}$ if and only the lift $\mathsf{F}_\sigma^{\mathrm{Fil}_{\mathbb{R}}}$ is trivial, i.e., for every $V \in \mathrm{Rep}(\mathsf{G})^{c,\heartsuit}$

$$\mathrm{gr}_r(\mathsf{F}_\sigma(V)) = 0 \text{ for } r \neq 0. \tag{24.19}$$

This construction is functorial. In particular, if a group acts on F_σ in a way preserving its lift to $\mathsf{F}_\sigma^{\mathrm{Fil}_{\mathbb{R}}}$, then the action of this group on σ is induced by its action on σ_{P}.

Proof of Proposition 24.4.4. Let $\mathrm{Gal}(X, x)^W$ be the Tannakian pro-algebraic group corresponding to the (abelian) symmetric monoidal category of Weil local systems on X, equipped with the fiber functor given by ev_x.

The datum of σ can viewed as a datum of a symmetric monoidal functor

$$\mathsf{F}_\sigma : \mathrm{Rep}(\mathsf{G})^{c,\heartsuit} \to \mathrm{Vect}_{\mathsf{e}}^{c,\heartsuit},$$

acted on by $\mathrm{Gal}(X, x)^W$.

Recall the setting of Sect. 24.3.1. We obtain that the canonical weight filtration on the Weil local systems V_σ defines a reduction of σ to a parabolic P. Since σ was assumed irreducible, we obtain that $\mathsf{P} = \mathsf{G}$. By (24.19), this implies

$$\mathrm{gr}_r(V_\sigma) = 0 \text{ for } r \neq 0.$$

I.e., all V_σ are pure of weight 0 as required.

□

□[Theorem 23.1.6]

Appendix A. Formal affine schemes vs ind-schemes

In this section we will outline the proof of Theorem 3.1.4.

A.1. **Creating the ring.** In this subsection we will state (a particular case of) [Lu3, Theorem 18.2.3.2] and deduce from it our Theorem 3.1.4.

A.1.1. Ley $\mathcal{Y}$ be an ind-affine ind-scheme. Write it as

$$\mathcal{Y} \simeq \underset{\alpha \in A}{\operatorname{colim}}\, Y_\alpha,$$

where:

- A is a filtered index category;
- $Y_\alpha = \operatorname{Spec}(R_\alpha)$'s are derived affine schemes almost of finite type;
- The transition maps $Y_\alpha \to Y_\beta$ are closed embeddings, i.e., the corresponding maps $R_\beta \to R_\alpha$ induce surjective maps $H^0(R_\beta) \twoheadrightarrow H^0(R_\alpha)$.

We can form a commutative ring

$$R := \lim_{\alpha \in A} R_\alpha.$$

However, in general, we would not be able to say much about this R; in particular, we do not know that it is connective.

A.1.2. Assume now that $\mathcal{Y}$ is as in Theorem 3.1.4, i.e.,

- ${}^{\mathrm{red}}\mathcal{Y}$ is an affine scheme (to be denoted $Y_{\mathrm{red}} = \operatorname{Spec}(R_{\mathrm{red}})$)[45];
- $\mathcal{Y}$ admits a corepresentable deformation theory, i.e., for any $(S, y) \in \mathrm{Sch}^{\mathrm{aff}}_{/\mathcal{Y}}$, the cotangent space $T^*_y(\mathcal{Y}) \in \operatorname{Pro}(\operatorname{QCoh}(S)^{\leq 0})$ actually belongs to $\operatorname{QCoh}(S)^{\leq 0}$.

In this case we claim:

Theorem A.1.3.

(a) *The ring R is connective. Furthermore, for every n, the natural map*

$$\tau^{\geq -n}(R) \to \lim_{\alpha \in A} \tau^{\geq -n}(R_\alpha)$$

is an isomorphism.

(b) *The ideal $I := \ker(H^0(R) \to R_{\mathrm{red}})$ is finitely generated.*

(c) *The map*

$$\mathcal{Y} \to \operatorname{Spec}(R)^\wedge_{\operatorname{Spec}(R_{\mathrm{red}})}$$

is an isomorphism.

In the above formula, for a prestack $\mathcal{W}$ and a classical reduced prestack $\mathcal{W}^0 \to {}^{\mathrm{red}}\mathcal{W}$, we denote by $\mathcal{W}^\wedge_{\mathcal{W}^0}$ the completion of $\mathcal{W}$ along $\mathcal{W}_0$, i.e.,

$$\operatorname{Maps}(S, \mathcal{W}^\wedge_{\mathcal{W}^0}) = \operatorname{Maps}(S, \mathcal{W}) \underset{\operatorname{Maps}({}^{\mathrm{red}}S, {}^{\mathrm{red}}\mathcal{W})}{\times} \operatorname{Maps}({}^{\mathrm{red}}S, \mathcal{W}^0).$$

A.1.4. The assertion of Theorem A.1.3 implies that of Theorem 3.1.4. Indeed, the possibility to write $\mathcal{Y}$ as a colimit (1.8) is the content of [GR3, Proposition 6.7.4].

The rest of this section is devoted to the proof of Theorem A.1.3.

A.2. **Analysis of the classical truncation.**

---

[45] Note that $Y_{\mathrm{red}} = {}^{\mathrm{red}}Y_\alpha$ for α large, but it is *not* ${}^{\mathrm{red}}\operatorname{Spec}(R)$.

A.2.1. With no restriction of generality, we can replace A be a cofinal subcategory consisting of indices α for which

$$^{\mathrm{red}}Y_\alpha \to \mathrm{Spec}(R_{\mathrm{red}})$$

is an isomorphism.

For an index $\alpha \in A$, let I_α denote the ideal

$$\ker(H^0(R_\alpha) \to R_{\mathrm{red}}).$$

For an integer n, we can consider its n-th power $I_\alpha^n \subset H^0(R_\alpha)$. We claim:

Proposition A.2.2. *For every n, the A-family*

$$\alpha \mapsto H^0(R_\alpha)/I_\alpha^n$$

stabilizes.

Proof. It is clear that the assertion of the proposition for a given n implies it for all $n' \leq n$. Hence, it is enough to prove it for integers n of the form 2^m. The proof proceeds by induction on m. We first consider the base case $m = 1$, so $n = 2$.

Thus, we wish to show that the family

$$\alpha \mapsto I_\alpha/I_\alpha^2$$

stabilizes.

For every α, consider

$$\mathrm{Fib}(T^*(Y_\alpha)|_{Y_{\mathrm{red}}} \to T^*(Y_{\mathrm{red}})) \in \mathrm{QCoh}(Y_{\mathrm{red}})^{\leq 0}.$$

By the assumption on $\mathcal{Y}$, the inverse system

$$\alpha \mapsto \mathrm{Fib}(T^*(Y_\alpha)|_{Y_{\mathrm{red}}} \to T^*(Y_{\mathrm{red}}))$$

is equivalent to a constant object of $\mathrm{QCoh}(Y_{\mathrm{red}})^{\leq 0}$.

Hence, the inverse system

$$\alpha \mapsto H^0\left(\mathrm{Fib}(T^*(Y_\alpha)|_{Y_{\mathrm{red}}} \to T^*(Y_{\mathrm{red}}))\right)$$

is equivalent to a constant object of $\mathrm{QCoh}(Y_{\mathrm{red}})^{\heartsuit}$.

However

$$H^0\left(\mathrm{Fib}(T^*(Y_\alpha)|_{Y_{\mathrm{red}}} \to T^*(Y_{\mathrm{red}}))\right) \simeq I_\alpha/I_\alpha^2$$

(see, e.g., [GR2, Chapter 1, Lemma 5.4.3(a)]), and the transition maps

$$I_\beta/I_\beta^2 \to I_\alpha/I_\alpha^2$$

are surjective.

This implies the stabilization assertion for $n = 2$. The induction step is carried out by the same argument:

Assume that the assertion holds for $n \leq 2^{m-1}$. Let $R_{n,\mathrm{cl}}$ denote the resulting ring (the eventual value of $H^0(R_\alpha)/I_\alpha^n$). Since A is filtered, we can assume that $\mathrm{Spec}(R_{n,\mathrm{cl}})$ maps to all the Y_α. Then we run the same argument as above with Y_{red} replaced by $\mathrm{Spec}(R_{n,\mathrm{cl}})$ for $n = 2^m$.

□

A.2.3. Let $R_{n,\mathrm{cl}}$ as above. Let $J_n := \ker(R_{n,\mathrm{cl}} \to R_{\mathrm{red}})$. By construction, for $m \leq n$, we have

$$R_{n,\mathrm{cl}}/(J_n)^m \simeq R_{m,\mathrm{cl}}.$$

Set

$$R_{\mathrm{cl}} := \lim_n R_{n,\mathrm{cl}}$$

Let J denote the ideal $\ker(R_{\mathrm{cl}} \to R_{\mathrm{red}})$.

By the almost of finite type assumption, the ideal

$$J_2 \subset R_{2,\mathrm{cl}}$$

is finitely generated; choose generators $\overline{f}_1, ...\overline{f}_m$. Let $f_1, ..., f_m$ be their lifts to elements of J.

The following is a standard convergence argument:

Lemma A.2.4.

(a) *The elements $f_1, ..., f_m$ generate J.*

(b) *For any n, the ideal $J^n \subset R_{\mathrm{cl}}$ is closed in the J-adic topology on R_{cl}, and the inclusion*

$$J^n \subset \ker(R_{\mathrm{cl}} \to R_{n,\mathrm{cl}})$$

is an equality.

A.2.5. By construction, we obtain that the map

$${}^{\mathrm{cl}}\mathcal{Y} \to \mathrm{Spec}(R_{\mathrm{cl}})^{\wedge}_{Y_{\mathrm{red}}}$$

is an isomorphism.

A.3. **Derived structure: a stabilization claim.**

A.3.1. For $k \geq 0$ consider the k-th coconnective truncation of $\mathcal{Y}$, denoted ${}^{\leq k}\mathcal{Y}$. Write

$${}^{\leq k}\mathcal{Y} = \underset{\alpha \in A}{\mathrm{colim}}\, Y_{\alpha,k},$$

where $Y_{\alpha,k} = \mathrm{Spec}(R_{\alpha,k}) \in {}^{\leq k}\mathrm{Sch}^{\mathrm{aff}}_{/\mathsf{e}}$ and $R_{\alpha,k} = \tau^{\geq -k}(R_\alpha)$.

Set

$$R_k := \lim_{\alpha \in A} R_{\alpha,k}.$$

Using induction on k, we will prove the following statements:

- (i) The ring R_k connective and for any $k' \leq k$, the map $\tau^{\geq -k'}(R_k) \to R_{k'}$ is an isomorphism. In particular, the map
$$H^0(R_k) \to R_{\mathrm{cl}}$$
is an isomorphism.
- (ii) The map ${}^{\leq k}\mathcal{Y} \to \mathrm{Spec}(R_k)^{\wedge}_{Y_{\mathrm{red}}}$ is an isomorphism.

Once we prove this, the assertion of Theorem A.1.3 will follow by taking the limit over k.

A.3.2. The base of the induction is the case $k = 0$, which has been considered in Sect. A.2. We will now carry out the induction step. Thus, we will assume that the statement is true for k and prove it for $k + 1$.

Thus, we write

$${}^{\leq k+1}\mathcal{Y} = \underset{\alpha \in A}{\mathrm{colim}}\, Y_{\alpha,k+1}.$$

We have $Y_{\alpha,k} := {}^{\leq k}Y_{\alpha,k+1}$, and consider the resulting fiber sequence

$$I_{\alpha,k+1}[k+1] \to R_{\alpha,k+1} \to R_{\alpha,k}, \quad I_{\alpha,k+1} \in \mathrm{QCoh}(Y_{\alpha,k})^{\heartsuit}.$$

A.3.3. For each α, let $J_{\alpha,k+1}$ denote the object in

$$\mathrm{Pro}(R_{\alpha,k}\text{-mod}^{\heartsuit}) \simeq \mathrm{Pro}(R_{\alpha,\mathrm{cl}}\text{-mod}^{\heartsuit})$$

given by

$$\text{``}\lim_{\beta \geq \alpha}\text{''}\, H^0\left(I_{\beta,k+1} \underset{R_{\beta,k}}{\otimes} R_{\alpha,k}\right).$$

We claim:

Proposition A.3.4. *The object $J_{\alpha,k+1}$ belongs to $R_{\alpha,k}$-mod$^{\heartsuit} \subset \mathrm{Pro}(R_{\alpha,k}\text{-mod}^{\heartsuit})$.*

The rest of this subsection is devoted to the proof of Proposition A.3.4.

A.3.5. For a pair of indices $\beta \geq \alpha$, consider

$$\mathrm{Fib}\left(T^*(Y_{\beta,k+1})|_{Y_{\alpha,k}} \to T^*(Y_{\beta,k})|_{Y_{\alpha,k}}\right). \tag{A.1}$$

By [GR2, Chapter 1, Lemma 5.4.3(b)], the object (A.1) lives in cohomological degrees $\leq -(k+1)$, and we we have

$$H^{-(k+1)}\left(\mathrm{Fib}(T^*(Y_{\beta,k+1})|_{Y_{\alpha,k}} \to T^*(Y_{\beta,k})|_{Y_{\alpha,k}})\right) \simeq H^0\left(I_{\beta,k+1} \underset{R_{\beta,k}}{\otimes} R_{\alpha,k}\right) =: J_{\alpha,k+1}.$$

Hence, it suffices to show that the object

$$\text{``}\lim_{\beta \geq \alpha}\text{''}\, H^{-(k+1)}\left(\mathrm{Fib}(T^*(Y_{\beta,k+1})|_{Y_{\alpha,k}} \to T^*(Y_{\beta,k})|_{Y_{\alpha,k}})\right) \in \mathrm{Pro}(R_{\alpha,k}\text{-mod}^{\heartsuit})$$

belongs to $R_{\alpha,k}$-mod$^{\heartsuit} \subset \mathrm{Pro}(R_{\alpha,k}\text{-mod}^{\heartsuit})$.

A.3.6. Again by [GR2, Chapter 1, Lemma 5.4.3(b)], the maps

$$\mathrm{Fib}(T^*(Y_\beta)|_{Y_{\alpha,k}} \to T^*(Y_{\beta,k})|_{Y_{\alpha,k}}) \to \mathrm{Fib}(T^*(Y_{\beta,k+1})|_{Y_{\alpha,k}} \to T^*(Y_{\beta,k})|_{Y_{\alpha,k}})$$

induce isomorphisms on the cohomology in degree $-(k+1)$,

Hence, it suffices to show that the object

$$\text{``}\lim_{\beta \geq \alpha}\text{''}\ \mathrm{Fib}(T^*(Y_\beta)|_{Y_{\alpha,k}} \to T^*(Y_{\beta,k})|_{Y_{\alpha,k}}) \in \mathrm{Pro}(R_{\alpha,k}\text{-mod}) \tag{A.2}$$

actually belongs to $\mathrm{QCoh}(Y_{\alpha,k})$.

A.3.7. Note that the object (A.2) identifies with

$$\mathrm{Fib}(T^*(\mathcal{Y})|_{Y_{\alpha,k}} \to T^*({}^{\leq k}\mathcal{Y})|_{Y_{\alpha,k}}).$$

Now, $T^*(\mathcal{Y})|_{Y_{\alpha,k}}$ belongs to $\mathrm{QCoh}(Y_{\alpha,k})$, by the assumption on $\mathcal{Y}$.

The object $T^*({}^{\leq k}\mathcal{Y})|_{Y_{\alpha,k}}$ also belongs to $\mathrm{QCoh}(Y_{\alpha,k})$, since ${}^{\leq k}\mathcal{Y}$ is a formal completion of an affine scheme, by the inductive hypothesis.

□[Proposition A.3.4]

A.4. The induction step, assertion (i).

A.4.1. To prove assertion (i) in the induction step, we only have to show that

$$\lim_{\alpha \in A} I_{\alpha,k+1} \tag{A.3}$$

lives in cohomological degree 0.

A.4.2. First, we claim that the index category A can be chosen to be the poset $\mathbb{N}$ of natural numbers. Indeed, this follows from [GR3, Proposition 5.2.3].

A.4.3. We have

$$\lim_{\alpha\in A} I_{\alpha,k+1} \simeq \lim_{\alpha\in A}\lim_{\beta\geq\alpha} H^0\left(I_{\beta,k+1} \underset{R_{\beta,k}}{\otimes} R_{\alpha,k}\right) \simeq \lim_{\alpha\in A}\lim_{\beta\geq\alpha} H^0\left(I_{\beta,k+1} \underset{R_{\beta,\mathrm{cl}}}{\otimes} R_{\alpha,\mathrm{cl}}\right).$$

By Proposition A.3.4, we can rewrite this further as

$$\lim_{\alpha\in A} J_{\alpha,k+1},$$

and we claim that the latter object indeed lives in cohomological degree 0.

A.4.4. Note that for $\alpha''\geq\alpha'$, we have

$$J_{\alpha',k+1}\simeq H^0(J_{\alpha'',k+1,\alpha} \underset{R_{\alpha'',\mathrm{cl}}}{\otimes} R_{\alpha',\mathrm{cl}}).$$

Hence, for $\alpha''\geq\alpha'$, the transition map $J_{\alpha'',k+1}\to J_{\alpha',k+1}$ is surjective. Since the category of induces is $\mathbb{N}$, this implies the desired assertion.

A.5. **The induction step, assertion (ii).**

A.5.1. We now proceed to the proof of assertion (ii) in the induction step, i.e., we wish to show that

$$ {}^{\leq k+1}\mathcal{Y}\to \mathrm{Spec}(R_{k+1})^\wedge_{Y_{\mathrm{red}}} \tag{A.4}$$

is an isomorphism.

Since both sides are $(k+1)$-coconnective as prestacks, it is enough to show that (A.4) is an isomorphism when evaluated on $(k+1)$-coconnective affine schemes. By the inductive hypothesis, we know that it is an isomorphism when evaluated on k-coconnective affine schemes.

A.5.2. Thus, let us be given a pair (S,S'), where S' is a $(k+1)$-coconnective affine scheme and S is its k-coconnective truncation. The morphism $S\to S'$ has a canonical structure of square-zero extension corresponding to $J\in \mathrm{QCoh}(S)^\heartsuit[k+1]$ and a map

$$T^*(S)\to J[1].$$

Let us be given a map $y:S\to {}^{\leq k+1}\mathcal{Y}$ and an extension of the composition

$$S\to {}^{\leq k+1}\mathcal{Y}\to \mathrm{Spec}(R_{k+1}),$$

denoted $\widetilde{y}$, to a map $\widetilde{y}':S'\to\mathrm{Spec}(R_{k+1})$. We need to show that y can be uniquely extended to a map $y':S'\to {}^{\leq k+1}\mathcal{Y}$, so that $\widetilde{y}'$ equals the composition

$$S'\overset{y'}{\to} {}^{\leq k+1}\mathcal{Y}\to \mathrm{Spec}(R_{k+1}).$$

A.5.3. Let α be an index such that the map y factors as

$$S\to Y_{\alpha,k}\to {}^{\leq k+1}\mathcal{Y}.$$

Taking the push-out

$$Y'_{\alpha,k}:=S'\underset{S}{\sqcup} Y_{\alpha,k}$$

we can assume that $S=Y_{\alpha,k}$.

Then the space of sought-for extensions y' is the colimit over $\beta\geq\alpha$ of spaces, denoted $\mathcal{S}_\beta$, of maps

$$y'_\beta: Y'_{\alpha,k}\to Y_{\beta,k+1}$$

equipped with an isomorphism between

$$Y'_{\alpha,k}\overset{y'_\beta}{\longrightarrow} Y_{\beta,k+1}\to \mathrm{Spec}(R_{k+1})$$

and the given map

$$\widetilde{y}': Y'_{\alpha,k}\to \mathrm{Spec}(R_{k+1}).$$

We need to show that the space

$$\underset{\beta\geq\alpha}{\mathrm{colim}}\, \mathcal{S}_\beta \tag{A.5}$$

is contractible.

A.5.4. For each $\beta \geq \alpha$, we obtain a map

$$\mathrm{Fib}(T^*(Y_{\beta,k})|_{Y_{\alpha,k}} \to \mathcal{F}_{\beta,k+1}|_{Y_{\alpha,k}}[k+2]) \to T^*(Y_{\beta,k})|_{Y_{\alpha,k}} \to T^*(Y_{\alpha,k}) \to J[1], \tag{A.6}$$

where $\mathcal{F}_{\beta,k+1}$ is the object of $\mathrm{QCoh}(Y_{\beta,k})^\heartsuit$ corresponding to the $R_{\beta,k}$-module $I_{\beta,k+1}$, so that $\mathcal{F}_{\beta,k+1}|_{Y_{\alpha,k}}$ corresponds to the $R_{\alpha,k}$-module $I_{\beta,k+1} \underset{R_{\beta,k}}{\otimes} R_{\alpha,k}$.

The datum of extension of

$$Y_{\alpha,k} \to Y_{\beta,k} \to Y_{\beta,k+1}$$

to a map $Y_{\alpha,k} \to Y_{\beta,k+1}$ is equivalent to the datum of null-homotopy of the map (A.6).

A.5.5. Let $I_{k+1} \in R_k\text{-mod}^\heartsuit$ denote the object that fits into the fiber sequence

$$I_{k+1}[k+1] \to R_{k+1} \to R_k.$$

Let $\mathcal{F}_{k+1}$ denote the corresponding object in $\mathrm{QCoh}(\mathrm{Spec}(R_k))^\heartsuit$.

The datum of $\widetilde{y}'$ amounts to a null-homotopy of the composition

$$\mathrm{Fib}(T^*(\mathrm{Spec}(R_k))|_{Y_{\alpha,k}} \to \mathcal{F}_{k+1}|_{Y_{\alpha,k}}[k+2]) \to T^*(\mathrm{Spec}(R_k))|_{Y_{\alpha,k}} \to T^*(Y_{\alpha,k}) \to J[1].$$

Denote

$$\begin{aligned}&\mathcal{G}_{\beta,k+1} = \\ &= \mathrm{Fib}\Big(\mathrm{Fib}(T^*(\mathrm{Spec}(R_k))|_{Y_{\alpha,k}} \to \mathcal{F}_{k+1}|_{Y_{\alpha,k}}[k+2]) \to \mathrm{Fib}(T^*(Y_{\beta,k})|_{Y_{\alpha,k}} \to \mathcal{F}_{\beta,k+1}|_{Y_{\alpha,k}}[k+2])\Big) \in \\ &\qquad \in \mathrm{QCoh}(Y_{\alpha,k}).\end{aligned}$$

We obtain a map $\mathcal{G}_{\beta,k+1} \to J$, and the space $\mathcal{S}_\beta$ is isomorphic to the space of null-homotopies of this map.

Hence, in order to show that the space (A.5) is contractible, it is sufficient to show that the object

$$\tau^{\geq -(k+1)}\left(\text{“}\lim_{\beta\geq\alpha}\text{”}\, \mathcal{G}_{\beta,k+1}\right) \in \mathrm{Pro}(\mathrm{QCoh}(Y_{\alpha,k}))$$

is zero.

A.5.6. We rewrite $\text{“}\lim\limits_{\beta\geq\alpha}\text{”}\, \mathcal{G}_{\beta,k+1}$ as the fiber of the map

$$\text{“}\lim_{\beta\geq\alpha}\text{”}\ \mathrm{Fib}(T^*(\mathrm{Spec}(R_k))|_{Y_{\alpha,k}} \to T^*(Y_{\beta,k})|_{Y_{\alpha,k}}) \to \text{“}\lim_{\beta\geq\alpha}\text{”}\ \mathrm{Fib}(\mathcal{F}_{k+1}|_{Y_{\alpha,k}} \to \mathcal{F}_{\beta,k+1}|_{Y_{\alpha,k}})[k+2].$$

However, we note that

$$\begin{aligned}&\text{“}\lim_{\beta\geq\alpha}\text{”}\ \mathrm{Fib}(T^*(\mathrm{Spec}(R_k))|_{Y_{\alpha,k}} \to T^*(Y_{\beta,k})|_{Y_{\alpha,k}}) \simeq \\ &\qquad \simeq \mathrm{Fib}\left(T^*(\mathrm{Spec}(R_k))|_{Y_{\alpha,k}} \to \text{“}\lim_{\beta\geq\alpha}\text{”}\, T^*(Y_{\beta,k})|_{Y_{\alpha,k}}\right) \simeq \\ &\qquad\qquad \simeq \mathrm{Fib}\left(T^*(\mathrm{Spec}(R_k))|_{Y_{\alpha,k}} \to T^*({}^{\leq k}\mathcal{Y})|_{Y_{\alpha,k}}\right)\end{aligned}$$

vanishes, by the induction hypothesis.

Hence, it suffices to show that the map

$$\mathcal{F}_{k+1}|_{Y_{\alpha,k}} \to \text{“}\lim_{\beta\geq\alpha}\text{”}\, \mathcal{F}_{\beta,k+1}|_{Y_{\alpha,k}},$$

where both sides are viewed as objects in $\mathrm{Pro}(\mathrm{QCoh}(Y_{\alpha,k}))$, induces an isomorphism on H^0.

A.5.7. Thus, we wish to show that

$$H^0(I_{k+1} \underset{R_k}{\otimes} R_{\alpha,k}) \to \text{“}\lim_{\beta\geq\alpha}\text{”}\, H^0\left(I_{\beta,k+1} \underset{R_{\beta,k}}{\otimes} R_{\alpha,k}\right),$$

viewed as objects in

$$\text{Pro}(\text{QCoh}(Y_{\alpha,k})^\heartsuit) \simeq \text{Pro}(\text{QCoh}(Y_{\alpha,\text{cl}})^\heartsuit),$$

is an isomorphism.

A.5.8. Note that

$$H^0(I_{k+1} \underset{R_k}{\otimes} R_{\alpha,k}) \simeq H^0(I_{k+1} \underset{R_{\text{cl}}}{\otimes} R_{\alpha,\text{cl}}).$$

Now, by Lemma A.2.4(a), the ring $R_{\alpha,\text{cl}}$ is finitely presented as a R_{cl}-module (indeed, $R_{\alpha,\text{cl}}$ is a quotient of some $R_{n,\text{cl}}$, while each $R_{n,\text{cl}}$ is of finite type and hence Noetherian).

Therefore, the functor

$$M \mapsto H^0(M \underset{R_{\text{cl}}}{\otimes} R_{\alpha,\text{cl}}), \quad R_{\text{cl}}\text{-mod}^\heartsuit \to \text{Vect}_{\mathsf{e}}^\heartsuit$$

commutes with filtered limits.

Hence, since

$$I_{k+1} \simeq \lim_{\beta\geq\alpha} I_{\beta,k+1},$$

we can rewrite $H^0(I_{k+1} \underset{R_{\text{cl}}}{\otimes} R_{\alpha,\text{cl}})$ as

$$\lim_{\beta\geq\alpha} H^0\left(I_{\beta,k+1} \underset{R_{\text{cl}}}{\otimes} R_{\alpha,\text{cl}}\right),$$

and further as

$$\lim_{\beta\geq\alpha} H^0\left(I_{\beta,k+1} \underset{R_{\beta,k}}{\otimes} R_{\alpha,k}\right).$$

A.5.9. Thus, we wish to show that the tautological map

$$\lim_{\beta\geq\alpha} H^0\left(I_{\beta,k+1} \underset{R_{\beta,k}}{\otimes} R_{\alpha,k}\right) \to \text{“}\lim_{\beta\geq\alpha}\text{”}\, H^0\left(I_{\beta,k+1} \underset{R_{\beta,k}}{\otimes} R_{\alpha,k}\right)$$

in $\text{Pro}(\text{QCoh}(Y_{\alpha,k})^\heartsuit)$ is an isomorphism.

However, this is the content of Proposition A.3.4.

Appendix B. Colimits over TwArr(fSet)

The goal of this section is to prove Lemma 11.9.2 and related statements that involve colimits over the category Lemma 11.9.2.

B.1. Operadic left Kan extensions.

B.1.1. Let O and O' be symmetric monoidal categories. We will denote by

$$\text{Funct}^\otimes(O,O'),\ \text{Funct}^{\otimes\text{-rlax}}(O,O'),\ \text{Funct}^{\otimes\text{-llax}}(O,O')$$

respectively, for categories strict, right-lax, and left-lax symmetric monoidal functors.

B.1.2. We now recall a particular aspect of the theory of operadic left Kan extensions [Lu2, Sect. 3.1].

In what follows, we will say that a symmetric monoidal category $\mathbf{A}$ is a cocomplete symmetric monoidal category if the underlying category $\mathbf{A}$ is cocomplete and the tensor product commutes with colimits in each variable.

The following is a special case of [Lu2, Corollary 3.1.3.5][46].

Theorem B.1.3. *Suppose that $F : O \to O'$ is a symmetric monoidal functor between (small) symmetric monoidal categories. For any cocomplete symmetric monoidal category $\mathbf{A}$, the restriction functor*

$$\mathrm{Res}_F : \mathrm{Funct}^{\otimes\text{-rlax}}(O', \mathbf{A}) \to \mathrm{Funct}^{\otimes\text{-rlax}}(O, \mathbf{A})$$

admits a left adjoint $\mathrm{LKE}_F^{\otimes}$ (the "operadic left Kan extension") such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Funct}^{\otimes\text{-rlax}}(O, \mathbf{A}) & \xrightarrow{\mathrm{LKE}_F^{\otimes}} & \mathrm{Funct}^{\otimes\text{-rlax}}(O', \mathbf{A}) \\ \downarrow{\scriptstyle \mathbf{oblv}} & & \downarrow{\scriptstyle \mathbf{oblv}} \\ \mathrm{Funct}(O, \mathbf{A}) & \xrightarrow{\mathrm{LKE}_F} & \mathrm{Funct}(O', \mathbf{A}) \end{array} \tag{B.1}$$

In particular, the left Kan extension of a right-lax symmetric monoidal functor along a symmetric monoidal functor is canonically right-lax symmetric monoidal.

B.1.4. Unraveling the definitions, given a right-lax symmetric monoidal functor $\Phi : O \to \mathbf{A}$, and $o_1, o_2 \in O'$, the structure map

$$(\mathrm{LKE}_F\, \Phi)(o_1') \otimes (\mathrm{LKE}_F\, \Phi)(o_2') \to (\mathrm{LKE}_F\, \Phi)(o_1' \otimes o_2')$$

is the composite

$$\underset{o_1 \in O/o_1'}{\mathrm{colim}}\, \Phi(o_1) \otimes \underset{o_2 \in O/o_2'}{\mathrm{colim}}\, \Phi(o_2) \overset{\sim}{\leftarrow} \underset{(o_1,o_2) \in O/o_1' \times O/o_2'}{\mathrm{colim}} \Phi(o_1) \otimes \Phi(o_2) \to$$

$$\to \underset{(o_1,o_2) \in O/o_1' \times O/o_2'}{\mathrm{colim}} \Phi(o_1 \otimes o_2) \to \underset{o \in O/(o_1' \otimes o_2')}{\mathrm{colim}} \Phi(o),$$

where the first map is an isomorphism since the tensor product in $\mathbf{A}$ commutes with colimits in each variable, the middle map is the right-lax structure of Φ and the last map is induced by the functor $O/o_1' \times O/o_2' \to O/(o_1' \otimes o_2')$ given by tensor product (using the fact that F is strictly symmetric monoidal).

B.1.5. From the above discussion we obtain:

Proposition B.1.6. *Let $F : O \to O'$ be a symmetric monoidal functor such that for any pair of objects $o_1', o_2' \in O'$ the functor*

$$O/o_1' \times O/o_2' \to O/(o_1' \otimes o_2')$$

given by tensor product is cofinal. Then for any cocomplete symmetric monoidal category $\mathbf{A}$ and any (strict) symmetric monoidal functor $\Phi : O \to \mathbf{A}$, the operadic left Kan extension $\mathrm{LKE}_F^{\otimes}(\Phi) : O' \to \mathbf{A}$ is strictly symmetric monoidal.

[46] In the notation of [Lu2], the commutativity of the diagram (B.1) follows from the fact that for a symmetric monoidal functor $O \to O'$, and any object $o' \in O'$, the functor $O/o' \to (O^{\otimes}_{\mathrm{act}})/o'$ is cofinal.

B.1.7. We now specialize Theorem B.1.3 to the case that $O' = \{*\}$. By definition, we have

$$\mathrm{ComAlg}(\mathbf{A}) = \mathrm{Funct}^{\otimes\text{-rlax}}(\{*\}, \mathbf{A}).$$

Given any symmetric monoidal category O, restriction along the terminal symmetric monoidal functor $O \to \{*\}$ gives a diagonal functor

$$\mathrm{ComAlg}(\mathbf{A}) \simeq \mathrm{Funct}^{\otimes\text{-rlax}}(\{*\}, \mathbf{A}) \to \mathrm{Funct}^{\otimes\text{-rlax}}(O, \mathbf{A}) \tag{B.2}$$

In this case, Theorem B.1.3 gives:

Corollary B.1.8. *Let O be a symmetric monoidal category. For any cocomplete symmetric monoidal category $\mathbf{A}$, the diagonal functor* (B.2) *admits a left adjoint*

$$\mathrm{colim}^{\otimes}_{O} : \mathrm{Funct}^{\otimes\text{-rlax}}(O, \mathbf{A}) \to \mathrm{ComAlg}(\mathbf{A})$$

which on underlying objects is given by colimit along O. In particular, the colimit of a right-lax symmetric monoidal functor is canonically a commutative algebra object.

B.2. **Proof of Lemma 11.9.2.** The proof we present was communicated to us by J. Campbell.

B.2.1. Suppose we have a symmetric monoidal functor $\Phi : \mathbf{A}' \to \mathbf{A}$ between cocomplete symmetric monoidal categories. The functor Φ induces a functor

$$\mathrm{ComAlg}(\mathbf{A}') \to \mathrm{ComAlg}(\mathbf{A}). \tag{B.3}$$

Now suppose that the underlying functor fo Φ admits a left adjoint Φ^L. Our present goal is to formulate and prove Proposition B.2.9 which uses Φ^L to give a description of the left adjoint to (B.3). Note that in general, Φ^L itself is only a *left-lax* symmetric monoidal functor and therefore does not induce a functor between commutative algebras.

B.2.2. Recall [Lu2, Construction 2.2.4.1 and Proposition 2.2.4.9] that given a symmetric monoidal category O, there exist universal symmetric monoidal categories $\mathrm{RLax}(O)$ and $\mathrm{LLax}(O)$ equipped with, respectively, right-lax and left-lax symmetric monoidal functors

$$O \to \mathrm{RLax}(O) \quad \text{and} \quad O \to \mathrm{LLax}(O)$$

which induce equivalences

$$\mathrm{Funct}^{\otimes}(\mathrm{RLax}(O), \mathbf{A}) \simeq \mathrm{Funct}^{\otimes\text{-rlax}}(O, \mathbf{A}) \text{ and } \mathrm{Funct}^{\otimes}(\mathrm{LLax}(O), \mathbf{A}) \simeq \mathrm{Funct}^{\otimes\text{-llax}}(O, \mathbf{A})$$

for any symmetric monoidal category $\mathbf{A}$. Evidently,

$$\mathrm{LLax}(O) \simeq \mathrm{RLax}(O^{\mathrm{op}})^{\mathrm{op}}.$$

B.2.3. We have $\mathrm{RLax}(\{*\}) = \mathrm{fSet}$, with the symmetric monoidal structure given by disjoint union. In particular, this gives the equivalence

$$\mathrm{ComAlg}(\mathbf{A}) \simeq \mathrm{Funct}^{\otimes}(\mathrm{fSet}, \mathbf{A})$$

for any symmetric monoidal category $\mathbf{A}$.

B.2.4. Now, suppose we have a symmetric monoidal functor $\Phi : \mathbf{A}' \to \mathbf{A}$ which admits a left adjoint Φ^L. Since Φ^L is canonically left-lax symmetric monoidal, we obtain a functor

$$\mathrm{ComAlg}(\mathbf{A}) \simeq \mathrm{Funct}^{\otimes\text{-rlax}}(\{*\}, \mathbf{A}) \simeq \mathrm{Funct}^{\otimes}(\mathrm{fSet}, \mathbf{A}) \hookrightarrow \tag{B.4}$$
$$\hookrightarrow \mathrm{Funct}^{\otimes\text{-llax}}(\mathrm{fSet}, \mathbf{A}) \overset{\Phi^L\circ}{\to} \mathrm{Funct}^{\otimes\text{-llax}}(\mathrm{fSet}, \mathbf{A}').$$

B.2.5. Let $\mathrm{TwArr}(\mathrm{fSet})$ denote the twisted arrow category of fSet with tensor product given by disjoint union. Applying [Lu2, Construction 2.2.4.1] to (the opposite category of) fSet, we obtain:

Proposition B.2.6. *The left-lax symmetric monoidal functor*

$$\mathrm{fSet} \to \mathrm{TwArr}(\mathrm{fSet})$$

given by $I \mapsto (I \to \{*\})$ induces an equivalence

$$\mathrm{LLax}(\mathrm{fSet}) \simeq \mathrm{TwArr}(\mathrm{fSet}).$$

B.2.7. Composing the functor (B.4) with the equivalence of Proposition B.2.6, we obtain a functor

$$\text{ComAlg}(\mathbf{A}) \to \text{Funct}^{\otimes}(\text{TwArr}(\text{fSet}), \mathbf{A}'). \tag{B.5}$$

Explicitly, given $R \in \text{ComAlg}(\mathbf{A})$, the corresponding functor

$$\text{TwArr}(\text{fSet}) \to \mathbf{A}'$$

is given by

$$(I \overset{\psi}{\to} J) \mapsto \underset{j\in J}{\otimes} \Phi^L(R^{\otimes \psi^{-1}(j)}). \tag{B.6}$$

B.2.8. The following is a more precise version of Lemma 11.9.2:

Proposition B.2.9. *Let $\Phi : \mathbf{A}' \to \mathbf{A}$ be a symmetric monoidal functor between cocomplete symmetric monoidal categories which admits a left adjoint Φ^L. Then the induced functor*

$$\text{ComAlg}(\mathbf{A}') \to \text{ComAlg}(\mathbf{A})$$

admits a left adjoint given by the composite

$$\text{ComAlg}(\mathbf{A}) \xrightarrow{(B.5)} \text{Funct}^{\otimes}(\text{TwArr}(\text{fSet}), \mathbf{A}') \xrightarrow{\text{colim}^{\otimes}} \text{ComAlg}(\mathbf{A}'),$$

where $\text{colim}^{\otimes}$ *is the composition*

$$\text{Funct}^{\otimes}(\text{TwArr}(\text{fSet}), \mathbf{A}') \hookrightarrow \text{Funct}^{\otimes\text{-rlax}}(\text{TwArr}(\text{fSet}), \mathbf{A}') \overset{\text{colim}_{\text{TwArr}(\text{fSet})}}{\to} \text{ComAlg}(\mathbf{A}'), \tag{B.7}$$

where $\text{colim}_{\text{TwArr}(\text{fSet})}$ *as as in Corollary B.1.8.*

B.2.10. Before proving Proposition B.2.9, we establish the following:

Proposition B.2.11. *Let $\mathbf{A}$ be a cocomplete symmetric monoidal category. Then the inclusion functor*

$$\text{ComAlg}(\mathbf{A}) \simeq \text{Funct}^{\otimes}(\text{fSet}, \mathbf{A}) \hookrightarrow \text{Funct}^{\otimes\text{-llax}}(\text{fSet}, \mathbf{A}) \simeq \text{Funct}^{\otimes}(\text{TwArr}(\text{fSet}), \mathbf{A})$$

admits a left adjoint given by

$$\text{colim}^{\otimes} : \text{Funct}^{\otimes}(\text{TwArr}(\text{fSet}), \mathbf{A}) \to \text{ComAlg}(\mathbf{A}),$$

where $\text{colim}^{\otimes}$ *is as in* (B.7).

Proof. The inclusion functor is given by restriction along the symmetric monoidal functor

$$s : \text{TwArr}(\text{fSet}) \to \text{fSet}$$

given by $(I \to J) \mapsto I$. By Theorem B.1.3, we have an adjunction

$$\text{LKE}^{\otimes}_s : \text{Funct}^{\otimes\text{-rlax}}(\text{TwArr}(\text{fSet}), \mathbf{A}) \rightleftarrows \text{Funct}^{\otimes\text{-rlax}}(\text{fSet}, \mathbf{A}) : \text{Res}_s .$$

Since s is strictly symmetric monoidal, the restriction functor Res_s preserves strictly symmetric monoidal functors. Furthermore, for every $I_1, I_2 \in \text{fSet}$, the functor

$$\text{TwArr}(\text{fSet})_{/I_1} \times \text{TwArr}(\text{fSet})_{/I_2} \to \text{TwArr}(\text{fSet})_{/I_1 \sqcup I_2}$$

is cofinal. Therefore, by Proposition B.1.6, $\text{LKE}^{\otimes}_s$ also preserves strictly symmetric monoidal functors. Thus, the desired left adjoint is given by

$$\text{Funct}^{\otimes}(\text{TwArr}(\text{fSet}), \mathbf{A}) \overset{\text{LKE}^{\otimes}_s}{\to} \text{Funct}^{\otimes}(\text{fSet}, \mathbf{A}) \simeq \text{ComAlg}(\mathbf{A}).$$

It remains to show that this functor is canonically isomorphic to $\text{colim}^{\otimes}$. The functor $\text{colim}^{\otimes}_{\text{TwArr}}$ is given by the operadic left Kan extension along the composite

$$\text{TwArr}(\text{fSet}) \overset{s}{\to} \text{fSet} \overset{p}{\to} \{*\}$$

Thus, it suffices to show that the composite functor

$$\text{ComAlg}(\mathbf{A}) \simeq \text{Funct}^{\otimes}(\text{fSet}, \mathbf{A}) \hookrightarrow \text{Funct}^{\otimes\text{-rlax}}(\text{fSet}, \mathbf{A}) \overset{\text{LKE}^{\otimes}_p}{\to} \text{Funct}^{\otimes\text{-rlax}}(\{*\}, \mathbf{A}) \simeq \text{ComAlg}(\mathbf{A})$$

is canonically isomorphic to the identity. By [Lu2, Corollary 7.3.2.7], since p is symmetric monoidal, $\mathrm{LKE}_p^{\otimes}$ is given by restriction along the right adjoint

$$\{*\} \hookrightarrow \mathrm{fSet},$$

which is the universal right-lax symmetric monoidal functor from $\{*\}$. This gives the desired result. □

B.2.12. *Proof of Proposition B.2.9.* Since Φ is symmetric monoidal, by [Lu2, Corollary 7.3.2.7], we obtain an adjunction

$$\Phi^L \circ (-) : \mathrm{Funct}^{\otimes\text{-llax}}(\mathrm{fSet}, \mathbf{A}) \rightleftarrows \mathrm{Funct}^{\otimes\text{-llax}}(\mathrm{fSet}, \mathbf{A}') : \Phi \circ (-).$$

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Funct}^{\otimes}(\mathrm{fSet}, \mathbf{A}') & \xrightarrow{\Phi\circ(-)} & \mathrm{Funct}^{\otimes}(\mathrm{fSet}, \mathbf{A}) \\ \downarrow & & \downarrow \\ \mathrm{Funct}^{\otimes\text{-llax}}(\mathrm{fSet}, \mathbf{A}') & \xrightarrow{\Phi\circ(-)} & \mathrm{Funct}^{\otimes\text{-llax}}(\mathrm{fSet}, \mathbf{A}) \end{array}$$

in which the vertical functors are fully faithful. The desired result now follows from Proposition B.2.11. □

B.2.13. We will show that the colimit expression (8.5) indeed produces the object $\underline{\mathrm{coHom}}(A, B)$.

We have the symmetric monoidal functor

$$(-) \otimes B^\vee : \mathbf{O} \to B^\vee\text{-mod}(\mathbf{O}),$$

which induces a functor

$$\mathrm{ComAlg}(\mathbf{O}) \to \mathrm{ComAlg}(B^\vee\text{-mod}(\mathbf{O})) \tag{B.8}$$

By definition, $\underline{\mathrm{coHom}}(A, B)$ is the left adjoint of (B.8) applied to

$$A \otimes B^\vee \in \mathrm{ComAlg}(B^\vee\text{-mod}(\mathbf{O})).$$

Hence, by Proposition B.2.9 $\underline{\mathrm{coHom}}(A, B)$, can be written as (8.5). □

B.3. Proof of Lemma 8.2.7.

B.3.1. By definition, $\underline{\mathrm{coHom}}(A, B)$ is the left adjoint of the functor

$$(-) \otimes B : \mathrm{ComAlg}(\mathbf{O}) \to \mathrm{ComAlg}(\mathbf{O}) \tag{B.9}$$

applied to $A \in \mathrm{ComAlg}(\mathbf{O})$.

Consider the right-lax symmetric monoidal functor

$$T_B : \mathbf{O} \to \mathrm{Funct}(\mathrm{fSet}, \mathbf{O}) \tag{B.10}$$

given by $X \mapsto (I \mapsto X \otimes (B)^{\otimes I})$. The functor (B.9) factors as

$$\mathrm{ComAlg}(\mathbf{O}) \overset{\mathrm{ComAlg}(T_B)}{\longrightarrow} \mathrm{ComAlg}(\mathrm{Funct}(\mathrm{fSet}, \mathbf{O})) \simeq \mathrm{Funct}(\mathrm{fSet}, \mathrm{ComAlg}(\mathbf{O})) \overset{\mathrm{ev}_{\{*\}}}{\to} \mathrm{ComAlg}(\mathbf{O}),$$

where the last functor is evaluation at $\{*\} \in \mathrm{fSet}$. Thus we have that

$$\underline{\mathrm{coHom}}(A, B) \simeq \mathrm{ComAlg}(T_B)^L \circ \mathrm{ev}_{\{*\}}^L(A).$$

The functor $\mathrm{ev}_{\{*\}}^L$ is given by left Kan extension along the inclusion $\{*\} \hookrightarrow \mathrm{fSet}$ and so

$$\mathrm{ev}_{\{*\}}^L(A) \simeq (I \mapsto A^{\otimes I}) \in \mathrm{Funct}(\mathrm{fSet}, \mathrm{ComAlg}(\mathbf{O})).$$

B.3.2. Thus, we obtain that Lemma 8.2.7 follows from the following two assertions:

(a) The natural map
$$\mathrm{AssocAlg}(T_B)^L(\mathbf{oblv}_{\mathrm{Com}\to\mathrm{Assoc}}) \to \mathbf{oblv}_{\mathrm{Com}\to\mathrm{Assoc}}(\mathrm{ComAlg}(T_B)^L)$$
of functors $\mathrm{ComAlg}(\mathrm{Funct}(\mathrm{fSet}, \mathbf{O})) \to \mathrm{AssocAlg}(\mathbf{O})$ is an isomorphism.

(b) The natural map
$$T_B^L(\mathbf{oblv}_{\mathrm{Com}}) \to \mathbf{oblv}_{\mathrm{Com}}(\mathrm{ComAlg}(T_B)^L)$$
of functors $\mathrm{ComAlg}(\mathrm{Funct}(\mathrm{fSet}, \mathbf{O})) \to \mathbf{O}$ is an isomorphism.

Both assertions follow from the assertion that the left adjoint T_B^L, which is a priori left-lax symmetric monoidal, is strictly symmetric monoidal. Indeed, in this case, by [Lu2, Corollary 7.3.2.7], we have
$$\mathrm{ComAlg}(T_B)^L \simeq \mathrm{ComAlg}(T_B^L) \text{ and } \mathrm{AssocAlg}(T_B)^L \simeq \mathrm{AssocAlg}(T_B^L).$$

B.3.3. It remains to prove that the left-lax symmetric monoidal structure on T_B^L is strict. By definition, we have
$$\mathrm{Maps}_{\mathbf{O}}(T_B^L(F), X) \simeq \mathrm{Maps}_{\mathrm{Funct}(\mathrm{fSet},\mathbf{O})}(F, T_B(X)).$$
However, (see e.g. [GKRV, Lemma 1.3.12]), the latter expression is canonically identified with
$$\lim_{(I\to J)\in\mathrm{TwArr}(\mathrm{fSet})^{\mathrm{op}}} \mathrm{Maps}_{\mathbf{O}}(F(I), X \otimes B^{\otimes J}) \simeq \lim_{(I\to J)\in\mathrm{TwArr}(\mathrm{fSet})^{\mathrm{op}}} \mathrm{Maps}_{\mathbf{O}}(F(I) \otimes C^{\otimes J}, X).$$
Thus, we have
$$T_B^L(F) \simeq \operatorname*{colim}_{(I\to J)\in\mathrm{TwArr}(\mathrm{fSet})} F(I) \otimes C^{\otimes J}.$$
Moreover, the left-lax structure map
$$T_B^L(F_1 \otimes F_2) \simeq \operatorname*{colim}_{(I\to J)\in\mathrm{TwArr}(\mathrm{fSet})} F_1(I) \otimes F_2(I) \otimes C^{\otimes J} \to \tag{B.11}$$
$$\to T_B^L(F_1) \otimes T_B^L(F_2) \simeq \operatorname*{colim}_{(I_1\to J_1, I_2\to J_2)\in\mathrm{TwArr}(\mathrm{fSet})^{\times 2}} F_1(I_1) \otimes F_2(I_2) \otimes C^{\otimes J_1} \otimes C^{\otimes J_2}$$
is induced by the maps
$$\mathrm{Id} \otimes \mathrm{comult}^{\otimes J} : F_1(I) \otimes F_2(I) \otimes C^{\otimes J} \to F_1(I) \otimes F_2(I) \otimes C^{\otimes J} \otimes C^{\otimes J}.$$

B.3.4. Consider the category
$$\mathrm{TwArr}(\mathrm{fSet}) \underset{\mathrm{fSet}}{\times} \mathrm{fSet}^{\times 2},$$
where the functor $\mathrm{fSet}^{\times 2} \to \mathrm{fSet}$ is given by coproduct. In other words, an object of this category consists of three finite sets I_1, I_2, J and a map $I_1 \sqcup I_2 \to J$.

The functor $\mathrm{TwArr}(\mathrm{fSet})^{\times 2} \to \mathbf{O}$ in the right-hand side of (B.11) is given by restriction along the functor
$$\mathrm{TwArr}(\mathrm{fSet})^{\times 2} \to \mathrm{TwArr}(\mathrm{fSet}) \underset{\mathrm{fSet}}{\times} \mathrm{fSet}^{\times 2} \tag{B.12}$$
given by $(I_1 \to J_1, I_2 \to J_2) \mapsto (I_1, I_2, I_1 \sqcup I_2 \to J_1 \sqcup J_2)$. The functor (B.12) admits a left adjoint given by $(I_1 \sqcup I_2 \to J) \mapsto (I_1 \to J, I_2 \to J)$ and is therefore cofinal.

Hence, we can rewrite the right-hand side of (B.11) as
$$\operatorname*{colim}_{(I_1\sqcup I_2\to J)\in\mathrm{TwArr}(\mathrm{fSet})^{\times 2} \underset{\mathrm{fSet}}{\times} \mathrm{fSet}^{\times 2}} F_1(I_1) \otimes F_2(I_2) \otimes C^{\otimes J}.$$

Furthermore, the map (B.11) is induced by the functor
$$\mathrm{TwArr}(\mathrm{fSet}) \to \mathrm{TwArr}(\mathrm{fSet}) \underset{\mathrm{fSet}}{\times} \mathrm{fSet}^{\times 2}$$
given by $(I \to J) \mapsto (I \sqcup I \to J)$. This functor also admits a left adjoint and is therefore also cofinal.

□[Lemma 8.2.7]

Appendix C. Semi-rigid symmetric monoidal categories

C.1. Definition and examples.

C.1.1. Let $\mathbf{A}$ be a (unital) symmetric monoidal DG category. We shall say that $\mathbf{A}$ is *semi-rigid* if the following two conditions are satisfied:

- (i) The functor $\text{mult}_{\mathbf{A}} : \mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$ admits a continuous right adjoint (to be denoted $\text{comult}_{\mathbf{A}}$) and the structure on $\text{comult}_{\mathbf{A}}$ of right-lax compatibility with the $\mathbf{A}$-bimodule structure is strict.
- (ii) $\mathbf{A}$ is dualizable as a DG category.

C.1.2. Of course, a rigid symmetric monoidal category is semi-rigid.

The one property that distinguishes rigid from semi-rigid is that in the former case we require that the unit object $\mathbf{1}_{\mathbf{A}} \in \mathbf{A}$ be compact.

As we will see shortly, semi-rigid categories behave in a way very similar to rigid ones with respect to 2-categorical properties, i.e., from the point of view of module categories over them.

However, their internal structure is very different in that dualizable objects are not necessarily compact. However, in a compactly generated semi-rigid category, compact objects are still dualizable, see Sect. C.3.

C.1.3. A key example for this paper of a semi-rigid category is $\text{QCoh}(\mathcal{Y})$, where $\mathcal{Y}$ is a formal affine scheme.

Condition (i) in Sect. C.1.1 holds because the diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is affine.

Condition (ii) in Sect. C.1.1 holds because $\text{QCoh}(\mathcal{Y})$ is compactly generated.

C.1.4. Let $\mathcal{Y}$ now be of the form $\mathcal{Y}'/\mathsf{G}$, where $\mathcal{Y}'$ is a formal affine scheme. Then $\text{QCoh}(\mathcal{Y})$ is semi-rigid, see Corollary 7.8.9(a).

C.1.5. Here is another way to deduce that $\text{QCoh}(\mathcal{Y})$ is semi-rigid when $\mathcal{Y}$ is a formal affine scheme, realized as a formal completion of an affine scheme.

Suppose that $\mathbf{A}$ is a semi-rigid symmetric monoidal category, and let $\Phi : \mathbf{A} \to \mathbf{A}'$ be a symmetric monoidal functor. Assume that Φ admits a left adjoint, to be denoted Φ^L, such that:

- Φ^L is fully faithful;
- The left-lax monoidal structure on Φ^L is strict (but not necessarily unital).

Then $\mathbf{A}'$ is also semi-rigid.

C.1.6. A completely different example of a semi-rigid category is $\text{Shv}^{\text{all}}(Y)$ for a scheme Y (or more generally, for a Hausdorff locally compact topological space).

Indeed, in this case, the functor $\text{comult}_{\mathbf{A}}$ identifies with

$$(\Delta_Y)_* \simeq (\Delta_Y)_!.$$

The fact that $\text{Shv}^{\text{all}}(Y)$ is dualizable is also known, see Sect. G.1.3.

C.2. Properties of semi-rigid categories.

In this subsection we let $\mathbf{A}$ be a semi-rigid symmetric monoidal category.

C.2.1. Set

$$\mathsf{R}_{\mathbf{A}} := \text{comult}_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}) \in \mathbf{A} \otimes \mathbf{A}.$$

This object has a canonical structure of commutative algebra in $\mathbf{A} \otimes \mathbf{A}$.

C.2.2. Here is the first observation:

Proposition C.2.3. *Let* $\mathbf{A}$ *be semi-rigid*[47]*. Then for a* $\mathbf{A}$*-bimodule category* $\mathbf{P}$*, there is a canonical equivalence*

$$\mathrm{Funct}_{(\mathbf{A}\otimes\mathbf{A})\text{-mod}}(\mathbf{A},\mathbf{P}) \simeq \mathbf{P} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} \mathbf{A}.$$

Proof. The adjunction

$$\text{(C.1)} \qquad \mathrm{mult}_{\mathbf{A}} : \mathbf{A}\otimes\mathbf{A} \rightleftarrows \mathbf{A} : \mathrm{comult}_{\mathbf{A}}$$

as $\mathbf{A}$-bimodules gives rise to an adjunction

$$\mathbf{P} \simeq \mathrm{Funct}_{(\mathbf{A}\otimes\mathbf{A})\text{-mod}}(\mathbf{A}\otimes\mathbf{A},\mathbf{P}) \rightleftarrows \mathrm{Funct}_{(\mathbf{A}\otimes\mathbf{A})\text{-mod}}(\mathbf{A},\mathbf{P}),$$

where the right adjoint identifies with the forgetful functor

$$\mathrm{Funct}_{(\mathbf{A}\otimes\mathbf{A})\text{-mod}}(\mathbf{A},\mathbf{P}) \to \mathrm{Funct}_{\mathrm{cont}}(\mathbf{A},\mathbf{P}) \to \mathrm{Funct}_{\mathrm{cont}}(\mathrm{Vect}_{\mathsf{e}},\mathbf{P}) = \mathbf{P};$$

in particular, it is conservative, and hence monadic. Furthermore, the resulting monad on $\mathbf{P}$ is given by the action of the object $\mathsf{R}_{\mathbf{A}} \in \mathbf{A}\otimes\mathbf{A}$.

Now, tensoring (C.1) we obtain an adjunction

$$\mathbf{P} \simeq \mathbf{P} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} (\mathbf{A}\otimes\mathbf{A}) \rightleftarrows \mathbf{P} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} \mathbf{A},$$

where the essential image of the left adjoint generates the target category, and hence the right adjoint is conservative and hence monadic. The resulting monad on $\mathbf{P}$ also identifies with one given the action of the object $\mathsf{R}_{\mathbf{A}} \in \mathbf{A}\otimes\mathbf{A}$.

Hence, we have identified both categories in the statement of the proposition with

$$\mathsf{R}_{\mathbf{A}}\text{-mod}(\mathbf{P}).$$

□

In the course of the proof, we have also shown:

Corollary C.2.4. *Suppose that* $\mathbf{P}$ *is dualizable as a plain DG category. Then so is* $\mathbf{P} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} \mathbf{A}$*.*

Proof. The category $\mathbf{P} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} \mathbf{A}$ is equivalent to that of modules over a monad in a dualizable category, and hence is dualizable (see [GKRV, Lemma 1.6.3]). □

C.2.5. Let $\mathbf{M}$ be an $\mathbf{A}$-module category, dualizable as a plain DG category, and consider

$$\mathbf{M}^\vee \simeq \mathrm{Funct}_{\mathrm{cont}}(\mathbf{M},\mathrm{Vect}_{\mathsf{e}})$$

as an $\mathbf{A}$-module via the action on the source.

Corollary C.2.6. *For* $\mathbf{M}$ *as above and another* $\mathbf{A}$*-module category* $\mathbf{N}$*, we have a canonical identification*

$$\mathrm{Funct}_{\mathbf{A}\text{-mod}}(\mathbf{M},\mathbf{N}) \simeq \mathbf{M}^\vee \underset{\mathbf{A}}{\otimes} \mathbf{N}.$$

Proof. Apply Proposition C.2.3 to $\mathbf{P} = \mathbf{M}^\vee \otimes \mathbf{N}$.

□

Corollary C.2.7. *If an* $\mathbf{A}$*-module category* $\mathbf{M}$ *is dualizable as a plain DG category, then it is dualizable as an* $\mathbf{A}$*-module category; moreover we have a canonical equivalence*

$$\text{(C.2)} \qquad \mathrm{Funct}_{\mathbf{A}\text{-mod}}(\mathbf{M},\mathbf{A}) \simeq \mathbf{M}^\vee.$$

[47]In fact we only need condition (i).

Taking $\mathbf{M} = \mathbf{A}$ in (C.2) we obtain a canonical identification

$$\mathbf{A} \simeq \mathbf{A}^\vee \tag{C.3}$$

as $\mathbf{A}$-modules.

We claim:

Lemma C.2.8. *The unit of the self-duality* (C.3) *is given by the object* $\mathsf{R}_\mathbf{A} \in \mathbf{A} \otimes \mathbf{A}$.

Proof. Let us denote by ϕ the identification (C.3). By construction, we have a commutative diagram

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\phi} & \mathbf{A}^\vee \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{Funct}_{\mathbf{A}\text{-mod}}(\mathbf{A},\mathbf{A}) & & \mathbf{A}^\vee \underset{\mathbf{A}}{\otimes} \mathbf{A} \\ \downarrow & & \downarrow \\ \mathbf{A}^\vee \otimes \mathbf{A} & \xrightarrow{\mathrm{Id}\otimes\mathrm{Id}} & \mathbf{A}^\vee \otimes \mathbf{A}, \end{array}$$

where the lower right vertical arrow is the right adjoint to the tautological projection. The image of $\mathbf{1}_\mathbf{A}$ along the composite left vertical arrow is the counit in $\mathbf{A}^\vee \otimes \mathbf{A}$.

Concatenating with the commutative diagram

$$\begin{array}{ccc} \mathbf{A}^\vee & \xrightarrow{\phi^{-1}} & \mathbf{A} \\ \downarrow & & \downarrow \mathrm{comult}_\mathbf{A} \\ \mathbf{A}^\vee \otimes \mathbf{A} & \xrightarrow{\phi^{-1}\otimes\mathrm{Id}} & \mathbf{A} \otimes \mathbf{A}, \end{array}$$

we obtain a diagram

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\mathrm{Id}} & \mathbf{A} \\ \downarrow & & \downarrow \mathrm{comult}_\mathbf{A} \\ \mathbf{A}^\vee \otimes \mathbf{A} & \xrightarrow{\phi^{-1}\otimes\mathrm{Id}} & \mathbf{A} \otimes \mathbf{A}. \end{array}$$

The evaluating both routes on $\mathbf{1}_\mathbf{A} \in \mathbf{A}$, we obtain the desired result. □

C.2.9. Let $\Gamma_{!,\mathbf{A}} : \mathbf{A} \to \mathrm{Vect}_\mathsf{e}$ denote the functor dual to the unit functor

$$\mathrm{Vect}_\mathsf{e} \overset{\mathbf{1}_\mathbf{A}}{\to} \mathbf{A}$$

with respect to the above self-duality.

We claim

Lemma C.2.10. *The counit of the self-duality on* $\mathbf{A}$ *is given by*

$$\mathbf{A} \otimes \mathbf{A} \overset{\mathrm{mult}_\mathbf{A}}{\longrightarrow} \mathbf{A} \overset{\Gamma_{!,\mathbf{A}}}{\to} \mathrm{Vect}_\mathsf{e}\,.$$

Proof. This is a general feature of an algebra object $\mathbf{A}$ in a symmetric monoidal category O, which is equipped with a datum of self-duality given by an object in

$$\mathrm{Maps}_O(\mathbf{A}, \mathbf{A}\otimes\mathbf{A}) \xrightarrow{\mathbf{1}_\mathbf{A}} \mathrm{Maps}_O(\mathbf{1}_O, \mathbf{A}\otimes\mathbf{A}).$$

□

The above lemma says that $(\mathbf{A}, \Gamma_{!,\mathbf{A}})$ is a *Frobenius algebra object* in the symmetric monoidal category DGCat.

C.2.11. Finally, we claim:

Lemma C.2.12. *Let* $\mathbf{M}$ *be an* $\mathbf{A}$*-module category. Then it is dualizable as such if* and only if *it is dualizable as a plain DG category.*

Proof. One direction has been proved in Corollary C.2.7. For the other direction, this is a general property of algebras dualizable as objects in an ambient category.

□

C.3. **Compactness and dualizability.** The material in this subsection will not be needed in the sequel.

Let $\mathbf{A}$ be a semi-rigid symmetric monoidal category. For the duration of this subsection we will assume that $\mathbf{A}$ is compactly generated as a plain DG category.

C.3.1. Let $\mathbf{a} \in \mathbf{A}$ be a compact object. Let

$$\mathbb{D}(\mathbf{a}) \in \mathbf{A}$$

be its abstract dual with respect to the canonical self-duality of $\mathbf{A}$, i.e.,

$$\mathcal{H}om_{\mathbf{A}}(\mathbf{a}, \mathbf{b}) = \Gamma_{!,\mathbf{A}}(\mathbb{D}(\mathbf{a}) \otimes \mathbf{b}). \tag{C.4}$$

Equivalently,

$$\mathbb{D}(\mathbf{a}) \simeq (\mathcal{H}om_{\mathbf{A}}(\mathbf{a}, -) \otimes \mathrm{Id}_{\mathbf{A}})(\mathsf{R}_{\mathbf{A}}). \tag{C.5}$$

C.3.2. We claim:

Proposition C.3.3. *The object* $\mathbb{D}(\mathbf{a})$ *identifies canonically with the monoidal dual of* $\mathbf{a}$.

Proof. We need to establish a canonical isomorphism

$$\mathcal{H}om_{\mathbf{A}}(\mathbf{a} \otimes \mathbf{b}, \mathbf{c}) \simeq \mathcal{H}om_{\mathbf{A}}(\mathbf{b}, \mathbb{D}(\mathbf{a}) \otimes \mathbf{c}), \quad \mathbf{b}, \mathbf{c} \in \mathbf{A}. \tag{C.6}$$

With no restriction of generality, we can assume that $\mathbf{b}$ is compact. We rewrite the left-hand side as

$$\mathcal{H}om_{\mathbf{A}\otimes\mathbf{A}}(\mathbf{a} \boxtimes \mathbf{b}, \mathrm{comult}_{\mathbf{A}}(\mathbf{c})) \simeq \mathcal{H}om_{\mathbf{A}\otimes\mathbf{A}}(\mathbf{a} \boxtimes \mathbf{b}, \mathsf{R}_{\mathbf{A}} \otimes (\mathbf{1}_{\mathbf{A}} \boxtimes \mathbf{c})),$$

and further as

$$\mathcal{H}om_{\mathbf{A}}\left(\mathbf{b}, (\mathcal{H}om_{\mathbf{A}}(\mathbf{a}, -) \otimes \mathrm{Id}_{\mathbf{A}}) \circ (\mathrm{Id}_{\mathbf{A}} \otimes (- \otimes \mathbf{c}))(\mathsf{R}_{\mathbf{A}})\right) \simeq$$
$$\simeq \mathcal{H}om_{\mathbf{A}}\left(\mathbf{b}, (- \otimes \mathbf{c}) \circ (\mathcal{H}om_{\mathbf{A}}(\mathbf{a}, -) \otimes \mathrm{Id}_{\mathbf{A}})(\mathsf{R}_{\mathbf{A}})\right).$$

Using (C.5), we rewrite the latter expression as

$$\mathcal{H}om_{\mathbf{A}}(\mathbf{b}, \mathbb{D}(\mathbf{a}) \otimes \mathbf{c}),$$

as desired.

□

Corollary C.3.4. *In a compactly generated semi-rigid category, compact objects are dualizable with respect to the monoidal structure.*

C.3.5. Next, we claim:

Corollary C.3.6. *The subcategory of compact objects in* $\mathbf{A}$ *is closed under the monoidal operation.*

Proof. Follows from the fact that (in any monoidal category) the tensor product of a compact object and a dualizable object is compact.

□

C.3.7. Let $\mathbf{a}$ be again a compact object. By Proposition C.3.3, we have

$$\mathcal{H}om_{\mathbf{A}}(\mathbb{D}(\mathbf{a}), \mathbf{1}_{\mathbf{A}}) \simeq \mathcal{H}om_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}, \mathbf{a}).$$

Combining with (C.4), we obtain:

Corollary C.3.8. *For a compact* $\mathbf{a}$*, we have a canonical isomorphism*

$$\Gamma_{!,\mathbf{A}}(\mathbf{a}) \simeq \mathcal{H}om_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}, \mathbf{a}).$$

Remark C.3.9. The last corollary means that the functor $\Gamma_{!,\mathbf{A}}$ can be thought of as a renormalized version of the functor $\mathcal{H}om_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}, -)$ in the following sense:

The functor $\Gamma_{!,\mathbf{A}}$ is the ind-extension of the restriction of $\mathcal{H}om_{\mathbf{A}}(\mathbf{1}_{\mathbf{A}}, -)$ to the subcategory of compact objects.

C.4. **Lax vs strict compatibility.** In this subsection we let $\mathbf{A}$ be a semi-rigid symmetric monoidal DG category.

C.4.1. Let $\mathbf{M}$ be an $\mathbf{A}$-module category. Consider the action functor

$$\mathrm{act}_{\mathbf{M}} : \mathbf{A} \otimes \mathbf{M} \to \mathbf{M}.$$

Let

$$\mathrm{coact}_{\mathbf{M}} : \mathbf{M} \to \mathbf{A} \otimes \mathbf{M}$$

denote the functor

$$\mathbf{M} \overset{\mathsf{R}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{M} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathrm{act}_{\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{M},$$

i.e., this is the $\mathbf{A}$-dual map of $\mathrm{act}_{\mathbf{M}}$, with respect to the canonical self-duality of $\mathbf{A}$.

The functor $\mathrm{act}_{\mathbf{M}}$ is recovered from $\mathrm{coact}_{\mathbf{M}}$ as

$$\mathbf{A} \otimes \mathbf{M} \overset{\mathrm{Id}_{\mathbf{A}} \otimes \mathrm{coact}_{\mathbf{M}}}{\longrightarrow} \mathbf{A} \otimes \mathbf{A} \otimes \mathbf{M} \overset{\mathrm{counit}_{\mathbf{A}} \otimes \mathrm{Id}_{\mathbf{M}}}{\longrightarrow} \mathbf{M}.$$

C.4.2. We claim:

Lemma C.4.3. *The functor* $\mathrm{coact}_{\mathbf{M}}$ *is canonically isomorphic to the right adjoint of* $\mathrm{act}_{\mathbf{M}}$*.*

Proof. It suffices to establish the adjunction

$$\mathrm{act}_{\mathbf{A}} : \mathbf{A} \otimes \mathbf{A} \rightleftarrows \mathbf{A} : \mathrm{coact}_{\mathbf{A}}$$

in a way compatible with the right action of $\mathbf{A}$.

However, in this case $\mathrm{act}_{\mathbf{A}} = \mathrm{mult}_{\mathbf{A}}$, and it easy to see that $\mathrm{coact}_{\mathbf{A}}$ identifies with $\mathrm{comult}_{\mathbf{A}}$. □

C.4.4. We now claim:

Proposition C.4.5. *Let* $T : \mathbf{M}_1 \to \mathbf{M}_2$ *be a map of* $\mathbf{A}$*-module categories. Suppose that* T *admits a continuous right adjoint as a functor between plain DG categories. Then the right-lax structure of compatibility with* $\mathbf{A}$*-actions on* T^R *is strict.*

Proof. We need to show that the diagram

$$\begin{array}{ccc} \mathbf{A} \otimes \mathbf{M}_1 & \xrightarrow{\mathrm{act}_{\mathbf{M}_1}} & \mathbf{M}_1 \\ {\scriptstyle \mathrm{Id}_{\mathbf{A}} \otimes T^R} \uparrow & & \uparrow {\scriptstyle T^R} \\ \mathbf{A} \otimes \mathbf{M}_2 & \xrightarrow{\mathrm{act}_{\mathbf{M}_2}} & \mathbf{M}_2 \end{array}$$

commutes.

This is equivalent to the commutation of the $\mathbf{A}$-dual diagram

$$\begin{array}{ccc} \mathbf{M}_1 & \xrightarrow{\text{coact}_{\mathbf{M}_1}} & \mathbf{A}\otimes \mathbf{M}_1 \\ {\scriptstyle T^R}\Big\uparrow & & \Big\uparrow{\scriptstyle \text{Id}_{\mathbf{A}}\otimes T^R} \\ \mathbf{M}_2 & \xrightarrow{\text{coact}_{\mathbf{M}_2}} & \mathbf{A}\otimes \mathbf{M}_2. \end{array}$$

However, by Lemma C.4.3, the latter diagram can be obtained from the commutative diagram

$$\begin{array}{ccc} \mathbf{M}_1 & \xleftarrow{\text{act}_{\mathbf{M}_1}} & \mathbf{A}\otimes \mathbf{M}_1 \\ {\scriptstyle T}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{Id}_{\mathbf{A}}\otimes T} \\ \mathbf{M}_2 & \xleftarrow{\text{act}_{\mathbf{M}_2}} & \mathbf{A}\otimes \mathbf{M}_2. \end{array}$$

by passing to right adjoints.

□

C.4.6. Finally, we claim:

Proposition C.4.7. *Let $T : \mathbf{M}_1 \to \mathbf{M}_2$ be a map of $\mathbf{A}$-module categories. Suppose that T admits a left adjoint as a functor between plain DG categories. Then the left-lax structure of compatibility with $\mathbf{A}$-actions on T^L is strict.*

Proof. We wish to show that the diagram

$$\begin{array}{ccc} \mathbf{A}\otimes \mathbf{M}_1 & \xrightarrow{\text{act}_{\mathbf{M}_1}} & \mathbf{M}_1 \\ {\scriptstyle \text{Id}_{\mathbf{A}}\otimes T^L}\Big\uparrow & & \Big\uparrow{\scriptstyle T^L} \\ \mathbf{A}\otimes \mathbf{M}_2 & \xrightarrow{\text{act}_{\mathbf{M}_2}} & \mathbf{M}_2 \end{array}$$

commutes.

By passing to right adjoints, this is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} \mathbf{A}\otimes \mathbf{M}_1 & \xleftarrow{\text{coact}_{\mathbf{M}_1}} & \mathbf{M}_1 \\ {\scriptstyle \text{Id}_{\mathbf{A}}\otimes T}\Big\downarrow & & \Big\downarrow{\scriptstyle T} \\ \mathbf{A}\otimes \mathbf{M}_2 & \xleftarrow{\text{coact}_{\mathbf{M}_2}} & \mathbf{M}_2. \end{array}$$

However, the latter follows from the fact that T is compatible with $\mathbf{A}$-actions.

□

C.5. **Persistence of semi-rigidity.**

C.5.1. Let $\mathbf{A}$ be a semi-rigid symmetric monoidal category. We claim:

Proposition C.5.2. *The tautological functor*

$$\mathbf{A} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} \mathbf{A} \to \mathbf{A} \tag{C.7}$$

admits a continuous right adjoint, strictly compatible with the $\mathbf{A}$-bimodule structures.

Proof. First, we note that once we show that the right adjoint in question is continuous, the strict compatibility would follow by Proposition C.4.5.

Consider the projection

$$\mathbf{A}\otimes\mathbf{A} \to \mathbf{A} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} \mathbf{A}.$$

It admits a right adjoint, that is continuous and conservative (say, by Proposition C.2.3). Hence in order to prove that the right adjoint to (C.7) is continuous, it suffices to show that the right adjoint of composite functor

$$\mathbf{A}\otimes\mathbf{A}\to\mathbf{A}\underset{\mathbf{A}\otimes\mathbf{A}}{\otimes}\mathbf{A}\to\mathbf{A}$$

is continuous.

However, the above composite functor is $\text{mult}_{\mathbf{A}}$, so the assertion follows from the definition of semi-rigidity.

□

C.5.3. We now claim:

Proposition C.5.4. *Let $\mathbf{A}_1\leftarrow\mathbf{A}\to\mathbf{A}_2$ be a diagram of semi-rigid symmetric monoidal categories. Then the symmetric monoidal category $\mathbf{A}_1\underset{\mathbf{A}}{\otimes}\mathbf{A}_2$ is also semi-rigid.*

Proof. The fact that $\mathbf{A}_1\underset{\mathbf{A}}{\otimes}\mathbf{A}_2$ is dualizable follows from Corollary C.2.4.

We now show that $\text{mult}_{\mathbf{A}_1\underset{\mathbf{A}}{\otimes}\mathbf{A}_2}$ admits a continuous right adjoint, strictly compatible with the $(\mathbf{A}_1\underset{\mathbf{A}}{\otimes}\mathbf{A}_2)$-bimodule structure. Note that for the latter, it suffices to show that it is strictly compatible with the bimodule structure with respect to $\mathbf{A}_1\otimes\mathbf{A}_2$, and the latter would follow by Proposition C.4.5, once we establish the continuity.

We write $\text{mult}_{\mathbf{A}_1\underset{\mathbf{A}}{\otimes}\mathbf{A}_2}$ as

$$((\mathbf{A}_1\otimes\mathbf{A}_1)\otimes(\mathbf{A}_2\otimes\mathbf{A}_2))\underset{(\mathbf{A}\otimes\mathbf{A})\otimes(\mathbf{A}\otimes\mathbf{A})}{\otimes}(\mathbf{A}\otimes\mathbf{A})\to(\mathbf{A}_1\otimes\mathbf{A}_2)\underset{\mathbf{A}\otimes\mathbf{A}}{\otimes}\mathbf{A}.$$

Denote

$$\mathbf{M}_1=\mathbf{A}_1\otimes\mathbf{A}_1,\ \mathbf{M}_2=\mathbf{A}_2\otimes\mathbf{A}_2,\ \mathbf{M}'_1=\mathbf{A}_1,\ \mathbf{M}'_1=\mathbf{A}_2,\ \widetilde{\mathbf{A}}:=\mathbf{A}\otimes\mathbf{A}.$$

So the above functor is

$$(\mathbf{M}_1\otimes\mathbf{M}_2)\underset{\widetilde{\mathbf{A}}\otimes\widetilde{\mathbf{A}}}{\otimes}\widetilde{\mathbf{A}}\to(\mathbf{M}'_1\otimes\mathbf{M}'_2)\underset{\widetilde{\mathbf{A}}\otimes\widetilde{\mathbf{A}}}{\otimes}\widetilde{\mathbf{A}}\simeq(\mathbf{M}'_1\otimes\mathbf{M}'_2)\underset{\mathbf{A}\otimes\mathbf{A}}{\otimes}(\mathbf{A}\underset{\widetilde{\mathbf{A}}}{\otimes}\mathbf{A})\to(\mathbf{M}'_1\otimes\mathbf{M}'_2)\underset{\mathbf{A}\otimes\mathbf{A}}{\otimes}\mathbf{A}.$$

We claim that both arrows in the above composition admit right adjoints with the required properties. Indeed, for the first arrow this follows from the fact that the corresponding property of the functors

$$\mathbf{M}_1\to\mathbf{M}'_1 \text{ and } \mathbf{M}_2\to\mathbf{M}'_2$$

(by the semi-rigidity of $\mathbf{A}_1$ and $\mathbf{A}_2$).

For the second arrow, this follows from Proposition C.5.2.

□

C.6. Hochschild chains of semi-rigid categories.

C.6.1. Let $\mathbf{A}$ be a semi-rigid symmetric monoidal category, and let $F_{\mathbf{A}}$ be a symmetric monoidal endofunctor of $\mathbf{A}$. Consider the corresponding category of Hochschild chains

$$\text{HH}_\bullet(F_{\mathbf{A}},\mathbf{A}):=\mathbf{A}\underset{\text{mult},\mathbf{A}\otimes\mathbf{A},\text{mult}\circ(\text{Id}\otimes F_{\mathbf{A}})}{\otimes}\mathbf{A}.$$

Note that $\text{HH}_\bullet(F_{\mathbf{A}},\mathbf{A})$ is also semi-rigid, by Proposition C.5.4.

C.6.2. Let now $\mathbf{A}_1$ and $\mathbf{A}_2$ be a pair of semi-rigid symmetric monoidal categories, and let

$$\Phi : \mathbf{A}_1 \to \mathbf{A}_2$$

be a symmetric monoidal functor.

Let $F_{\mathbf{A}_1}$ and $F_{\mathbf{A}_2}$ be symmetric monoidal endofunctors of $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively, and let us be given an isomorphism

$$F_{\mathbf{A}_2} \circ \Phi \simeq \Phi \circ F_{\mathbf{A}_1}. \tag{C.8}$$

Then we obtain a functor

$$\mathrm{HH}_\bullet(F_{\mathbf{A}_1}, \mathbf{A}_1) \to \mathrm{HH}_\bullet(F_{\mathbf{A}_2}, \mathbf{A}_2), \tag{C.9}$$

to be denoted $\mathrm{HH}_\bullet(F, \Phi)$.

C.6.3. Let $\mathbf{M}_2$ be an $\mathbf{A}_2$-module category. Let $F_{\mathbf{M}}$ denote and endofunctor of $\mathbf{M}_2$ compatible with $F_{\mathbf{A}_2}$ (see [GKRV, Sect. 3.8.2]).

Assume that $\mathbf{M}_2$ is dualizable as a plain DG category. Then by Corollary C.2.6, $\mathbf{M}_2$ is dualizable also as an $\mathbf{A}_2$-module, and by [GKRV, Sect. 3.8.2] we can attach to it an object

$$\mathrm{Tr}^{\mathrm{enh}}_{\mathbf{A}_2}(F_{\mathbf{M}}, \mathbf{M}_2) \in \mathrm{HH}_\bullet(F_{\mathbf{A}_2}, \mathbf{A}_2).$$

C.6.4. Let $\mathbf{M}_1 := \mathrm{Res}_\Phi(\mathbf{M}_2) \in \mathbf{A}_1\text{-mod}$ be the $\mathbf{A}_1$-module category obtained from $\mathbf{M}_2$ be restriction along Φ.

The data of compatibility of $F_{\mathbf{M}}$ and $F_{\mathbf{A}_2}$, combined with (C.8) defines a data of compatibility of $F_{\mathbf{M}}$ and $F_{\mathbf{A}_1}$. Hence, we can consider the object

$$\mathrm{Tr}^{\mathrm{enh}}_{\mathbf{A}_1}(F_{\mathbf{M}}, \mathbf{M}_1) \in \mathrm{HH}_\bullet(F_{\mathbf{A}_1}, \mathbf{A}_1).$$

The main result of this section is the following:

Theorem C.6.5. *There exists a canonical isomorphism*

$$\mathrm{Tr}^{\mathrm{enh}}_{\mathbf{A}_1}(F_{\mathbf{M}}, \mathbf{M}_1) \simeq (\mathrm{HH}_\bullet(F, \Phi))^\vee \left(\mathrm{Tr}^{\mathrm{enh}}_{\mathbf{A}_2}(F_{\mathbf{M}}, \mathbf{M}_2)\right),$$

where $(\mathrm{HH}_\bullet(F, \Phi))^\vee$ *is the functor dual to* $\mathrm{HH}_\bullet(F, \Phi)$ *with respect to the canonical self-dualities on* $\mathrm{HH}_\bullet(F_{\mathbf{A}_i}, \mathbf{A}_i)$*,* $i = 1, 2$ *of* (C.3) *for semi-rigid categories.*

Remark C.6.6. This theorem is a generalization of [GKRV, Theorem 3.10.6], where $\mathbf{A}_1$ and $\mathbf{A}_2$ were assumed rigid.

Note in *loc. cit.*, instead of the functor $(\mathrm{HH}_\bullet(F, \Phi))^\vee$, one considered the functor $(\mathrm{HH}_\bullet(F, \Phi))^R$. However, it follows from the proof of Theorem C.6.5 that when $\mathbf{A}_1$ and $\mathbf{A}_2$ are rigid, we have a canonical equivalence

$$(\mathrm{HH}_\bullet(F, \Phi))^R \simeq (\mathrm{HH}_\bullet(F, \Phi))^\vee.$$

By contrast, in the semi-rigid case, the functor $(\mathrm{HH}_\bullet(F, \Phi))^R$ may be discontinuous (e.g., it corresponds to $\Gamma(\mathcal{Y}, -)$ on a formal affine scheme).

C.6.7. Consider the particular case when $\mathbf{A}_1 = \mathrm{Vect}_{\mathsf{e}}$ and $F_{\mathbf{A}_1} = \mathrm{Id}$. Denote $(\mathbf{A}_2, F_{\mathbf{A}_2})$ by $(\mathbf{A}, F_{\mathbf{A}})$ and $\mathbf{M}_2$ by $\mathbf{M}$. We obtain:

Corollary C.6.8. *There exists a canonical isomorphism*

$$\mathrm{Tr}(F_{\mathbf{M}}, \mathbf{M}) \simeq \Gamma_{!,\mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A})}\left(\mathrm{Tr}^{\mathrm{enh}}_{\mathbf{A}}(F_{\mathbf{M}}, \mathbf{M})\right).$$

C.6.9. Finally, we observe that Theorem 7.10.6 is a particular case of Corollary C.6.8.

C.7. **Proof of Theorem C.6.5.**

C.7.1. First, we recall that if $\Psi : \mathbf{A}' \to \mathbf{A}''$ is a monoidal functor between monoidal categories, we have an adjoint pair of 2-functors[48]

$$\mathrm{Ind}_\Psi : \mathbf{A}'\text{-mod} \rightleftarrows \mathbf{A}''\text{-mod} : \mathrm{Res}_\Psi,$$

where

$$\mathrm{Ind}_\Psi(\mathbf{M}) = \mathbf{A}'' \underset{\mathbf{A}'}{\otimes} \mathbf{M}.$$

In particular, the induction 2-functor Ind_Ψ always admits a right adjoint.

Suppose now that $\mathbf{A}''$ is dualizable as a left $\mathbf{A}'$-module category. Then the 2-functor Res_Ψ admits a right adjoint, denoted coInd_Ψ,

$$\mathrm{coInd}_\Psi(\mathbf{M}) = \mathrm{Funct}_{\mathbf{A}'}(\mathbf{A}'', \mathbf{M}).$$

C.7.2. Let $\mathbf{A}$ a monoidal category. Let as assume that:

- $\mathbf{A}$ is dualizable as a $\mathbf{A} \otimes \mathbf{A}$-module;
- $\mathbf{A}$ is dualizable as a plain DG category.

The first condition implies that the 2-functor

$$\mathrm{Res}_{\mathrm{mult}_\mathbf{A}} : \mathbf{A}\text{-mod} \to (\mathbf{A} \otimes \mathbf{A})\text{-mod}$$

admits a right adjoint, and the second condition implies that the 2-functor

$$\mathbf{oblv}_\mathbf{A} : \mathbf{A}\text{-mod} \to \mathrm{DGCat}$$

admits a right adjoint.

In particular, the 2-functors

$$\mathrm{DGCat} \overset{\mathbf{A}}{\to} \mathbf{A}\text{-mod} \overset{\mathrm{Res}_{\mathrm{mult}_\mathbf{A}}}{\longrightarrow} (\mathbf{A} \otimes \mathbf{A})\text{-mod} \tag{C.10}$$

and

$$(\mathbf{A} \otimes \mathbf{A})\text{-mod} \overset{\mathrm{Ind}_{\mathrm{mult}_\mathbf{A}}}{\longrightarrow} \mathbf{A}\text{-mod} \overset{\mathbf{oblv}_\mathbf{A}}{\longrightarrow} \mathrm{DGCat} \tag{C.11}$$

admit right adjoints.

C.7.3. Let $F_\mathbf{A}$ be a monoidal endofunctor of $\mathbf{A}$. Then by [GKRV, Sects. 3.3.4 and 3.7.1-3.7.2], the 2-functors (C.10) and (C.11) give rise to functors

$$\mathrm{Vect}_\mathsf{e} \to \mathrm{HH}_\bullet(F_\mathbf{A}, \mathbf{A}) \otimes \mathrm{HH}_\bullet(F_\mathbf{A}, \mathbf{A}) \tag{C.12}$$

and

$$\mathrm{HH}_\bullet(F_\mathbf{A}, \mathbf{A}) \otimes \mathrm{HH}_\bullet(F_\mathbf{A}, \mathbf{A}) \to \mathrm{Vect}_\mathsf{e}\,. \tag{C.13}$$

Furthermore, since the functors (C.10) and (C.11) define a unit and a counit of a self-duality on $\mathbf{A}$-mod (in the symmetric monoidal category of DG 2-categories, see [GKRV, Sect. 3.6][49]), the functors (C.12) and (C.13) define a unit and a counit of a self-duality on $\mathrm{HH}_\bullet(F_\mathbf{A}, \mathbf{A})$.

C.7.4. Let $\mathbf{A}$ be a semi-rigid symmetric monoidal category. Note that it automatically satisfies the conditions of Sect. C.7.2. We will prove:

Proposition C.7.5. *The data of self-duality on* $\mathrm{HH}_\bullet(F_\mathbf{A}, \mathbf{A})$ *defined by the functors* (C.12) *and* (C.13) *coincides with the data of self-duality on* $\mathrm{HH}_\bullet(F_\mathbf{A}, \mathbf{A})$ *as a semi-rigid symmetric monoidal category of* (C.3).

The proof of Proposition C.7.5 will be given in Sect. C.8. Let us assume it for now, and use it in order to prove Theorem C.6.5.

---

[48]We use the terminology "2-functor" for 1-morphisms in the $(\infty, 3)$-category of DG 2-categories.

[49]In [GKRV, Sect. 3.6], this symmetric monoidal category is denoted Morita(DGCat).

C.7.6. Let $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ be a monoidal functor between monoidal categories. Let $F_{\mathbf{A}_1}$ and $F_{\mathbf{A}_2}$ be monoidal endofunctors of $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively. We will denote by the same symbol $F_{\mathbf{A}_i}$ the resulting 2-endomorphism of $\mathbf{A}_i$-mod, $i = 1, 2$.

Let us be given an isomorphism

$$F_{\mathbf{A}_2} \circ \Phi \simeq \Phi \circ F_{\mathbf{A}_1} \tag{C.14}$$

as monoidal functors.

The isomorphism (C.14) induces an isomorphism

$$F_{\mathbf{A}_1} \circ \mathrm{Res}_\Phi \simeq \mathrm{Res}_\Phi \circ F_{\mathbf{A}_2},$$

and by adjunction a morphism

$$\mathrm{Ind}_\Phi \circ F_{\mathbf{A}_1} \to F_{\mathbf{A}_2} \circ \mathrm{Ind}_\Phi .$$

Then by [GKRV, Sects. 3.3.4 and 3.7.1-3.7.2], the 2-functor

$$\mathrm{Ind}_\Phi : \mathbf{A}_1\text{-mod} \to \mathbf{A}_2\text{-mod}$$

induces a functor

$$\mathrm{Tr}(F, \mathrm{Ind}_\Phi) : \mathrm{HH}_\bullet(F_{\mathbf{A}_1}, \mathbf{A}_1) \to \mathrm{HH}_\bullet(F_{\mathbf{A}_2}, \mathbf{A}_2).$$

C.7.7. Assume now that $\mathbf{A}_2$ is dualizable as an $\mathbf{A}_1$-module, so that the 2-functor Res_Φ also admits a right adjoint. Then again by [GKRV, Sects. 3.3.4 and 3.7.1-3.7.2], we obtain a functor

$$\mathrm{HH}_\bullet(F_{\mathbf{A}_2}, \mathbf{A}_2) \to \mathrm{HH}_\bullet(F_{\mathbf{A}_1}, \mathbf{A}_1),$$

which we will denote by $\mathrm{Tr}(F, \mathrm{Res}_\Phi)$.

C.7.8. Suppose now that $\mathbf{A}_1$ and $\mathbf{A}_2$ satisfy the assumptions of Sect. C.7.2. (In particular, in this case, $\mathbf{A}_2$ is automatically dualizable as an $\mathbf{A}_1$-module.)

We claim that we have a canonical identification

$$\mathrm{Tr}(F, \mathrm{Res}_\Phi) \simeq (\mathrm{Tr}(F, \mathrm{Ind}_\Phi))^\vee, \tag{C.15}$$

with respect to the self-dualities of (C.12) and (C.13).

Indeed, this follows by taking traces from the identification of the 2-functors

$$\mathrm{Res}_\Phi \simeq (\mathrm{Ind}_\Phi)^\vee$$

with respect to the self-dualities (C.10) and (C.11).

C.7.9. Assume now that $\mathbf{A}_1, \mathbf{A}_2$ are *symmetric* monoidal and semi-rigid, that the functors $F_{\mathbf{A}_1}, F_{\mathbf{A}_2}, \Phi$ are *symmetric* monoidal, and that the data of compatibility (C.14) respects the symmetric monoidal structures.

In order to prove Theorem C.6.5, it suffices to show that we have a canonical identification

$$\mathrm{Tr}(F, \mathrm{Res}_\Phi) \simeq (\mathrm{HH}_\bullet(F, \Phi))^\vee, \tag{C.16}$$

with respect to the canonical self-dualities $\mathrm{HH}_\bullet(F_{\mathbf{A}_i}, \mathbf{A}_i)$, $i = 1, 2$ of (C.3) for semi-rigid categories.

However, a straightforward calculation shows that we have a canonical identification

$$\mathrm{Tr}(F, \mathrm{Ind}_\Phi) \simeq \mathrm{HH}_\bullet(F, \Phi). \tag{C.17}$$

Hence, the identification (C.16) follows from (C.15), since by Proposition C.7.5, the above self-dualities equal those given by (C.12) and (C.13),

□[Theorem C.6.5]

C.8. **Proof of Proposition C.7.5.**

C.8.1. It suffices to show that the unit functor (C.12) identifies canonically with

$$\mathrm{Vect}_{\mathsf{e}} \to \mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A}) \overset{\mathrm{comult}_{\mathrm{HH}_\bullet(F_{\mathbf{A}},\mathbf{A})}}{\longrightarrow} \mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A}) \otimes \mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A}).$$

For this, it suffices to show that the latter functor is obtained from (C.10) by taking traces.

This is obvious for the first arrow, i.e.,

$$\mathrm{DGCat} \to \mathbf{A}\text{-mod}$$

(see (C.17)).

C.8.2. For the second arrow, i.e.,

$$\mathbf{A}\text{-mod} \overset{\mathrm{Res}_{\mathrm{mult}_{\mathbf{A}}}}{\longrightarrow} (\mathbf{A} \otimes \mathbf{A})\text{-mod}$$

we argue as follows:

Note that the trace $\mathrm{Tr}(F, \mathrm{Ind}_{\mathrm{mult}_{\mathbf{A}}})$ of the 2-functor

$$(\mathbf{A} \otimes \mathbf{A})\text{-mod} \overset{\mathrm{Ind}_{\mathrm{mult}_{\mathbf{A}}}}{\longrightarrow} \mathbf{A}\text{-mod}$$

identifies with $\mathrm{mult}_{\mathrm{HH}_\bullet(F_{\mathbf{A}},\mathbf{A})}$, again by (C.17).

Hence, it suffices to show that the functor

$$\mathrm{Tr}(F, \mathrm{Res}_{\mathrm{mult}_{\mathbf{A}}}) : \mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A}) \to \mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A}) \otimes \mathrm{HH}_\bullet(F_{\mathbf{A}}, \mathbf{A})$$

identifies with the right adjoint of $\mathrm{Tr}(F, \mathrm{Ind}_{\mathrm{mult}_{\mathbf{A}}})$.

C.8.3. Recall the setting of [GKRV, Sect. 3.9]. Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be a pair of DG 2-categories, each equipped with an endofunctor F_i, $i = 1, 2$. Consider the corresponding categories

$$\mathrm{Tr}(F_1, \mathfrak{T}_1) \text{ and } \mathrm{Tr}(F_2, \mathfrak{T}_2).$$

Let $\mathbf{\Phi} : \mathfrak{T}_1 \to \mathfrak{T}_2$ be a 2-functor that admits a right adjoint. Let us be given a natural transformation

$$\alpha : \mathbf{\Phi} \circ F_1 \to F_2 \circ \mathbf{\Phi}. \tag{C.18}$$

Then by [GKRV, Sect. 3.3.4], we obtain a functor

$$\mathrm{Tr}(F, \mathbf{\Phi}) : \mathrm{Tr}(F_1, \mathfrak{T}_1) \to \mathrm{Tr}(F_2, \mathfrak{T}_2).$$

Suppose now that we are given two 2-functors $\mathbf{\Phi}', \mathbf{\Phi}'' : \mathfrak{T}_1 \to \mathfrak{T}_2$ as above, and let us be given a 2-morphism

$$\beta : \mathbf{\Phi}' \to \mathbf{\Phi}'',$$

equipped with a natural 3-transformation γ from

$$\mathbf{\Phi}' \circ F_1 \overset{\alpha'}{\to} F_2 \circ \mathbf{\Phi}' \overset{\beta}{\to} F_2 \circ \mathbf{\Phi}''$$

to

$$\mathbf{\Phi}' \circ F_1 \overset{\beta}{\to} \mathbf{\Phi}'' \circ F_1 \overset{\alpha''}{\to} F_2 \circ \mathbf{\Phi}''.$$

Finally assume that the 2-morphism β admits a right adjoint. Then, by [GKRV, Sect. 3.9.4], we obtain a natural transformation

$$\mathrm{Tr}(F, \mathbf{\Phi}') \overset{\mathrm{Tr}(F,\beta)}{\longrightarrow} \mathrm{Tr}(F, \mathbf{\Phi}''). \tag{C.19}$$

C.8.4. Let $\mathbf{\Phi} : \mathfrak{T}_1 \to \mathfrak{T}_2$ be as above, and suppose that the 2-functor $\mathbf{\Phi}$ is the left adjoint of a 2-functor

$$\mathbf{\Psi} : \mathfrak{T}_2 \to \mathfrak{T}_1,$$

equipped with an *isomorphism*

$$\mathbf{\Psi} \circ F_2 \to F_1 \circ \mathbf{\Psi}, \tag{C.20}$$

so that (C.18) arises from (C.20) by adjunction.

Note that in this case, the unit and the counit of the adjunction

$$\mathrm{Id}_{\mathfrak{T}_1} \to \mathbf{\Psi} \circ \mathbf{\Phi} \text{ and } \mathbf{\Phi} \circ \mathbf{\Psi} \to \mathrm{Id}_{\mathfrak{T}_2} \tag{C.21}$$

automatically come equipped with the data of 3-morphisms γ as above, which are in fact *isomorphisms*.

Assume that $\mathbf{\Psi}$ itself admits and a right adjoint (which is a 1-morphism) and that the 2-morphisms (C.21) admit right adjoints (which are also 2-morphisms). We obtain that the natural transformation (C.19) applied to (C.21) defines an adjunction between $\mathrm{Tr}(F, \mathbf{\Phi})$ and $\mathrm{Tr}(F, \mathbf{\Psi})$.

C.8.5. We apply the material of Sect. C.8.4 to the situation when

$$\mathfrak{T}_1 := \mathbf{A}_1\text{-mod},\ \mathfrak{T}_2 := \mathbf{A}_2\text{-mod},\ \mathbf{\Phi} := \mathrm{Ind}_\Phi,\ \mathbf{\Psi} := \mathrm{Res}_\Phi,$$

for a monoidal functor $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$. The datum of (C.20) is supplied by (C.14).

Assume that $\mathbf{A}_2$ is dualizable as an $\mathbf{A}_1$-module. We obtain that the functors

$$\mathrm{Tr}(F, \mathrm{Ind}_\Phi) : \mathrm{HH}_\bullet(F_{\mathbf{A}_1}, \mathbf{A}_1) \rightleftarrows \mathrm{HH}_\bullet(F_{\mathbf{A}_2}, \mathbf{A}_2) : \mathrm{Tr}(F, \mathrm{Res}_\Phi)$$

are an adjoint pair if the following conditions are satisfied:

- The functor $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ admits a right adjoint as a map of $\mathbf{A}_1$-bimodule categories;
- The functor $\mathbf{A}_2 \underset{\mathbf{A}_1}{\otimes} \mathbf{A}_2 \to \mathbf{A}_2$ admits a right adjoint as a map of $\mathbf{A}_2$-bimodule categories.

C.8.6. We apply this to $\mathbf{A}_1 = \mathbf{A} \otimes \mathbf{A}$, $\mathbf{A}_2 = \mathbf{A}$ and $\Phi = \mathrm{mult}_\mathbf{A}$.

Now, the existence of the right adjoint to $\mathrm{mult}_\mathbf{A}$ follows from the semi-rigidity condition.

The existence of the right adjoint to

$$\mathbf{A} \underset{\mathbf{A}\otimes\mathbf{A}}{\otimes} \mathbf{A} \to \mathbf{A}$$

follows from Proposition C.5.2.

□[Proposition C.7.5]

Appendix D. The dimension of the global nilpotent cone

In this section we prove that under certain the restrictions on $\mathrm{char}(k)$, the global nilpotent cone

$$\mathrm{Nilp} \subset T^*(\mathrm{Bun}_G),$$

viewed as a *classical* algebraic stack has dimension equal to $\dim(\mathrm{Bun}_G) = \dim(G) \cdot (g-1)$.

D.1. **The Faltings-Ginzburg argument.** We first explain Faltings' proof that Nilp is isotropic (see [Fa, Theorem III.2]) which was conceptualized by V. Ginzburg in [Gi]. This argument is valid in characteristic 0, and requires a certain assumption in positive characteristic. This assumption will be satisfied if $\mathrm{char}(k)$ is "very good", see Sect. D.2.6.

D.1.1. We first formulate the assumption on $\mathfrak{g}$ and $\mathrm{char}(k)$ that we need for the Faltings-Ginzburg proof:

Let $\widetilde{k} \supset k$ be a not necessarily algebraically closed field extension, and let n be a nilpotent element in $\widetilde{k} \underset{k}{\otimes} \mathfrak{g}$. We need there to exist a parabolic $\widetilde{P} \subset G$ defined over $\widetilde{k}$ such that n belongs to the Lie algebra of its unipotent radical.

When $\mathrm{char}(k) = 0$, this is trivially satisfied; we can take P to be a Borel subgroup. Indeed, the element n generates a copy of $\mathbb{G}_a \subset G$; take an arbitrary point on the flag variety, consider the resulting map $\mathbb{G}_a \to G/B$, and complete it to a map $\mathbb{P}^1 \to G/B$. Then the image of $\infty \in \mathbb{P}^1$ is $\mathbb{G}_a$-invariant, and hence the Lie algebra of the corresponding Borel subgroup contains n.

When $\mathrm{char}(k) > 0$, we were able to prove the validity of this assumption for "very good" characteristics, using (a variant of) the Jacobson-Morozov theory, see Sect. D.2.6.

D.1.2. Let $\mathcal{Y}$ be a smooth algebraic stack. For a Zariski locally closed subset $\mathcal{N} \subset T^*(\mathcal{Y})$, and a smooth map $S \to \mathcal{Y}$, where S is a scheme, denote by $\mathcal{N}_S \subset T^*(S)$ the image of

$$\mathcal{N} \underset{\mathcal{Y}}{\times} S \subset T^*(\mathcal{Y}) \underset{\mathcal{Y}}{\times} S$$

under the codifferential map

$$T^*(\mathcal{Y}) \underset{\mathcal{Y}}{\times} S \to T^*(S).$$

We say that $\mathcal{N}$ is half-dimensional if $\dim(\mathcal{N}_S) \leq \dim(S)$ for every S as above (equivalently, for a collection of affine schemes S that smoothly cover $\mathcal{Y}$).

We will say a Zariski locally closed subset of $Z \subset T^*(S)$ is isotropic if for every irreducible component of Z, some non-empty smooth open subset Z° of this irreducible component is isotropic (i.e., the symplectic form vanishes on its tangent spaces).

We will say that a Zariski locally closed subset of $Z \subset T^*(S)$ is *strongly* isotropic if for every point of $z \in Z$, the symplectic vanishes on $H^0(T_z(Z))$.

We will say that a Zariski locally closed subset $\mathcal{N} \subset T^*(\mathcal{Y})$ is isotropic (resp., strongly isotropic) if $\mathcal{N}_S$ is isotropic (resp., strongly isotropic) for every S as above (equivalently, for a collection of schemes S that smoothly cover $\mathcal{Y}$).

Clearly, if $\mathcal{N}$ is isotropic then it is half-dimensional.

Remark D.1.3. One can also consider a notion intermediate between isotropic and strongly isotropic: one can require that for any smooth scheme Z' mapping to Z, the pullback of the symplectic form to Z' vanishes.

D.1.4. Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a schematic map between smooth stacks. Let K_f denote the kernel of the codifferential, i.e.,

$$\mathrm{Ker}\left(T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \overset{df^*}{\longrightarrow} T^*(\mathcal{Y}_1)\right),$$

viewed as a Zariski-closed subset in $T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1$.

Let $\mathcal{N} \subset T^*(\mathcal{Y}_2)$ be a Zariski-closed subset. We have the following assertion:

Proposition D.1.5. *Assume that the projection*

$$T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to T^*(\mathcal{Y}_2)$$

maps K_f to $\mathcal{N}$, such that the following holds:

There exists an open dense subset $\mathcal{N}^\circ \subset \mathcal{N}$ such that for every (not necessarily algebraically closed) field extension $k' \supset k$ and every k'-point n of $\mathcal{N}^\circ$, there exists a separable field extension $k'' \supset k'$ and a lift of n to a k''-point of K_f.

Then $\mathcal{N}$ is isotropic.

Remark D.1.6. If $\mathrm{char}(k) = 0$, the condition in Proposition D.1.5 amounts to the requirement that the map $K_f \to \mathcal{N}$ be dominant.

Proof. By base change, we can assume that $\mathcal{Y}_2 = Y_2$ is a scheme. Then, since f is schematic, $\mathcal{Y}_1 = Y_1$ is a scheme as well. Consider the product

$$T^*(Y_1) \times T^*(Y_2) \simeq T^*(Y_1 \times Y_2)$$

with its symplectic structure.

We have a natural embedding

$$T^*(Y_2) \underset{Y_2}{\times} Y_1 \hookrightarrow T^*(Y_1) \times T^*(Y_2),$$

whose image is a smooth Lagrangian. We can view K_f as the intersection of this Lagrangian with $\{0\} \times T^*(Y_2)$.

Hence, K_f is *strongly* isotropic as a subset of $T^*(Y_1) \times T^*(Y_2)$. We wish to show that its image along the projection

$$T^*(Y_1) \times T^*(Y_2) \to T^*(Y_2)$$

is isotropic.

More precisely, let Z be a smooth open subset of an irreducible component of $\mathcal{N}^\circ$. We wish to show that some non-empty open subset Z° of any such Z is isotropic.

Let k' be the field of fractions of Z. By assumption, there exists a separable field extension $k'' \supset k'$ and a k''-point of K_f such that

$$\mathrm{Spec}(k'') \to K_f \to \mathcal{N}$$

equals

$$\mathrm{Spec}(k'') \to \mathrm{Spec}(k') \to Z \to \mathcal{N}.$$

Hence, we can find a (non-empty) scheme $\widetilde{Z}$ equipped with an étale map $\widetilde{Z} \to Z$ together with a lift of this map to a map

$$\widetilde{Z} \to K_f.$$

Let $Z^\circ \subset Z$ denote the image of the map $\widetilde{Z} \to Z$. We claim that the symplectic form vanishes on Z°.

Indeed, for every $z \in Z^\circ$, let $\widetilde{z}$ be its lift to a point of $\widetilde{Z}$. Let (w, z) denote the resulting point of $K_f \subset T^*(Y_1) \times T^*(Y_2)$.

By étaleness, the map $T_{\widetilde{z}}(\widetilde{Z}) \to T_z(Z)$ is surjective. Hence, the map

$$H^0(T_{(w,z)}(K_f)) \to T_z(Z)$$

is surjective.

Now, the restriction of the symplectic form on $T^*(Y_1) \times T^*(Y_2)$ along

$$H^0(T_{(w,z)}(K_f)) \to T_{(w,z)}(T^*(Y_1) \times T^*(Y_2))$$

equals the restriction of the symplectic form on $T^*(Y_2)$ along

$$H^0(T_{(w,z)}(K_f)) \to T_z(T^*(Y_2)).$$

Indeed, the two restrictions are already equal on $T_{(w,z)}(\{0\} \times T^*(Y_2))$. □

Remark D.1.7. The above proof shows that $\mathcal{N}$ is actually semi-strongly isotropic, see Remark D.1.3 for what this means.

D.1.8. We are now ready to prove that Nilp is half-dimensional. We are assuming that the condition from Sect. D.1.1 holds, and also that $\mathfrak{g}$ admits a non-degenerate G-invariant bilinear form.

We are going to apply Proposition D.1.5 in the following situation:

We take $\mathcal{Y}_2 = \mathrm{Bun}_G$ and $\mathcal{N} = \mathrm{Nilp}$. We we take $\mathcal{Y}_1$ to be the union of Bun_P over the set of standard parabolics $P \subset G$.

Using an invariant form on $\mathfrak{g}$, we identify $T^*(\mathrm{Bun}_G)$ with the stack that classifies pairs $(\mathcal{P}_G, A)$, where $\mathcal{P}_G$ is a G-bundle on X and A is a section of $\mathfrak{g}_{\mathcal{P}_G} \otimes \omega_X$.

Then, for a given parabolic, the stack K_f classifies pairs $(\mathcal{P}_P, A)$, where $\mathcal{P}_P$ is a P-bundle on X, and A is a section of $\mathfrak{n}(P)_{\mathcal{P}_G} \otimes \omega_X$.

Clearly, the projection $K_f \to T^*(\mathrm{Bun}_G)$ has its image contained in Nilp. Thus, in order to prove that Nilp is isotropic, we have to show that the condition of Proposition D.1.5 is satisfied.

Let $k' \supset k$ be a field extension. Let $(\mathcal{P}'_G, A')$ be a k'-point of $T^*(\mathrm{Bun}_G)$ with A' nilpotent. By [DS], there exists a separable field extension $k'' \supset k'$ such that the pullback $\mathcal{P}''_G$ of $\mathcal{P}'_G$ to the curve

$$X'' := X \underset{\mathrm{Spec}(k)}{\times} \mathrm{Spec}(k'')$$

can be trivialized at the generic η point of X''. Let $\widetilde{k}$ denote the field of rational functions on X''. Denote $A'' := A'|_{X''}$ and $\widetilde{A} := A''|_\eta$.

Up to trivializing $\omega_{X''}$ and $\mathcal{P}''_G$ generically, we can think of $\widetilde{A}$ as a nilpotent element in $\widetilde{k} \underset{k}{\otimes} \mathfrak{g}$. Let $\widetilde{P}$ be a parabolic defined over $\widetilde{k}$ such that $\widetilde{A} \in \mathfrak{n}(\widetilde{P})$. It exists by the assumption in Sect. D.1.1.

Let P be the standard parabolic conjugate to $\widetilde{P}$. Then we can think of $\widetilde{P}$ as a reduction $\mathcal{P}''_P$ of $\mathcal{P}''_G$ to P at η, so that A'' is a section of $\mathfrak{n}(P)_{\mathcal{P}''_P} \otimes \omega_{X''}$.

By the valuative criterion, the reduction $\mathcal{P}''_P$ of $\mathcal{P}''_G$ (uniquely) extends to the entire X'', and the section A'' belongs to $\mathfrak{n}(P)_{\mathcal{P}''_P} \otimes \omega_{X'}$ (because it does so generically).

The resulting pair $(\mathcal{P}''_P, A'')$ is the sought-for lift of $(\mathcal{P}''_G, A'')$ to a k''-point of K_f.

$\square$[Isotropy of Nilp]

D.2. **Adaptation of the Jacobson-Morozov theory.** In this subsection, we will assume that the characteristic of k is "very good" for $\mathfrak{g}$ (this excludes very small primes for every isomorphism class of root data of G).

D.2.1. We first summarize the results that of [Pre] that we will need.

Let n be a nilpotent element of $\mathfrak{g}$. Then there exists a homomorphism $\lambda : \mathbb{G}_m \to G$ with the following properties:

Denote by

$$\mathfrak{g} = \underset{i}{\oplus}\, \mathfrak{g}_i$$

the weight decomposition of $\mathfrak{g}$ for the induced adjoint action of $\mathbb{G}_m$. Denote by

$$\mathfrak{g}^i := \underset{j \geq i}{\oplus}\, \mathfrak{g}_j$$

the corresponding filtration.

We have:

- $n \in \mathfrak{g}_2$;
- $\mathfrak{g}^0 =: \mathfrak{p}$ is a Lie algebra of a parabolic subgroup (to be denoted P);
- The map $\mathfrak{p} \xrightarrow{\mathrm{ad}_n} \mathfrak{g}^2$ is surjective.
- $\mathfrak{z}_\mathfrak{g}(n) \subset \mathfrak{p}$ and $Z_G(n) \subset P$.

The above is (part of) the content of [Pre, Theorem A][50].

D.2.2. Note that the surjectivity of the map

$$\mathfrak{p} \xrightarrow{\mathrm{ad}_n} \mathfrak{g}^2 \tag{D.1}$$

implies that the Ad_P-orbit of n, denoted $\overset{\circ}{\mathfrak{g}}{}^2$, is a Zariski open subset in $\mathfrak{g}^2$.

D.2.3. Note also that the fact that $\mathfrak{z}_n(\mathfrak{g}) \subset \mathfrak{p}$ implies that the map

$$\mathrm{ad}_n : \mathfrak{g}_{-1} \to \mathfrak{g}_1 \tag{D.2}$$

is injective.

Since $\dim(\mathfrak{g}_{-1}) = \dim(\mathfrak{g}_1)$ (the bilinear form on $\mathfrak{g}$ restricts to a perfect pairing on $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1$), we obtain that (D.2) is an isomorphism.

D.2.4. Let $\mathbf{O} \subset \mathfrak{g}$ denote the orbit of n under the adjoint action. Denote by Y the partial flag variety $Y = G/P$.

Let $\widetilde{\mathbf{O}} \to Y$ be the total space of a G-equivariant vector bundle over Y, whose fiber over $P \in Y$ is the space $\mathfrak{g}^2$, equipped with the natural action of P.

Let

$$\widetilde{\mathbf{O}}^\circ \subset \widetilde{\mathbf{O}}$$

be the G-invariant open subscheme whose fiber over P is

$$\overset{\circ}{\mathfrak{g}}{}^2 \subset \mathfrak{g}^2.$$

The natural projection

$$\widetilde{\mathbf{O}} \to \mathfrak{g}$$

restricts to a map

$$\widetilde{\mathbf{O}}^\circ \to \mathbf{O}. \tag{D.3}$$

Lemma D.2.5. *The map* (D.3) *is an isomorphism.*

Proof. Since G acts transitively on both $\widetilde{\mathbf{O}}^\circ$ and $\mathbf{O}$, it suffices to show that the stabilizers are equal.

However, this follows from the fact that $Z_G(n) \subset P$.

□

D.2.6. We are now ready to prove the property from Sect. D.1.1.

The assumption that $\mathrm{char}(k)$ is very good implies that G acts on the nilpotent cone of $\mathfrak{g}$ with finitely many orbits. Hence, the nilpotent cone of $\mathfrak{g}$ is a finite union of its locally closed subsets $\mathbf{O}$.

Hence, given a field extension $\widetilde{k} \supset k$ and a nilpotent element $n \in \mathfrak{g}(\widetilde{k})$, there exists a nilpotent G-orbit $\mathbf{O}$ defined over k such that $n \in \mathbf{O}(\widetilde{k})$.

Now the assertion follows from Lemma D.2.5 by taking $\widetilde{k}$-points.

D.3. **The Beilinson-Drinfeld argument.** In this subsection we will give another proof of the fact that Nilp is half-dimensional (under the same assumptions as above).

We will explicitly write Nilp as a union of algebraic stacks of dimension $\leq \dim(G) \cdot (g-1)$.

This argument is wholly borrowed from [BD2, Sect. 2.10.3]. We include it here for completeness.

---

[50]For the validity of this theorem, one only needs $\mathrm{char}(k)$ to be "good". The "very good" assumption will be used later.

D.3.1. Let $\mathbf{O}$ be a nilpotent conjugacy class. Let $\mathrm{Nilp}_{\mathbf{O}} \subset \mathrm{Nilp}$ be the locally closed substack consisting of pairs $(\mathcal{P}_G, A)$, where A generically belongs to $\mathbf{O}$.

By the same reasoning as in Sect. D.2.6 above, we have

$$\mathrm{Nilp} = \underset{\mathbf{O}}{\cup} \mathrm{Nilp}_{\mathbf{O}}.$$

Therefore, it is enough show that each $\mathrm{Nilp}_{\mathbf{O}}$ has dimension $\leq \dim(G) \cdot (g-1)$.

D.3.2. Consider the algebraic stack $\mathbf{Maps}(X, \widetilde{\mathbf{O}}_{\omega_X}/G)$, where $\widetilde{\mathbf{O}}_{\omega_X}$ denotes the twist of the constant bundle over X with fiber $\widetilde{\mathbf{O}}$ by ω_X viewed as a $\mathbb{G}_m$-torsor, using the $\mathbb{G}_m$-action on $\widetilde{\mathbf{O}}$ by fiber-wise dilations.

Let

$$\mathbf{Maps}(X, \widetilde{\mathbf{O}}_{\omega_X}/G)^{\circ} \subset \mathbf{Maps}(X, \widetilde{\mathbf{O}}_{\omega_X}/G)$$

be the open substack consisting of maps that generically land in $\widetilde{\mathbf{O}}^{\circ}$.

The isomorphism of Lemma D.2.5 and the valuative criterion imply that the map

$$\mathbf{Maps}(X, \widetilde{\mathbf{O}}_{\omega_X}/G)^{\circ} \to \mathrm{Nilp}_{\mathbf{O}},$$

is bijective on geometric points.

Hence, it suffices to show that $\mathbf{Maps}(X, \widetilde{\mathbf{O}}_{\omega_X}/G)^{\circ}$ has dimension $\leq \dim(G) \cdot (g-1)$.

We will show that $\mathbf{Maps}(X, \widetilde{\mathbf{O}}_{\omega_X}/G)^{\circ}$ is a smooth algebraic stack and that its (stacky) tangent spaces at k-points have Euler characteristics $\leq \dim(G) \cdot (g-1)$.

D.3.3. Let $(\mathcal{P}_P, A)$ be a k-point of $\mathbf{Maps}(X, \widetilde{\mathbf{O}}_{\omega_X}/G)^{\circ}$. Its stacky tangent space is given by

$$\Gamma(X, E_{-1} \overset{\mathrm{ad}_A}{\to} E_0),$$

where $E_{-1} = \mathfrak{g}^0_{\mathcal{P}_P}$ and $E_0 = \mathfrak{g}^2_{\mathcal{P}_P} \otimes \omega_X$.

Note that since A generically belongs to $\overset{\circ}{\mathfrak{g}}{}^2$ and the map (D.1) is surjective, we obtain that the map

$$E_{-1} \overset{\mathrm{ad}_A}{\to} E_0$$

is generically surjective, i.e., its cokernel is a torsion sheaf on X.

This implies that $T^*_{(\mathcal{P}_P,A)}(\mathbf{Maps}(X, \widetilde{\mathbf{O}}_{\omega_X}/G)^{\circ})$ is acyclic is cohomological degrees > 0. This implies that $\mathbf{Maps}(X, \widetilde{\mathbf{O}}_{\omega_X}/G)^{\circ}$ is smooth.

D.3.4. We have

$$\chi\left(T^*_{(\mathcal{P}_P,A)}(\mathbf{Maps}(X, \widetilde{\mathbf{O}}_{\omega_X}/G)^{\circ})\right) = \chi(\Gamma(X, E_0)) - \chi(\Gamma(X, E_{-1})).$$

Using the non-degenerate G-invariant form on $\mathfrak{g}$, we identify $\mathfrak{g}^2_{\mathcal{P}_P}$ with the dual vector bundle of $(\mathfrak{g}/\mathfrak{g}^{-1})_{\mathcal{P}_G}$. Hence, by Serre duality

$$\chi(\Gamma(X, E_0)) = -\chi(\Gamma(X, (\mathfrak{g}/\mathfrak{g}^{-1})_{\mathcal{P}_G})).$$

Hence, we obtain

$$\chi(T^*_{(\mathcal{P}_P,A)}(\mathbf{Maps}(X, \widetilde{\mathbf{O}}_{\omega_X}/G)^{\circ})) = -\chi(\Gamma(X, (\mathfrak{g}/\mathfrak{g}^{-1})_{\mathcal{P}_P})) - \chi(\Gamma(X, \mathfrak{g}^0_{\mathcal{P}_P})) =$$

$$= -\chi(\Gamma(X, \mathfrak{g}_{\mathcal{P}_P})) + \chi(\Gamma(X, (\mathfrak{g}^{-1}/\mathfrak{g}^0)_{\mathcal{P}_P}) = \dim(\mathrm{Bun}_G) + \chi(\Gamma(X, (\mathfrak{g}^{-1}/\mathfrak{g}^0)_{\mathcal{P}_P}).$$

It remains to show that $\chi(\Gamma(X, (\mathfrak{g}^{-1}/\mathfrak{g}^0)_{\mathcal{P}_P}) \leq 0$.

D.3.5. Again, by Serre duality, we have

$$\chi(\Gamma(X, (\mathfrak{g}^{-1}/\mathfrak{g}^0)_{\mathcal{P}_P}) = -\chi(\Gamma(X, (\mathfrak{g}^0/\mathfrak{g}^2)_{\mathcal{P}_P} \otimes \omega_X).$$

Hence,

$$2\chi(\Gamma(X, (\mathfrak{g}^{-1}/\mathfrak{g}^0)_{\mathcal{P}_P}) = \chi(\Gamma(X, (\mathfrak{g}^{-1}/\mathfrak{g}^0)_{\mathcal{P}_P}) - \chi(\Gamma(X, (\mathfrak{g}^0/\mathfrak{g}^2)_{\mathcal{P}_P} \otimes \omega_X) =$$

$$= -\chi\left(\Gamma(X, (\mathfrak{g}^{-1}/\mathfrak{g}^0)_{\mathcal{P}_P} \overset{\mathrm{ad}_A}{\to} (\mathfrak{g}^0/\mathfrak{g}^2)_{\mathcal{P}_P} \otimes \omega_X\right).$$

Hence, it is enough to show that

$$\chi\left(\Gamma(X, (\mathfrak{g}^{-1}/\mathfrak{g}^0)_{\mathcal{P}_P} \overset{\mathrm{ad}_A}{\to} (\mathfrak{g}^0/\mathfrak{g}^2)_{\mathcal{P}_P} \otimes \omega_X\right) \geq 0.$$

Note, however, that the map

$$\mathrm{ad}_n : \mathfrak{g}^{-1}/\mathfrak{g}^0 \to \mathfrak{g}^1/\mathfrak{g}^2$$

is an isomorphism, since (D.2) is an isomorphism.

Therefore, the map

$$(\mathfrak{g}^{-1}/\mathfrak{g}^0)_{\mathcal{P}_P} \overset{\mathrm{ad}_A}{\to} (\mathfrak{g}^0/\mathfrak{g}^2)_{\mathcal{P}_P} \otimes \omega_X$$

is generically an isomorphism. Hence, it is injective and its cokernel is torsion. Hence, the Euler characteristic of its cone is non-negative.

□

Appendix E. Ind-constructible sheaves on schemes

Algebro-geometric objects in this section will be quasi-compact schemes over k, assumed almost[51] of finite type. Let $\mathrm{Shv}(-)^{\mathrm{constr}}$ be one of the sheaf-theoretic contexts from Sect. 1.1.1.

E.1. **The left completeness theorem.**

E.1.1. Recall that for a (quasi-compact) scheme Y we define

$$\mathrm{Shv}(Y) := \mathrm{Ind}(\mathrm{Shv}(Y)^{\mathrm{constr}}).$$

The goal of this subsection is to prove Theorem 1.1.6. The proof will be obtained as a combination of the following two statements:

Theorem E.1.2. *The canonical functor*

$$D^b(\mathrm{Perv}(Y)) \to \mathrm{Shv}(Y)^{\mathrm{constr}}$$

is an equivalence.

Theorem E.1.3. *Let $\mathcal{A}$ be a small abelian category of finite cohomological dimension. Then the DG category $\mathrm{Ind}(D^b(\mathcal{A}))$ is left-complete in its t-structure.*

Theorem E.1.2 is a theorem of A. Beilinson, and it is proved in [Be1]. The rest of this subsection is devoted to the proof of Theorem E.1.3.

[51] Since we are dealing with $\mathrm{Shv}(-)$, we lose nothing by only considering classical schemes, i.e., derived algebraic geometry over k will play no role.

E.1.4. Recall that for a DG category $\mathbf{C}$ equipped with a t-structure, we denote by $\mathbf{C}^\wedge$ its left completion, i.e.,

$$\mathbf{C}^\wedge := \lim_n \mathbf{C}^{\geq -n}.$$

We will think of objects of $\mathbf{C}^\wedge$ as compatible collections

$$\{\mathbf{c}^n \in \mathbf{C}^{\geq -n}\}.$$

We have the tautological functor

$$\mathbf{C} \to \mathbf{C}^\wedge, \quad \mathbf{c} \mapsto \{\tau^{\geq -n}(\mathbf{c}) \in \mathbf{C}^{\geq -n}\} \tag{E.1}$$

and its right adjoint given by

$$\{\mathbf{c}^n \in \mathbf{C}^{\geq -n}\} \mapsto \lim_n \mathbf{c}^n, \tag{E.2}$$

where the limit is taken in $\mathbf{C}$.

E.1.5. We shall say that $\mathbf{C}$ has *convergent Postnikov towers* if (E.1) is fully faithful. Equivalently, if for $\mathbf{c} \in \mathbf{C}$, the natural map

$$\mathbf{c} \to \lim_n \tau^{\geq -n}(\mathbf{c})$$

is an isomorphism.

E.1.6. We shall say that an object $\mathbf{c} \in \mathbf{C}$ has cohomological dimension $\leq n$ if

$$\mathrm{Hom}_{\mathbf{C}}(\mathbf{c}, \mathbf{c}') = 0 \text{ for all } \mathbf{c}' \in \mathbf{C}^{< -n}.$$

We claim:

Proposition E.1.7. *Let $\mathbf{C}$ be generated by compact objects of finite cohomological dimension. Then $\mathbf{C}$ has convergent Postnikov towers. Furthermore, the right adjoint to* (E.1) *is continuous.*

Proof. It is enough to show that for every $\mathbf{c}_0 \in \mathbf{C}^c$, the functor

$$\mathbf{C}^\wedge \overset{\text{(E.2)}}{\longrightarrow} \mathbf{C} \overset{\mathcal{H}om_{\mathbf{C}}(\mathbf{c}_0,-)}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}} \tag{E.3}$$

is continuous, and that its precomposition with (E.1) is isomorphic to $\mathcal{H}om_{\mathbf{C}}(\mathbf{c}_0, -)$.

Now, the functor (E.3) sends

$$\{\mathbf{c}^n \in \mathbf{C}^{\geq -n}\} \in \mathbf{C}^\wedge$$

to

$$\lim_n \mathcal{H}om_{\mathbf{C}}(\mathbf{c}_0, \mathbf{c}^n),$$

while in the above limit, each individual cohomology group stabilizes due to the assumption on $\mathbf{c}_0$.

This implies both claims.

□

E.1.8. Applying Proposition E.1.7 to $\mathrm{Ind}(D^b(\mathcal{A}))$, we obtain that it has convergent Postnikov towers. It remains to show that the functor (E.2) is fully faithful.

We claim:

Lemma E.1.9. *The functor* (E.2) *is fully faithful if and only if the following condition is satisfied:*

For a family of objects of $\mathbf{C}^{\leq 0}$ indexed by $\mathbb{N}$

$$n \mapsto \mathbf{c}_n$$

such that for every N the family $\tau^{\geq -N}(\mathbf{c}_n)$ stabilizes, we have

$$\lim_n \mathbf{c}_n \in \mathbf{C}^{\leq 0}.$$

Proof. Let $\{\mathbf{c}^m \in \mathbf{C}^{\geq -m}\}$ be an object of $\mathbf{C}^\wedge$, and set

$$\mathbf{c} := \lim_m \mathbf{c}^m.$$

We need to show that for any n, the map

$$\tau^{\geq -n}(\mathbf{c}) \to \mathbf{c}^n$$

is an isomorphism.

We have a fiber sequence

$$\lim_{m\geq n+1} \tau^{<-(n+1)}(\mathbf{c}^m) \to \mathbf{c} \to \lim_{m\geq n+1} \tau^{\geq -(n+1)}(\mathbf{c}^m),$$

where the left-most term belongs to $\mathbf{C}^{<-(n+1)}$ by assumption, and the right-most term is $\mathbf{c}^{n+1}$, since the corresponding inverse family is constant with value $\mathbf{c}^{n+1}$.

From here, we obtain that

$$\tau^{\geq -n}(\mathbf{c}) \to \tau^{\geq -n}(\mathbf{c}^{n+1}) \simeq \mathbf{c}^n$$

is an isomorphism.

□

E.1.10. Let us show that Lemma E.1.9 is applicable to $\mathrm{Ind}(D^b(\mathcal{A}))$. Let

$$n \mapsto \mathbf{c}_n$$

be a family of objects in $\mathrm{Ind}(D^b(\mathcal{A}))$ as in Lemma E.1.9. Let d be the cohomological dimension of $\mathcal{A}$. Considering the fiber sequence

$$\tau^{\leq -d}(\mathbf{c}_n) \to \mathbf{c}_n \to \tau^{>-d}(\mathbf{c}_n)$$

and taking into account that the family $n \mapsto \tau^{>-d}(\mathbf{c}_n)$ stabilizes, we obtain that we can assume that $\mathbf{c}_n \in \mathrm{Ind}(D^b(\mathcal{A}))^{\leq -d}$.

We claim that for any $a \in \mathcal{A}$

$$\mathcal{H}om_{\mathrm{Ind}(D^b(\mathcal{A}))}(a, \lim_n \mathbf{c}_n) \simeq \lim_n \mathcal{H}om_{\mathrm{Ind}(D^b(\mathcal{A}))}(a, \mathbf{c}_n) \in \mathrm{Vect}_{\mathsf{e}}^{\leq 0}.$$

Indeed, in the family

$$n \mapsto \mathcal{H}om_{\mathrm{Ind}(D^b(\mathcal{A}))}(a, \mathbf{c}_n),$$

the terms belong to $\mathrm{Vect}_{\mathsf{e}}^{\leq 0}$, and for any N, the family

$$n \mapsto \tau^{\geq -N}(\mathcal{H}om_{\mathrm{Ind}(D^b(\mathcal{A}))}(a, \mathbf{c}_n))$$

stabilizes.

Now the fact that $\lim_n \mathbf{c}_n \in \mathrm{Ind}(D^b(\mathcal{A}))^{\leq 0}$ follows from the next assertion:

Lemma E.1.11. *Let $\mathcal{A}$ be a small abelian category of finite cohomological dimension. Let $\mathbf{c} \in \mathrm{Ind}(D^b(\mathcal{A}))$ be an object such that*

$$\mathcal{H}om_{\mathrm{Ind}(D^b(\mathcal{A}))}(a, \mathbf{c}) \in \mathrm{Vect}_{\mathsf{e}}^{\leq 0} \text{ for all } a \in \mathcal{A} \subset \mathrm{Ind}(D^b(\mathcal{A})).$$

Then $\mathbf{c} \in \mathrm{Ind}(D^b(\mathcal{A}))^{\leq 0}$.

Proof. Suppose that $\tau^{>0}(\mathbf{c}) \neq 0$. Let $k > 0$ be the smallest integer such that $H^k(\mathbf{c}) \neq 0$. Then

$$\operatorname*{colim}_a H^0\left(\mathcal{H}om_{\mathrm{Ind}(D^b(\mathcal{A}))}(a[-k], \tau^{\geq k}(\mathbf{c}))\right) \neq 0,$$

where the colimit goes over the (filtered) category, whose objects are objects of $\mathcal{A}$, and whose morphisms are surjections.

Note that for any fixed n, the map

$$\operatorname*{colim}_a H^0\left(\mathcal{H}om_{\mathrm{Ind}(D^b(\mathcal{A}))}(a[-k], \tau^{\geq k-n}(\mathbf{c}))\right) \to \operatorname*{colim}_a H^0\left(\mathcal{H}om_{\mathrm{Ind}(D^b(\mathcal{A}))}(a[-k], \tau^{\geq k}(\mathbf{c}))\right)$$

is surjective (by the definition of the derived category).

Let d be the cohomological dimension of $\mathcal{A}$. Note that for any a, the map

$$H^0\left(\mathcal{H}om_{\mathrm{Ind}(D^b(\mathcal{A}))}(a[-k],\mathbf{c})\right)\to H^0\left(\mathcal{H}om_{\mathrm{Ind}(D^b(\mathcal{A}))}(a[-k],\tau^{\geq k-d}(\mathbf{c}))\right)$$

is surjective.

Hence, we obtain that the map

$$\underset{a}{\mathrm{colim}}\, H^0\left(\mathcal{H}om_{\mathrm{Ind}(D^b(\mathcal{A}))}(a[-k],\mathbf{c})\right)\to \underset{a}{\mathrm{colim}}\, H^0\left(\mathcal{H}om_{\mathrm{Ind}(D^b(\mathcal{A}))}(a[-k],\tau^{\geq k}(\mathbf{c}))\right)$$

is surjective.

In particular, we obtain that for some a,

$$H^0\left(\mathcal{H}om_{\mathrm{Ind}(D^b(\mathcal{A}))}(a[-k],\mathbf{c})\right)\neq 0.$$

However, this contradicts the assumption on $\mathbf{c}$.

□

E.2. **Categorical $K(\pi,1)$'s.**

E.2.1. Recall the subcategories

$$\mathrm{Lisse}(Y)\subset \mathrm{IndLisse}(Y)\subset \mathrm{QLisse}(Y),$$

see Sects. 1.2-1.3.

Definition E.2.2. *We shall say that Y is a* categorical $K(\pi,1)$ *if the naturally defined functor*

$$D^b(\mathrm{Lisse}(Y)^\heartsuit)\to \mathrm{Lisse}(Y)$$

is an equivalence.

Note that from Theorem E.1.3 and Sect. 1.3.2 we obtain:

Corollary E.2.3. *If Y is a categorical $K(\pi,1)$, then the inclusion*

$$\mathrm{IndLisse}(Y)\subset \mathrm{QLisse}(Y) \tag{E.4}$$

is an equality.

E.2.4. Let $\mathrm{Lisse}(Y)_0\subset \mathrm{Lisse}(Y)$ be the full subcategory, consisting of objects whose cohomologies (with respect to the usual t-structure) are extensions of the constant sheaf $\underline{\mathsf{e}}_Y$. Let

$$\mathrm{IndLisse}(Y)_0\subset \mathrm{IndLisse}(Y) \text{ and } \mathrm{QLisse}(Y)_0\subset \mathrm{QLisse}(Y)$$

denote the corresponding subcategories.

Definition E.2.5. *We shall say that Y is a* unipotent categorical $K(\pi,1)$ *if the naturally defined functor*

$$D^b(\mathrm{Lisse}(Y)_0^\heartsuit)\to \mathrm{Lisse}(Y)_0$$

is an equivalence.

E.2.6. An easy example of Y, which is neither a categorical $K(\pi,1)$ nor a unipotent categorical $K(\pi,1)$ is $Y=\mathbb{P}^1$.

First, note that in this case, the embedding

$$\mathrm{Lisse}(Y)_0\hookrightarrow \mathrm{Lisse}(Y)$$

is an equivalence. So, the statements about categorical $K(\pi,1)$ vs. unipotent categorical $K(\pi,1)$ are equivalent.

We will show that the functor (E.4) is *not* an equivalence.

Indeed, the category $\mathrm{IndLisse}(Y)$ is generated by one object, namely, $\underline{\mathsf{e}}_{\mathbb{P}^1}$, whose algebra of endomorphisms is

$$A:=\mathsf{e}[\eta]/\eta^2=0,\ \deg(\eta)=2.$$

Hence,

$$\text{IndLisse}(Y) \simeq A\text{-mod}.$$

By Koszul duality, we have

$$A\text{-mod} \simeq B\text{-mod}_0,$$

where

$$B = \mathsf{e}\langle \xi \rangle,\ \deg(\xi) = -1$$

is the free *associative* algebra on one generator in degree -1, and

$$B\text{-mod}_0 \subset B\text{-mod} \tag{E.5}$$

is the full subcategory consisting of objects on which ξ acts nilpotently.

The t-structure on $\text{IndLisse}(Y)$ corresponds to the usual t-structure on B-mod, for which the forgetful functor to Vect_{e} is t-exact.

Now it is easy to see that the embedding (E.5) realizes B-mod as the left completion of $B\text{-mod}_0$.

E.2.7. We now claim:

Theorem E.2.8.
(a) *All connected algebraic curves other than $\mathbb{P}^1$ are categorical $K(\pi,1)$'s.*
(b) *All connected algebraic curves other than $\mathbb{P}^1$ are unipotent categorical $K(\pi,1)$'s.*

E.2.9. We observe:

Lemma E.2.10. *Let $\mathbf{C}_0$ be a small DG category equipped with a* bounded *t-structure, and consider the functor*

$$D^b(\mathbf{C}_0^\heartsuit) \to \mathbf{C}_0. \tag{E.6}$$

(a) *Suppose every object of $\mathbf{c}_0 \in \mathbf{C}_0^\heartsuit$ admits a non-zero map to an* injective *object $\mathbf{c} \in \text{Ind}(\mathbf{C}_0^\heartsuit)$ that satisfies*

$$\text{Hom}_{\text{Ind}(\mathbf{C}^0)}(\mathbf{c}_0', \mathbf{c}[k]) = 0, \quad \forall\, \mathbf{c}_0' \in \mathbf{C}_0^\heartsuit,\ \forall\, k > 0.$$

Then (E.6) *is an equivalence.*
(b) *Suppose that*

$$\text{Hom}_{\mathbf{C}_0}(\mathbf{c}_0', \mathbf{c}_0[k]) = 0 \textit{ for } k > 2 \textit{ for all } \mathbf{c}_0, \mathbf{c}_0' \in \mathbf{C}_0^\heartsuit.$$

Then (E.6) *is an equivalence if and only if for every $\mathbf{c}_0, \mathbf{c}_0'$ as above, the (a priori injective) map*

$$\text{Ext}^2_{\mathbf{C}_0^\heartsuit}(\mathbf{c}_0', \mathbf{c}_0) \to \text{Hom}_{\mathbf{C}_0}(\mathbf{c}_0', \mathbf{c}_0[2])$$

is surjective.

Proof. Point (a) is standard: the assumption allows us to compute $\text{Hom}_{\mathbf{C}^0}(\mathbf{c}_0', \mathbf{c}_0)$ via (ind)-injective resolutions in $\mathbf{C}_0$. Point (b) follows formally from point (a).

□

Remark E.2.11. Note that Lemma E.2.10(b) implies the assertion of Theorem E.2.8 when X is *affine*, as in this case

$$\text{Hom}_{\text{QLisse}(Y)}(E, E'[2]) = 0$$

for any pair of local systems E and E'.

E.3. **Proof of Theorem E.2.8 for complete curves.** Let X be a complete algebraic curve of genus > 0. Let $E_0 \in \text{Lisse}(X)^\heartsuit$ denote the trivial local system.

E.3.1. We will first show that point (b) of Theorem E.2.8 implies point (a).

By Lemma E.2.10(b), we have to show that for $E_1, E \in \text{Lisse}(X)^{\heartsuit}$, any element

$$\alpha \in \text{Hom}_{\text{Lisse}(X)}(E_1, E[2])$$

can be written as a cup product of classes

$$\beta \in \text{Ext}^1(E_1, \widetilde{E}) \text{ and } \gamma \in \text{Ext}^1(\widetilde{E}, E)$$

for some $\widetilde{E} \in \text{Lisse}(X)^{\heartsuit}$.

Dualizing E_1, we can assume that $E_1 = E_0$, so we can think of α as an element of $H^2(X, E)$.

Write

$$0 \to E'' \to E \to E' \to 0,$$

where E' is an extension of copies of E_0, and E'' does not have trivial quotients. Note that the map

$$H^2(X, E) \to H^2(X, E')$$

is an isomorphism, since $H^2(X, E'') = 0$. Let α' be the image of α in $H^2(X, E')$

Assuming point (b), we can write α' as a cup product of classes

$$\beta \in H^1(X, \widetilde{E}) \text{ and } \gamma' \in \text{Ext}^1(\widetilde{E}, E')$$

for some $\widetilde{E} \in \text{Lisse}(X)_0^{\heartsuit}$ (i.e., $\widetilde{E}$ is also an extension of copies of E_0).

It suffices to show that γ' can be lifted to an element $\gamma \in \text{Ext}^1(\widetilde{E}, E)$. However, the obstruction to such a lift lies in $\text{Ext}^2(\widetilde{E}, E'')$, which embeds into $\text{Hom}_{\text{Lisse}(X)}(\widetilde{E}, E''[2])$, and the latter vanishes since

$$\text{Hom}_{\text{Lisse}(X)}(E_0, E''[2]) = H^2(X, E'') = 0.$$

E.3.2. We now prove point (b) of Theorem E.2.8. By Lemma E.2.10, it suffices to construct an object $E_0^{\text{cofree}_x} \in \text{IndLisse}(X)$ with the following properties:

- (i) $E_0^{\text{cofree}_x} \in \text{IndLisse}(X)^{\heartsuit}$;
- (ii) $E_0^{\text{cofree}_x} \in \text{IndLisse}(X)_0$;
- (iii) $\text{Hom}(E_0, E_0^{\text{cofree}_x}) \neq 0$;
- (iv) $\text{Hom}_{\text{IndLisse}(X)}(E_0, E_0^{\text{cofree}_x}[k]) = 0$ for $k > 0$.

Indeed, note that point (iv) for $k = 1$ implies that $E_0^{\text{cofree}_x}$ is injective as an object of $\text{IndLisse}_0^{\heartsuit}$, and combining with point (iii), we obtain that any object in $\text{Lisse}_0^{\heartsuit}$ admits a non-zero map to $E_0^{\text{cofree}_x}$.

E.3.3. Choose a point $x \in X$, and consider the corresponding augmentation map

$$\text{C}^{\cdot}(X) \to \mathsf{e}.$$

Set

$$E_0^{\text{cofree}_x} := \mathsf{e} \underset{\text{C}^{\cdot}(X)}{\otimes} E_0,$$

where we regard $\text{C}^{\cdot}(X)$ as $\mathcal{H}om_{\text{IndLisse}(X)}(E_0, E_0)$.

Item (ii) follows by construction. Items (iii) and (iv) follow from the fact that

$$\mathcal{H}om_{\text{IndLisse}(X)}(E_0, E_0^{\text{cofree}_x}) \overset{E_0 \text{ is compact}}{\simeq} \mathsf{e} \underset{\text{C}^{\cdot}(X)}{\otimes} \mathcal{H}om_{\text{IndLisse}(X)}(E_0, E_0) \simeq \mathsf{e}.$$

It remains to establish item (i).

E.3.4. Taking the fiber of $E_0^{\text{cofree}_x}$ at x, property (i) is equivalent to the fact that the object

$$\mathsf{e} \underset{\text{C}^{\cdot}(X)}{\otimes} \mathsf{e} \in \text{Vect}_{\mathsf{e}} \tag{E.7}$$

is acyclic off degree 0.

This fact is probably well-known. We will supply a proof for completeness.

E.3.5. First, the manipulation in Sects. 9.5-9.6 allows us to reduce the assertion to the case when our sheaf-theoretic context is Betti (and $k = \mathbb{C}$).

Remark E.3.6. Note that in Sects. 9.5-9.6 we appealed to Theorem E.2.8 (which we are still in the process of proving) in order to compare the categories $\mathrm{QLisse}(X)$ in the Betti context to the the étale context in characteristic 0 and further to the étale context in characteristic p.

However, there is no circularity in the argument, because for the purposes of Sect. E.3.5, we only need to compare the algebras of cochains $\mathrm{C}^\cdot(X)$ in the three contexts, and the fact that the corresponding maps are isomorphisms is standard.

E.3.7. In the Betti context, it is known that the algebra $\mathrm{C}^\cdot(X)$ is formal (by [DGMS, Main Theorem, Sect. 6]). I.e., it is isomorphic to the DG algebra A with

$$A_0 = \mathsf{e},\ A_1 = V,\ A_2 = \mathsf{e},\ A_n = 0 \text{ for } n > 2,$$

where V is a symplectic vector space and the multiplication $V \otimes V \to \mathsf{e}$ is given by a symplectic form.

If $V \neq 0$ (it is here that we use the assumption that the genus of X is > 0), this algebra is quadratic and hence Koszul[52], see [PP, Chapter 5, Sect. 5(1)]. Hence,

$$H^k(\mathsf{e} \underset{A}{\otimes} \mathsf{e}) = 0 \text{ for } k \neq 0.$$

□[Theorem E.2.8]

E.4. **The dual of** $\mathrm{QLisse}(Y)$**.** In this subsection we let Y be a *smooth* scheme.

E.4.1. We give the following definition:

Definition E.4.2. *We shall say that* $\mathrm{QLisse}(Y)$ *is Verdier-compatible if the functor*

$$\mathrm{QLisse}(Y) \otimes \mathrm{QLisse}(Y) \to \mathrm{Vect}_{\mathsf{e}}, \quad E_1, E_2 \mapsto \mathrm{C}^\cdot(Y, E_1 \overset{!}{\otimes} E_2). \tag{E.8}$$

is the counit of a self-duality.

E.4.3. We claim:

Proposition E.4.4. *Assume that* $\mathrm{IndLisse}(Y) \to \mathrm{QLisse}(Y)$ *is an equivalence. Then* $\mathrm{QLisse}(Y)$ *is Verdier-compatible.*

Proof. Since $\mathrm{IndLisse}(Y) \to \mathrm{QLisse}(Y)$ is an equivalence, the category $\mathrm{QLisse}(Y)$ is compactly generated by $\mathrm{Lisse}(Y)$. Now, *naive* duality defines a contravariant equivalence

$$\mathrm{Lisse}(Y) \simeq \mathrm{Lisse}(Y)^{\mathrm{op}}.$$

Since Y is smooth, the above naive duality coincides with Verdier duality, up to a shift. Hence, the latter defines an identification

$$\mathrm{QLisse}(Y) \simeq \mathrm{QLisse}(Y)^\vee.$$

Its counit is given by (E.8) by definition.

□

Remark E.4.5. Note that the above argument shows that for any smooth Y, the pairing

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathrm{C}^\cdot(Y, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2)$$

defines a self-duality on $\mathrm{IndLisse}(Y)$.

[52] We are grateful for L. Positselski for explaining this to us.

E.4.6. We claim:

Corollary E.4.7. *If X is a smooth curve, then* $\mathrm{QLisse}(X)$ *is Verdier-compatible.*

Proof. The case of curves different from $\mathbb{P}^1$ follows from Theorem E.2.8(a) and Proposition E.4.4.

The case of $\mathbb{P}^1$ follows by direct inspection: in terms of the equivalence

$$\mathrm{QLisse}(\mathbb{P}^1) \simeq B\text{-mod}$$

(see Sect. E.2.6), the pairing (E.8) corresponds (up to a shift) to the canonical pairing

$$B\text{-mod} \otimes B^{\mathrm{op}}\text{-mod} \to \mathrm{Vect}_{\mathsf{e}},$$

corresponding to the isomorphism

$$B \simeq B^{\mathrm{op}}, \quad \xi \mapsto -\xi.$$

□

E.5. **Specifying singular support.** In this subsection we let Y be a *smooth* scheme. In Sect. E.6 below we will explain how to extend the discussion to the case when Y is not necessarily smooth.

E.5.1. Let Y be a scheme and $\mathcal{N}$ a conical Zariski-closed subset of $T^*(Y)$. In this case we have a well-defined full abelian subcategory

$$\mathrm{Perv}_{\mathcal{N}}(Y) \subset \mathrm{Perv}(Y),$$

see [GKRV, Sect. A.3.1].

A key property of $\mathrm{Perv}_{\mathcal{N}}(Y)$ is that it is a Serre subcategory of $\mathrm{Perv}(Y)$, i.e., it is stable under the operations of taking sub- and quotient objects.

E.5.2. Another basic feature of this subcategory is that the Verdier self-duality

$$\mathbb{D} : \mathrm{Perv}(Y) \to \mathrm{Perv}(Y)$$

sends $\mathrm{Perv}_{\mathcal{N}}(Y)$ to itself.

This follows from the geometric characterization of singular support in [Be2].

E.5.3. Consider the abelian category

$$\mathrm{Ind}(\mathrm{Perv}_{\mathcal{N}}(Y)) \subset \mathrm{Ind}(\mathrm{Perv}(Y)) \simeq \mathrm{Shv}(Y)^{\heartsuit}.$$

We let

$$\mathrm{Shv}_{\mathcal{N}}(Y) \subset \mathrm{Shv}(Y)$$

be the full subcategory consisting of objects whose cohomologies belong to $\mathrm{Ind}(\mathrm{Perv}_{\mathcal{N}}(Y))$.

Since the t-structure on $\mathrm{Shv}(Y)$ is compatible with filtered colimits, we obtain that $\mathrm{Shv}_{\mathcal{N}}(Y)$ is closed under filtered colimits.

By construction, $\mathrm{Shv}_{\mathcal{N}}(Y)$ inherits a t-structure so that its embedding into $\mathrm{Shv}(Y)$ is t-exact. By Theorem 1.1.6, the category $\mathrm{Shv}_{\mathcal{N}}(Y)$ is left-complete in its t-structure.

E.5.4. Set

$$\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{constr}} := \mathrm{Shv}_{\mathcal{N}}(Y) \cap \mathrm{Shv}(Y)^{\mathrm{constr}} \subset \mathrm{Shv}(Y).$$

This is the full subcategory of $\mathrm{Shv}(Y)^{\mathrm{constr}}$ consisting of objects whose cohomologies belong to $\mathrm{Perv}_{\mathcal{N}}(Y)$. By construction, $\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{constr}}$ inherits a t-structure so that its embedding into $\mathrm{Shv}(Y)^{\mathrm{constr}}$ is t-exact.

Set

$$\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}} := \mathrm{Ind}(\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{constr}}).$$

Ind-extension of the tautological embedding

$$\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{constr}} \hookrightarrow \mathrm{Shv}_{\mathcal{N}}(Y)$$

defines a functor

$$\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}} \to \mathrm{Shv}_{\mathcal{N}}(Y). \tag{E.9}$$

The functor (E.9) is fully faithful: indeed, its composition with the embedding $\mathrm{Shv}_{\mathcal{N}}(Y) \hookrightarrow \mathrm{Shv}(Y)$ preserves compactness and is fully faithful on compacts.

The t-structure on $\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{constr}}$ extends to a unique t-structure on $\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}}$ compatible with filtered colimits. The functor (E.9) is t-exact, since the t-structure on $\mathrm{Shv}(Y)$ (and hence $\mathrm{Shv}_{\mathcal{N}}(Y)$) is also compatible with filtered colimits.

It is easy to see that the functor (E.9) induces an equivalence on the hearts. Hence, it induces an equivalence

$$(\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}})^{\geq -n} \to (\mathrm{Shv}_{\mathcal{N}}(Y))^{\geq -n}$$

for any n. From here it follows that the functor (E.9) identifies $\mathrm{Shv}_{\mathcal{N}}(Y)$ with the left completion of $\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}}$.

From the fact that $\mathrm{Shv}_{\mathcal{N}}(Y)$ is left-complete, we obtain that the functor (E.9) realizes $\mathrm{Shv}_{\mathcal{N}}(Y)$ realizes as a left completion of $\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}}$.

E.5.5. *Example.* Let Y be smooth and take $\mathcal{N} = \{0\}$. Then $\mathrm{Shv}_{\mathcal{N}}(Y)$ is what we have previously denoted by $\mathrm{QLisse}(Y)$ and $\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}} = \mathrm{IndLisse}(Y)$.

Remark E.5.6. Note that the process of left completion in (E.9) is in general non-trivial, i.e., the category $\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}}$ is not necessarily left-complete, see Sect. E.2.6.

Remark E.5.7. Our conventions are different from those of [GKRV]. In *loc.cit.* we denoted by $\mathrm{Shv}_{\mathcal{N}}(Y)$ what we denote here by $\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}}$.

E.6. **Singular support on schemes that are not necessarily smooth.** Let Y be a scheme of finite type that is not necessarily smooth. In this subsection we explain what we mean by a closed subset of $T^*(Y)$, and how to assign singular support to objects of $\mathrm{Shv}(Y)$. (The same discussion applies when instead of $\mathrm{Shv}(Y)$ we consider $\mathrm{D\text{-}mod}(Y)$ and $\mathrm{Shv}^{\mathrm{all}}(Y)$.)

The discussion is Zariski-local, so we can assume that Y is affine.

E.6.1. Let Y be a scheme of finite type, and let E be a (classical) coherent sheaf on Y. Write E as a quotient of a map $E_{-1} \to E_0$, where E_{-1} and E_0 are vector bundles. Let $\mathrm{Tot}(E)$ be the algebraic stack over Y equal to

$$\mathrm{Tot}(E_0)/\mathrm{Tot}(E_{-1}),$$

where we regard $\mathrm{Tot}(E_{-1})$ as a group-scheme over Y that acts on $\mathrm{Tot}(E_0)$.

Of course, $\mathrm{Tot}(E)$ as defined above depends on the presentation. However, it is well-defined as an object of the category obtained by localizing algebraic stacks with respect to morphisms, whose fibers are of the form pt/V, where V is a vector group.

In particular, the notion of a (locally) closed subset in $\mathrm{Tot}(E)$ is well-defined. By definition, this is the same as a (locally) closed subset in $\mathrm{Tot}(E_0)$ that is $\mathrm{Tot}(E_{-1})$-invariant for every $\mathrm{Tot}(E_{-1})$ as above.

One can talk about (isomorphism classes of) k-points of $\mathrm{Tot}(E)$. They are in bijection with pairs (y,ξ), where $y \in Y(k)$, and ξ is an element in the (classical) fiber of E at y.

E.6.2. We apply the above discussion to $E = \Omega(Y)$, the (classical) sheaf of Kähler differentials. Denote the resulting stack $\mathrm{Tot}(\Omega(Y))$ by $T^*(Y)$.

By definition, its k-points are pairs (y,ξ), where $y \in Y(k)$ and $\xi \in T^*_y(Y)$.

Note that for every embedding $Y \overset{j}{\hookrightarrow} Y'$, where Y' is smooth, we obtain a presentation of $T^*(Y)$ as a quotient of the vector bundle $T^*(Y')|_Y$. Denote by dj^* the resulting projection

$$T^*(Y')|_Y \to T^*(Y).$$

E.6.3. We now claim that to every object $\mathcal{F} \in \mathrm{Shv}(Y)^{\mathrm{constr}}$ we can attach a closed subset

$$\mathrm{SingSupp}(\mathcal{F}) \subset T^*(Y)$$

with the following property:

For every closed embedding $Y \overset{j}{\hookrightarrow} Y'$ with Y' is a smooth affine scheme, we have

$$(dj^*)^{-1}(\mathrm{SingSupp}(\mathcal{F})) = \mathrm{SingSupp}(j_*(\mathcal{F})).$$

Proof. We only need to show that for every $y \in Y$, the intersection

$$T^*_y(Y') \cap \mathrm{SingSupp}(j_*(\mathcal{F}))$$

is invariant under translations by the elements of

$$\ker(T^*_y(Y') \to T^*_y(Y)).$$

Note that if

$$Y \overset{j_1}{\hookrightarrow} Y_1' \text{ and } Y \overset{j_2}{\hookrightarrow} Y_2'$$

are two embeddings as above, we can always complete them to a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{j_1} & Y_1 \\ {\scriptstyle j_2}\big\downarrow & & \big\downarrow{\scriptstyle j_2'} \\ Y_2 & \xrightarrow{j_1'} & Y_{1,2}, \end{array}$$

where j_1' and j_2' are also closed embeddings and $Y_{1,2}$ is smooth.

From here it is easy to see that if for a given (Y_1', j_1) and a pair of cotangent vectors $\xi_1', \xi_1'' \in T^*_y(Y_1)$ that project to the same vector in $T^*_y(Y)$ one can find a commutative diagram as above and a pair of cotangent vectors $\xi_{1,2}', \xi_{1,2}'' \in T^*_y(Y_{1,2})$ that project to ξ_1', ξ_1'', respectively under $T^*_y(Y_{1,2}) \to T^*_y(Y_1)$ and that project to the same element, to be denoted $\xi_2 \in T^*_y(Y_2)$ under $T^*_y(Y_{1,2}) \to T^*_y(Y_2)$.

Since our assertion holds for smooth Y, we have

$$\xi_1' \in \mathrm{SingSupp}((j_1)_*(\mathcal{F})) \Leftrightarrow \xi_{1,2}' \in \mathrm{SingSupp}((j_2')_* \circ (j_1)_*(\mathcal{F})) \Leftrightarrow \xi_{1,2}' \in \mathrm{SingSupp}((j_1')_* \circ (j_2)_*(\mathcal{F})) \Leftrightarrow$$
$$\Leftrightarrow \xi_2 \in \mathrm{SingSupp}((j_2)_*(\mathcal{F})) \Leftrightarrow$$
$$\Leftrightarrow \xi_{1,2}'' \in \mathrm{SingSupp}((j_1')_* \circ (j_2)_*(\mathcal{F})) \Leftrightarrow \xi_{1,2}'' \in \mathrm{SingSupp}((j_2')_* \circ (j_1)_*(\mathcal{F})) \Leftrightarrow \xi_1'' \in \mathrm{SingSupp}((j_1)_*(\mathcal{F})),$$

as required.

□

E.6.4. The above construction of $\mathrm{SingSupp}(\mathcal{F})$ implies that it has the usual functoriality property of singular support.

For example, if $f : Y_1 \to Y_2$ is a closed embedding and $\mathcal{F}_1 \in \mathrm{Shv}(Y_1)$, we have

$$\mathrm{SingSupp}(f_*(\mathcal{F}_1)) = (df^*)^{-1}(\mathrm{SingSupp}(\mathcal{F}_1),$$

where df^* denotes the map

$$T^*(Y_2) \supset Y_1 \underset{Y_2}{\times} T^*(Y_2) \to T^*(Y_1).$$

E.6.5. *The case of ind-schemes.* In Sect. 19.4 we need to also consider the notion of singular support on ind-schemes (of ind-finite type).

If $\mathcal{Y}$ is an ind-scheme, and let $Z \subset \mathcal{Y}$ be a closed-subscheme. We will consider $T^*(\mathcal{Y})|_Z$ as a pro-object in the above localization of the category of stacks. Namely,

$$T^*(\mathcal{Y})|_Z = \text{“}\lim\text{”}\, T^*(Y_i)|_Z,$$

where Y_i is a (cofinal) family of closed subschemes of $\mathcal{Y}$ such that $Z \subset Y_i$ and

$$\mathcal{Y} = \operatorname{colim} Y_i.$$

In particular, a k-point of $T^*(\mathcal{Y})$ is a pair (y, ξ), where $y \in Y(k)$ and ξ is an element in the classical pro-cotangent space to $\mathcal{Y}$ at y, viewed as a pro-finite dimensional vector space.

To an object $\mathcal{F} \in \operatorname{Shv}(\mathcal{Y})$ supported on Z we can attach its singular support, which is a subset in $T^*(\mathcal{Y})|_Z$, which projects to a closed in each $T^*(Y_i)|_Z$ above.

This justifies the manipulations with singular support of sheaves on ind-schemes/stacks in Sect. 19.4.

E.7. **The external tensor product functor.** For a pair of schemes Y_1 and Y_2, consider the external tensor product functor

$$\operatorname{Shv}(Y_1) \otimes \operatorname{Shv}(Y_2) \to \operatorname{Shv}(Y_1 \times Y_2). \tag{E.10}$$

The functor (E.10) sends compacts to compacts, and is fully faithful, but *not an equivalence* (unless one of the schemes is a disjoint union of set-theoretic points).

E.7.1. Recall that for a pair of DG categories, each equipped with a t-structure, their tensor product acquires a t-structure, see Sect. 1.4.1.

We claim:

Proposition E.7.2. *The functor* (E.10) *is t-exact.*

The proof is given in Sect. E.8 below.

E.7.3. Given $\mathcal{N}_i \subset T^*(Y_i)$ and $\mathcal{F}_i \in \operatorname{Perv}_{\mathcal{N}_i}(Y_i)$,

$$\mathcal{F}_1 \boxtimes \mathcal{F}_2 \in \operatorname{Shv}_{\mathcal{N}_1 \times \mathcal{N}_2}(Y_1 \times Y_2).$$

From here, it follows that the same is true for arbitrary $\mathcal{F}_i \in \operatorname{Shv}_{\mathcal{N}_i}(Y_i)$.

Hence, we obtain a functor

$$\operatorname{Shv}_{\mathcal{N}_1}(Y_1) \otimes \operatorname{Shv}_{\mathcal{N}_2}(Y_2) \overset{\boxtimes}{\to} \operatorname{Shv}_{\mathcal{N}_1 \times \mathcal{N}_2}(Y_1 \times Y_2). \tag{E.11}$$

If one of the categories $\operatorname{Shv}_{\mathcal{N}_i}(Y_i)$, $i = 1, 2$, is dualizable, the functor (E.11) is fully faithful:

Indeed, if, say, $\operatorname{Shv}_{\mathcal{N}_1}(Y_1)$ is dualizable, write the composition

$$\operatorname{Shv}_{\mathcal{N}_1}(Y_1) \otimes \operatorname{Shv}_{\mathcal{N}_2}(Y_2) \to \operatorname{Shv}_{\mathcal{N}_1 \times \mathcal{N}_2}(Y_1 \times Y_2) \hookrightarrow \operatorname{Shv}(Y_1 \times Y_2)$$

as

$$\operatorname{Shv}_{\mathcal{N}_1}(Y_1) \otimes \operatorname{Shv}_{\mathcal{N}_2}(Y_2) \to \operatorname{Shv}_{\mathcal{N}_1}(Y_1) \otimes \operatorname{Shv}(Y_2) \to \operatorname{Shv}(Y_1) \otimes \operatorname{Shv}(Y_2) \to \operatorname{Shv}(Y_1 \times Y_2). \tag{E.12}$$

E.7.4. We will say that a t-structure on a DG category $\mathbf{C}$ is *compactly generated*, if it satisfies the following two conditions:

- $\mathbf{C}^{\leq 0}$ is generated under filtered colimits by objects in $\mathbf{C}^{\leq 0} \cap \mathbf{C}^c$;
- All of $\mathbf{C}$ is generated under filtered colimits by shifts of objects in $\mathbf{C}^{\leq 0} \cap \mathbf{C}^c$.

This is equivalent to:

- If $\mathcal{H}om_{\mathbf{C}}(c_0, c) \in \mathrm{Vect}_{\mathsf{e}}^{>0}$ for all $c_0 \in \mathbf{C}^{\leq 0} \cap \mathbf{C}^c$, then $c \in \mathbf{C}^{>0}$;
- If $\mathcal{H}om_{\mathbf{C}}(c_0, c) = 0$ for all $c_0 \in \mathbf{C}^{\leq 0} \cap \mathbf{C}^c$, then $c = 0$.

For example, for any Y, the t-structure on $\mathrm{Shv}(Y)$ is compactly generated.

It is easy to see that if a t-structure on $\mathbf{C}$ is compactly generated, then it is compatible with filtered colimits and is right-complete.

We have:

Lemma E.7.5. *Let $\mathbf{C}_i$ be DG categories, each equipped with a t-structure, and let $F : \mathbf{C}_1 \to \mathbf{C}_2$ be a t-exact functor. Let $\mathbf{C}$ be another DG category, equipped with a compactly generated t-structure. Then the functor*

$$F \otimes \mathrm{Id}_{\mathbf{C}} : \mathbf{C}_1 \otimes \mathbf{C} \to \mathbf{C}_2 \otimes \mathbf{C}$$

is t-exact.

E.7.6. Combining Proposition E.7.2 and Lemma E.7.5 (applied to the maps in (E.12)), we obtain:

Corollary E.7.7. *Suppose that for one of the categories $\mathrm{Shv}_{\mathcal{N}_i}(Y_i)$, $i = 1, 2$ its t-structure is compactly generated. Then the functor* (E.11) *is t-exact.*

E.7.8. *Example.* Note that the conditions of the corollary are satisfied for $(Y, \mathcal{N}) = (X, \{0\})$, where X is a smooth curve.

Indeed, when X is a connected curve different from $\mathbb{P}^1$, this follows from Theorem E.2.8(a). When $X \simeq \mathbb{P}^1$, this follows from the explicit description of the category $\mathrm{QLisse}(\mathbb{P}^1)$ in Sect. E.2.6.

Remark E.7.9. In fact, one can show that the functor (E.11) is t-exact for any $(Y_i, \mathcal{N}_i)$. In fact the assertion of Lemma E.7.5 holds without the assumption that the t-structure on $\mathbf{C}$ be compactly generated. This is a rather non-trivial assertion, proved in [Lu3, Proposition C.4.4.1].

Note that the conditions of the corollary are satisfied for $(Y, \mathcal{N}) = (X, \{0\})$, where X is a smooth curve.

E.8. **Proof of Proposition E.7.2.**

E.8.1. The functor (E.10) is right t-exact by construction. Hence, it remains to show that it is left t-exact.

E.8.2. *Sheaves at the generic point.* Let Z be an irreducible scheme of finite type, and let η_Z be its generic point.

Set

$$\mathrm{Shv}(\eta_Z) := \underset{U}{\mathrm{colim}}\ \mathrm{Shv}(U),$$

where U runs the (filtered) category of non-empty open subschemes of Z, and the transition functors $\mathrm{Shv}(U_1) \to \mathrm{Shv}(U_2)$ for $U_2 \subset U_1$ are given by restriction. The category $\mathrm{Shv}(\eta_Z)$ carries a naturally defined t-structure.

Define $\mathrm{IndLisse}(\eta_Z)$ by a similar procedure. We have a tautological functor

$$\mathrm{IndLisse}(\eta_Z) \to \mathrm{Shv}(\eta_Z), \tag{E.13}$$

and we claim that it is actually an equivalence.

Indeed, the functor (E.13) is fully faithful because for every U, the category $\mathrm{IndLisse}(U)$ is compactly generated and the functor $\mathrm{IndLisse}(U) \to \mathrm{Shv}(U)$ preserves compactness, and the category of indices involved in the colimit is filtered.

The fact that (E.13) is essentially surjective follows from the definition of constructibility.

E.8.3. Let Z' be another scheme. In a similar way, we define the category $\mathrm{Shv}(\eta_Z \times Z')$, and if Z' is also irreducible, the category $\mathrm{Shv}(\eta_Z \times \eta_{Z'})$.

E.8.4. Let Y be a scheme of finite type. For an irreducible subvariety $Z \overset{\mathbf{i}_Z}{\hookrightarrow} Y$ with generic point η_Z, let $\mathbf{i}^!_{\eta_Z}$ denote the functor

$$\mathrm{Shv}(Y) \overset{\mathbf{i}^!_Z}{\to} \mathrm{Shv}(Z) \to \mathrm{Shv}(\eta_Z).$$

It is easy to see that an object $\mathcal{F} \in \mathrm{Shv}(Y)$ is coconnective if and only if $\mathbf{i}^!_{\eta_Z}(\mathcal{F}) \in \mathrm{Shv}(\eta_Z)$ is coconnective for every Z.

E.8.5. We now return to the proof of the fact that the functor Proposition E.7.2 is left t-exact.

By the above, it suffices to show that for every irreducible $Z \overset{\mathbf{i}_Z}{\hookrightarrow} Y_1 \times Y_2$, the composite functor

$$\mathrm{Shv}(Y_1) \otimes \mathrm{Shv}(Y_2) \overset{\boxtimes}{\to} \mathrm{Shv}(Y_1 \times Y_2) \overset{\mathbf{i}^!_{\eta_Z}}{\longrightarrow} \mathrm{Shv}(\eta_Z) \tag{E.14}$$

is left t-exact.

Let

$$Z_1 \overset{\mathbf{i}_{Z_1}}{\hookrightarrow} Y_1 \text{ and } Z_2 \overset{\mathbf{i}_{Z_2}}{\hookrightarrow} Y_2$$

be the closures of the images of Z in Y_1 and Y_2, respectively.

The functor (E.14) factors as

$$\mathrm{Shv}(Y_1) \otimes \mathrm{Shv}(Y_2) \overset{\mathbf{i}^!_{\eta_{Z_1}} \boxtimes \mathbf{i}^!_{\eta_{Z_2}}}{\longrightarrow} \mathrm{Shv}(\eta_{Z_1}) \otimes \mathrm{Shv}(\eta_{Z_2}) \to \mathrm{Shv}(\eta_Z),$$

where the first arrow is t-exact by Lemma E.7.5. Hence, it suffices to show that the functor

$$\mathrm{Shv}(\eta_{Z_1}) \otimes \mathrm{Shv}(\eta_{Z_2}) \to \mathrm{Shv}(\eta_Z) \tag{E.15}$$

is left t-exact.

E.8.6. Using the equivalence (E.13), we rewrite the functor (E.15) as

$$\mathrm{IndLisse}(\eta_{Z_1}) \otimes \mathrm{IndLisse}(\eta_{Z_2}) \to \mathrm{IndLisse}(\eta_Z).$$

Hence, it is enough to show that for any open *smooth* $U_1 \subset Z_1$ and $U_2 \subset Z_1$ and

$$U \subset Z \cap (U_1 \times U_2),$$

the functor

$$\mathrm{IndLisse}(U_1) \otimes \mathrm{IndLisse}(U_2) \overset{\boxtimes}{\to} \mathrm{IndLisse}(U_1 \times U_2) \overset{\mathbf{i}^!_Z}{\to} \mathrm{IndLisse}(U) \tag{E.16}$$

is left t-exact, where $\mathrm{IndLisse}(-)$ are considered in the *perverse* t-structure. (In the above formula, by a slight abuse of notation we denote by $\mathbf{i}_Z$ the locally closed embedding $U \to U_1 \times U_2$.)

E.8.7. Let $\mathrm{pt} \overset{\mathbf{i}_z}{\to} U$ be the embedding corresponding to a closed point $z \in U$. By the definition of the perverse t-structure on $\mathrm{IndLisse}(U)$ and Sect. 1.2.8, it suffices to show that the composition

$$\mathrm{IndLisse}(U_1) \otimes \mathrm{IndLisse}(U_2) \overset{\boxtimes}{\to} \mathrm{IndLisse}(U_1 \times U_2) \overset{\mathbf{i}^!_Z}{\to} \mathrm{IndLisse}(U) \overset{\mathbf{i}^!_z[\dim(Z)]}{\to} \mathrm{Vect}_{\mathsf{e}} \tag{E.17}$$

is t-exact.

Let z_1 and z_2 be the images of z in U_1 and U_2, respectively. Let $\mathbf{i}_{z_i}$, $i = 1, 2$ denote the corresponding embeddings $\mathrm{pt} \to U_i$. The functor (E.17) identifies with

$$\mathrm{IndLisse}(U_1) \otimes \mathrm{IndLisse}(U_2) \overset{\mathbf{i}^!_{z_1}[\dim(Z_1)] \otimes \mathbf{i}^!_{z_2}[\dim(Z_2)]}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}} \overset{[\dim(Z) - \dim(Z_1) - \dim(Z_2)]}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}}.$$

In the above composition, the first arrow is t-exact by Lemma E.7.5, and the second arrow is left t-exact because $\dim(Z_1) + \dim(Z_2) \geq \dim(Z)$.

$\square$[Proposition E.7.2]

E.9. **The tensor product theorems.** In this subsection we will discuss several variants of the tensor product result [GKRV, Theorem A.3.8].

E.9.1. First, we have the following result, which is [GKRV, Theorem A.3.8].

Theorem E.9.2. *Assume that X is smooth and proper. Let $\mathcal{N} \subset T^*(Y)$ be half-dimensional. Then the resulting functor*

$$\text{IndLisse}(X) \otimes \text{Shv}_{\mathcal{N}}(Y)^{\text{access}} \to \text{Shv}_{\{0\}\times\mathcal{N}}(X \times Y)^{\text{access}} \tag{E.18}$$

is an equivalence.

Remark E.9.3. It is natural to ask whether the functor

$$\text{QLisse}(X) \otimes \text{Shv}_{\mathcal{N}}(Y) \to \text{Shv}_{\{0\}\times\mathcal{N}}(X \times Y)$$

is an equivalence.

Unfortunately, we do not have an answer to this, except in the cases covered by Theorems E.9.5 and E.9.9 below. Namely, we did not find a way to determine when the tensor product

$$\text{QLisse}(X) \otimes \text{Shv}_{\mathcal{N}}(Y)$$

is left-complete in its t-structure.

E.9.4. Next, we claim:

Theorem E.9.5. *Let X be smooth and proper. Then the functor*

$$\text{QLisse}(X) \otimes \text{Shv}(Y) \to \text{Shv}(X \times Y) \tag{E.19}$$

is an equivalence onto the full subcategory that consists of objects $\mathcal{F}$ with the following property: For every m and every constructible sub-object $\mathcal{F}'$ of $H^m(\mathcal{F})$, the singular support of $\mathcal{F}'$ is contained in a subset of the form $\{0\} \times \mathcal{N}$, where $\mathcal{N} \subset T^*(Y)$ is half-dimensional.

The proof will use the following variant of Theorem E.1.3 (the proof is given in Sect. E.9.10):

Theorem E.9.6. *Let $\mathcal{A}$ be a small abelian category of finite cohomological dimension. Let $\mathbf{C}$ be a DG category equipped with a t-structure in which it is left-compete. Then*

$$\text{Ind}(D^b(\mathcal{A})) \otimes \mathbf{C}$$

is left-complete in its t-structure.

Proof of Theorem E.9.5. First, we observe that the functor (E.19) is fully faithful, being a composition of

$$\text{QLisse}(X) \otimes \text{Shv}(Y) \to \text{Shv}(X) \otimes \text{Shv}(Y) \to \text{Shv}(X \times Y),$$

where the first arrow is fully faithful because $\text{Shv}(Y)$ is dualizable.

Thus, it remains to show that (E.19) is essentially surjective onto the specified subcategory.

First, we claim that every *bounded below* object in $\text{Shv}(X \times Y)$ with the specified condition belongs to the essential image of (E.19).

Indeed, by devissage we can assume that the object in question is also bounded above; then that it is in the heart of the t-structure, and then that it is contained in $\text{Perv}(X \times Y)$, and has singular support of the form $\{0\} \times \mathcal{N}$ with $\mathcal{N} \subset T^*(Y)$ is half-dimensional. However, such an object is contained in the essential image of (E.18), by Theorem E.9.2.

Now, the assertion of the theorem follows, as both sides are left-complete in their respective t-structures: the right-hand side by Theorem 1.1.6, and the left-hand side by Theorem E.9.6.

□

Corollary E.9.7. *Suppose that* $\text{char}(k) = 0$. *Then the functor* (E.19) *is an equivalence onto a subcategory consisting of objects whose singular support is contained in $\{0\} \times T^*(Y) \subset T^*(X \times Y)$.*

Proof. It suffices to show for every constructible object $\mathcal{F} \in \mathrm{Shv}(X \times Y)$, with $\mathrm{SingSupp}(\mathcal{F}) \subset \{0\} \times T^*(Y) \subset T^*(X \times Y)$, the singular support of $\mathcal{F}$ is in fact contained in a subset of the form $\{0\} \times \mathcal{N}$ for some half-dimensional $\mathcal{N} \subset T^*(Y)$.

However, since $\mathrm{char}(k) = 0$, the singular support of $\mathcal{F}$ is a *Lagrangian* subset of $T^*(X \times Y)$,

We claim that any irreducible Lagrangian subspace $\mathcal{L} \subset T^*(X \times Y)$ contained in $\{0\} \times T^*(Y)$ is of required form.

Indeed, at its generic point $\mathcal{L}$ is the conormal of some $Z \subset Y \times X$. However, if

$$N^*_{Z/Y \times X} \subset T^*(Y) \times \{0\},$$

then Z is of the form $Y' \times X$ for $Y' \subset Y$.

□

E.9.8. Finally, we claim:

Theorem E.9.9. *Let X be smooth and proper. Assume also that* $\mathrm{QLisse}(X)$ *is Verdier-compatible (see Sect. E.4). Let $\mathcal{N} \subset T^*(Y)$ be half-dimensional. Then the resulting functor*

$$\mathrm{QLisse}(X) \otimes \mathrm{Shv}_{\mathcal{N}}(Y) \to \mathrm{Shv}_{\{0\} \times \mathcal{N}}(X \times Y) \tag{E.20}$$

is an equivalence.

Proof. Since $\mathrm{QLisse}(X)$ is dualizable, the functor

$$\mathrm{QLisse}(X) \otimes \mathrm{Shv}_{\mathcal{N}}(Y) \to \mathrm{QLisse}(X) \otimes \mathrm{Shv}(Y) \tag{E.21}$$

is fully faithful.

Given Theorem E.9.5, it suffices to show that any object

$$\mathcal{F} \in \mathrm{QLisse}(X) \otimes \mathrm{Shv}(Y)$$

whose image $\mathcal{F}' \in \mathrm{Shv}(X \times Y)$ has singular support in $\{0\} \times \mathcal{N}$, belongs to the essential image of (E.21).

Since $\mathrm{QLisse}(X)$ is Verdier-compatible, it suffices to show that for any $E \in \mathrm{QLisse}(X)$, the object

$$(\mathrm{C}^{\cdot}(X, -) \otimes \mathrm{Id})(E \overset{!}{\otimes} \mathcal{F}) \in \mathrm{Shv}(Y)$$

belongs to $\mathrm{Shv}_{\mathcal{N}}(Y)$.

However, the latter object is the same as

$$(p_Y)_*(p_X^!(E) \overset{!}{\otimes} \mathcal{F}'),$$

where p_X and p_Y are the two projections from $X \times Y$ to X and Y, respectively.

The latter object indeed belongs to $\mathrm{Shv}_{\mathcal{N}}(Y)$, due to the assumption on the singular support of $\mathcal{F}'$ and the fact that X is proper (see Lemma 18.4.8).

□

E.9.10. *Proof of Theorem E.9.6.* The proof repeats the argument of Theorem E.1.3, using the following variants of Proposition E.1.7 and Lemma E.1.11, respectively:

Proposition E.9.11. *Let $\mathbf{C}$ be compactly generated by compact objects of finite cohomological dimension. Then for any $\mathbf{C}_1$ equipped with a t-structure in which it is left-complete, the functor*

$$\mathbf{C} \otimes \mathbf{C}_1 \to (\mathbf{C} \otimes \mathbf{C}_1)^{\wedge} \tag{E.22}$$

is fully faithful and its right adjoint is continuous.

Lemma E.9.12. *Let $\mathcal{A}$ be as in Theorem E.9.6, and let $\mathbf{C}_1$ be equipped with a t-structure. Let $\mathbf{c} \in \mathrm{Ind}(D^b(\mathcal{A})) \otimes \mathbf{C}_1$ be an object satisfying*

$$(\mathcal{H}om_{\mathrm{Ind}(D^b(\mathcal{A}))}(a, -) \otimes \mathrm{Id})(\mathbf{c}) \in (\mathbf{C}_1)^{\leq 0} \text{ for all } a \in \mathcal{A} \subset \mathrm{Ind}(D^b(\mathcal{A})).$$

Then $\mathbf{c} \in (\mathrm{Ind}(D^b(\mathcal{A})) \otimes \mathbf{C}_1)^{\leq 0}$.

Both these statements are proved in a way mimicking the original arguments.

Appendix F. Constructible sheaves on an algebraic stack

As in Sect. E, in this section we let $\mathrm{Shv}(-)^{\mathrm{constr}}$ be a constructible sheaf theory. All algebro-geometric objects will be assumed (locally) of finite type over the ground field k.

F.1. **Generalities.**

F.1.1. Let $\mathcal{Y}$ be a prestack. Recall that we define

$$\mathrm{Shv}(\mathcal{Y}) := \lim_{S} \mathrm{Shv}(S),$$

where the index category is that of affine schemes equipped with a map to $\mathcal{Y}$, and the transition functors are given by !-pullback.

Since we are in the constructible context, the !-pullback functor admits a left adjoint, given by !-pushforward. Hence, using [GR1, Chapter 1, Proposition 2.5.7], we can rewrite

$$\mathrm{Shv}(\mathcal{Y}) \simeq \operatorname*{colim}_{S} \mathrm{Shv}(S), \tag{F.1}$$

where the transition functors are given by !-pushforward.

In particular, we obtain that $\mathrm{Shv}(\mathcal{Y})$ is compactly generated: the compact generators are of the form

$$\mathrm{ins}_{f_0}(\mathcal{F}_{S_0}), \quad \mathcal{F}_{S_0} \in \mathrm{Shv}(S_0)^c, \tag{F.2}$$

where for an affine scheme S_0 equipped with a map $f_0 : S_0 \to \mathcal{Y}$, we denote by ins_{f_0} the corresponding tautological functor

$$\mathrm{Shv}(S_0) \to \operatorname*{colim}_{S} \mathrm{Shv}(S) \simeq \mathrm{Shv}(\mathcal{Y}).$$

F.1.2. Suppose for a moment that $\mathcal{Y}$ is an algebraic stack[53]. Then the above index category can be replaced by its non-full subcategory, where we allow as objects affine schemes that are smooth over $\mathcal{Y}$, and as morphisms smooth maps between those.

Furthermore, formula (F.2) has a more explicit meaning: for $S \xrightarrow{f} \mathcal{Y}$, where S is an affine scheme, we have

$$\mathrm{ins}_f \simeq f_!,$$

where the functor

$$f_! : \mathrm{Shv}(S) \to \mathrm{Shv}(\mathcal{Y})$$

is defined because the morphism f is schematic.

Thus, $\mathrm{Shv}(\mathcal{Y})$ is compactly generated by objects of the form

$$f_!(\mathcal{F}_S), \quad \mathcal{F}_S \in \mathrm{Shv}(S)^c.$$

Moreover, as above, it is sufficient to consider only the pairs (S, f) with f smooth.

F.1.3. Recall that for a quasi-compact scheme Y, Verdier duality defines a contravariant equivalence

$$(\mathrm{Shv}(Y)^{\mathrm{constr}})^{\mathrm{op}} \xrightarrow{\mathbb{D}} \mathrm{Shv}(Y)^{\mathrm{constr}}.$$

Since

$$\mathrm{Shv}(Y) := \mathrm{Ind}(\mathrm{Shv}(Y)^{\mathrm{constr}}),$$

we obtain that the category $\mathrm{Shv}(Y)$ is canonically self-dual with the counit

$$\mathrm{Shv}(Y) \otimes \mathrm{Shv}(Y) \to \mathrm{Vect}_{\mathsf{e}}$$

given by

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathrm{C}^{\cdot}(Y, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2).$$

[53]As per our conventions, we will assume that $\mathcal{Y}$ has an affine diagonal.

F.1.4. In particular, by [DrGa2, Proposition 1.8.3] and (F.1), we obtain that for a prestack $\mathcal{Y}$, the category $\text{Shv}(\mathcal{Y})$ is dualizable, and

$$\text{Shv}(\mathcal{Y})^\vee \simeq \underset{S}{\text{colim}}\, \text{Shv}(S),$$

where the transition functors are given by *-pushforward.

Remark F.1.5. Note that there is no a priori reason for $\text{Shv}(\mathcal{Y})^\vee$ to be equivalent to the original $\text{Shv}(\mathcal{Y})$.

We will see that there is a canonical such equivalence when $\mathcal{Y}$ is a quasi-compact algebraic stack (at least when $\mathcal{Y}$ is locally a quotient). However, for more general $\mathcal{Y}$ (e.g., for non-quasi-compact algebraic stacks) such an equivalence would reflect a particular feature of $\mathcal{Y}$, for example its property of being *miraculous*, see [Ga3, Sect. 6.7].

F.2. **Constructible vs compact.**

F.2.1. Let $\mathcal{Y}$ be an algebraic stack. Let

$$\text{Shv}(\mathcal{Y})^{\text{constr}} \subset \text{Shv}(\mathcal{Y})$$

be the full subcategory consisting of objects that pullback to an object of

$$\text{Shv}(S)^{\text{constr}} = \text{Shv}(S)^c \subset \text{Shv}(S)$$

for any affine scheme S mapping to $\mathcal{Y}$.

It is easy to see that this condition is enough to test on smooth maps $S \to \mathcal{Y}$. In the latter case, we can use either !- or *- pullback, as they differ by a cohomological shift.

F.2.2. Using the definition of the constructible subcategory via *-pullbacks along smooth maps, we obtain that we have an inclusion

$$\text{Shv}(\mathcal{Y})^c \subset \text{Shv}(\mathcal{Y})^{\text{constr}}. \tag{F.3}$$

Indeed, for $f : S \to \mathcal{Y}$, the functor f^* sends compacts to compacts, since its right adjoint, namely f_*, is continuous.

However, the inclusion (F.3) is typically *not* an equality. For example, the constant sheaf

$$\underline{\mathsf{e}}_{\mathcal{Y}} \in \text{Shv}(\mathcal{Y})^{\text{constr}}$$

is *not* compact for $\mathcal{Y} = B(\mathbb{G}_m)$.

F.2.3. That said, we have the following assertion:

Proposition F.2.4. *Let $\mathcal{Y}$ be quasi-compact. Then an object $\mathcal{F} \in \text{Shv}(\mathcal{Y})^{\text{constr}} \cap \text{Shv}(\mathcal{Y})^{\geq n}$ is compact as an object of $\text{Shv}(\mathcal{Y})^{\geq m}$ for any $m \leq n$.*

Proof. It suffices to show that for any k, the functor

$$\mathcal{F}' \mapsto \tau^{\leq k}\left(\mathcal{H}om_{\text{Shv}(\mathcal{Y})}(\mathcal{F}, \mathcal{F}')\right)$$

commutes with filtered colimits as $\mathcal{F}'$ ranges over $\text{Shv}(\mathcal{Y})^{\geq m}$ for some fixed m.

Choose a smooth covering $f : S \to \mathcal{Y}$, where S is an affine scheme, and let $S^\bullet$ be its Čech nerve; let $f^n : S^n \to \mathcal{Y}$ denote the resulting maps.

For $\mathcal{F}' \in \text{Shv}(\mathcal{Y})$, we can calculate $\mathcal{H}om_{\text{Shv}(\mathcal{Y})}(\mathcal{F}, \mathcal{F}')$ as the totalization of the cosimplicial complex whose n-simplices are

$$\mathcal{H}om_{\text{Shv}(S^n)}((f^n)^!(\mathcal{F}), (f^n)^!(\mathcal{F}')).$$

Assume that $\mathcal{F} \in \text{Shv}(\mathcal{Y})^{\leq N}$. Then for all n, the objects $\mathcal{H}om_{\text{Shv}(S^n)}((f^n)^!(\mathcal{F}), (f^n)^!(\mathcal{F}'))$ belong to $\text{Vect}_{\mathsf{e}}^{\geq -N+m}$. Hence,

$$\tau^{\leq k}\left(\lim_{[n]} \mathcal{H}om_{\text{Shv}(S^n)}((f^n)^!(\mathcal{F}), (f^n)^!(\mathcal{F}'))\right)$$

maps isomorphically to the limit

$$\tau^{\leq k}\left(\lim_{[n],n\leq k+N-m+1} \mathcal{H}om_{\mathrm{Shv}(S^n)}((f^n)^!(\mathcal{F}),(f^n)^!(\mathcal{F}'))\right)$$

(e.g., by [Lu2, Proposition 1.2.4.5(4)]), which is a *finite* limit.

Since finite limits commute with filtered colimits, it suffices to show that for every n, the functor

$$\mathcal{F}'\mapsto \mathcal{H}om_{\mathrm{Shv}(S^n)}((f^n)^!(\mathcal{F}),(f^n)^!(\mathcal{F}'))$$

commutes with filtered colimits. However, this follows from the fact that $(f^n)^!(\mathcal{F})$ is compact.

□

F.2.5. Verdier duality defines a contravariant equivalence

$$(\mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}})^{\mathrm{op}} \overset{\mathbb{D}}{\to} \mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}.$$

If $\mathcal{Y}$ is not quasi-compact, the functor $\mathbb{D}$ will typically *not* send $\mathrm{Shv}(\mathcal{Y})^c$ to $\mathrm{Shv}(\mathcal{Y})^c$.

F.2.6. Assume that $\mathcal{Y}$ is quasi-compact. We will say that $\mathcal{Y}$ is *duality-adapted* if the functor $\mathbb{D}$ sends $\mathrm{Shv}(\mathcal{Y})^c$ to $\mathrm{Shv}(\mathcal{Y})^c$.

Based in [DrGa1, Corollary 8.4.2], we conjecture:

Conjecture F.2.7. *Any quasi-compact algebraic stack with an affine diagonal is duality-adapted.*

We are going to prove:

Theorem F.2.8. *Let $\mathcal{Y}$ be such that it can be covered by open subsets each of which has the form Y/G, where Y is a quasi-compact scheme and G is an algebraic group. Then $\mathcal{Y}$ is duality-adapted.*

F.3. **Proof of Theorem F.2.8.**

F.3.1. *A reduction step.* Let us reduce the assertion to the case when $\mathcal{Y}$ is globally a quotient, i.e., is of the form Y/G.

Indeed, suppose $\mathcal{Y}$ can be covered by open substacks $\mathcal{U}_i \overset{j_i}{\hookrightarrow} \mathcal{Y}$, such that each $\mathcal{U}_i$ is duality-adapted. We will show that $\mathcal{Y}$ is duality-adapted.

Since $\mathcal{Y}$ was assumed quasi-compact, we can assume that the above open cover is finite. Now the assertion follows from the fact that an object $\mathcal{F}\in \mathrm{Shv}(\mathcal{Y})$ is compact if and only if all $j_i^*(\mathcal{F})$ are compact.

Indeed, the implication

$$\mathcal{F}\in \mathrm{Shv}(\mathcal{Y})^c \;\Rightarrow\; j_i^*(\mathcal{F})\in \mathrm{Shv}(\mathcal{U}_i)^c$$

follows from the fact that j_i^* admits a continuous right adjoint, namely, $(j_i)_*$.

The opposite implication follows from the fact that $\mathcal{H}om_{\mathrm{Shv}(\mathcal{Y})}(\mathcal{F},-)$ can be expressed as a finite limit in terms of $\mathcal{H}om_{\mathrm{Shv}(\mathcal{U}_i)}(j_i^*(\mathcal{F}),-)$ and finite intersections of these opens.

F.3.2. *Explicit generators for a global quotient.* Thus, we can assume that $\mathcal{Y}$ has the form Y/G.

It suffices to find a system of compact generators of $\mathrm{Shv}(\mathcal{Y})$ that are sent to compact objects by the functor $\mathbb{D}$.

Let π_Y denote the map

$$Y\to Y/G=\mathcal{Y}.$$

Note that for any $\mathcal{F}\in \mathrm{Shv}(Y/G)^{\mathrm{constr}}$, the object

$$(\pi_Y)_!\circ(\pi_Y)^*(\mathcal{F})$$

is compact. Hence, it suffices to show that:

(I) Such objects generate $\mathrm{Shv}(Y/G)$;

(II) They are sent to compact objects by Verdier duality.

F.3.3. *Digression: cochains on the group.* Consider the map

$$\pi_{\mathrm{pt}} : \mathrm{pt} \to \mathrm{pt}/G$$

and the objects

$$(\pi_{\mathrm{pt}})_*(\mathsf{e}), (\pi_{\mathrm{pt}})_!(\mathsf{e}) \in \mathrm{Shv}(\mathrm{pt}/G).$$

Note that that

$$(\pi_{\mathrm{pt}})^* \circ (\pi_{\mathrm{pt}})_*(\mathsf{e}) \simeq \mathrm{C}^\cdot(G).$$

Note also that

$$(\pi_{\mathrm{pt}})_*(\mathsf{e}) \simeq (\pi_{\mathrm{pt}})_!(\mathsf{e})[d], \tag{F.4}$$

where for

$$1 \to G_{\mathrm{unip}} \to G \to G_{\mathrm{red}} \to 1,$$

we have

$$d = 2\dim(G_{\mathrm{unip}}) + \dim(G_{\mathrm{red}}).$$

The isomorphism (F.4) follows from the fact that for a reductive group G, the DG algebra of cochains $\mathrm{C}^\cdot(G)$ is a Frobenius algebra (in fact, a symmetric algebra on generators in odd degrees), and so $\mathrm{C}^\cdot_c(G)[2\dim(G)] \simeq (\mathrm{C}^\cdot(G))^\vee$ is isomorphic to $\mathrm{C}^\cdot(G)$ up to a shift by $[d]$.

F.3.4. *Verification of Property II.* Denote by q the map $Y/G \to \mathrm{pt}/G$. For any $\mathcal{F}' \in \mathrm{Shv}(Y/G)$ we have:

$$(\pi_Y)_! \circ (\pi_Y)^*(\mathcal{F}') \simeq \mathcal{F}' \overset{*}{\otimes} q^*((\pi_{\mathrm{pt}})_!(\mathsf{e}))$$

and

$$(\pi_Y)_* \circ (\pi_Y)^!(\mathcal{F}') \simeq \mathcal{F}' \overset{!}{\otimes} q^!((\pi_{\mathrm{pt}})_*(\mathsf{e})) \simeq \mathcal{F}' \overset{*}{\otimes} q^*((\pi_{\mathrm{pt}})_*(\mathsf{e}))[2\dim(G)] \simeq$$
$$\simeq \mathcal{F}' \overset{*}{\otimes} q^*((\pi_{\mathrm{pt}})_!(\mathsf{e}))[2\dim(G)+d] \simeq (\pi_Y)_! \circ (\pi_Y)^*(\mathcal{F}')[2\dim(G)+d]$$

Hence,

$$\mathbb{D}((\pi_Y)_! \circ (\pi_Y)^*(\mathcal{F})) \simeq (\pi_Y)_* \circ (\pi_Y)^!(\mathbb{D}(\mathcal{F})) \simeq (\pi_Y)_! \circ (\pi_Y)^*(\mathbb{D}(\mathcal{F}))[2\dim(G)+d].$$

This proves Property (II).

F.3.5. *Verification of Property I.* To prove Property (I), let $\mathcal{F}'$ be a non-zero object of $\mathrm{Shv}(Y/G)$, and let us find $\mathcal{F} \in \mathrm{Shv}(Y/G)^{\mathrm{constr}}$ so that

$$\mathcal{H}om_{\mathrm{Shv}(Y/G)}((\pi_Y)_! \circ (\pi_Y)^*(\mathcal{F}), \mathcal{F}') \neq 0.$$

This can be done for $Y \overset{\pi_Y}{\to} Y/G$ replaced by any pair $Y \overset{f}{\to} \mathcal{Z}$, where $\mathcal{Z}$ is an algebraic stack and f is a smooth covering map.

Indeed, for any $\mathcal{F}, \mathcal{F}' \in \mathrm{Shv}(\mathcal{Z})$, we have

$$\mathcal{H}om_{\mathrm{Shv}(\mathcal{Z})}(f_! \circ f^*(\mathcal{F}), \mathcal{F}') \simeq \mathcal{H}om_{\mathrm{Shv}(Y)}(f^*(\mathcal{F}), f^!(\mathcal{F}')) \simeq \mathcal{H}om_{\mathrm{Shv}(\mathcal{Z})}(\mathcal{F}, f_* \circ f^!(\mathcal{F}')). \tag{F.5}$$

Since $\mathrm{Shv}(\mathcal{Z})$ is compactly generated, and $\mathrm{Shv}(\mathcal{Z})^c \subset \mathrm{Shv}(\mathcal{Z})^{\mathrm{constr}}$, it suffices to show that if $\mathcal{F}' \neq 0$, then $f_* \circ f^!(\mathcal{F}') \neq 0$.

Applying (F.5) to $\mathcal{F} = \mathcal{F}'$, it suffices to show that $\mathcal{H}om_{\mathrm{Shv}(Y)}(f^*(\mathcal{F}), f^!(\mathcal{F}')) \neq 0$. However, $f^!$ is isomorphic to f^* up to a shift, so the assertion follows from the fact that $f^*(\mathcal{F}) \neq 0$.

☐[Theorem F.2.8]

F.4. **Verdier duality on stacks.** In this subsection $\mathcal{Y}$ will be a duality-adapted quasi-compact algebraic stack.

F.4.1. The assumption that $\mathcal{Y}$ is duality-adapted implies that the Verdier duality functor defines a contravariant equivalence

$$(\mathrm{Shv}(\mathcal{Y})^c)^{\mathrm{op}} \to \mathrm{Shv}(\mathcal{Y})^c.$$

Hence, we obtain a canonical identification

$$\mathrm{Shv}(\mathcal{Y})^\vee \simeq \mathrm{Shv}(\mathcal{Y}).$$

By construction, the corresponding pairing

$$\mathrm{Shv}(\mathcal{Y})^c \times \mathrm{Shv}(\mathcal{Y})^c \to \mathrm{Vect}_{\mathsf{e}}$$

sends

$$\mathcal{F}_1, \mathcal{F}_2 \to \mathrm{C}^\cdot(\mathcal{Y}, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2).$$

F.4.2. Let

$$\mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, -) : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Vect}$$

be the functor dual to the functor

$$\mathrm{Vect}_{\mathsf{e}} \to \mathrm{Shv}(\mathcal{Y}), \quad \mathsf{e} \mapsto \omega_{\mathcal{Y}},$$

see [DrGa1, Sect. 9.1]. This functor is the ind-extension of the restriction of the functor

$$\mathrm{C}^\cdot(\mathcal{Y}, -) : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Vect}$$

to $\mathrm{Shv}(\mathcal{Y})^c \subset \mathrm{Shv}(\mathcal{Y})$.

In particular, we have a natural transformation

$$\mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, -) \to \mathrm{C}^\cdot(\mathcal{Y}, -), \tag{F.6}$$

which is an equivalence when evaluated on compact objects.

Furthermore, the duality pairing on all of $\mathrm{Shv}(\mathcal{Y}) \otimes \mathrm{Shv}(\mathcal{Y})$ can be written as

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2).$$

We have a map

$$\mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2) \to \mathrm{C}^\cdot(\mathcal{Y}, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2),$$

which is an isomorphism when one of the objects $\mathcal{F}_1$ or $\mathcal{F}_2$ is compact.

F.4.3. We observe:

Lemma F.4.4. *For $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})^c$ and $\mathcal{F}' \in \mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}$, both*

$$\mathcal{F} \overset{*}{\otimes} \mathcal{F}' \text{ and } \mathcal{F} \overset{!}{\otimes} \mathcal{F}'$$

are compact.

Proof. The assertion for $\mathcal{F} \overset{*}{\otimes} \mathcal{F}'$ follows from the fact that

$$\mathcal{H}om(\mathcal{F} \overset{*}{\otimes} \mathcal{F}', \mathcal{F}'') \simeq \mathcal{H}om(\mathcal{F}, \mathbb{D}(\mathcal{F}') \overset{!}{\otimes} \mathcal{F}'').$$

The assertion for $\mathcal{F} \overset{!}{\otimes} \mathcal{F}'$ follows by Verdier duality (and the assumption that $\mathcal{Y}$ is duality-adapted). □

F.4.5. For future reference, we record the following properties of duality-adapted prestacks, borrowed from [DrGa1, Theorem 10.2.9] (we will omit the proof as it repeats verbatim the arguments from *loc. cit.*):

Proposition F.4.6. *Assume that* $\mathcal{Y}$ *is duality-adapted. Then the following properties of an object* $\mathcal{F} \in \text{Shv}(\mathcal{Y})^{\text{constr}}$ *are equivalent:*

(i) $\mathcal{F}$ *is compact;*

(i') $\mathbb{D}(\mathcal{F})$ *is compact;*

(ii) $\mathcal{F}$ *belongs to the smallest (non cocomplete) DG subcategory of* $\text{Shv}(\mathcal{Y})$ *closed under taking direct summands that contains objects of the form* $f_!(\mathcal{F}_S)$*, where* $f : S \to \mathcal{Y}$ *with* S *an affine scheme and* $\mathcal{F}_S \in \text{Shv}(S)^{\text{constr}}$*;*

(ii') $\mathcal{F}$ *belongs to the smallest (non cocomplete) DG subcategory of* $\text{Shv}(\mathcal{Y})$ *closed under taking direct summands that contains objects of the form* $f_*(\mathcal{F}_S)$*, where* $f : S \to \mathcal{Y}$ *with* S *an affine scheme and* $\mathcal{F}_S \in \text{Shv}(S)^{\text{constr}}$*;*

(iii) *The functor*

$$\mathcal{F}' \mapsto \mathrm{C}^\cdot(\mathcal{Y}, \mathcal{F} \overset{!}{\otimes} \mathcal{F}'), \quad \text{Shv}(\mathcal{Y}) \to \text{Vect}_{\mathsf{e}}$$

is continuous;

(iv) *The functor*

$$\mathcal{F}' \mapsto \mathrm{C}^\cdot(\mathcal{Y}, \mathcal{F} \overset{!}{\otimes} \mathcal{F}'), \quad \text{Shv}(\mathcal{Y}) \to \text{Vect}_{\mathsf{e}}$$

is cohomologically bounded on the right;

(v) *The functor*

$$\mathcal{F}' \mapsto \mathrm{C}^\cdot_c(\mathcal{Y}, \mathcal{F} \overset{*}{\otimes} \mathcal{F}'), \quad \text{Shv}(\mathcal{Y}) \to \text{Vect}_{\mathsf{e}}$$

is cohomologically bounded on the left;

(vi) *The functor*

$$\mathcal{F}' \mapsto \mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, \mathcal{F} \overset{!}{\otimes} \mathcal{F}'), \quad \text{Shv}(\mathcal{Y}) \to \text{Vect}_{\mathsf{e}}$$

is cohomologically bounded on the left;

(vii) *The natural transformation*

$$\mathrm{C}^\cdot_\blacktriangle(\mathcal{Y}, \mathcal{F} \overset{!}{\otimes} \mathcal{F}') \to \mathrm{C}^\cdot(\mathcal{Y}, \mathcal{F} \overset{!}{\otimes} \mathcal{F}'), \quad \mathcal{F}' \in \text{Shv}(\mathcal{Y})$$

is an isomorphism;

(viii) *For any schematic quasi-compact morphism* $g : \mathcal{Y}' \to \mathcal{Y}$ *and* $f : \mathcal{Y}' \to S$ *where* S *is a scheme, the object* $f_* \circ g^!(\mathcal{F})$ *is cohomologically bounded above;*

(viii') *For any schematic quasi-compact morphism* $g : \mathcal{Y}' \to \mathcal{Y}$ *and* $f : \mathcal{Y}' \to S$ *where* S *is a scheme, the object* $f_! \circ g^*(\mathcal{F})$ *is cohomologically bounded below;*

(ix) *Same as* (viii) *but* g *is a finite étale map onto a locally closed substack of* $\mathcal{Y}$*;*

(ix') *Same as* (viii') *but* g *is a finite étale map onto a locally closed substack of* $\mathcal{Y}$*.*

F.5. **The renormalized category of sheaves.**

F.5.1. Let $\mathcal{Y}$ be a quasi-compact algebraic stack.

We define the renormalized version of the category of sheaves on $\mathcal{Y}$, denoted $\text{Shv}(\mathcal{Y})^{\text{ren}}$ to be

$$\text{Ind}(\text{Shv}(\mathcal{Y})^{\text{constr}}).$$

The t-structure on $\text{Shv}(\mathcal{Y})^{\text{constr}}$ induces a unique t-structure on $\text{Shv}(\mathcal{Y})^{\text{ren}}$ compatible with filtered colimits.

Ind-extension of the tautological embedding $\text{Shv}(\mathcal{Y})^{\text{constr}} \hookrightarrow \text{Shv}(\mathcal{Y})$ defines a functor

$$\text{un-ren}_{\mathcal{Y}} : \text{Shv}(\mathcal{Y})^{\text{ren}} \to \text{Shv}(\mathcal{Y}).$$

The functor $\text{un-ren}_{\mathcal{Y}}$ is t-exact by construction.

F.5.2. Note that for every fixed n, the functor

$$\text{un-ren}_{\mathcal{Y}} : (\text{Shv}(\mathcal{Y})^{\text{ren}})^{\geq -n} \to (\text{Shv}(\mathcal{Y}))^{\geq -n} \tag{F.7}$$

preserves compactness (by Proposition F.2.4), and hence is fully faithful.

Moreover, since the functor $\text{un-ren}_{\mathcal{Y}}$ is essentially surjective on the hearts, we obtain that (F.7) is actually an equivalence.

From here, we obtain that the functor $\text{un-ren}_{\mathcal{Y}}$ identifies $\text{Shv}(\mathcal{Y})$ with the left completion of $\text{Shv}(\mathcal{Y})^{\text{ren}}$ with respect to its t-structure.

F.5.3. This construction of the pair $(\text{Shv}(\mathcal{Y})^{\text{ren}}, \text{un-ren}_{\mathcal{Y}})$ mimics the construction of how one defines $\text{IndCoh}(S)$ for an eventually coconnective affine scheme (see [GR1, Chapter 4, Sect. 1.2]), and shares its formal properties:

- Ind-extension of the tautological embedding $\text{Shv}(\mathcal{Y})^c \hookrightarrow \text{Shv}(\mathcal{Y})^{\text{constr}}$ defines a fully faithful functor
$$\text{ren}_{\mathcal{Y}} : \text{Shv}(\mathcal{Y}) \to \text{Shv}(\mathcal{Y})^{\text{ren}},$$
which is the left adjoint of $\text{un-ren}_{\mathcal{Y}}$.
- The functor $\text{un-ren}_{\mathcal{Y}}$ realizes $\text{Shv}(\mathcal{Y})$ as the co-localization of $\text{Shv}(\mathcal{Y})^{\text{ren}}$ with respect to the subcategory consisting of objects all of whose cohomologies with respect to the above t-structure vanish.
- The operation of *-tensor product makes $\text{Shv}(\mathcal{Y})^{\text{ren}}$ into a symmetric monoidal category, and $\text{Shv}(\mathcal{Y})$ into a module category over it (see Lemma F.4.4). The same is true for the !-tensor product provided that $\mathcal{Y}$ is duality-adapted.

F.5.4. Note that Verdier duality

$$\mathbb{D} : (\text{Shv}(\mathcal{Y})^{\text{constr}})^{\text{op}} \to \text{Shv}(\mathcal{Y})^{\text{constr}}$$

defines an identification

$$\text{Shv}(\mathcal{Y})^{\text{ren}} \simeq (\text{Shv}(\mathcal{Y})^{\text{ren}})^{\vee}.$$

Assume for a moment that $\mathcal{Y}$ is duality-adapted. In particular, we have also the identification

$$\text{Shv}(\mathcal{Y}) \simeq \text{Shv}(\mathcal{Y})^{\vee}.$$

The functors $\text{ren}_{\mathcal{Y}}$ and $\text{un-ren}_{\mathcal{Y}}$ are mutually dual with respect to these identifications.

F.5.5. Let us consider the example of $\mathcal{Y} = \text{pt}/G$. In this case $\text{Shv}(\text{pt}/G)^{\text{ren}}$ is compactly generated by the object $\underline{\mathsf{e}}_{\text{pt}/G}$. Hence, we obtain a canonical equivalence

$$\text{Shv}(\text{pt}/G)^{\text{ren}} \simeq \text{C}^{\cdot}(\text{pt}/G)\text{-mod}. \tag{F.8}$$

Under this equivalence, the symmetric monoidal structure on $\text{Shv}(\text{pt}/G)^{\text{ren}}$ given by *-tensor product corresponds to the usual symmetric monoidal structure on the category of modules over a commutative algebra.

Recall that $\text{C}^{\cdot}(\text{pt}/G)$ is isomorphic to a polynomial algebra on generators in even degrees. The canonical point

$$\pi_{\text{pt}} : \text{pt} \to \text{pt}/G$$

defines an augmentation module

$$\mathsf{e} \in \text{C}^{\cdot}(\text{pt}/G)\text{-mod}.$$

Note that under the equivalence (F.8), we have

$$\mathsf{e} \in \text{C}^{\cdot}(\text{pt}/G)\text{-mod} \leftrightarrow \text{ren}_{\text{pt}/G}((\pi_{\text{pt}})_*(\mathsf{e})) \in \text{Shv}(\text{pt}/G)^{\text{ren}}.$$

Hence, under (F.8), the (isomorphic) essential image of the functor $\text{ren}_{\text{pt}/G}$ corresponds to the full subcategory

$$\text{C}^{\cdot}(\text{pt}/G)\text{-mod}_0 \subset \text{C}^{\cdot}(\text{pt}/G)\text{-mod}$$

be the full subcategory generated by the the augmentation module e.

F.5.6. Let $\mathcal{Y}$ be of the form Y/G, where Y is a quasi-compact scheme. The functor of *- (resp., !-) pullback

$$\operatorname{Shv}(\operatorname{pt}/G)^{\operatorname{ren}} \to \operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}}$$

has a natural symmetric monoidal structure with respect to the *- (resp., !-) tensor product operation.

We claim:

Proposition F.5.7. *The co-localization*

$$\text{un-ren}_{\mathcal{Y}} : \operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}} \leftrightarrows \operatorname{Shv}(\mathcal{Y}) : \operatorname{ren}_{\mathcal{Y}}$$

identifies with the co-localization

$$\operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}} \simeq \operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}} \underset{\operatorname{Shv}(\operatorname{pt}/G)^{\operatorname{ren}}}{\otimes} \operatorname{Shv}(\operatorname{pt}/G)^{\operatorname{ren}} \leftrightarrows \operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}} \underset{\operatorname{Shv}(\operatorname{pt}/G)^{\operatorname{ren}}}{\otimes} \operatorname{Shv}(\operatorname{pt}/G)$$

*(for either *- or !- monoidal structures).*

Proof. The functor

$$\text{un-ren}_{\mathcal{Y}} : \operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}} \to \operatorname{Shv}(\mathcal{Y})$$

clearly factors as

$$\operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}} \simeq \operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}} \underset{\operatorname{Shv}(\operatorname{pt}/G)^{\operatorname{ren}}}{\otimes} \operatorname{Shv}(\operatorname{pt}/G)^{\operatorname{ren}} \overset{\operatorname{Id}\otimes \text{un-ren}_{\operatorname{pt}/G}}{\longrightarrow}$$
$$\to \operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}} \underset{\operatorname{Shv}(\operatorname{pt}/G)^{\operatorname{ren}}}{\otimes} \operatorname{Shv}(\operatorname{pt}/G) \to \operatorname{Shv}(\mathcal{Y}).$$

Hence, to prove the proposition it suffices to show that the essential image of

$$\operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}} \underset{\operatorname{Shv}(\operatorname{pt}/G)^{\operatorname{ren}}}{\otimes} \operatorname{Shv}(\operatorname{pt}/G) \overset{\operatorname{Id}\otimes \operatorname{ren}_{\operatorname{pt}/G}}{\longrightarrow} \operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}} \underset{\operatorname{Shv}(\operatorname{pt}/G)^{\operatorname{ren}}}{\otimes} \operatorname{Shv}(\operatorname{pt}/G)^{\operatorname{ren}} \simeq \operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}}$$

is contained in that of

$$\operatorname{Shv}(\mathcal{Y}) \overset{\operatorname{ren}_{\mathcal{Y}}}{\hookrightarrow} \operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}}.$$

For that end it suffices to show that for $\mathcal{F} \in \operatorname{Shv}(\mathcal{Y})^{\operatorname{constr}}$, we have

$$\mathcal{F} \overset{*}{\otimes} q^*((\pi_{\operatorname{pt}})_*(\mathsf{e})) \in \operatorname{Shv}(\mathcal{Y})^c,$$

where $q : \mathcal{Y} \to \operatorname{pt}/G$. However, this follows from (F.4).

□

F.5.8. Let now $\mathcal{Y}$ be a not necessarily quasi-compact algebraic stack. We let

$$\operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}} := \lim_{\mathcal{U}} \operatorname{Shv}(\mathcal{U})^{\operatorname{ren}}, \tag{F.9}$$

where the limit is taken over the index category of quasi-compact open substacks $\mathcal{U} \subset \mathcal{Y}$, and the transition functors are given by restriction.

The properties and structures listed in Sect. F.5.1 for the opens $\mathcal{U}$ induce the corresponding properties and structures on $\mathcal{Y}$. In particular, we have an adjunction

$$\text{un-ren}_{\mathcal{Y}} : \operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}} \leftrightarrows \operatorname{Shv}(\mathcal{Y}) : \operatorname{ren}_{\mathcal{Y}},$$

with $\operatorname{ren}_{\mathcal{Y}}$ fully faithful, a t-structure on $\operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}}$, etc.

Note also that the transition functors in forming the limit (F.9) admit left adjoints, given by !-extension. Hence, we can rewrite $\operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}}$ as

$$\operatorname{colim}_{\mathcal{U}} \operatorname{Shv}(\mathcal{U})^{\operatorname{ren}},$$

where the transition functors are given by !-extension.

In particular, we obtain that $\operatorname{Shv}(\mathcal{Y})^{\operatorname{ren}}$ is compactly generated by objects of the form $j_!(\mathcal{F})$, where

$$j : \mathcal{U} \hookrightarrow \mathcal{Y}$$

with $\mathcal{U}$ quasi-compact and $\mathcal{F} \in \operatorname{Shv}(\mathcal{U})^{\operatorname{constr}}$.

F.6. **Singular support on stacks.** Let $\mathcal{Y}$ be a quasi-compact algebraic stack.

F.6.1. Let $\mathcal{N}$ be a conical Zariski-closed subset in $T^*(\mathcal{Y})$, see [GKRV, Sect. A.3.4] for what this means.

We define the full subcategory

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \subset \mathrm{Shv}(\mathcal{Y})$$

to consist of those $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$ whose pullback under any smooth map $S \to \mathcal{Y}$ (with S a scheme) belongs to $\mathrm{Shv}_{\mathcal{N}_S}(S)$ (see Sect. E.5.3), where

$$\mathcal{N}_S := \mathcal{N} \underset{\mathcal{Y}}{\times} S \subset T^*(\mathcal{Y}) \underset{\mathcal{Y}}{\times} S \subset T^*(S).$$

F.6.2. Since pullbacks with respect to smooth morphisms are t-exact up to a cohomological shift, we obtain that the t-structures on $\mathrm{Shv}_{\mathcal{N}_S}(S)$ induce a t-structure on $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$. It follows automatically that $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ is complete in its t-structure.

It is easy to see that an object $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$ belongs to $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ if and only if for every m and every constructible sub-object $\mathcal{F}'$ of $H^m(\mathcal{F})$, the object $\mathcal{F}'$ belongs to

$$\mathrm{Perv}_{\mathcal{N}}(\mathcal{Y}) := \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \cap \mathrm{Perv}(\mathcal{Y}).$$

Note also that $\mathrm{Perv}_{\mathcal{N}}(\mathcal{Y})$ is a Serre subcategory of $\mathrm{Perv}(\mathcal{Y})$ and

$$\mathrm{Ind}(\mathrm{Perv}_{\mathcal{N}}(\mathcal{Y})) \simeq \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\heartsuit}.$$

Remark F.6.3. The category $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ defined above is different from the category denoted by the same symbol in [GKRV]. In our current notations, the category in *loc.cit.* is

$$\lim_{S} \mathrm{Shv}_{\mathcal{N}_S}(S)^{\mathrm{access}},$$

where the limit taken over the category of affine schemes S smooth over $\mathcal{Y}$ and smooth maps between such, and where $\mathrm{Shv}_{\mathcal{N}_S}(S)^{\mathrm{access}}$ is as in Sect. E.5.4.

Since for schemes, the functor $\mathrm{Shv}_{\mathcal{N}_S}(S)^{\mathrm{access}} \to \mathrm{Shv}_{\mathcal{N}_S}(S)$ is fully faithful, the category in [GKRV] embeds fully faithfully into our $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$.

F.6.4. Set

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{constr}} := \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \cap \mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}$$

and define

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} := \mathrm{Ind}(\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{constr}}).$$

Note that we have a tautologically defined functor

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} \to \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}). \tag{F.10}$$

The category $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{constr}}$ inherits a t-structure, and the latter uniquely extends to a t-structure on $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}}$, compatible with filtered colimits. The functor (F.10) is t-exact, by construction.

As in Sect. F.5.2 one shows that for every n, the functor (F.10) induces an equivalence

$$(\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}})^{\geq -n} \simeq (\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}))^{\geq -n}. \tag{F.11}$$

From here it follows that the functor (F.10) realizes $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ as the left completion of $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}}$ in its t-structure.

F.6.5. Note also that we have a fully faithful t-exact functor

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} \to \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}.$$

Remark F.6.6. Note that $\mathcal{Y} = Y$ is a quasi-compact scheme, the category that denoted above by $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}}$ was denoted by $\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}}$ in Sect. E.5.4.

In the case of stacks, the notation $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$ will have a different meaning, see Sect. F.7.1 below.

By contrast, when $\mathcal{N}$ is all of $T^*(\mathcal{Y})$, the category $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}}$ is the category $\mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}$ introduced in Sect. F.5.

Note also that, unlike the case of schemes, the functor (F.10) does not preserves compactness, and hence fails to be fully faithful.

F.7. **The accessible category on stacks.** We continue to assume that $\mathcal{Y}$ is a quasi-compact algebraic stack.

F.7.1. Denote by $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$ the full subcategory in $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ generated under colimits by the essential image of the functor (F.10) above.

Thus, we have the functors

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} \overset{\text{un-ren}_{\mathcal{Y},\mathcal{N}}}{\longrightarrow} \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} \hookrightarrow \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}). \tag{F.12}$$

Since the functors (F.11) are equivalences, we obtain that the subcategory

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} \hookrightarrow \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \tag{F.13}$$

is preserved by the truncation functors acting on $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$. In particular, $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$ inherits a t-structure, and the embeddings

$$(\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}})^{\geq -n} \hookrightarrow (\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}))^{\geq -n}$$

are equivalences.

In particular, the embedding (F.13) realizes $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ as the left completion of $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$ in its t-structure.

F.7.2. *Examples.* When $\mathcal{Y} = Y$ is a scheme, the functor (F.10) is fully faithful, so the functor

$$\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{ren}} \overset{\text{un-ren}_{Y,\mathcal{N}}}{\longrightarrow} \mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{access}}$$

is an equivalence.

When $\mathcal{Y}$ is a stack but $\mathcal{N} = T^*(\mathcal{Y})$, the embedding (F.13) is an equivalence.

F.7.3. We give the following definitions:

Definition F.7.4. *We shall say that the pair* $(\mathcal{Y}, \mathcal{N})$ *is* renormalization-adapted *if the category* $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$ *is generated by objects that are compact as objects of* $\mathrm{Shv}(\mathcal{Y})$.

Definition F.7.5. *We shall say that the pair* $(\mathcal{Y}, \mathcal{N})$ *is* constraccessible *if the inclusion* (F.13) *is an equality.*

Remark F.7.6. We emphasize again that a pair $(\mathcal{Y}, \mathcal{N})$ *may not be* constraccessible even if $\mathcal{Y} = S$ is a scheme and $\mathcal{N} = \{0\}$ (see Remark E.5.6). But it is tautologically renormalization-adapted.

F.7.7. We make the following few observations:

(I) If a pair $(\mathcal{Y}, \mathcal{N})$ is renormalization-adapted, then $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$ equals the full subcategory of $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ generated by $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \cap \mathrm{Shv}(\mathcal{Y})^c$.

(II) A pair $(\mathcal{Y}, \mathcal{N})$ is both renormalization-adapted and constraccessible if and only if $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ is generated by objects that are compact in $\mathrm{Shv}(\mathcal{Y})$.

(III) If a pair $(\mathcal{Y}, \mathcal{N})$ is renormalization-adapted, the functor $\text{un-ren}_{\mathcal{Y},\mathcal{N}}$ admits a left adjoint, to be denoted $\mathrm{ren}_{\mathcal{Y},\mathcal{N}}$. Moreover, this left adjoint is fully faithful, so the adjunction

$$\mathrm{ren}_{\mathcal{Y},\mathcal{N}} : \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} \rightleftarrows \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} : \text{un-ren}_{\mathcal{Y},\mathcal{N}}$$

realizes $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$ as a co-localization of $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}}$.

(IV) If $(\mathcal{Y}, \mathcal{N})$ is renormalization-adapted and $\mathcal{Y}$ is duality-adapted, then the category $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$ is naturally self-dual, with the pairing

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} \to \mathrm{Vect}_{\mathsf{e}}$$

given by

$$\mathcal{F}_1, \mathcal{F}_2 \mapsto \mathrm{C}^{\cdot}_{\blacktriangle}(\mathcal{Y}, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2),$$

and the corresponding contravariant equivalence on compact objects is given by the Verdier duality functor.

F.7.8. We claim:

Proposition F.7.9. *Suppose that $\mathcal{Y}$ is a global quotient, i.e., $\mathcal{U} = Y/G$, where Y is a quasi-compact scheme and G an algebraic group. Then $(\mathcal{Y}, \mathcal{N})$ is renormalization-adapted for any $\mathcal{N}$.*

Proof. Follows from the argument in Sect. F.3.5.

□

Based on the above proposition, we propose:

Conjecture F.7.10. *For any quasi-compact algebraic stack with an affine diagonal, and any $\mathcal{N} \subset T^*(\mathcal{Y})$, the pair $(\mathcal{Y}, \mathcal{N})$ is renormalization-adapted.*

F.8. **Singular support condition for non quasi-compact stacks.** In this subsection we let $\mathcal{Y}$ be an algebraic stack, locally of finite type, but not necessarily quasi-compact.

F.8.1. We define

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) := \lim_{\mathcal{U}} \mathrm{Shv}_{\mathcal{N}}(\mathcal{U}),$$

where the index category is the poset of quasi-compact open substacks of $\mathcal{Y}$, and the transition functors used in forming the limit are given by restriction.

The t-structures for the individual $\mathcal{U}'s$ induces a t-structure on $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$.

The fully faithful embeddings $\mathrm{Shv}_{\mathcal{N}}(\mathcal{U}) \to \mathrm{Shv}(\mathcal{U})$ give rise to a t-exact fully faithful embedding

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y}).$$

F.8.2. We define the categories

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} \text{ and } \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} \tag{F.14}$$

similarly:

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} := \lim_{\mathcal{U}} \mathrm{Shv}_{\mathcal{N}}(\mathcal{U})^{\mathrm{ren}} \text{ and } \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} := \lim_{\mathcal{U}} \mathrm{Shv}_{\mathcal{N}}(\mathcal{U})^{\mathrm{access}}.$$

The t-structures for the individual $\mathcal{U}'s$ induces t-structures on the above categories.

We have a t-exact functor

$$\text{un-ren}_{\mathcal{Y},\mathcal{N}} : \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} \to \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$$

and a t-exact fully faithful embedding

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} \hookrightarrow \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}). \tag{F.15}$$

Both these functors induce equivalences on the n-coconnective subcategories for any n. This implies both these functors identify $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ with the left completion of both $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}}$ and $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$.

F.8.3. The fully faithful embeddings $\mathrm{Shv}_{\mathcal{N}}(\mathcal{U})^{\mathrm{ren}} \hookrightarrow \mathrm{Shv}(\mathcal{U})^{\mathrm{ren}}$ give rise to a t-exact fully faithful embedding

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} \hookrightarrow \mathrm{Shv}(\mathcal{Y})^{\mathrm{ren}}.$$

F.8.4. We shall say that $(\mathcal{Y}, \mathcal{N})$ is renormalization-adapted if $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$ is generated by objects that are compact in $\mathrm{Shv}(\mathcal{Y})$.

We shall say that $(\mathcal{Y}, \mathcal{N})$ is constraccessible if the inclusion (F.15) is an equality.

Observations (I), (II) and (III) from Sect. F.7.7 remain valid for non quasi-compact stacks as well.

Remark F.8.5. Note that the notions of being renormalization-adapted and constraccessible, as defined above, are of global nature, i.e., in general, they do not translate into statements that can be checked on quasi-compact open substacks of $\mathcal{Y}$.

We will now introduce a condition on $\mathcal{Y}$ that allows us to check these properties locally.

F.8.6. We shall say that a pair $(\mathcal{Y}, \mathcal{N})$ is *truncatable*[54] if $\mathcal{Y}$ can be written as a filtered union of quasi-compact open substacks $\mathcal{U}_i$, such that for every i and the corresponding open embedding

$$\mathcal{U}_i \overset{j_i}{\hookrightarrow} \mathcal{Y},$$

the functor

$$(j_i)_! : \mathrm{Shv}(\mathcal{U}_i)^{\mathrm{constr}} \to \mathrm{Shv}(\mathcal{Y})^{\mathrm{constr}}$$

sends

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_i)^{\mathrm{constr}} \to \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{constr}}.$$

F.8.7. Here are some equivalent ways to rewrite the truncatability condition:

(A) We can require that the functors $(j_{i_1,i_2})_!$ send $\mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_{i_1})^{\mathrm{constr}} \to \mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_{i_2})^{\mathrm{constr}}$ for $U_1 \overset{j_{i_1,i_2}}{\hookrightarrow} U_2$.

(B) We can require that the functors $(j_{i_1,i_2})_!$ send $\mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_{i_1}) \to \mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_{i_2})$. Indeed, (B) $\Rightarrow$ (A) tautologically. For the converse implication, using the fact that the functor $(j_{i_1,i_2})_!$ has a finite cohomological amplitude, it is enough to show that $(j_{i_1,i_2})_!$ sends $\mathrm{Perv}_{\mathcal{N}}(\mathcal{U}_{i_1})$ to $\mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_{i_2})$, which follows from (A).

(C) We can require that the functors $(j_i)_!$ send $\mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_i) \to \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ (indeed, this is easily seen to be equivalent to (B));

(D) Same as all of the above with $(-)_!$ replaced by $(-)_*$. Indeed, for the constructible categories, this follows by Verdier duality, and for the entire $\mathrm{Shv}_{\mathcal{N}}(-)$, this follows as in the equivalence (A) $\Leftrightarrow$ (B) above.

[54]The terminology is borrowed from [DrGa2, Sect. 4].

F.8.8. Let $\mathcal{Y}$ and $\mathcal{U}_i$ be as in Sect. F.8.6. Note that in this case, for $U_1 \overset{j_{i_1,i_2}}{\hookrightarrow} U_2$ the functors

$$(j_{i_1,i_2})^* : \mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_{i_2}) \to \mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_{i_1}),$$

$$(j_{i_1,i_2})^* : \mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_{i_2})^{\mathrm{ren}} \to \mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_{i_1})^{\mathrm{ren}}$$

and

$$(j_{i_1,i_2})^* : \mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_{i_2})^{\mathrm{access}} \to \mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_{i_1})^{\mathrm{access}}$$

all admit fully faithful left adjoints, to be denoted $(j_{i_1,i_2})_!$ in all three cases.

This implies that the corresponding categories

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}),\ \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} \text{ and } \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$$

can also be written as *colimits*

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \simeq \underset{i}{\mathrm{colim}}\ \mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_i),$$

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} \simeq \underset{i}{\mathrm{colim}}\ \mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_i)^{\mathrm{ren}}$$

and

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} \simeq \underset{i}{\mathrm{colim}}\ \mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_i)^{\mathrm{access}},$$

where the transition functors are given by given by !-extension.

F.8.9. The following assertion is proved by considering the $((j_{i_1,i_2})_!, (j_{i_1,i_2})^*)$ and $((j_{i_1,i_2})^*, (j_{i_1,i_2})_*)$ adjunctions:

Lemma F.8.10. *Let $\mathcal{Y}$ and $\mathcal{U}_i$ be as in Sect. F.8.6.*

(a) *The category $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}}$ is compactly generated.*

(b) *The category $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$ is compactly generated if and only if each $\mathrm{Shv}_{\mathcal{N}}(\mathcal{U}_i)$ is compactly generated.*

(c) *The pair $(\mathcal{Y}, \mathcal{N})$ is renormalization-adapted if and only if each $(\mathcal{U}_i, \mathcal{N}|_{\mathcal{U}_i})$ is renormalization-adapted.*

(d) *The pair $(\mathcal{Y}, \mathcal{N})$ is constraccessible if and only if each $(\mathcal{U}_i, \mathcal{N}|_{\mathcal{U}_i})$ is constraccessible.*

In particular, combining with Proposition F.7.9, we obtain:

Corollary F.8.11. *Let $\mathcal{Y}$ and $\mathcal{U}_i$ be as in Sect. F.8.6, and suppose that each $\mathcal{U}_i$ is a global quotient. Then the pair $(\mathcal{Y}, \mathcal{N})$ is renormalization-adapted.*

F.9. **Product theorems for stacks.** In this subsection we will prove versions of Theorems E.9.2 and E.9.9 for stacks.

F.9.1. Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be a pair of algebraic stacks. First, by [GKRV, Proposition A.2.10], the external tensor product functor

$$\mathrm{Shv}(\mathcal{Y}_1) \otimes \mathrm{Shv}(\mathcal{Y}_2) \to \mathrm{Shv}(\mathcal{Y}_1 \times \mathcal{Y}_2) \tag{F.16}$$

is fully faithful. It follows from Proposition E.7.2 that it is t-exact.

An argument parallel to *loc. cit.* shows that the functor

$$\mathrm{Shv}(\mathcal{Y}_1)^{\mathrm{ren}} \otimes \mathrm{Shv}(\mathcal{Y}_2)^{\mathrm{ren}} \to \mathrm{Shv}(\mathcal{Y}_1 \times \mathcal{Y}_2)^{\mathrm{ren}} \tag{F.17}$$

is also fully faithful. It is also t-exact (this follows formally from the case of schemes, i.e., Proposition E.7.2).

F.9.2. Let $\mathcal{N} \subset T^*(\mathcal{Y})$ a Zariski-closed conical subset. From the fact that (F.17) is fully faithful and using the fact that $\mathrm{IndLisse}(X)$ is dualizable, we obtain that for a scheme X, the functor

$$\mathrm{IndLisse}(X) \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{ren}} \to \mathrm{Shv}_{\{0\}\times\mathcal{N}}(X \times \mathcal{Y})^{\mathrm{ren}} \tag{F.18}$$

is also fully faithful. It is t-exact by Lemma E.7.5.

F.9.3. From now on let us assume that $\mathcal{N}$ is half-dimensional.

First, we claim:

Theorem F.9.4. *Let X be smooth and proper. Then the functor* (F.18) *is an equivalence.*

Proof. Follows from Theorem E.9.2 by passing to the limit (using the fact that $\mathrm{IndLisse}(X)$ is dualizable, so $\mathrm{IndLisse}(X) \otimes -$ commutes with limits). □

F.9.5. Next, we claim:

Theorem F.9.6. *Let X be smooth and proper. Then the functor*

$$\mathrm{IndLisse}(X) \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} \to \mathrm{Shv}_{\{0\}\times\mathcal{N}}(X \times \mathcal{Y})^{\mathrm{access}} \tag{F.19}$$

is an equivalence.

Proof. Since $\mathrm{IndLisse}(X)$ is dualizable, by passing to the limit, we reduce to the case when $\mathcal{Y}$ is quasi-compact.

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{IndLisse}(X) \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}} & \longrightarrow & \mathrm{Shv}_{\{0\}\times\mathcal{N}}(X \times \mathcal{Y})^{\mathrm{access}} \\ \downarrow & & \downarrow \\ \mathrm{IndLisse}(X) \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) & & \mathrm{Shv}_{\{0\}\times\mathcal{N}}(X \times \mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathrm{IndLisse}(X) \otimes \mathrm{Shv}(\mathcal{Y}) & \longrightarrow & \mathrm{Shv}(X \times \mathcal{Y}) \end{array}$$

with vertical arrows being fully faithful (for the left column this again uses the fact that $\mathrm{IndLisse}(X)$ is dualizable). The bottom horizontal arrow is also fully faithful (because $\mathrm{Shv}(\mathcal{Y})$ is dualizable and the functor (F.16) is fully faithful). This implies that the top horizontal arrow, i.e., our functor (F.19), is fully faithful.

Thus, it suffices to show that the right adjoint of (F.19) is conservative. Let us describe this right adjoint explicitly (it will be continuous by construction).

Recall that the category $\mathrm{IndLisse}(X)$ is canonically self-dual (by Verdier), see Remark E.4.5. In terms of this self-duality, the right adjoint to (F.19) corresponds to the functor

$$\mathrm{IndLisse}(X) \otimes \mathrm{Shv}_{\{0\}\times\mathcal{N}}(X \times \mathcal{Y})^{\mathrm{access}} \to \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{access}}$$

given by

$$E, \mathcal{F} \mapsto (p_Y)_*(p_X^!(E) \overset{!}{\otimes} \mathcal{F}).$$

where p_X and p_Y are the two projections from $X \times Y$ to X and Y, respectively.

This description of the right adjoint that it commutes with pullbacks for smooth maps $\mathcal{Y}' \to \mathcal{Y}$. This allows us to replace $\mathcal{Y}$ by an affine scheme covering it. In the latter case, the assertion that right adjoint to (F.19) is conservative follows from Theorem E.9.2.

□

F.9.7. Finally, we claim:

Theorem F.9.8. *Let X be smooth and proper. Assume also that* $\mathrm{QLisse}(X)$ *Verdier-compatible (see Sect. E.4). Let $\mathcal{N} \subset T^*(\mathcal{Y})$ be half-dimensional. Then the resulting functor*

$$\mathrm{QLisse}(X) \otimes \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \to \mathrm{Shv}_{\{0\}\times\mathcal{N}}(X \times \mathcal{Y})$$

is an equivalence.

Proof. Since $\mathrm{QLisse}(X)$ is assumed dualizable, the assertion reduces to the case of schemes. In the latter case, it is given by Theorem E.9.9.

□

Appendix G. Sheaves in the Betti context

In this section we will work the ground field $k = \mathbb{C}$. All our algebro-geometric objects will be of finite type over $\mathbb{C}$.

The coefficients of our sheaves will be an arbitrary algebraically closed field of characteristic 0, denoted e.

G.1. **Betti sheaves on schemes.**

G.1.1. Let Y be a scheme of finite type over $\mathbb{C}$. We denote by $\mathrm{Shv}^{\mathrm{all}}(Y)$ the category of all sheaves of e-vector spaces on the (topological space) underlying S.

We will consider $\mathrm{Shv}^{\mathrm{all}}(Y)$ as equipped with the usual t-structure, in which it is left-complete (see [Lu1, Theorem 7.2.3.6 and Proposition 7.2.1.10]).

G.1.2. For a pair of affine schemes, we have a naturally defined functor

$$\mathrm{Shv}^{\mathrm{all}}(Y_1) \otimes \mathrm{Shv}^{\mathrm{all}}(Y_2) \overset{\boxtimes}{\to} \mathrm{Shv}^{\mathrm{all}}(Y_1 \times Y_2). \tag{G.1}$$

This functor is known to be an equivalence (see, e.g., [Lu1, Theorem 7.3.3.9, Proposition 7.3.1.11] and [Lu2, Proposition 4.8.1.17]).

G.1.3. The category $\mathrm{Shv}^{\mathrm{all}}(Y)$ is known to be dualizable, and in fact, self-dual, with the duality data given by

$$(\Delta_Y)_!(\underline{\mathsf{e}}_Y) \in \mathrm{Shv}^{\mathrm{all}}(Y \times Y) \simeq \mathrm{Shv}^{\mathrm{all}}(Y) \otimes \mathrm{Shv}^{\mathrm{all}}(Y),$$

and

$$\mathrm{Shv}^{\mathrm{all}}(Y) \otimes \mathrm{Shv}^{\mathrm{all}}(Y) \overset{\boxtimes}{\to} \mathrm{Shv}^{\mathrm{all}}(Y \times Y) \overset{\Delta_Y^*}{\to} \mathrm{Shv}^{\mathrm{all}}(Y) \overset{\Gamma_c(Y,-)}{\to} \mathrm{Vect}_{\mathsf{e}}. \tag{G.2}$$

However, $\mathrm{Shv}^{\mathrm{all}}(Y)$ is *not* compactly generated (unless Y is a disjoint union of points).

G.1.4. We have an embedding

$$\mathrm{Shv}(Y)^{\mathrm{constr}} \hookrightarrow \mathrm{Shv}^{\mathrm{all}}(Y), \tag{G.3}$$

compatible with t-structures, whose ind-extension is a t-exact functor

$$\mathrm{Shv}(Y) \to \mathrm{Shv}^{\mathrm{all}}(Y). \tag{G.4}$$

However, the functor (G.4) is *not* fully faithful, because (G.4) does not preserves compactness.

G.1.5. We have a fully faithful (continuous) embedding

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(Y) \hookrightarrow \mathrm{Shv}^{\mathrm{all}}(Y). \tag{G.5}$$

We claim:

Lemma G.1.6. *The functor* (G.5) *admits a left adjoint.*

Proof. Since the categories involved are presentable, the assertion of the proposition is equivalent to the fact that the subcategory (G.5) is closed under limits.

Thus, let E_i be objects of $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(Y)$, where i runs over some index category I, and we wish to show that

$$\mathcal{F} := \lim_i E_i \in \mathrm{Shv}^{\mathrm{all}}(Y)$$

also belongs to $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(Y)$.

For every $\mathbb{C}$-point $y \in Y$, let $U_y \subset Y(\mathbb{C})$ be a contractible *analytic* neighborhood. We wish to show that $\mathcal{F}|_{U_y}$ is constant.

The functor of restriction $\mathrm{Shv}^{\mathrm{all}}(Y) \to \mathrm{Shv}^{\mathrm{all}}(U_y)$ commutes with limits, it is enough to show that the essential image of

$$\mathrm{Vect}_{\mathsf{e}} \simeq \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(U_y) \hookrightarrow \mathrm{Shv}^{\mathrm{all}}(U_y)$$

is closed under limits.

However, the latter is evident: an inverse limit of constant sheaves is a constant sheaf.

□

Remark G.1.7. Let U be a (real) ball of dimension d. Then the left adjoint to the embedding

$$\mathrm{Vect}_{\mathsf{e}} \simeq \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(U) \hookrightarrow \mathrm{Shv}^{\mathrm{all}}(U)$$

can be described explicitly as follows: it sends

$$\mathcal{F} \mapsto \mathrm{C}^{\cdot}_c(U, \mathcal{F})[d] \otimes \ell,$$

where ℓ is e-line, induced from the ± 1-torsor of orientations on U.

G.2. **Singular support for Betti sheaves.**

G.2.1. Let Y be as above, a scheme of finite type over $\mathbb{C}$. Let $\mathcal{N} \subset T^*(Y)$ be a conical Lagrangian. Following [KS, Sect. 8], we introduce a full subcategory

$$\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y) \subset \mathrm{Shv}^{\mathrm{all}}(Y). \tag{G.6}$$

G.2.2. One can describe the subcategory (G.6) more explicitly as follows.

thanks to [KS, Corollary 8.3.22], we can choose a μ-stratification

$$Y_i \overset{j_i}{\hookrightarrow} Y(\mathbb{C}),$$

such that $\mathcal{N}$ is contained in the union of the conormals to the strata (the latter will be denote by $\mathcal{N}_\mu$).

The property of being a μ-stratification implies that for a pair of strata

$$Y_{i_1} \overset{j_{i_1}}{\hookrightarrow} Y \overset{j_{i_2}}{\hookleftarrow} Y_{i_2}$$

and $\mathcal{F}_1 \in \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(Y_{i_1})$, the object

$$(j_{i_2})^* \circ (j_{i_1})_*(\mathcal{F}_1) \in \mathrm{Shv}^{\mathrm{all}}(Y_{i_2})$$

belongs to $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(Y_{i_2})$.

This formally implies that for $\mathcal{F} \in \mathrm{Shv}^{\mathrm{all}}(Y)$ the following conditions are equivalent:

$$(*) \qquad j_i^*(\mathcal{F}) \in \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(Y_i),\ \forall\, i \ \Leftrightarrow\ j_i^!(\mathcal{F}) \in \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(Y_i),\ \forall\, i;$$

G.2.3. Let

$$\mathrm{Shv}^{\mathrm{all}}_{\mu}(Y) \subset \mathrm{Shv}^{\mathrm{all}}(Y) \tag{G.7}$$

be the subcategory of sheaves satisfying (*). Then

$$\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y) \subset \mathrm{Shv}^{\mathrm{all}}_{\mu}(Y). \tag{G.8}$$

We will now explain, following [NY2, Sect. A.1.3], how to single out the subcategory (G.8) by an explicit procedure.

G.2.4. Choose a *smooth* point $(\xi, y) \in \mathcal{N}_\mu$. Let B be a sufficiently small open ball around y in $Y(\mathbb{C})$, and let g be a C^∞ real-valued function on B such that:

- $g(y) = 0$;
- $dg_y = \xi$;
- $\mathrm{Graph}(dg) \cap \mathcal{N}_\mu = \{(\xi, y)\}$ and the intersection is transversal.

Consider the corresponding loci

$$B_{\leq 0} \overset{q^-}{\hookrightarrow} B \overset{q^+}{\hookleftarrow} B_{\geq 0}.$$

Define the functors

$$\Phi^{+,!}_\xi, \Phi^{-,!}_\xi, \Phi^{+,*}_\xi, \Phi^{-,*}_\xi : \mathrm{Shv}(Y) \to \mathrm{Vect}_{\mathsf{e}}$$

by

$$\Phi^{+,!}_\xi(\mathcal{F}) := \mathrm{C}^\cdot_c(B_{\geq 0}, (q^+)^*(\mathcal{F}|_B)), \quad \Phi^{-,!}_\xi(\mathcal{F}) := \mathrm{C}^\cdot_c(B_{\leq 0}, (q^-)^*(\mathcal{F}|_B)),$$
$$\Phi^{+,*}_\xi(\mathcal{F}) = \mathrm{C}^\cdot(B_{\geq 0}, (q^+)^!(\mathcal{F}|_B)), \quad \Phi^{-,*}_\xi(\mathcal{F}) := \mathrm{C}^\cdot(B_{\leq 0}, (q^-)^!(\mathcal{F}|_B)).$$

For $\mathcal{F} \in \mathrm{Shv}_\mu(Y)$, the above functors are independent of the choice of g (i.e., they only depend on ξ) and satisfy

$$\Phi^{+,!}_\xi(\mathcal{F}) \simeq \Phi^{-,*}_\xi(\mathcal{F}) \text{ and } \Phi^{-,!}_\xi(\mathcal{F}) \simeq \Phi^{+,*}_\xi(\mathcal{F}).$$

Note also that we have, tautologically,

$$\Phi^{+,!}_\xi(\mathcal{F}) \simeq \Phi^{-,!}_{-\xi}(\mathcal{F}) \text{ and } \Phi^{+,*}_\xi(\mathcal{F}) \simeq \Phi^{-,*}_{-\xi}(\mathcal{F}).$$

G.2.5. Now, an object $\mathcal{F} \in \mathrm{Shv}_\mu(Y)$ belongs to $\mathrm{Shv}_{\mathcal{N}}(Y)$ if and only if

$$\Phi^{+,*}_\xi(\mathcal{F}) = 0$$

for all smooth $(\xi, y) \in \mathcal{N}_\mu - \mathcal{N}$.

For future reference we note that since $\mathcal{N}$ is a complex subvariety of $T^*(\mathcal{Y})$

$$(\xi, y) \in \mathcal{N} \;\Leftrightarrow\; (-\xi, y) \in \mathcal{N}.$$

Hence, the condition that $\mathcal{F} \in \mathrm{Shv}_\mu(Y)$ belongs to $\mathrm{Shv}_{\mathcal{N}}(Y)$ is equivalent to all

$$\Phi^{+,!}_\xi(\mathcal{F}),\; \Phi^{-,!}_\xi(\mathcal{F}),\; \Phi^{+,*}_\xi(\mathcal{F}),\; \Phi^{-,*}_\xi(\mathcal{F})$$

being 0.

G.2.6. We now claim:

Proposition G.2.7. *The subcategory* (G.6) *is closed under limits and colimits.*

Proof. First, we claim that the subcategory $\mathrm{Shv}^{\mathrm{all}}_\mu(Y) \subset \mathrm{Shv}(Y)$ is closed under limits and colimits. Indeed, this follows from Lemma G.1.6, combined with the fact that the functors $j_i^!$ commute with limits and the functors j_i^* commute with colimits.

Now the assertion for $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$ follows using its characterization in Sect. G.2.5, since the functors $\Phi^{\pm,*}_\xi$ commutes with limits and the functors $\Phi^{\pm,!}_\xi$ commute with colimits (on all of $\mathrm{Shv}(Y)$). □

G.2.8. Let ι^{all} denote the tautological embedding corresponding to (G.6).

From Proposition G.2.7 we see that the category $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$ is cocomplete (and the functor ι^{all} is continuous). In addition, we obtain;

Corollary G.2.9. *The functor ι^{all} admits a left adjoint.*

G.3. **Compact generators of the category $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$.**

G.3.1. We now claim:

Proposition G.3.2. *The category* $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$ *is compactly generated. Moreover, it admits a* finite *set of compact generators.*

The rest of this subsection is devoted to the proof of Proposition G.3.2. In fact, we will exhibit a particular finite set of compact generators.

G.3.3. For a point $y \in \mathcal{Y}$, let $\delta_y \in \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y})$ be the !-δ function, i.e., $\delta_y := (\mathbf{i}_y)_!(\mathsf{e})$, where

$$\mathbf{i}_y : \mathrm{pt} \to \mathcal{Y}$$

is the morphism corresponding to y.

G.3.4. Choose a μ-stratification as in Sect. G.2.2. It follows formally from its properties that the (*discontinuous*) functors

$$j_i^! : \mathrm{Shv}^{\mathrm{all}}(Y) \to \mathrm{Shv}^{\mathrm{all}}(Y_i)$$

are *continuous* when restricted to $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$.

Choose a point y_i on each connected component of each stratum. We will prove:

Proposition G.3.5. *The objects* $(\iota^{\mathrm{all}})^L(\delta_{y_i})$ *compactly generate* $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$.

Proof. We have to show that the functors

$$\mathcal{F} \mapsto \mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)}((\iota^{\mathrm{all}})^L(\delta_{y_i}), \mathcal{F}), \quad \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y) \to \mathrm{Vect}_{\mathsf{e}}$$

are continuous, and of they all vanish on a given $\mathcal{F}$, then $\mathcal{F} = 0$.

By adjunction, we rewrite the above functors as

$$\mathcal{F} \mapsto \mathbf{i}^!_{y_i} \circ \iota^{\mathrm{all}}(\mathcal{F}),$$

and further as

$$\mathcal{F} \mapsto \widetilde{\mathbf{i}}^!_{y_i} \circ j_i^! \circ \iota^{\mathrm{all}}(\mathcal{F}),$$

where

$$\mathrm{pt} \overset{\widetilde{\mathbf{i}}_{y_i}}{\to} Y_i.$$

Now, the assertion follows from the combination of the following facts:

- The functors $j_i^! : \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y) \to \mathrm{Shv}_{\mathrm{loc.const.}}(Y_i)$ are continuous;
- If $j_i^!(\mathcal{F}) = 0$ for some $\mathcal{F} \in \mathrm{Shv}^{\mathrm{all}}(Y)$ for all i, then $\mathcal{F} = 0$;
- For every i, the functor $\widetilde{\mathbf{i}}^!_{y_i}$ is continuous and conservative on $\mathrm{Shv}_{\mathrm{loc.const.}}(Y_i)$.

□

G.4. **Constructible sheaves vs all sheaves.**

G.4.1. As was mentioned above, the functor (G.4) does not preserve compactness. However, we claim:

Proposition G.4.2. *Let* $\mathcal{N} \subset T^*(Y)$ *be a conical Lagrangian. Then the objects of* $\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{constr}}$ *are compact as objects of* $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$.

As a corollary, we obtain:

Corollary G.4.3. *The functor*

$$\mathrm{Shv}^{\mathrm{access}}_{\mathcal{N}}(Y) \to \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$$

is fully faithful.

Remark G.4.4. We do not know whether the functor

$$\mathrm{Shv}_{\mathcal{N}}(Y) \to \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$$

preserves compactness or is fully faithful. (Note that in our usual counterexample of Sect. E.2.6, this functor is actually an equivalence.)

G.4.5. *Proof of Proposition G.4.2.* Let $\mathcal{F}$ be an object of $\mathrm{Shv}_{\mathcal{N}}(Y)^{\mathrm{constr}}$. Choose a stratification as in Sect. G.3.4. Then for $\mathcal{F}' \in \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$, we can express

$$\mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)}(\mathcal{F},\mathcal{F}')$$

as a *finite* colimit with terms

$$\mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y_i)}(j_i^*(\mathcal{F}), j_i^!(\mathcal{F}')), \tag{G.9}$$

functorially in $\mathcal{F}'$.

Hence, it suffices to show that the expression in (G.9), viewed as a functor of $\mathcal{F}'$, is continuous. Since the functor $j_i^!$ is continuous on $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$, and

$$j_i^*(\mathcal{F}) \in \mathrm{Lisse}(Y_i), \quad j_i^!(\mathcal{F}') \in \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(Y_i),$$

it suffices to show that for a given $E \in \mathrm{Lisse}(Y_i)$, the functor

$$E' \mapsto \mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(Y_i)}(E, E'), \quad E' \in \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(Y_i)$$

is continuous, which is obvious.

□[Proposition G.4.2]

G.5. **Duality on the category of Betti sheaves with prescribed singular support.**

G.5.1. The goal of this subsection is to prove the following assertion:

Theorem G.5.2.

(a) *The pairing*

$$\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)\otimes \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y) \overset{\iota^{\mathrm{all}}\otimes\iota^{\mathrm{all}}}{\longrightarrow} \mathrm{Shv}^{\mathrm{all}}(Y)\otimes \mathrm{Shv}^{\mathrm{all}}(Y) \overset{\mathrm{C}^\cdot_c(Y,-\overset{*}{\otimes}-)}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}}$$

is the counit of a self-duality.

(b) *With respect to the above identification* $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)^\vee \simeq \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$ *and the identification* $\mathrm{Shv}^{\mathrm{all}}(Y)^\vee \simeq \mathrm{Shv}^{\mathrm{all}}(Y)$ *of Sect. G.1.3, the natural map*

$$(\iota^{\mathrm{all}})^\vee \to (\iota^{\mathrm{all}})^L$$

is an isomorphism.

The assertion of Theorem G.5.2 is formally equivalent to the following:

Theorem G.5.3. *For* $\mathcal{F} \in \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$ *and* $\mathcal{F}' \in \mathrm{Shv}^{\mathrm{all}}(Y)$, *the unit of the* $((\iota^{\mathrm{all}})^L, \iota^{\mathrm{all}})$*-adjunction defines an* isomorphism

$$\mathrm{C}^\cdot_c(Y, \iota^{\mathrm{all}}(\mathcal{F}) \overset{*}{\otimes} \mathcal{F}') \to \mathrm{C}^\cdot_c(Y, \iota^{\mathrm{all}}(\mathcal{F}) \overset{*}{\otimes} (\iota^{\mathrm{all}} \circ (\iota^{\mathrm{all}})^L(\mathcal{F}'))).$$

G.5.4. Let

$$\mathbb{D} : (\mathrm{Shv}^{\mathrm{all}}(Y))^{\mathrm{op}} \to \mathrm{Shv}^{\mathrm{all}}(Y),$$

defined by

$$\mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}(Y)}(\mathcal{F}, \mathbb{D}(\mathcal{F}')) = \mathrm{C}^\cdot_c(Y, \mathcal{F} \overset{*}{\otimes} \mathcal{F}')^\vee, \quad \mathcal{F}, \mathcal{F}' \in \mathrm{Shv}^{\mathrm{all}}(Y).$$

G.5.5. The assertion of Theorem G.5.3 is in turn equivalent to the following:

Theorem G.5.6. *The (contravariant) functor* $\mathbb{D}$ *preserves the subcategory* $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$.

The rest of this section is devoted to the proof of Theorem G.5.6.

G.5.7. *Reduction to a μ-stratification.* Choose a stratification as in Sect. G.2.2. We claim that it suffices to show that the functor $\mathbb{D}$ preserves the corresponding subcategory $\mathrm{Shv}^{\mathrm{all}}_{\mu}(Y)$ (see Sect. G.2.3).

Indeed, suppose that we know that for $\mathcal{F} \in \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y) \subset \mathrm{Shv}^{\mathrm{all}}_{\mu}(Y)$, the object $\mathbb{D}(\mathcal{F})$ belongs to $\mathrm{Shv}^{\mathrm{all}}_{\mu}(Y)$.

By Sect. G.2.5, we have to show that for all smooth $\xi \in \mathcal{N}_{\mu} - \mathcal{N}$,

$$\Phi^{+,*}_{\xi}(\mathbb{D}(\mathcal{F})) = 0.$$

However, for any $\mathcal{F} \in \mathrm{Shv}^{\mathrm{all}}(Y)$, we have

$$\Phi^{+,*}_{\xi}(\mathbb{D}(\mathcal{F})) \simeq (\Phi^{+,!}_{\xi}(\mathcal{F}))^{\vee}. \tag{G.10}$$

Now,

$$\Phi^{+,!}_{\xi}(\mathcal{F}) \simeq \Phi^{-,*}_{\xi}(\mathcal{F}) \simeq \Phi^{+,*}_{-\xi}(\mathcal{F}) = 0.$$

G.5.8. *Reduction to a stratum.* Let $Y_i \overset{j_i}{\hookrightarrow} Y$ be the embedding of a stratum. For any $\mathcal{F} \in \mathrm{Shv}^{\mathrm{all}}(Y)$ we have

$$j_i^!(\mathbb{D}_Y(\mathcal{F})) \simeq \mathbb{D}_{Y_i}(j_i^*(\mathcal{F})).$$

This reduces the assertion of Theorem G.5.6 to the case when $Y(\mathbb{C})$ is a manifold and

$$\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y) = \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(Y).$$

G.5.9. *Reduction to a ball.* Let $U \subset Y(\mathbb{C})$ be an open ball. For $\mathcal{F} \in \mathrm{Shv}^{\mathrm{all}}(U)$, we have

$$\mathbb{D}_Y(\mathcal{F})|_U \simeq \mathbb{D}_U(\mathcal{F}|_U).$$

Hence, it remains to show that the functor $\mathbb{D}_U$ preserves the full subcategory

$$\mathrm{Vect}_{\mathsf{e}} \simeq \mathrm{Shv}^{\mathrm{all}}_{\mathrm{loc.const.}}(U) \subset \mathrm{Shv}^{\mathrm{all}}(U).$$

However, for $V \in \mathrm{Vect}_{\mathsf{e}}$,

$$\mathbb{D}_U(\underline{\mathsf{e}}_U \otimes V) \simeq \underline{\mathsf{e}}_U \otimes V^{\vee}[2\dim(U)].$$

□[Theorem G.5.6]

G.6. **A smoothness property of the category of Betti sheaves with restricted singular support.** The material in this subsection is not needed in the rest of the paper.

G.6.1. Recall that a dualizable DG category $\mathbf{C}$ is said to be *smooth* if the unit object

$$\mathrm{u}_{\mathbf{C}} \in \mathbf{C} \otimes \mathbf{C}^{\vee}$$

is compact.

We are going to prove:

Proposition G.6.2. *Let Y be a scheme and let $\mathcal{N} \subset T^*(Y)$ be a Zariski-closed conical Lagrangian subset. Then the category $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)$ is smooth.*

The rest of this subsection is devoted to the proof of this proposition[55].

[55] We have learned this assertion from D. Nadler.

G.6.3. Let $\mathrm{u}_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)}$ denote the unit of the self-duality specified by Theorem G.5.2. We have have to check that the functor

$$(E \in \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y) \otimes \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)) \mapsto \mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)\otimes \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)}(\mathrm{u}_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)}, E) \tag{G.11}$$

is continuous.

By Theorem G.5.2(b), object $\mathrm{u}_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y)}$ can be explicitly described as

$$((\iota^{\mathrm{all}})^L \otimes (\iota^{\mathrm{all}})^L)((\Delta_Y)_!(\underline{\mathsf{e}}_Y)).$$

Hence, the expression in (G.11) can be rewritten as

$$\mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}(Y)\otimes \mathrm{Shv}^{\mathrm{all}}(Y)}((\Delta_Y)_!(\underline{\mathsf{e}}_Y), (\iota^{\mathrm{all}} \otimes \iota^{\mathrm{all}})(E)). \tag{G.12}$$

Denote by $\widetilde{E}$ the image of E along

$$\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y) \otimes \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y) \overset{\boxtimes}{\hookrightarrow} \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}\times\mathcal{N}}(Y \times Y).$$

Thus, we can rewrite (G.12) as

$$\mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}(Y\times Y)}((\Delta_Y)_!(\underline{\mathsf{e}}_Y), \iota^{\mathrm{all}}_{Y\times Y}(\widetilde{E})) \simeq \mathrm{C}^{\cdot}(Y, \Delta_Y^! \circ \iota^{\mathrm{all}}_{Y\times Y}(\widetilde{E})),$$

where $\iota^{\mathrm{all}}_{Y\times Y}$ is the embedding

$$\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}\times\mathcal{N}}(Y \times Y) \hookrightarrow \mathrm{Shv}^{\mathrm{all}}(Y \times Y).$$

Thus, it is sufficient to show that the functor

$$\mathcal{F} \mapsto \mathrm{C}^{\cdot}(Y, \Delta_Y^! \circ \iota^{\mathrm{all}}_{Y\times Y}(\mathcal{F})), \quad \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}\times\mathcal{N}}(Y \times Y) \to \mathrm{Vect}_{\mathsf{e}} \tag{G.13}$$

is continuous. (Note, that the functor $\mathrm{C}^{\cdot}(Y, \Delta_Y^!(-))$ itself is *discontinuous*.)

G.6.4. Using a stratification as in Sect. G.3.4, we can write the functor (G.13) as a *finite* limit of functors of the form

$$\mathcal{F} \mapsto \mathrm{C}^{\cdot}(Y_i, \Delta_{Y_i}^! \circ (j_i \times j_i)^!(\mathcal{F})),$$

so it suffices to show that each of the latter functors is continuous on $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}\times\mathcal{N}}(Y \times Y)$.

As in Sect. G.3.4, each of the functors $(j_i \times j_i)^!$ is continuous on $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}\times\mathcal{N}}(Y \times Y)$ and maps to

$$\mathrm{Shv}_{\mathrm{loc.const}}(Y_i \times Y_i).$$

Next, the functor $\Delta_{Y_i}^!$, when restricted to $\mathrm{Shv}_{\mathrm{loc.const}}(Y_i \times Y_i)$, is isomorphic to $\Delta_{Y_i}^*$ up to a shift, and hence is also continuous, and maps to $\mathrm{Shv}_{\mathrm{loc.const}}(Y_i)$.

Finally, the functor $\mathrm{C}^{\cdot}(Y_i, -)$ is continuous, when restricted to $\mathrm{Shv}_{\mathrm{loc.const}}(Y_i)$.

□[Proposition G.6.2]

G.7. **Betti sheaves on stacks.** For the purposes of this paper, we do *not* need the category $\mathrm{Shv}^{\mathrm{all}}(-)$ for arbitrary prestacks, but only for algebraic stacks.

G.7.1. Thus, let $\mathcal{Y}$ be an algebraic stack. We define

$$\mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}) := \lim_{S} \mathrm{Shv}^{\mathrm{all}}(S), \tag{G.14}$$

where the limit is taken over the category of affine schemes almost of finite type, equipped with a smooth map to $\mathcal{Y}$, and where we allow only smooth maps $f : S_1 \to S_2$ over $\mathcal{Y}$. The transition functors are given by

$$\mathrm{Shv}^{\mathrm{all}}(S_2) \xrightarrow{f^!} \mathrm{Shv}^{\mathrm{all}}(S_1).$$

These functors are continuous because the morphisms f were assumed smooth.

This limit can be rewritten as a colimit

$$\mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}) := \operatorname*{colim}_{S} \mathrm{Shv}^{\mathrm{all}}(S), \tag{G.15}$$

where the transition functors are give by !-puhsforward.

G.7.2. We define the functor

$$\mathrm{C}^\cdot_c(\mathcal{Y}, -) : \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}) \to \mathrm{Vect}_{\mathsf{e}}$$

to correspond to the (compatible) system of functors

$$\mathrm{C}^\cdot_c(S, -) : \mathrm{Shv}^{\mathrm{all}}(S) \to \mathrm{Vect}_{\mathsf{e}}$$

in the presentation (G.15).

G.7.3. Let $\mathcal{N} \subset T^*(\mathcal{Y})$ be a conical Lagrangian subset, which by definition means that for any affine scheme equipped with a smooth map $S \to \mathcal{Y}$, the subset

$$\mathcal{N}_S \subset T^*(\mathcal{Y}),$$

defined as in Sect. F.6.1, is Lagrangian.

Define the full subcategory

$$\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y}) \overset{\iota^{\mathrm{all}}}{\hookrightarrow} \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}) \tag{G.16}$$

as

$$\lim_S \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}_S}(S) \subset \lim_S \mathrm{Shv}^{\mathrm{all}}(S).$$

G.7.4. Since the functors of pullback along smooth morphisms commute with limits and colimits, from Proposition G.2.7:

Corollary G.7.5. *The subcategory* (G.16) *is closed under limits and colimits.*

In particular, we obtain that the category $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})$ is presentable, and the functor ι^{all} commutes with limits and colimits. In particular, we obtain:

Corollary G.7.6. *The functor ι^{all} of* (G.16) *admits a left adjoint.*

G.7.7. We now claim:

Proposition G.7.8. *The category $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})$ is compactly generated. If $\mathcal{Y}$ is quasi-compact, then is admits a finite set of compact generators.*

As in the case of Proposition G.3.2, we will exhibit a particular set of compact generators for $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})$.

Proof of Proposition G.7.8. Let $f_\alpha : S_\alpha \to \mathcal{Y}$ be a covering collection of smooth maps, where S_α are affine schemes. (Note that when $\mathcal{Y}$ is quasi-compact, we can take a single affine scheme S.)

For each index α consider the category

$$\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}_{S_\alpha}}(S_\alpha),$$

and let $s_{\alpha,i} \in S_\alpha$ be a finite collection of points such that the objects

$$(\iota^{\mathrm{all}}_{S_\alpha})^L(\delta_{s_{\alpha,i},S_\alpha}) \in \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}_{S_\alpha}}(S_\alpha)$$

(compactly) generate $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}_{S_\alpha}}(S_\alpha)$.

Let $y_{\alpha,i}$ denote the image of $s_{\alpha,i}$ in $\mathcal{Y}$. We claim that the objects

$$(\iota^{\mathrm{all}}_{\mathcal{Y}})^L(\delta_{y_{\alpha,i},\mathcal{Y}}) \in \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})$$

are compact and generate $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})$.

Indeed, for every α and i and $\mathcal{F} \in \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})$, we have

$$\mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})}((\iota^{\mathrm{all}}_{\mathcal{Y}})^L(\delta_{y_{\alpha,i},\mathcal{Y}}), \mathcal{F}) \simeq \mathbf{i}^!_{y_{\alpha,i}} \circ \iota^{\mathrm{all}}_{\mathcal{Y}}(\mathcal{F}) \simeq \mathbf{i}^!_{s_{\alpha,i}} \circ f^!_\alpha \circ \iota^{\mathrm{all}}_{\mathcal{Y}}(\mathcal{F}) \simeq$$

$$\simeq \mathbf{i}^!_{s_{\alpha,i}} \circ \iota^{\mathrm{all}}_S \circ f^!_\alpha(\mathcal{F}) \simeq \mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}_{S_\alpha}}(S_\alpha)}((\iota^{\mathrm{all}}_{S_\alpha})^L(\delta_{s_{\alpha,i},S_\alpha}), f^!_\alpha(\mathcal{F})),$$

where in the second line, $f_\alpha^!$ is regarded as a functor

$$\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y}) \to \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}_{S_\alpha}}(S_\alpha). \tag{G.17}$$

This implies the compactness statement, since the functors (G.17) are continuous. This also implies the generation statements as

$$\mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})}((\iota^{\mathrm{all}}_{\mathcal{Y}})^L(\delta_{y_{\alpha,i},\mathcal{Y}}), \mathcal{F}) = 0, \ \forall\, \alpha \text{ and } i$$

implies

$$f_\alpha^!(\mathcal{F}) = 0, \ \forall\, \alpha \ \Rightarrow \mathcal{F} = 0.$$

□

G.7.9. Finally, we have the following extension of Proposition G.4.2:

Proposition G.7.10. *Suppose that $\mathcal{Y}$ can be covered by a filtered family of quasi-compact substacks, each of which is a global quotient (i.e., of the form Y/G, where Y is a quasi-compact scheme). Then the functor*

$$\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y}) \to \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})$$

sends objects that are compact in $\mathrm{Shv}(\mathcal{Y})$ *to objects that are compact in* $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})$.

Remark G.7.11. We expect that the assertion of Proposition G.7.10 holds for any algebraic stack $\mathcal{Y}$, without the local global quotient condition.

G.8. **Proof of Proposition G.7.10.**

G.8.1. Since compact objects in $\mathrm{Shv}(\mathcal{Y})$ are !-extensions from quasi-compact substacks, we can assume that $\mathcal{Y}$ is quasi-compact.

By assumption, we can assume that $\mathcal{Y}$ is of the form Y/G, where Y is a quasi-compact scheme and G is an algebraic group.

G.8.2. For a pair of objects $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Shv}^{\mathrm{all}}(Y/G)$, let $\mathcal{F}'_1, \mathcal{F}'_2$ denote their *-pullbacks to Y. On the one hand, we can consider the object

$$\mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}(Y)}(\mathcal{F}'_1, \mathcal{F}'_2) \in \mathrm{Vect}_{\mathsf{e}}.$$

On the other hand, we can consider the object

$$\underline{\mathcal{H}om}_{\mathrm{Shv}^{\mathrm{all}}(Y)}(\mathcal{F}'_1, \mathcal{F}'_2) \in \mathrm{Shv}^{\mathrm{all}}(\mathrm{pt}/G) \simeq \mathrm{Shv}(\mathrm{pt}/G)$$

defined by the requirement that

$$\mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}(\mathrm{pt}/G)}(V, \underline{\mathcal{H}om}_{\mathrm{Shv}^{\mathrm{all}}(Y)}(\mathcal{F}'_1, \mathcal{F}'_2)) \simeq \mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}(Y/G)}(q^*(V) \overset{*}{\otimes} \mathcal{F}_1, \mathcal{F}_2), \quad V \in \mathrm{Shv}(\mathrm{pt}/G),$$

where $q : Y/G \to \mathrm{pt}/G$.

It is easy to see that

$$\pi^*_{\mathrm{pt}}(\underline{\mathcal{H}om}_{\mathrm{Shv}^{\mathrm{all}}(Y)}(\mathcal{F}'_1, \mathcal{F}'_2)) \simeq \mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}(Y)}(\mathcal{F}'_1, \mathcal{F}'_2),$$

where $\pi_{\mathrm{pt}} : \mathrm{pt} \to \mathrm{pt}/G$.

Note that

$$\mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}(Y/G)}(\mathcal{F}_1, \mathcal{F}_2) \simeq \mathrm{C}^\cdot(\mathrm{pt}/G, \underline{\mathcal{H}om}_{\mathrm{Shv}^{\mathrm{all}}(Y)}(\mathcal{F}'_1, \mathcal{F}'_2)). \tag{G.18}$$

G.8.3. We wish to show that for $\mathcal{F}_1 \in \mathrm{Shv}_{\mathcal{N}}(Y/G) \cap \mathrm{Shv}(Y/G)^c$, the functor

$$\mathcal{F}_2 \mapsto \mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}(Y/G)}(\mathcal{F}_1, \mathcal{F}_2), \quad \mathrm{Shv}^{\mathrm{all}}(Y/G) \to \mathrm{Vect}_{\mathsf{e}} \tag{G.19}$$

commutes with colimits when restricted to $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y/G)$.

First, we claim that for $\mathcal{F}_1 \in \mathrm{Shv}_{\mathcal{N}}(Y/G)^{\mathrm{constr}}$, the functor

$$\mathcal{F}_2 \mapsto \underline{\mathcal{H}om}_{\mathrm{Shv}^{\mathrm{all}}(Y)}(\mathcal{F}'_1, \mathcal{F}'_2), \quad \mathrm{Shv}^{\mathrm{all}}(Y/G) \to \mathrm{Shv}(\mathrm{pt}/G) \tag{G.20}$$

commutes with colimits when restricted to $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y/G)$.

Indeed, since the functor π^*_{pt} commutes with colimits and is conservative, it suffices to show that the composition of (G.20) with π^*_{pt} commutes with colimits when restricted to $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y/G)$.

However, the latter composition is the functor

$$\mathcal{F}_2 \mapsto \mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}(Y)}(\mathcal{F}'_1, \mathcal{F}'_2),$$

and it commutes with colimits when restricted to $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(Y/G)$ by Proposition G.4.2.

G.8.4. Recall now the natural transformation

$$\mathrm{C}^{\cdot}_{\blacktriangle}(\mathrm{pt}/G, -) \to \mathrm{C}^{\cdot}(\mathrm{pt}/G, -),$$

see (F.6). Given (G.18), it suffices to show that for $\mathcal{F}_1 \in \mathrm{Shv}(Y/G)^c$ and any $\mathcal{F}_2$, the map

$$\mathrm{C}^{\cdot}_{\blacktriangle}(\mathrm{pt}/G, \underline{\mathcal{H}om}_{\mathrm{Shv}^{\mathrm{all}}(Y)}(\mathcal{F}'_1, \mathcal{F}'_2)) \to \mathrm{C}^{\cdot}(\mathrm{pt}/G, \underline{\mathcal{H}om}_{\mathrm{Shv}^{\mathrm{all}}(Y)}(\mathcal{F}'_1, \mathcal{F}'_2))$$

is an isomorphism.

G.8.5. By Sect. F.3.2, we can assume that $\mathcal{F}_1$ is of the form $(\pi_Y)_!(\widetilde{\mathcal{F}})$ for $\widetilde{\mathcal{F}} \in \mathrm{Shv}^{\mathrm{all}}(Y)$.

In this case, it is easy to see that

$$\underline{\mathcal{H}om}_{\mathrm{Shv}^{\mathrm{all}}(Y)}(\mathcal{F}'_1, \mathcal{F}'_2) \simeq (\pi_{\mathrm{pt}})_*(\mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}(Y)}(\widetilde{\mathcal{F}}, (\pi_Y)^!(\mathcal{F}_2))).$$

G.8.6. Now, we claim that for any $V \in \mathrm{Vect}_{\mathsf{e}}$, the map

$$\mathrm{C}^{\cdot}_{\blacktriangle}(\mathrm{pt}/G, (\pi_{\mathrm{pt}})_*(V)) \to \mathrm{C}^{\cdot}(\mathrm{pt}/G, (\pi_{\mathrm{pt}})_*(V))$$

is an isomorphism.

Indeed, it suffices to show that the right-hand side commutes with colimits in V. However, this follows from the fact that

$$\mathrm{C}^{\cdot}(\mathrm{pt}/G, (\pi_{\mathrm{pt}})_*(V)) = \mathcal{H}om_{\mathrm{Shv}(\mathrm{pt}/G)}(\underline{\mathsf{e}}_{\mathrm{pt}/G}, (\pi_{\mathrm{pt}})_*(V)) \simeq \mathcal{H}om_{\mathrm{Shv}(\mathrm{pt})}(\pi^*_{\mathrm{pt}}(\underline{\mathsf{e}}_{\mathrm{pt}/G}), V) = V.$$

□[Proposition G.7.10]

G.9. **Duality for Betti sheaves on stacks.**

G.9.1. Let $\mathcal{Y}_1, \mathcal{Y}_2$ be a pair of algebraic stacks. It follows from (G.15) and the equivalence (G.1) for schemes that the functor

$$\mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}_1) \otimes \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}_2) \overset{\boxtimes}{\to} \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}_1 \times \mathcal{Y}_2)$$

is an equivalence.

From here it follows formally that for an algebraic stack $\mathcal{Y}$, the data of

$$(\Delta_{\mathcal{Y}})_!(\underline{\mathsf{e}}_{\mathcal{Y}}) \in \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y} \times \mathcal{Y}) \simeq \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}) \otimes \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}),$$

and

$$\mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}) \otimes \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}) \overset{\boxtimes}{\to} \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y} \times \mathcal{Y}) \overset{\Delta^*_{\mathcal{Y}}}{\to} \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}) \overset{\Gamma_c(\mathcal{Y}, -)}{\to} \mathrm{Vect}_{\mathsf{e}}\,. \tag{G.21}$$

define a self-duality on $\mathrm{Shv}^{\mathrm{all}}(\mathcal{Y})$.

G.9.2. Let $\mathcal{N} \subset T^*(\mathcal{Y})$ be as in Sect. G.7.3. We are going to deduce from Theorem G.5.6 the following assertion:

Corollary G.9.3.

(a) *The pairing*

$$\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y}) \otimes \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y}) \overset{\iota^{\mathrm{all}} \otimes \iota^{\mathrm{all}}}{\longrightarrow} \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}) \otimes \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}) \overset{\mathrm{C}^{\cdot}_c(\mathcal{Y}, - \overset{*}{\otimes} -)}{\longrightarrow} \mathrm{Vect}_{\mathsf{e}}$$

is the counit of a self-duality.

(b) *With respect to the above identification* $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})^\vee \simeq \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})$ *and the identification* $\mathrm{Shv}^{\mathrm{all}}(\mathcal{Y})^\vee \simeq \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y})$ *of Sect. G.9.1, the resulting map*

$$(\iota^{\mathrm{all}})^\vee \to (\iota^{\mathrm{all}})^L$$

is an isomorphism.

Proof. As in Sect. G.5, the assertion of the corollary is formally equivalent to the fact that the functor

$$\mathbb{D} : (\mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}))^{\mathrm{op}} \to \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}),$$

defined by

$$\mathcal{H}om_{\mathrm{Shv}^{\mathrm{all}}(\mathcal{Y})}(\mathcal{F}, \mathbb{D}(\mathcal{F}')) = \mathrm{C}^{\cdot}_c(\mathcal{Y}, \mathcal{F} \overset{*}{\otimes} \mathcal{F}')^\vee, \quad \mathcal{F}, \mathcal{F}' \in \mathrm{Shv}^{\mathrm{all}}(\mathcal{Y})$$

preserves the subcategory $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})$.

Let S be a scheme equipped with a smooth map $f : S \to \mathcal{F}$. By the definition of $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})$, we need to show that for $\mathcal{F} \in \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}}(\mathcal{Y})$, we have

$$f^!(\mathbb{D}_{\mathcal{Y}}(\mathcal{F})) \in \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}_S}(S).$$

However, we have

$$f^!(\mathbb{D}_{\mathcal{Y}}(\mathcal{F})) \simeq \mathbb{D}_S(f^*(\mathcal{F})),$$

for any $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$.

Since f is smooth, $f^*(\mathcal{F}) \in \mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}_S}(S)$. Now the fact that $\mathbb{D}_S(f^*(\mathcal{F}))$ belongs to $\mathrm{Shv}^{\mathrm{all}}_{\mathcal{N}_S}(S)$ follows from Theorem G.5.6.

□

Appendix H. Proof of Theorem 19.1.3

In this section, we will prove Theorem 19.1.3. We will work in a constructible étale, Betti or regular holonomic sheaf-theoretic context.

Note, however, that in order to treat the Betti case, it is sufficient to treat the regular holonomic one, by Riemann-Hilbert. So, from now on we will assume that our sheaf-theoretic context is either étale or regular holonomic (this will be needed for a change of fields manipulation in Sect. H.4.)

H.1. **Method of proof.**

H.1.1. *Reduction to the case of a proper morphism.* First, with no restriction of generality, we can assume that $\mathcal{Y}_2$ is a smooth scheme (and hence $\mathcal{Y}_1$ is a scheme as well, since f was assumed schematic).

The assumption on $\mathcal{F}_1$ is local on $\mathcal{Y}_1$ around the point y_1. Hence, we can assume that f is proper: indeed choose a relative compactification of f

$$\mathcal{Y}_1 \overset{j}{\hookrightarrow} \overline{\mathcal{Y}}_1 \overset{\overline{f}}{\to} \mathcal{Y}_2,$$

and replace the initial $\mathcal{F}_1$ with $j_*(\mathcal{F}_1)$.

H.1.2. Recall that for a (smooth) scheme $\mathcal{Y}$ and $\mathcal{F} \in \operatorname{Shv}(\mathcal{F})$, we define

$$\operatorname{SingSupp}(\mathcal{F}) \subset T^*(\mathcal{Y})$$

as the union of irreducible closed subsets $\mathcal{N}$ that appear as irreducible components of constructible sub-objects $\mathcal{F}'$ of $H^m(\mathcal{F})$ for all m.

By [Be2], such $\mathcal{N}$ all have dimension equal to $\dim(\mathcal{Y})$. (But in the étale setting, when $\operatorname{char}(k) \neq 0$, they are not necessarily Lagrangian.)

Thus, an irreducible subvariety Z of $T^*(\mathcal{Y})$ belongs to $\operatorname{SingSupp}(\mathcal{F})$ if it is contained in one of the irreducible components $\mathcal{N}$ as above.

H.1.3. By assumption (i) in Theorem 19.1.3, we can find an irreducible half-dimensional subvariety $\mathcal{N}_1 \subset \operatorname{SingSupp}(\mathcal{F}_1)$ with $(df^*(\xi_2), y_1) \in \mathcal{N}_1$. We will fix it from now on.

Consider the subscheme

$$(df^*)^{-1}(\mathcal{N}_1) \subset T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1.$$

Let $\widetilde{\mathcal{N}}_1$ be its irreducible component that contains the point (ξ_2, y_1). By assumption,

$$\dim(\widetilde{\mathcal{N}}_1) = \dim(\mathcal{Y}_2),$$

and the map

$$\widetilde{\mathcal{N}}_1 \hookrightarrow T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to T^*(\mathcal{Y}_2)$$

is quasi-finite at (ξ_2, y_1).

Let $\mathcal{N}_2 \subset T^*(\mathcal{Y}_2)$ be the closure of the image of the above map. This is an irreducible subvariety of $T^*(\mathcal{Y}_2)$ of dimension equal to $\dim(\mathcal{Y}_2)$, and

$$(\xi_2, y_2) \in \mathcal{N}_2.$$

H.1.4. Let

$$\operatorname{Shv}_{\widetilde{\mathcal{N}}_1\text{-q.f.}}(\mathcal{Y}_1) \subset \operatorname{Shv}(\mathcal{Y}_1)$$

be the full subcategory of objects $\mathcal{F}'_1$ such that for every irreducible $\mathcal{N}'_1 \subset \operatorname{SingSupp}(\mathcal{F}'_1)$ such that

$$\widetilde{\mathcal{N}}_1 \subset (df^*)^{-1}(\mathcal{N}'_1),$$

the dimension of $(df^*)^{-1}(\mathcal{N}'_1)$ at the generic point of $\widetilde{\mathcal{N}}_1$ equals $\dim(\mathcal{Y}_2)$. This is equivalent to requiring that the map

$$(df^*)^{-1}(\mathcal{N}'_1) \hookrightarrow T^*(\mathcal{Y}_2) \underset{\mathcal{Y}_2}{\times} \mathcal{Y}_1 \to T^*(\mathcal{Y}_2)$$

be quasi-finite at the generic point of $\widetilde{\mathcal{N}}_1$. This is also equivalent to $\widetilde{\mathcal{N}}_1$ being actually a whole irreducible component of $(df^*)^{-1}(\mathcal{N}'_1)$ as a subset[56].

By assumption, our $\mathcal{F}_1$ belongs to $\operatorname{Shv}_{\widetilde{\mathcal{N}}_1\text{-q.f.}}(\mathcal{Y}_1)$.

H.1.5. We will construct a DG category $\mathbf{C}$ and functors

$$\Phi_1 : \operatorname{Shv}_{\widetilde{\mathcal{N}}_1\text{-q.f.}}(\mathcal{Y}_1) \to \mathbf{C}$$

and

$$\Phi_2 : \operatorname{Shv}(\mathcal{Y}_2) \to \mathbf{C}$$

with the following properties:

(1) If $\mathcal{F}_2 \in \operatorname{Shv}(\mathcal{Y}_2)$ and $\mathcal{N}_2 \not\subset \operatorname{SingSupp}(\mathcal{F}_2)$, then $\Phi_2(\mathcal{F}_2) = 0$;
(2) If $\mathcal{F}'_1 \in \operatorname{Shv}_{\widetilde{\mathcal{N}}_1\text{-q.f.}}(\mathcal{Y}_1)$ and $\mathcal{N}_1 \subset \operatorname{SingSupp}(\mathcal{F}'_1)$, then $\Phi_1(\mathcal{F}'_1) \neq 0$;
(3) Φ_1 is canonically isomorphic to a direct summand of the composition

$$\operatorname{Shv}_{\widetilde{\mathcal{N}}_1\text{-q.f.}}(\mathcal{Y}_1) \hookrightarrow \operatorname{Shv}(\mathcal{Y}_1) \xrightarrow{f_*} \operatorname{Shv}(\mathcal{Y}_2) \xrightarrow{\Phi_2} \mathbf{C}.$$

[56] I.e., we discard embedded components.

Clearly, the existence of a triple $(\mathbf{C}, \Phi_1, \Phi_2)$ as above implies Theorem 19.1.3.

H.2. **Isolated points and vanishing cycles, constructible case.**

H.2.1. Let $\mathcal{Y}$ be a smooth scheme and let $\mathcal{N} \subset T^*(\mathcal{Y})$ a Zariski-closed subset.

Let $g : \mathcal{Y} \to \mathbb{A}^1$ be a function with a non-vanishing differential. For a k-point y of $\mathcal{Y}$, we denote by $0 \neq dg_y \in T^*_y(\mathcal{Y})$ the differential of g at y.

We shall say that g is $\mathcal{N}$-*characteristic* at $y \in \mathcal{Y}$ if the element $dg_y \in T^*_y(\mathcal{Y})$ belongs to $\mathcal{N}$.

We shall say that g is *non-characteristic with respect to* $\mathcal{N}$ if it is not $\mathcal{N}$-characteristic at all $y \in \mathcal{Y}$.

We shall say that a point $y \in \mathcal{Y}$ is an *isolated* point for the pair $(\mathcal{N}, g)$ if:

- $g(y) = 0$ and g is $\mathcal{N}$-characteristic at y;
- There exists a Zariski neighborhood $y \in U \subset \mathcal{Y}$ such that g is non-characteristic with respect to $\mathcal{N}$ on $U - \{y\}$.

H.2.2. We record the following two geometric observations:

Lemma H.2.3. *Let $f' : \mathcal{Y}'_1 \to \mathcal{Y}'_2$ be a map between two smooth schemes. Let $\mathcal{N}'_1 \subset T^*(\mathcal{Y}'_1)$ and $\mathcal{N}'_2 \subset T^*(\mathcal{Y}'_2)$ be closed irreducible subvarieties and let $\widetilde{\mathcal{N}}'_1$ be an irreducible component of $(df'^*)^{-1}(\mathcal{N}'_1)$ that projects to $\mathcal{N}'_2$. Let g be a function on $\mathcal{Y}'_2$, and let y'_2 be an isolated point for $(\mathcal{N}'_2, g)$. Let $y'_1 \in \mathcal{Y}'_1$ be a point such that:*

(i) *$(dg_{y'_2}, y'_1) \in \widetilde{\mathcal{N}}'_1$ and it does not belong to other irreducible components of $(df'^*)^{-1}(\mathcal{N}'_1)$.*

(ii) *The map $\widetilde{\mathcal{N}}'_1 \to \mathcal{N}'_2$ is quasi-finite at $(dg_{y'_2}, y'_1)$.*

Then y'_1 is an isolated point for $(g \circ f', \mathcal{N}'_1)$.

The proof of the lemma is straightforward.

Proposition H.2.4. *Assume that* $\dim(\mathcal{N}) \leq \dim(\mathcal{Y})$*. Then for any non-zero vector $\xi \in T^*_y(\mathcal{Y}) \cap \mathcal{N}$ there exists a function g defined on a Zariski neighborhood of y, such that $dg_y = \xi$ and y is an isolated point for the pair $(\mathcal{N}, g)$.*

The proof of the proposition is given in Sect. H.5.

Remark H.2.5. In fact, we will only need Proposition H.2.4 in the case when (ξ, y) is a smooth point on $\mathcal{N}$ (see Sect. H.4.2), in which case the assertion is easy (at least when $\mathrm{char}(k) \neq 2$):

Linearizing, we can assume being given a finite-dimensional vector space V and a subspace

$$W \subset V \oplus V^\vee,$$

of dimension $\leq \dim(V)$, and we have to find a quadratic form Q on V, such that for the associated bilinear form, viewed as a map $V \to V^\vee$, its graph is transversal to W.

H.2.6. Given a function g, consider the vanishing cycles functor

$$\Phi_g : \mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y}_0),$$

where $\mathcal{Y}_0 := \mathcal{Y} \underset{\mathbb{A}^1}{\times} \{0\}$.

The following results from the definition of singular support:

Lemma H.2.7. *Let $\mathcal{N}$ be conical and suppose that g is non-characteristic with respect to $\mathcal{N}$. Then the functor Φ_g vanishes on* $\mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$.

H.2.8. Let now $y \in \mathcal{Y}$ be isolated for $(\mathcal{N}, g)$.

Then by Lemma H.2.7, for $\mathcal{F} \in \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$, the object

$$\Phi_g(\mathcal{F}) \in \mathrm{Shv}(\mathcal{Y}_0),$$

canonically splits as a direct sum

$$\Phi_g(\mathcal{F}) \simeq \Phi_{g,y}(\mathcal{F}) \oplus \Phi_{g,y\text{-disj}}(\mathcal{F}), \tag{H.1}$$

where $\Phi_{g,y}(\mathcal{F})$ is supported at $\{y\}$ and $\Phi_{g,y\text{-disj}}(\mathcal{F})$ is supported on a closed subset of $\mathcal{Y}_0$ disjoint from the point y.

H.2.9. We have the following fundamental result:

Theorem H.2.10. *In the situation of Sect. H.2.8, if for $\mathcal{F} \in \mathrm{Perv}(\mathcal{Y}) \cap \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})$, we have $(dg_y, y) \in \mathrm{SingSupp}(\mathcal{F})$, then*

$$\Phi_{g,y}(\mathcal{F}) \neq 0.$$

In the context of étale sheaves, this theorem follows from [Sai, Theorem 5.9 (combined with Proposition 5.14)].

For constructible Betti sheaves, it follows easily from the definition of singular support in [KS], under the additional assumption that $(dg_y, y) \in T^*(\mathcal{Y})$ is a *smooth* point of $\mathcal{N}$ (which will be the case in our situation, see Sect. H.4.5 below.

In the case of regular holonomic D-modules, it follows from the Betti case, by Riemann-Hilbert (under the same smoothness assumption).

Remark H.2.11. The reason we did not formulate Theorem 19.1.3 for holonomic (but not necessarily regular) D-modules is that we are not certain about the status of an analog of Theorem H.2.10 in this context.

H.3. **Isolated points and vanishing cycles–beyond the constructible.**

H.3.1. Given $y \in \mathcal{Y}$ and a function g with $g(y) = 0$, let

$$\mathrm{Shv}_{(g,y)\text{-isol}}(\mathcal{Y}) \subset \mathrm{Shv}(\mathcal{Y}) \tag{H.2}$$

denote the full subcategory consisting of objects $\mathcal{F}$ such that, whenever $\mathcal{N}$ is a conical closed irreducible subvariety satisfying

$$\mathcal{N} \subset \mathrm{SingSupp}(\mathcal{F}) \text{ and } (dg_y, y) \in \mathcal{N},$$

then y is isolated for $(\mathcal{N}, g)$.

The subcategory (H.2) is compatible with the t-structure on the ambient category. In particular, $\mathrm{Shv}_{(g,y)\text{-isol}}(\mathcal{Y})$ acquires a t-structure.

H.3.2. We claim that the restriction of the functor Φ_g to $\mathrm{Shv}_{(g,y)\text{-isol}}(\mathcal{Y})$ also splits canonically as a direct sum

$$\Phi_g \simeq \Phi_{g,y} \oplus \Phi_{g,y\text{-disj}}, \tag{H.3}$$

where $\Phi_{g,y}$ maps to

$$\mathrm{Vect}_{\mathsf{e}} \simeq \mathrm{Shv}(\{y\}) \subset \mathrm{Shv}(\mathcal{Y}_0),$$

and the range of the functor $\Phi_{g,y\text{-disj}}$ consists of objects $\mathcal{F}'$ such that for every m, every constructible sub-object of $H^m(\mathcal{F}')$ is supported on a closed subset of $\mathcal{Y}_0$ that does not contain y.

H.3.3. Indeed, as in Sect. E.5.4, the category $\mathrm{Shv}_{(g,y)\text{-isol}}(\mathcal{Y})$ is the left-completion of its full subcategory

$$\mathrm{Shv}_{(g,y)\text{-isol}}(\mathcal{Y})^{\mathrm{access}} \subset \mathrm{Shv}_{(g,y)\text{-isol}}(\mathcal{Y})$$

equal to the ind-completion of

$$\underset{\mathcal{N}}{\cup}\, \mathrm{Shv}_{\mathcal{N}}(\mathcal{Y})^{\mathrm{constr}} \tag{H.4}$$

for closed $\mathcal{N} \subset T^*(\mathcal{Y})$ such that

$$(dg_y, y) \in \mathcal{N} \Rightarrow \; y \text{ is isolated for } (\mathcal{N}, g).$$

Since the functor Φ_g is t-exact, and the target category $\mathrm{Shv}(\mathcal{Y}_0)$ is left-complete in its t-structure, we obtain that in order to construct a decomposition (H.3), it suffices to construct it after restricting to $\mathrm{Shv}_{(g,y)\text{-isol}}(\mathcal{Y})^{\mathrm{access}}$. The latter amounts to a similar decomposition after restriction to (H.4), which, in turn, follows from (H.1).

H.3.4. From Theorem H.2.10 we deduce the following:

Corollary H.3.5. *Let $\mathcal{F}$ be an object of* $\mathrm{Shv}_{(g,y)\text{-isol}}(\mathcal{Y})$ *such that* $(dg_y, y) \in \mathrm{SingSupp}(\mathcal{F})$. *Then*

$$\Phi_{g,y}(\mathcal{F}) \neq 0.$$

Proof. Since the functor Φ_g is t-exact (on all of $\mathrm{Shv}(\mathcal{Y})$), so is its restriction to $\mathrm{Shv}_{(g,y)\text{-isol}}(\mathcal{Y})$, and hence so is $\Phi_{g,y}$. Hence, it suffices to show that for some m and for some constructible sub-object $\mathcal{F}'$ of $H^m(\mathcal{F})$. we have $\Phi_{g,y}(\mathcal{F}') \neq 0$.

Let m be such that $H^m(\mathcal{F})$ contains a constructible sub-object $\mathcal{F}'$ with $(dg_y, y) \in \mathrm{SingSupp}(\mathcal{F}')$. Then $\Phi_{g,y}(\mathcal{F}') \neq 0$ by Theorem H.2.10.

□

H.4. **Construction of $(\mathbf{C}, \Phi_1, \Phi_2)$.**

H.4.1. Let $\mathcal{N}_2$ be as in Sect. H.1.3. Let k' be the algebraic closure of the field of rational functions on $\mathcal{N}_2$.

Let us denote by

$$\mathcal{Y}_1',\; \mathcal{Y}_2',\; \mathcal{N}_2' \subset T^*(\mathcal{Y}_2'),$$

etc., the base change of all of our geometric objects to k'.

Let BC denote the corresponding base change functors

$$\mathrm{Shv}(\mathcal{Y}_1) \to \mathrm{Shv}(\mathcal{Y}_1'),\; \mathrm{Shv}(\mathcal{Y}_2) \to \mathrm{Shv}(\mathcal{Y}_2'),$$

etc.

The functor BC preserves singular support, in the sense that if for $\mathcal{F}_i \in \mathrm{Shv}(\mathcal{Y}_i)$ we have $\mathrm{SingSupp}(\mathcal{F}_i) = \mathcal{N}_i$, then

$$\mathrm{SingSupp}(\mathrm{BC}(\mathcal{F}_i)) = \mathcal{N}_i'.$$

H.4.2. The generic point of $\mathcal{N}_2$ gives rise to a k'-point of $\mathcal{N}_2'$; denote it by (ξ_2', y_2').

After shrinking $\mathcal{Y}_2'$ around y_2', choose a function g such that $dg_{y_2'} = \xi_2'$ and y_2' is an isolated point for $(\mathcal{N}_2', g)$. Such g exists by Proposition H.2.4[57].

Set

$$\mathcal{Y}_{2,0}' := \mathcal{Y}_2' \underset{\mathbb{A}^1_{k'}}{\times} \{0\}.$$

Let

$$\mathbf{C} := \underset{y_2' \in U \subset \mathcal{Y}_{2,0}'}{\mathrm{colim}}\, \mathrm{Shv}(U),$$

where the colimit is taken over the poset of Zariski neighborhoods of y_2' in $\mathcal{Y}_{2,0}'$.

[57]Note that the point (ξ_2', y_2') is a smooth point on $\mathcal{N}_2'$, so we are applying an easy case of Proposition H.2.4.

H.4.3. We let Φ_2 be the composition

$$\mathrm{Shv}(\mathcal{Y}_2) \overset{\mathrm{BC}}{\to} \mathrm{Shv}(\mathcal{Y}_2') \overset{\Phi_g}{\to} \mathrm{Shv}(\mathcal{Y}_{2,0}') \to \mathbf{C}.$$

We claim that Φ_2 satisfies Property (1) in Sect. H.1.5.

Indeed, it suffices to show that if Z is a *Zariski-closed* subset of $T^*(\mathcal{Y}_2)$ of dimension equal to $\dim(\mathcal{Y}_2)$ and

$$\mathcal{N}_2 \not\subset Z \text{ and } \mathcal{F}_2 \in \mathrm{Shv}_Z(\mathcal{Y}_2),$$

then

$$\Phi_g(\mathrm{BC}(\mathcal{F}_2))$$

vanishes on some neighborhood of y_2'.

However, for Z as above,

$$(\xi_2', y_2') \notin Z',$$

hence g is not Z'-characteristic at y_2', and hence is non-characteristic with respect to Z' on some neighborhood U of y_2'.

Since $\mathrm{SingSupp}(\mathrm{BC}(\mathcal{F}_2)) \subset Z'$, we obtain that

$$\Phi_g(\mathrm{BC}(\mathcal{F}_2))|_U = 0$$

by Lemma H.2.7.

H.4.4. Consider the function $g \circ f' : \mathcal{Y}_1' \to \mathbb{A}^1$. Set

$$\mathcal{Y}_{1,0}' := \mathcal{Y}_1' \underset{\mathbb{A}^1_{k'}}{\times} \{0\},$$

and let f_0' denote the induced map $\mathcal{Y}_{1,0}' \to \mathcal{Y}_{2,0}'$.

Recall the subscheme $\widetilde{\mathcal{N}}_1$, see Sect. H.1.3, and consider its base change

$$\widetilde{\mathcal{N}}_1' \subset T^*(\mathcal{Y}_2') \underset{\mathcal{Y}_2'}{\times} \mathcal{Y}_1'.$$

Let (ξ_2', y_1') be one (out of the finite and non-empty set) of k'-points of $\widetilde{\mathcal{N}}_1'$ that projects to (ξ_2', y_2') along

$$\widetilde{\mathcal{N}}_1' \to \mathcal{N}_2'.$$

By condition (iib) in Theorem 19.1.3, the point (ξ_2', y_1') does not belong to other irreducible components of $(df'^*)^{-1}(\mathcal{N}_1')$. Applying Lemma H.2.3, we obtain that the functor

$$\mathrm{BC} : \mathrm{Shv}(\mathcal{Y}_1) \to \mathrm{Shv}(\mathcal{Y}_1')$$

maps

$$\mathrm{Shv}_{\widetilde{\mathcal{N}}_1\text{-q.f.}}(\mathcal{Y}_1) \to \mathrm{Shv}_{(g\circ f', y_1')\text{-isol}}(\mathcal{Y}_1').$$

We let Φ_1 be the composition

$$\mathrm{Shv}_{\widetilde{\mathcal{N}}_1\text{-q.f.}}(\mathcal{Y}_1) \overset{\mathrm{BC}}{\longrightarrow} \mathrm{Shv}_{(g\circ f', y_1')\text{-isol}}(\mathcal{Y}_1') \overset{\Phi_{g\circ f', y_1'}}{\longrightarrow} \mathrm{Shv}(\{y_1'\}) \simeq \mathrm{Vect}_{\mathsf{e}} \simeq \mathrm{Shv}(\{y_2'\}) \hookrightarrow \mathrm{Shv}(Y_{2,0}') \to \mathbf{C}.$$

H.4.5. We claim that Φ_1 satisfies Property (2) in Sect. H.1.5.

Indeed, if $\mathcal{F}_1' \in \mathrm{Shv}_{\widetilde{\mathcal{N}}_1\text{-q.f.}}(\mathcal{Y}_1)$ is such that

$$\mathcal{N}_1 \subset \mathrm{SingSupp}(\mathcal{F}_1'),$$

then

$$\mathcal{N}_1' \subset \mathrm{SingSupp}(\mathrm{BC}(\mathcal{F}_1')),$$

and hence $(df'^*(\xi_2'), y_1') \in \mathrm{SingSupp}(\mathrm{BC}(\mathcal{F}_1'))$.

Hence, $\Phi_{g\circ f', y_1'}(\mathrm{BC}(\mathcal{F}_1')) \neq 0$ by Corollary H.3.5.

H.4.6. We will now prove that Φ_1 is canonically a direct summand of $\Phi_2 \circ f_*$, which is Property (3) in Sect. H.1.5.

We now use the fact that f is proper. This implies that we have a canonical isomorphism

$$\Phi_g \circ f'_* \simeq (f'_0)_* \circ \Phi_{g\circ f'},$$

where f'_0 is the induced map $\mathcal{Y}'_{1,0} \to \mathcal{Y}'_{2,0}$.

Hence, we can rewrite the composition

$$\mathrm{Shv}_{\widetilde{\mathcal{N}}_1\text{-q.f.}}(\mathcal{Y}_1) \hookrightarrow \mathrm{Shv}(\mathcal{Y}_1) \xrightarrow{f_*} \mathrm{Shv}(\mathcal{Y}_1) \xrightarrow{\Phi_2} \mathbf{C}$$

as

$$\mathrm{Shv}_{\widetilde{\mathcal{N}}_1\text{-q.f.}}(\mathcal{Y}_1) \xrightarrow{\mathrm{BC}} \mathrm{Shv}_{(g\circ f', y'_1)\text{-isol}}(\mathcal{Y}'_1) \xrightarrow{\Phi_{g\circ f'}} \mathrm{Shv}(\mathcal{Y}'_{1,0}) \xrightarrow{(f'_0)_*} \mathrm{Shv}(\mathcal{Y}'_{2,0}) \to \mathbf{C}.$$

By Sect. H.3.2, the above functor contains a direct summand isomorphic to

$$\mathrm{Shv}_{\widetilde{\mathcal{N}}_1\text{-q.f.}}(\mathcal{Y}_1) \xrightarrow{\mathrm{BC}} \mathrm{Shv}_{(g\circ f', y'_1)\text{-isol}}(\mathcal{Y}'_1) \xrightarrow{\Phi_{g\circ f', y'_1}} \mathrm{Shv}(\{y'_1\}) \hookrightarrow \mathrm{Shv}(\mathcal{Y}'_{1,0}) \xrightarrow{(f'_0)_*} \mathrm{Shv}(\mathcal{Y}'_{2,0}) \to \mathbf{C}.$$

However, the latter functor is the same as Φ_1. Indeed, the composition

$$\mathrm{Shv}(\{y'_1\}) \hookrightarrow \mathrm{Shv}(\mathcal{Y}'_{1,0}) \xrightarrow{(f'_0)_*} \mathrm{Shv}(\mathcal{Y}'_{2,0})$$

is the same as

$$\mathrm{Shv}(\{y'_1\}) \simeq \mathrm{Vect}_{\mathsf{e}} \simeq \mathrm{Shv}(\{y'_2\}) \hookrightarrow \mathrm{Shv}(Y'_{2,0}).$$

H.5. **Proof of Proposition H.2.4.**

H.5.1. The proof proceeds by induction on $\dim(\mathcal{Y})$. The base of induction is when $\dim(\mathcal{Y}) = 1$, which is easy and is left to the reader (cf. proof of Lemma H.5.2 below).

Without loss of generality, we assume that assume that $\mathcal{Y}$ admits a smooth map

$$f : \mathcal{Y} \to \mathcal{Y}'$$

of relative dimension 1, and that $\mathcal{Y}$ and $\mathcal{Y}'$ are affine. Denote $y' = f(y)$.

Let h be a function on $\mathcal{Y}$, to be chosen later. We define an automorphism $\mathrm{shr}_h : T^*(\mathcal{Y}) \to T^*(\mathcal{Y})$ by

$$\mathrm{shr}_h(\eta) = \eta + dh_y \qquad (y \in Y, \eta \in T^*_y(\mathcal{Y})).$$

Let τ_h denote the composite map

$$T^*(\mathcal{Y}') \underset{\mathcal{Y}'}{\times} \mathcal{Y} \hookrightarrow T^*(\mathcal{Y}) \xrightarrow{\mathrm{shr}_h} T^*(\mathcal{Y}).$$

Clearly, τ_h is a closed embedding.

Consider also the projection

$$(\mathrm{id} \times f) : T^*(\mathcal{Y}') \underset{\mathcal{Y}'}{\times} \mathcal{Y} \to T^*(\mathcal{Y}').$$

It is a smooth map of relative dimension one.

We now specify the choice of h. Choose $0 \neq \xi' \in T^*_{y'}(Y')$.

Lemma H.5.2. *There exists a function h on $\mathcal{Y}$ with $h(y) = 0$ such that:*

- $\xi = dh_y + df^*(\xi')$*, in particular,* $(\xi, y) \in \mathrm{Im}(\tau_h)$*;*
- *The dimension of* $\dim(\tau_h^{-1}(\mathcal{N})) \leq \dim(\mathcal{Y}) - 1$*;*
- *The restriction of the map* $(\mathrm{id} \times f)$ *to* $\tau_h^{-1}(\mathcal{N})$ *is quasi-finite at the point* $\tau_h^{-1}(\xi, y)$*.*

We prove Lemma H.5.2 below. Let us assume it, and perform the induction step.

H.5.3. Let h be as in Lemma H.5.2.

Let $\mathcal{N}'$ be the closure of the image $(\mathrm{id} \times f)(\tau_h^{-1}(\mathcal{N}))$. By the choice of h, we have $(\xi', y') \in \mathcal{N}'$, $\dim(\mathcal{N}') \leq \dim(\mathcal{Y}')$, and

$$\tau_h^{-1}(\mathcal{N}) \to \mathcal{N}'$$

given by $(\mathrm{id} \times f)$ is quasi-finite at (ξ', y).

Applying the induction hypothesis to $\mathcal{N}'$ and ξ', we can find a function g' on $\mathcal{Y}'$ such that the point y' is isolated for $(\mathcal{N}', g')$.

Take $g := g' \circ f + h$. It is easy to see that the point y is isolated for $(\mathcal{N}, g)$.

□[Proposition H.2.4]

H.5.4. *Proof of Lemma H.5.2.* Let us impose the following restrictions on h:

First require that

$$dh_y = \xi - df^*(\xi'),$$

i.e., we impose the first condition of Lemma H.5.2.

Consider the relative cotangent bundle $T^*(\mathcal{Y}/\mathcal{Y}')$. Let

$$r : T^*(\mathcal{Y}) \to T^*(\mathcal{Y}/\mathcal{Y}')$$

be the natural projection; r is smooth of relative dimension $n - 1$.

For an irreducible component $\mathcal{N}_i$ on $\mathcal{N}$, consider two cases.

Case (a): $r(\mathcal{N}_i) = \{r(\xi, y)\}$. In this case, $\dim(\tau_h^{-1}(\mathcal{N}_i)) \leq n - 1$ for any h.

Case (b): There exists a point $(\eta_i, y_i) \in r(\mathcal{N}_i)$, $\eta_i \in T^*_{y_i}(\mathcal{Y}/\mathcal{Y}')$ such that $(\eta_i, y_i) \neq r(\xi, y)$. In this case, we require that $\eta_i \neq r(dh_{y_i})$. This restriction guarantees that $\dim(\tau_h^{-1}(\mathcal{N}_i)) \leq n - 1$.

Imposing the above condition for for all irreducible components $\mathcal{N}_i$ in case (b), we obtain that h satisfies the second condition of Lemma H.5.2.

Since $\dim(\mathcal{N}) \leq n$, there exists a point y_0 that lies in the same connected component of $f^{-1}(y')$ as y and an element $\xi_0 \in T^*_{y_0}(\mathcal{Y})$ such that $(\xi_0 + df^*(\xi'), y_0) \notin \mathcal{N}$. Moreover, we can choose the pair (y_0, ξ_0) so that $y_0 \neq y$ and also $y_0 \neq y_i$ for all of the points y_i from the second step, case (b). We require that $dh_{y_0} = \xi_0$. This implies that $\tau_h^{-1}(\mathcal{N})$ does not entirely contain the connected component of $(\mathrm{id} \times f)^{-1}(\xi', y')$ to which the point (ξ', y) belongs, which is equivalent to the third condition of Lemma H.5.2 (since the morphism $(\mathrm{id} \times f)$ has relative dimension 1).

It is easy to see that it is possible to find a function h subject to the above restrictions.

□[Lemma H.5.2]

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