NEUTRALIZED LOCAL ENTROPY AND DIMENSION BOUNDS FOR INVARIANT MEASURES

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Abstract. We introduce a notion of a point-wise entropy of measures (i.e local entropy) called *neutralized local entropy*, and compare it with the Brin-Katok local entropy. We show that the neutralized local entropy coincides with Brin-Katok local entropy almost everywhere. Neutralized local entropy is computed by measuring open sets with a relatively simple geometric description. Our proof uses a measure density lemma for Bowen balls, and a version of a Besicovitch covering lemma for Bowen balls. As an application, we prove a lower point-wise dimension bound for invariant measures, complementing the previously established bounds for upper point-wise dimension.

1. Introduction and main results

One of the most useful tools in studying dynamical systems are dynamical covers and partitions of a measure. That is, given an invariant measure, a cover or a partition modulo the measure, whose elements have a significant structure w.r.t the dynamics. The pursuit for such objects creates a tension between searching for elements with a simple geometric description, with which it is easy to work, and searching for elements with significant dynamical structure, which allows to control their orbits and measure. Two charecteristic examples of this tension are geometric balls, and Bowen balls.

Geometric balls allow one to utilize strong geometric properties such as the Lebesgue density theorem, or the Besicovitch covering lemma. Bowen balls on the other hand, allow one to estimate their measure in terms of the entropy (see [BK83]), and control their image under the dynamics for a fixed amount of iterations; while having a possibly very complicated geometric shape.

Pesin theory allows one to linearize locally the action of the dynamics on typical orbits. However, the size of the neighborhood where the linearization is valid may deteriorate along the orbit, although in a sub-exponential rate. Given an ergodic measure with positive metric entropy, the Ruelle inequality says it must have some positive and some negative Lyapunov exponents ([Rue78]). When an orbit admits a central direction, i.e an invariant sub-space of the tangent space corresponding to 0 Lyapunov exponents, the central direction may not be integrable into an invariant manifold. In addition, the differential may contract or expand tangent vectors in the central direction, in a sub-exponential way. These effects make it very hard, and generally not attainable, to get a simple description of the set of points which remain close to the orbit for a fixed amount of steps.

Furthermore, even in the absence of 0 Lyapunov exponents, the decay of the size of the Pesin chart (i.e neighborhood with a local linearization of the dynamics) does not allow to control the set of points which remain close to the orbit by a fixed distance (i.e Bowen balls). Hence, phasing-out sub-exponential effects such as the central direction or the deterioration of the size of Pesin charts becomes very useful. Removing these effects allows one to treat the action of the dynamics along an orbit as if they were linear and hyperbolic, and so simplifying greatly the geometric description of the set of points which shadow the orbit (or part of it).

Our goal in this paper is to address exactly this difficulty. Our proof relies on a sub-exponential measure density lemma over Bowen balls (rather than geometric balls as in the Lebesgue density lemma).

In our setup M is a closed Riemannian manifold, $d = \dim M \geq 2$, and $f \in \mathrm{Diff}^{1+\beta}(M)$, $\beta > 0$. Let μ be an f-invariant Borel probability measure. The purpose of this paper is to compute the *neutralized local*

entropy defined by,

$$\mathcal{E}_\mu(x) := \lim_{r\to 0} \limsup_{n\to\infty} \frac{-1}{n} \log \mu(B(x,n,e^{-rn})), \tag{1}$$

where $B(x,n,e^{-rn}) := \{y \in M : d(f^i(y), f^i(y)(x)) \le e^{-rn}, \forall 0 \le i \le n\}$. One can check the following properties for the neutralized local entropy:

(a)

$$\mathcal{E}_\mu(x) \in [h_\mu^{\mathrm{BK}}(x), d \cdot \log M_f] \ \mu\text{-a.e}, \tag{2}$$

where $M_f := \max_{x\in M}\{\|d_x f\|, \|d_x f^{-1}\|\}$, and $h_\mu^{\mathrm{BK}}(x)$ is the local entropy at x given by the Brin-Katok formula, [BK83].[1] The upper bound is given by the classical lemma which states that $\lim_{s\to 0} \frac{\log \mu(B(x,s))}{\log s} \le d$ for μ-a.e x, and by the fact that $B(x,n,e^{-rn}) \supseteq B(x, M_f^{-n})$ for all $n \ge 0$ and sufficiently small $r > 0$.

(b)

$$\mathcal{E}_\mu \circ f = \mathcal{E}_\mu \ \mu\text{-a.e.} \tag{3}$$

This can be seen by the following two inequalities:

(I) $\mathcal{E}_\mu(f(x)) \ge \mathcal{E}_\mu(x)$: since

$$\mu(B(f^{-1}(x),n,e^{-rn})) = \mu(f[B(f^{-1}(x),n,e^{-rn})]) \le \mu(B(x,n,M_f e^{-rn})),$$

(II) $\mathcal{E}_\mu(f(x)) \le \mathcal{E}_\mu(x)$: since

$$\mu(B(f^{-1}(x),n,e^{-rn})) \ge \mu(f^{-1}[B(x,n+1,e^{-r\frac{n}{n+1}\cdot(n+1)})]) \ge \mu(B(x,n+1,e^{-r(n+1)})).$$

The significance of the neutralized entropy is that it estimates the asymptotic measure of sets with a distinctive geometric shape. Unlike the sequence $\{B(x,n,r)\}_{n\ge 0}$ which can develop a very complicated geometric shape for large n- due to a central direction, or even for a non-uniformly hyperbolic trajectory- the sequence $\{B(x,n,e^{-rn})\}_{n\ge 0}$ can have a nice description a.e for any $r > 0$, by neutralizing any sub-exponential effects. Sets with a more explicit geometric description are very useful for the construction of covers (or consequently even partitions), and so controlling the measure of such sets is important.

In fact, one can guess that by the lack of diversity for intrinsic dynamical invariants, the neutralized local entropy must coincide with other notions of local entropy (or the metric entropy in the ergodic case). This paper is dedicated to the proof of this statement for smooth systems.

Thieullen [Thi92a] studied a similar notion to the neutralized local entropy, called α-entropy, for certain systems on infinite-dimensional systems (see also [Thi92b, Thi91]). Other generalizations of entropy have also been studied in terms of ergodic theory for some systems by [TV02, TV03].

As an application of the neutralized local entropy, in Theorem 4.1 we prove a lower bound for the point-wise dimension of ergodic invariant measures: for almost every point

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \ge d^u + d^s,$$

where d^u and d^s are the point-wise dimension of the conditional measures of μ along the unstable and the stable laminations, resp. (see §4 for details). This extends the previous result in [BPS99] for hyperbolic measures, and complements the upper bound in [LY85b]. In particular, using the notion of neutralized local entropy, the proof of Theorem 4.1 is relatively elementary as it only uses covers and does not require constructing adapted partitions.

[1] $h_\mu^{\mathrm{BK}}(x) = \lim_{\epsilon\to 0} \limsup \frac{-1}{n} \log \mu(B(x,n,\epsilon)) = \lim_{\epsilon\to 0} \liminf \frac{-1}{n} \log \mu(B(x,n,\epsilon))$, where the limits exist, the equality holds μ-a.e, and $B(x,n,\epsilon) = \{y \in M : d(f^i(x), f^i(y)) \le \epsilon, \forall 0 \le i \le n\}$.

2. Covering, differentation, and Ergodic theorems

Let M be a closed Riemannian manifold, $d = \dim M \geq 2$, and let $f \in \mathrm{Diff}^{1+\beta}(M)$, $\beta > 0$.
Notation: Throughout the paper, given $A, B, C > 0$, we write $A = B \cdot e^{\pm C}$ to mean $Be^{-C} \leq A \leq Be^{C}$.

Definition 2.1 (Pesin blocks)**.** *Let μ be an f-invariant ergodic probability measure which admits a positive Lyapunov exponent. Denote by $\underline{\chi} = \underline{\chi}(\mu) := (\chi_1, m_1, \ldots, \chi_{\ell_{\underline{\chi}}}, m_{\ell_{\underline{\chi}}})$ the positive Lyapunov exponents, and dimensions of the corresponding Oseledec subspaces, of μ in a decreasing order.*

(1) Let $0 < \tau \leq \tau_{\underline{\chi}} := \frac{1}{100d} \min\{\chi_{\ell_{\underline{\chi}}}, \chi_i - \chi_{i+1} : i \leq \ell_{\underline{\chi}} - 1\}$, and let $C_{\underline{\chi},\tau}(\cdot)$ be the Lyapunov change of coordinates for points in $\mathrm{LR}_{\underline{\chi}} = \{$Lyapunov regular points with an index $\underline{\chi}\}$ (see [KM95]).

*(2) Let $\mathrm{PR}_{\underline{\chi}} = \{x \in \mathrm{LR}_{\underline{\chi}} : \limsup_{n\to\pm\infty} \frac{1}{n} \log \|C^{-1}_{\underline{\chi},\tau}(f^n(x))\| = 0, \forall 0 < \tau \leq \tau_{\underline{\chi}}\}$, the set of $\underline{\chi}$-*Pesin regular *points which carries μ. $\mathrm{PR} := \bigcup_{\underline{\chi}} \mathrm{PR}_{\underline{\chi}}$ is called the set of* Pesin regular *points.*

(3) Given $x \in \mathrm{PR}_{\underline{\chi}}$, let $E_j(x)$ be the Oseledec subspace of x corresponding to χ_j.

(4) A Pesin block *$\Lambda^{(\underline{\chi},\tau)}_{\ell}$ is a subset of $\bigcup_{|\underline{\chi}'-\underline{\chi}|_\infty<\tau} \mathrm{PR}_{\underline{\chi}'}$ which is a level set $[q_\tau \geq \frac{1}{l}]$ of a measurable function $q_\tau : \bigcup_{|\underline{\chi}'-\underline{\chi}|_\infty<\tau} \mathrm{PR}_{\underline{\chi}'} \to (0,1)$ s.t (a) $\frac{q_\tau \circ f}{q_\tau} = e^{\pm\tau}$, (b) $q_\tau(\cdot) \leq \frac{1}{\|C^{-1}_{\underline{\chi},\tau}(\cdot)\|^{\frac{d}{\beta}}}$. Often we omit the subscript ℓ when the dependence on ℓ is clear from the context.*

Lemma 2.2 (Besicovitch-Bowen covering lemma)**.** *Let $\Lambda^{(\underline{\chi},\tau)}$ $(0 < \tau \leq \tau_{\underline{\chi}})$ be a Pesin block, and let $x_0 \in \Lambda^{(\underline{\chi},\tau)}$. Let $B(x_0)$ be the Pesin chart of x_0 for $\Lambda^{(\underline{\chi},\tau)}$. Let $A \subseteq \Lambda^{(\underline{\chi},\tau)} \cap B(x_0)$ be a measurable subset. Then A can be covered by a cover of exponential Bowen balls of points in A (i.e $B(\cdot, n, e^{-n\epsilon})$), with multiplicity bounded by $e^{3d\tau n}$, where n is sufficiently large w.r.t $\Lambda^{(\underline{\chi},\tau)}$, and $\epsilon \geq 2\tau$.*

The idea of the proof follows the principle steps of the proof of the classical Besicovitch covering lemma (see [DiB02, § 18]), using a certain volume doubling property for exponential Bowen balls.

Proof. Let $n \in \mathbb{N}$ sufficiently large so $e^{-\epsilon n}$ is smaller than the Pesin chart size for $\Lambda^{(\underline{\chi},\tau)}$. Set $A_1 := A$, and choose $x_1 \in A_1$. Given k, set $A_{k+1} := A \setminus \bigcup_{j=1}^{k} B(x_j, n, e^{-n\epsilon})$ and choose $x_{k+1} \in A_{k+1}$. Continue in this process as long as $A \setminus \bigcup_{j=1}^{k} B(x_j, n, e^{-n\epsilon})$ is not empty.

Claim 1: $\forall k < j$, $B(x_j, n, \frac{1}{3}e^{-n\epsilon}) \cap B(x_k, n, \frac{1}{3}e^{-n\epsilon}) = \varnothing$.
Proof: Otherwise, $\forall i \leq n$, $d(f^i(x_k), f^i(x_j)) \leq \frac{2}{3}e^{-\epsilon n}$, whence $x_j \in B(x_k, n, e^{-n\epsilon})$, in contradiction to the choice of $\{x_l\}_{l\geq 1}$.

Claim 2: $\exists N \in \mathbb{N}$ s.t $\bigcup_{k=1}^{N} B(x_k, n, e^{-\epsilon n}) \supseteq A$.
Proof: For any $k \geq 1$, $\mathrm{Vol}(B(x_k, n, \frac{1}{3}e^{-\epsilon n})) = C^{\pm 1} \cdot e^{-\underline{\chi}^u n} \cdot e^{-d_{\mathrm{cs}}\epsilon n} e^{\pm d\tau n}$, where $e^{-\underline{\chi}^u n} = \prod_{i\leq \ell_{\underline{\chi}}} e^{-\chi_i n \cdot m_i}$, m_i is the multiplicity of χ_i, $d_{\mathrm{cs}} := \sum_{i:\chi_i\leq 0} \dim E_i$, and C is a constant depending on M, d, and $\Lambda^{(\underline{\chi},\epsilon)}$. Then $N \leq \frac{\mathrm{Vol}(M)}{C^{-1}\cdot e^{-\underline{\chi}^u n}\cdot e^{-d\tau n}\cdot e^{-d_{\mathrm{cs}}\epsilon n}}$.

Claim 3: $\bigcup_{k=1}^{N} A_k \supseteq A$.
Proof: The process continues unless A is covered.

Claim 4: $\forall k \leq N$, $\#\{j \leq N : B(x_j, n, e^{-n\epsilon}) \cap B(x_k, n, e^{-n\epsilon}) \neq \varnothing\} \leq C_d e^{2dn\tau}$, where C_d is a constant depending on M and on $\Lambda^{(\underline{\chi},\tau)}$.
Proof: Let $k \leq N$, and let $k \neq j \leq N$ s.t $B(x_j, n, e^{-n\epsilon}) \cap B(x_k, n, e^{-n\epsilon}) \neq \varnothing$. Then

$$B(x_j, n, \frac{1}{3}e^{-n\epsilon}) \subseteq B(x_j, n, e^{-n\epsilon}) \subseteq B(x_k, n, 3e^{-n\epsilon}).$$

Then

$$\#\{j \leq N : B(x_j, n, e^{-n\epsilon}) \cap B(x_k, n, e^{-n\epsilon}) \neq \varnothing\} \leq \frac{\max_{k\leq N} \mathrm{Vol}(B(x_k, n, 3e^{-n\epsilon}))}{\min_{j\leq N} \mathrm{Vol}(B(x_j, n, \frac{1}{3}e^{-n\epsilon}))} \leq C^2 3^{2d} e^{2d\tau n}.$$

Claim 5: We can divide $\{x_k\}_{k\leq N}$ into sub-collections $\mathcal{C}_i$, $i = 1, \ldots, \lceil C_d e^{2nd\tau} + 1\rceil$, where for any $i \leq \lceil C_d e^{2dn\tau} + 1\rceil$, $\{B(x, n, e^{-\epsilon n}) : x \in \mathcal{C}_i\}$ is a mutually disjoint collection. In particular, for n sufficiently large, $\lceil C_d e^{2n\tau} + 1\rceil \leq e^{3dn\tau}$.

Proof: Let $K_n := \lceil C_d e^{2nd\tau} + 1 \rceil$. We associate x_k with $\mathcal{C}_k$ for all $k \leq K_n$. For each $k > K_n$, we allocate it into one of the pre-existing collections in the following way.

Given $B(x_{K_n+1}, n, e^{-\epsilon n})$, by claim 4, there exists at least one $i_{K_n+1} \leq K_n$ s.t $B(x_{K_n+1}, n, e^{-\epsilon n}) \cap B(x_{i_{K_n+1}}, n, e^{-\epsilon n}) = \varnothing$; so allocate x_{K_n+1} to $\mathcal{C}_{i_{K_n+1}}$.

Next, consider $B(x_{K_n+2}, n, e^{-\epsilon n})$, at least two of the $K_n + 1$-first balls do not intersect it. If any of the balls $2, \ldots, K_n$ do not intersect $B(x_{K_n+2}, n, e^{-\epsilon n})$, allocate it to the first associated collection as such. If all of the balls of $1, \ldots, i_{K_n+1} - 1, i_{K_n+1} + 1, \ldots, K_n$ intersect it, then allocate it to $\mathcal{C}_{i_{K_n+1}}$ which now contains three disjoint balls.

We continue by induction. Assume that the balls $\{B(x_j, n, e^{-\epsilon n})\} j \leq K_n + l - 1$ have all been allocated into one of the K_n-many disjoint sub-collections. Consider $B(x_{K_n+l}, n, e^{-\epsilon n})$, which is disjoint from at least l balls in $\{B(x_j, n, e^{-\epsilon n})\} j \leq K_n + l - 1$. This implies that least one of the collections $\mathcal{C}_i$, $i \leq K_n$, is disjoint from $B(x_{K_n+l}, n, e^{-\epsilon n})$, to which we may allocate it. $\square$

Lemma 2.3 (Bowen-Lebesgue density lemma)**.** *Let μ be an f-invariant probability measure. Let A be a measurable set s.t $\mu(A) > 0$. Then for μ-a.e $x \in A$,*

$$\lim_{r \to 0} \limsup_{n \to \infty} \frac{-1}{n} \log \frac{\mu(B(x, n, e^{-nr}) \cap A)}{\mu(B(x, n, e^{-nr}))} = 0.$$

Proof. First, assume that $\exists \lambda > 0$ s.t (w.l.o.g) for μ-a.e $x \in A$,

$$\lim_{r \to 0} \limsup \frac{-1}{n} \log \frac{\mu(B(x, n, e^{-rn}) \cap A)}{\mu(B(x, n, e^{-rn}))} \geq \lambda.$$

Let $0 < r < \frac{1}{3d} \frac{\lambda}{4}$ s.t $\mu(A^{(0)}) > 0$ where

$$A^{(0)} := \{x \in A : \limsup \frac{-1}{n} \log \frac{\mu(B(x, n, e^{-rn}) \cap A)}{\mu(B(x, n, e^{-rn}))} \geq \frac{7\lambda}{8}\}.$$

Then it follows that $\mu(A^{(1)}) > 0$ where $A^{(1)} := A^{(0)} \cap \Lambda^{(\underline{\chi}, \tau)}$ for some index $\underline{\chi}$ and $\tau \leq \min\{\frac{r}{2}, \tau_{\underline{\chi}}\}$ (this can be achieved by first dividing the parameter space of $\underline{\chi}$ by the Oseledec dimensions, then by boxes around each $\underline{\chi}'$ of size $\frac{1}{2}\tau_{\underline{\chi}'}$, and then further by boxes of size $\tau := \min\{\frac{r}{2}, \frac{1}{2}\tau_{\underline{\chi}'}\}$; whence $\frac{1}{2}\tau_{\underline{\chi}'} \leq \tau_{\underline{\chi}}$).

Let $N \geq 1$ large and $x \in A^{(1)}$, and set

$$n_x^N := \min\{n \geq N : \frac{-1}{n} \log \frac{\mu(B(x, n, e^{-rn}) \cap A)}{\mu(B(x, n, e^{-rn}))} \geq \frac{6\lambda}{8}\}.$$

Set for all $n \geq N$,

$$A_n^{(1)} := \{x \in A^{(1)} : n_x^N = n\}.$$

By Lemma 2.2, we can choose a finite subset $\mathcal{C}_n \subseteq A_n^{(1)}$ s.t $\bigcup_{x \in \mathcal{C}_n} B(x, n, e^{-rn}) \supseteq A_n^{(1)}$ while $\{B(x, n, e^{-rn})\}_{x \in \mathcal{C}_n}$ has an overlap bound smaller than $e^{3d\tau n} \leq e^{3d\frac{r}{2}n} \leq e^{\frac{\lambda}{8}n}$ (for all n large enough depending on M and $\Lambda^{(\underline{\chi}, \tau)}$). Then,

$$\begin{aligned}\mu(A_n^{(1)}) =& \mu\left((\bigcup_{x \in \mathcal{C}_n} B(x, n, e^{-rn})) \cap A_n^{(1)} \right) \leq \sum_{x \in \mathcal{C}_n} \mu(B(x, n, e^{-rn}) \cap A_n^{(1)}) \leq \sum_{x \in \mathcal{C}_n} \mu(B(x, n, e^{-rn}) \cap A) \\ &\leq \sum_{x \in \mathcal{C}_n} \mu(B(x, n, e^{-rn})) e^{-\frac{6\lambda}{8}n} \leq e^{-\frac{6\lambda}{8}n} e^{\frac{\lambda}{8}n} \mu(\bigcup_{x \in \mathcal{C}_n} B(x, n, e^{-rn})) \leq e^{-\frac{5\lambda}{8}n}.\end{aligned}$$

Then,

$$\mu(A^{(1)}) = \sum_{n \geq N} \mu(A_n^{(1)}) \leq \sum_{n \geq N} e^{-\frac{5\lambda}{8}n} \leq (\sum_{n \geq 0} e^{-\frac{5\lambda}{8}n}) \cdot e^{-\frac{5\lambda}{8}N} \xrightarrow{N \to \infty} 0, \text{ a contradiction!}$$

$\square$

Remark: A possible heuristic way to interpret Lemma 2.3 is the following: we think of belonging to a measurable set as satisfying some property. Then, for almost any point which satisfies a certain property, more and more points which spend a long portion of their orbit close to this point inherit this property as well. That is, a portion of the exponential Bowen ball, with bigger and bigger exponential portion bounds, lies in the measurable set as well.

Corollary 2.4 (log-differentation lemma)**.** *Let μ be an f-invariant probability measure, and let $g \in \mathcal{M}(\mu)$ be a measurable function and $A \in \mathcal{B}$ be a measurable set with $\mu(A) > 0$. Then for μ-a.e $x \in A$,*

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{-1}{n}\log\left(\frac{1}{\mu(B(x,n,e^{-n\epsilon}))}\int_{B(x,n,e^{-n\epsilon})} e^{-n|g(y)-g(x)|}\mathbb{1}_A(y)d\mu(y)\right) = 0.$$

Proof. Let $\delta > 0$, and let $a \in \mathbb{R}$ s.t $\mu(E) > 0$ where $E := A \cap g^{-1}[[a-\frac{\delta}{2}, a+\frac{\delta}{2}]]$. Let $x \in E$, $n \geq 0$, and $\epsilon > 0$, then

$$\begin{aligned}\frac{1}{\mu(B(x,n,e^{-\epsilon n}))}\int_{B(x,n,e^{-\epsilon n})} e^{-n|g(y)-g(x)|}\mathbb{1}_A(y)d\mu(y) \geq& \frac{1}{\mu(B(x,n,e^{-\epsilon n}))}\int_{B(x,n,e^{-\epsilon n})\cap E} e^{-n\delta}d\mu(y)\\ =& e^{-n\delta}\frac{\mu(B(x,n,e^{-\epsilon n})\cap E)}{\mu(B(x,n,e^{-\epsilon n}))}.\end{aligned}$$

Then by Lemma 2.2, for μ-a.e $x \in E$, $\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{-1}{n}\log\frac{1}{\mu(B(x,n,e^{-n\epsilon}))}\int_{B(x,n,e^{-n\epsilon})} e^{-n|g(y)-g(x)|}\mathbb{1}_A(y)d\mu(y) \leq \delta$. Since $\delta > 0$ was arbitrary (and the limit is independent of a), we are done. □

Theorem 2.5 (Log-Ergodic Theorem)**.** *Let μ be an f-invariant probability measure, let A be a measurable set s.t $\mu(A) > 0$, and let $g \in L^1(\mu)$. Then for μ-a.e $x \in A$,*

$$\lim_{r\to 0}\limsup_{n\to\infty}\frac{-1}{n}\log\frac{1}{\mu(B(x,n,e^{-nr}))}\int_{B(x,n,e^{-nr})}\mathbb{1}_A(y)\cdot e^{-|\sum_{j=0}^{n-1} g\circ f^{-j}(y)-\sum_{j=0}^{n-1} g\circ f^{-j}(x)|}d\mu(y) = 0.$$

Proof. Let $\mu = \int \mu_x d\mu(x)$ be the ergodic decomposition of μ. Since $g \in L^1(\mu)$, for μ-a.e x, $g \in L^1(\mu_x)$. Then for μ-a.e x, $\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}(g-\int g d\mu_x)\circ f^{-j}(x) = 0$. Let $\delta > 0$, and let $n_\delta \geq 0$ s.t $\mu(A_\delta) \geq e^{-\delta}\mu(A)$ where

$$A_\delta := \left\{y \in A : \forall n \geq n_\delta, \left|\sum_{j=0}^{n-1}(g-\int g d\mu_y)\circ f^{-j}(y)\right| \leq n\delta\right\}.$$

Let $x \in A_\delta$, then for all $n \geq n_\delta$ and $r > 0$,

$$\begin{aligned}&\int_{B(x,n,e^{-nr})}\mathbb{1}_A(y)\cdot e^{-|\sum_{j=0}^{n-1} g\circ f^{-j}(y)-\sum_{j=0}^{n-1} g\circ f^{-j}(x)|}d\mu(y)\\ &\geq \int_{B(x,n,e^{-nr})}\mathbb{1}_{A_\delta}(y)\cdot e^{-|\sum_{j=0}^{n-1}(g-\int g d\mu_y)\circ f^{-j}(y)|-n|\int g d\mu_y-\int g d\mu_x|-|\sum_{j=0}^{n-1}(g-\int g d\mu_x)\circ f^{-j}(x)|}d\mu(y)\\ &\geq e^{-2\delta n}\int_{B(x,n,e^{-nr})}\mathbb{1}_{A_\delta}(y)\cdot e^{-n|G(y)-G(x)|}d\mu(y),\end{aligned}$$

where $G(y) := \int g d\mu_y$. Then by the log-differentiation lemma (Corollary 2.4) for G, and since $\delta > 0$ was arbitrary, we are done. □

3. Neutralized local entropy is entropy

Lemma 3.1. *Let μ be an f-invariant Borel probability, and let $\mu = \int \mu_x d\mu(x)$ be its ergodic decomposition. Then $h_\mu^{\mathrm{BK}}(x) = h_{\mu_x}(f)$ μ-a.e.*

Proof. Let $E_\lambda := \{x : h_\mu^{\mathrm{BK}}(x) \geq h_{\mu_x}(f) + \lambda\}$, and assume that there exists $\lambda > 0$ s.t $\mu(E_\lambda) > 0$ (notice that E_λ is f-invariant). Let $G_\lambda := \{x : \mu_x(E_\lambda) = 1\}$ with $a_\lambda := \mu(G_\lambda) > 0$ (o.w $\mu(E_\lambda) = 0$). Write $\mu_\lambda := \frac{1}{\mu(G_\lambda)}\int_{G_\lambda}\mu_x d\mu(x)$, and $\mu_c := \frac{1}{a_c}(1-\mu_\lambda)$ where $a_c := 1 - a_\lambda$. Then $\mu = a_\lambda\mu_\lambda + a_c\mu_c$ and for μ_λ-a.e x,

$$h^{\mathrm{BK}}_{\mu_\lambda}(x) \geq h^{\mathrm{BK}}_{\mu}(x) \geq h_{\mu_x}(f) + \lambda.$$

However, this is a contradiction, since integrating both sides by μ_λ admits $h_{\mu_\lambda}(f) \geq h_{\mu_\lambda}(f) + \lambda$, a contradiction! Hence $h^{\mathrm{BK}}_{\mu}(x) \leq h_{\mu_x}(f)$ μ-a.e, but $\int h^{\mathrm{BK}}_{\mu}(x)d\mu(x) = \int h_{\mu_x}(f)d\mu(x)$, hence $h^{\mathrm{BK}}_{\mu}(x) = h_{\mu_x}(f)$ μ-a.e. □

Lemma 3.2. *Let μ be an f-invariant Borel probability, and K be a μ-positive measure set, and let $\delta > 0$. Assume that $\exists \Delta > 0$ s.t $h^{\mathrm{BK}}_{\mu}(x) \leq h + \Delta$ for μ-a.e $x \in K$. Then for all n large enough w.r.t δ and K, there exist a measurable subset $K_\delta \subseteq K$ and a subset $\mathcal{A}_{n,\delta}$ and $0 < \rho \leq \delta$ s.t*

(1) $\bigcup_{x\in\mathcal{A}_{n,\delta}} B(x,n,\rho) \supseteq K_\delta$,
(2) $\frac{\mu(K_\delta)}{\mu(K)} \geq e^{-\delta}$,
(3) $\#\mathcal{A}_{n,\delta} \leq e^{n(h+\Delta+\delta)}$.

Proof. For μ-a.e $x \in K$, $\lim_{r\to 0}\limsup \frac{-1}{n}\log\mu(B(x,n,r)) \leq h+\Delta$. Set $K_\delta \subseteq K$ and $n_\delta \in \mathbb{N}$ s.t $\forall x \in K_\delta$, $\forall n \geq n_\delta$, $\mu(B(x,n,\frac{\rho}{3})) \geq e^{-n(h+\Delta+\delta)}$, and $\mu(K_\delta) \geq e^{-\delta}\mu(K)$, for some $0 < \rho \leq \delta$.

Let $n \geq n_\delta$. Set $K^1 := K_\delta$, and let $x_1 \in K^1$. $K^{i+1} := K^i \setminus B(x_i,n,\rho)$, and choose $x_{i+1} \in K^{i+1}$.

For any $i \neq j$, $B(x_i,n,\frac{\rho}{3}) \cap B(x_j,n,\frac{\rho}{3}) = \varnothing$, hence we have at most $e^{n(h+\Delta+\delta)}$-many elements in $\{x_i\}_i =: \mathcal{A}_{n,\delta}$. Moreover, one can check that $\bigcup_{x\in\mathcal{A}_{n,\delta}} B(x,n,\rho) \supseteq K_\delta$. □

Corollary 3.3. *Let $\chi_0 > 0$, $\tau \in (0, \frac{\chi_0}{100d})$, $\epsilon \geq 4\tau$ small (w.r.t χ_0), $\gamma \geq \sqrt{\epsilon}$, $\ell \in \mathbb{N}$, and let n' s.t $\mu(K) > 0$ where*

$$K \subseteq \Big\{x \text{ Lyap. reg. s.t } \chi^u_{\min}(x), \chi^s_{\min}(x) \geq \chi_0 : \forall n \geq n', x \in \Lambda_\ell^{(\underline{\chi}(x),\tau)} \cap \Big(\bigcup_{n(1+\gamma)\leq i\leq n(1+2\gamma)} f^{-i}[\Lambda_\ell^{(\underline{\chi}(x),\tau)}]\Big)\Big\}. \tag{4}$$

Assume that $h^{\mathrm{BK}}_{\mu} \leq h+\Delta$ for μ-a.e $x \in K$. Then there exists $K_\tau \subseteq K$ s.t $\mu(K_\tau) \geq e^{-\tau}\mu(K)$ and for all n large enough, $\exists \mathcal{A}_n \subseteq K_\tau$ s.t $\bigcup_{x\in\mathcal{A}} B(x,n,e^{-\epsilon n}) \supseteq K_\tau$ and $\#\mathcal{A}_n \leq e^{n(1+2\gamma)(h+\Delta+3d\epsilon)}$.

Proof. Let K_τ and n_τ as in Lemma 3.2, for some $0 < \rho \leq \tau$.

Let $n \geq \max\{n_\tau, n'\}$, and let $\mathcal{C}_n$ be a Besicovitch cover of K_τ by balls of radius $e^{-2\epsilon n}$.

For each such ball B, we cover $K_\tau \cap B$ with at most $e^{n(1+2\gamma)(h+\Delta+\tau)}$-many Bowen balls of the form $B(\cdot, \lfloor n(1+2\gamma)\rfloor, \rho)$ by Lemma 3.2. Hence in total we cover K_τ with at most $B_d C_M e^{2d\epsilon n} e^{n(1+2\gamma)(h+\Delta+\tau)}$-many elements, where B_d is the Besicovitch constant of M, and C_M is a constant s.t $\mathrm{Vol}(B(x,e^{-2\epsilon n})) \geq \frac{1}{C_M}e^{-2d\epsilon n}$ for all $x \in M$ and n large.

Let $B \in \mathcal{C}_n$, and let $\mathcal{A}^B_{n,\tau}$ as in Lemma 3.2 for $B \cap K_\tau$. Let $x \in \mathcal{A}^B_{n,\tau}$, and notice that for $j_x \in [\lfloor n(1+\gamma)\rfloor, \lfloor n(1+2\gamma)\rfloor]$,

$$B(x, \lfloor n(1+2\gamma)\rfloor, \rho) \cap B \subseteq B(x, j_x, \rho) \cap B(x, 2e^{-2\epsilon n}) \subseteq B(x,n,e^{-\epsilon n}), \tag{5}$$

for all n large enough depending on ℓ and ϵ (we prove (5) in the end of this lemma). Thus in total, $\{B(x,n,e^{-\epsilon n}) : x \in \mathcal{A}^B_{n,\tau}, B \in \mathcal{C}_n\}$ is a cover of K_τ by exponential Bowen balls, of cardinality bounded by $e^{n(1+2\gamma)(h+\Delta+3d\epsilon)}$ for all n large enough.

To prove (5), we work with Pesin charts, which is where we need the assumption from (4).

Let $x \in K$. We wish to show that for all n large enough (depending on ℓ), $\forall i \in [1,n]$, $\forall y \in B(x, j_x, \rho) \cap B(x, 2e^{-2\epsilon n})$, $f^i(y) \in B(f^i(x), e^{-\epsilon n})$. Letting ψ_i be the Pesin chart of $f^i(x)$, it is enough to show that $|\psi_i^{-1} \circ f^i \circ \psi_0(v_y)| \leq e^{-\frac{5}{4}\epsilon n}$, where $v_y := \psi_0^{-1}(y)$.

Write $\psi_i^{-1} \circ f^i \circ \psi_0(v_y) = v = v^s + v^c + v^u$, where $v^t \in E^t(x)$, $t \in \{s,c,u\}$. Set $f_i := \psi_{i+1}^{-1} \circ f \circ \psi_i$, and $F_i := f_{i-1} \circ \cdots \circ f_0$. We assume for the simplicity of presentation that all of the negative Lyapunov exponents of x are equal, and that all of the positive Lyapunov exponents of x are equal, otherwise decompose v_s and v_u into corresponding components.

A standard result of Pesin theory tells us that the maps f_i can be put in the form $f_i = \sum_{t\in\{s,c,u\}} D_t v_t + h_i^t(v)$, $\|h_i^t\|_{C^1} \leq \tau$, and where D_t are linear self-maps of E^t, and $e^{\chi^s(x)+\tau} \geq \|D_s^{-1}\|, \|D_s\| \leq e^{-\chi^s(x)+\tau}$, $e^{-\chi^u(x)+\tau} \geq \|D_u^{-1}\|, \|D_u\| \leq e^{\chi^u(x)+\tau}$, and $\|D_c^{-1}\|, \|D_c\| \leq e^{\tau}$.

Therefore the stable and central components of $F_n(v_y)$ remain small enough, and we are left to bound $v^u = (F_n(v_y))^u$. Since $f^{j_x}(y) \in B(f^{j_x}(x), \rho)$, similar contraction estimates hold for f^{-1}, and we get $|v_y^u| \leq e^{-(\chi^u(x)-2\tau)n(1+\gamma)}$. Thus, $|(F_n(v))^u| \leq e^{-(\chi^u(x)-2\tau)n(1+\gamma)} e^{(\chi^u(x)+2\tau)n} \leq e^{-\gamma n(\chi_0-4\tau)} \leq e^{-\frac{5}{4}\epsilon n}$ for $\epsilon > 0$ small enough (w.r.t χ_0). □

Theorem 3.4. *Let $f \in \mathrm{Diff}^{1+\beta}(M)$, where M is a closed Riemannian manifold with* $\dim M = d \geq 2$. *Let μ be an f-invariant Borel probability on M. Then*

$$\mathcal{E}_\mu(x) = h_\mu^{\mathrm{BK}}(x) \ \mu\text{-a.e.}$$

Proof. We start with a reduction. Let $\mu = a^+\mu^+ + a^0\mu^0$ where μ^+ admits a positive Lyapunov exponent a.e, and μ^0 has all exponent less or equal to 0 a.e. Then $\mathcal{E}_\mu \leq \mathcal{E}_{\mu^+}$, and for μ^0-a.e x, $\mathcal{E}_\mu(x) = 0 = h_\mu^{\mathrm{BK}}(x)$ (by dimension bounds). Therefore we may assume w.l.o.g that μ admits a positive Lyapunov exponent a.e. Moreover, write $\mu = \sum_{i\geq 1} a_i\mu_i$ where for every $i \geq 1$, μ_i-a.e, all non-zero Lyapunov exponents are greater than $\frac{1}{i}$ in absolute value. Thus if for every $i \geq 1$ we have $h_{\mu_i}^{\mathrm{BK}} \geq \mathcal{E}_{\mu_i}$ μ_i-a.e, then by Lemma 3.1,

$$h_\mu^{\mathrm{BK}} = h_{\mu_i}^{\mathrm{BK}} \geq \mathcal{E}_{\mu_i} \geq \mathcal{E}_\mu,\ \mu_i\text{-a.e}, \forall i \geq 1 \Rightarrow\ h_\mu^{\mathrm{BK}} \geq \mathcal{E}_\mu,\ \mu\text{-a.e.}$$

Then it is enough to assume that the non-zero Lyapunov exponents of μ are uniformly bounded from below in absolute value by a constant $\chi_0 > 0$.

It is enough to assume for contradiction that $\exists\lambda \in (0, \min\{\chi_0^3, \frac{1}{2}\})$ s.t $\mathcal{E}_\mu \geq h_\mu^{\mathrm{BK}} + 2\lambda$ μ-a.e, since if $\mu = a_\lambda\mu_\lambda + (1-a_\lambda)\mu_\lambda^-$ where $a_\lambda > 0$ and $\mathcal{E}_\mu \geq h_\mu^{\mathrm{BK}} + 2\lambda$ μ_λ-a.e , then once more by Lemma 3.1,

$$\mathcal{E}_{\mu_\lambda} \geq \mathcal{E}_\mu \geq h_\mu^{\mathrm{BK}} + 2\lambda = h_{\mu_\lambda}^{\mathrm{BK}} + 2\lambda,\ \mu_\lambda\text{-a.e.}$$

Finally, we make the following reduction: let $G_{(\mathcal{E},h)} := \{x : \mathcal{E}_\mu(x) = \mathcal{E} \pm \frac{\lambda}{2}, h_\mu^{\mathrm{BK}}(x) = h \pm \frac{\lambda}{2}\}$, then let $\mu = a_{(\mathcal{E},h)}\mu' + (\mu - a_{(\mathcal{E},h)}\mu')$, where $a_{(\mathcal{E},h)} > 0$ and μ' is carried by $G_{(\mathcal{E},h)}$. Then, μ'-a.e

$$\mathcal{E}_{\mu'} \geq \mathcal{E}_\mu \geq h_\mu^{\mathrm{BK}} + 2\lambda = h_{\mu'}^{\mathrm{BK}} + 2\lambda. \tag{6}$$

Moreover, similarly, we may assume w.l.o.g that $\mathcal{E}_\mu \gg \sqrt{\lambda}$ a.e. Therefore we may assume for contradiction that $\mathcal{E}_\mu$ and h_μ^{BK} are "almost constant" w.r.t to the gap between them which is uniformly bounded from below.

There exists $0 < r \leq \frac{\min\{1,\chi_0\}}{3d}\frac{\lambda^4}{4}$ s.t $\mu(A_1) \geq 1-\lambda^3$ where $A_1 := \{x \in A : \limsup \frac{-1}{n}\log\mu(B(x,n,e^{-rn})) > h_\mu^{\mathrm{BK}}(x) + \frac{7\lambda}{8}\}$.

We can then choose $0 < \tau \leq \min\{\frac{r}{400}\}$ and $\ell \in \mathbb{N}$ s.t $\mu(A_2) \geq 1 - 2\lambda^4$ where

$$A_2 := \{x \in A_1 \text{ Lyapunov reg.} : x \in \Lambda_\ell^{(\underline{\chi}(x),\tau)}\}.$$

Let $\mu = \int \mu_x d\mu(x)$ be the ergodic decomposition of μ. Then by the Markov inequality, $\mu(\{x : \mu_x(A_2) \geq 1 - \sqrt{2\lambda^4}\}) \geq 1 - \sqrt{2\lambda^4}$. Then $\mu(\{x \in A_2 : \mu_x(A_2) \geq 1 - \sqrt{2}\lambda^2\}) \geq 1 - 2\lambda^2$. So by the ergodic theorem $\exists n_0 \in \mathbb{N}$ s.t $\mu(A_3) \geq 1 - 3\lambda^2$ where

$$A_3 := \left\{x \in A_2 : \forall n \geq n_0 \exists j \in [n(1+4\lambda^2), n(1+8\lambda^2)] \text{ s.t } f^j(x) \in A_2\right\}.$$

Let $K_\tau \subseteq A_3$ as in Corollary 3.3 s.t $\mu(K_\tau) > 0$ (notice, the restrictions on K_τ in Corollary 3.3 are given by Lemma 3.2, which in turn are merely Brin-Katok estimates; which are inherited by subsets). Let $N \geq n_0$ large, then for all $x \in K_\tau$ set

$$n_x^N := \min\{n \geq N : \frac{-1}{n}\log\mu(B(x,n,e^{-rn})) > h + \frac{6\lambda}{8}\}.$$

For all $n \geq N$, set $K_n := \{x \in K_\tau : n_x^N = n\}$.

By Corollary 3.3, we can cover K_n with a cover whose cardinality is less or equal to $e^{n(1+8\lambda^2)(h+\frac{\lambda}{8}+3dr)}$, of exponential Bowen balls of the form $B(x,n,e^{-nr})$, $x \in K_n$. Hence, $\mu(K_n) \leq e^{n(1+8\lambda^2)(h+\frac{\lambda}{8}+3dr)} \cdot e^{-n(h+\frac{6\lambda}{8})}$, whence $0 < \mu(K_\tau) \leq \sum_{n\geq N} e^{-n\frac{\lambda}{8}} \xrightarrow{N\to\infty} 0$, a contradiction! Hence $h_\mu^{\mathrm{BK}} \leq \mathcal{E}_\mu \leq h_\mu^{\mathrm{BK}}$ μ-a.e. □

Remark: Theorem 3.4 implies that $\mathcal{E}_\mu(x) = \lim_{r\to 0}\liminf_{n\to\infty}\frac{-1}{n}\log\mu(B(x,n,e^{-rn}))$ for μ-a.e x since

$$h_\mu^{\mathrm{BK}}(x) = \lim_{r\to 0}\liminf_{n\to\infty}\frac{-1}{n}\log\mu(B(x,n,r)) \leq \lim_{r\to 0}\liminf_{n\to\infty}\frac{-1}{n}\log\mu(B(x,n,e^{-rn})) \leq \mathcal{E}_\mu(x) = h_\mu^{\mathrm{BK}}(x).$$

4. Lower-dimension bounds for invariant measures

In [LY85b], the authors consider an ergodic f-invariant measure μ, where f is a C^2 diffeomorphism of a closed Riemannian manifold M. The authors consider two important partitions- ξ^u and ξ^s sub-ordinated to the unstable and to the stable laminations respectively, and consider the conditional measures of μ w.r.t to these partitions. The authors then go on and prove that the conditional measures $\mu_{\xi^u(\cdot)}$ and $\mu_{\xi^s(\cdot)}$ are exact-dimensional for almost every point, with the dimensions being constant and denoted by d^u and d^s respectively. Moreover, the authors prove that the point-wise upper-dimension of μ-a.e point is bounded by

$$\overline{d} \leq d^u + d^s + d^c,$$

where $d^c = d^c(\mu) := \#\{0 \text{ Lyapunov exponent of } \mu\}$.

In [BPS99], the authors extend the results of [LY85b] by relaxing the regularity assumption of f to $C^{1+\beta}$ ($\beta > 0$) by proving the Lipschitz property of intermediate foliations; and bound from below the point-wise dimension of ergodic invariant measures under the additional assumption of hyperbolicity:

$$d^u + d^s \leq \underline{d} \leq \overline{d}.$$

In particular, it follows as a consequence that a hyperbolic measure is exact-dimensional, since $d^c = 0$.

The purpose of this section, is to extend these results, as an application of Theorem 3.4. We give lower bounds to the point-wise dimension of invariant measures, which coincide with the lower bounds from [BPS99] when the measure is hyperbolic. In general, invariant measures with 0 Lyapunov exponents need not be exact-dimensional, and the bounds from [LY85b] are tight. In particular, our proof is aimed to be short and accessible by using the neutralized local entropy instead of adapted partitions.

Given an ergodic f-invariant measure, let d^s and d^u be the point-wise dimension of the conditional measures $\{\mu_{\xi^s(\cdot)}\}$ and $\{\mu_{\xi^u(\cdot)}\}$, respectively, which are introduced in [LY85b].

Theorem 4.1. *Let μ be an f-invariant ergodic measure, where $f \in \mathrm{Diff}^{1+\beta}(M)$ ($\beta > 0$ and M a closed Riemannian manifold). Then for μ-a.e x,*

$$d^s + d^u \leq \underline{d}(x) := \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

Proof. We start with two simple reductions. Given $r > 0$, and $\chi > 0$, let $r' := \max\{e^{-n\chi} : e^{-n\chi} \leq r, n \in \mathbb{N}\}$. Then,

$$\begin{aligned}\frac{\log \mu(B(x,r))}{\log r} =& \frac{\log r'}{\log r} \cdot \frac{\log \mu(B(x,r))}{\log \mu(B(x,r'))} \cdot \frac{\log \mu(B(x,r'))}{\log r'} \\ \geq& (1 - \frac{\chi}{\log r}) \cdot \frac{\log \mu(B(x,r'))}{\log r'}.\end{aligned}$$

Then it is enough to prove $\liminf_{n\to\infty} \frac{\log \mu(B(x,e^{-n\chi}))}{-n\chi} \geq d^u + d^s$, for some $\chi > 0$. The second assumption we make is that μ admits both positive and negative Lyapunov exponents, since otherwise $d^s = d^u = 0$, and the statement is trivial.[2] Set $\chi := \frac{1}{3}\min\{|\chi_i(\mu)| : \chi_i(\mu) \neq 0\} > 0$.

We now can start with the construction for the proof. Assume that μ admits positive exponents, and so we prove $\underline{d} \geq d^u$. The case of the negative exponents is treated subsequently, and we improve the bound.

Let $\delta \in (0,\chi)$, and let $\epsilon \in (0, \frac{\delta^2}{8})$ and $n_\delta \in \mathbb{N}$ s.t $e^{-\chi n_\delta} \ll \frac{1}{\ell_\delta^3}$ and $\mu(A_\delta) \geq 1 - \delta$ where

$$\begin{aligned}A_\delta := \{x \in \Lambda_{\ell_\delta}^{(\underline{\chi},\tau)} : \forall n \geq n_\delta, &\mu_{\xi^u(x)}(B^u(x,e^{-\chi n})) \leq e^{-\chi d^u n + n\delta}, \\ &\mu(B(x,-n,e^{-\epsilon n})) = e^{-n\mathcal{E}_\mu \pm \delta n}, \mu(B(x,n,e^{-\epsilon n})) = e^{-n\mathcal{E}_\mu \pm \delta n}, \\ &\mu(B(x,-n,n,2e^{-\epsilon n})) = e^{-2\mathcal{E}_\mu n \pm \delta n}\},\end{aligned}$$

where $\underline{\chi} = \underline{\chi}(\mu)$ and $\tau \in (0, \frac{\epsilon}{100d})$, and $B(\cdot,-n,e^{-\epsilon n})$ denotes an exponential Bowen balls for f^{-1}, while $B(\cdot,-n,n,2e^{-\epsilon n})$ denotes a two-sided exponential Bowen ball.

[2] If there are either no positive or no negative Lyapunov exponents, then by the Ruelle inequality $h_\mu(f) = 0$. Therefore, by [LY85a, LY85b] ([BPS99, Appendix] guarantees that our $C^{1+\beta}$ setup is sufficient), $h_s(\mu) = h_u(\mu) = 0$ and $d^u = d^s = 0$.

By Lemma 2.3, there exists $A'_\delta \subseteq A_\delta$ and $m_\delta \geq n_\delta$ s.t $\mu(A'_\delta) \geq 1-2\delta$ where[3]

$$A'_\delta := \{x \in A_\delta : \forall n \geq m_\delta, \frac{1}{n}\log\frac{\mu(B(x,n,e^{-\epsilon n}))}{\mu(B(x,n,e^{-\epsilon n})\cap A_\delta)} \leq 48d\epsilon\}. \tag{7}$$

Let $x \in A'_\delta$ which is a Lebesgue density point s.t $\mu(B(x,e^{-n\chi})\cap A'_\delta) \geq e^{-\delta}\mu(B(x,e^{-n\chi}))$ for all $n \geq n_x \geq m_\delta$, and let $n \geq n_x$.

Then cover $A'_\delta \cap B(x,e^{-\chi n})$ as in Lemma 2.2, by a cover C^u of balls $B(\cdot,n,e^{-\epsilon n})$ with multiplicity bounded by $e^{3d\tau n}$.

Claim: For μ-a.e $x \in A'_\delta$, for all n large enough, $\mu(\bigcup C^u \cap A_\delta) \leq e^{-\chi d^u n + 2\sqrt{\delta}n}$.
Proof: For $x \in A'_\delta$, define

$$S_n(x) := \bigcup_{y \in A'_\delta \cap B(x,e^{-\chi n})} B(y,n,e^{-\epsilon n}) \cap A_\delta.$$

Assume that $\mu(\{x \in A'_\delta : \limsup \frac{-1}{n}\log\mu(S_n(x)) \leq \chi d^u - 2\sqrt{\delta}\}) > 0$. Then set $A_n := \{\{x \in A'_\delta : \frac{-1}{n}\log\mu(S_n(x)) \leq \chi d^u - \sqrt{\delta}\}\}$, and so by the Borel-Cantelli lemma there are infinitely many $n \in \mathbb{N}$ s.t $\mu(A_n) \geq e^{-\epsilon}$. Then there exists $x \in A'_\delta$ s.t $\mu_{\xi^u(x)}(A_n) \geq e^{-\epsilon n}$.

Cover $\xi^u(x) \cap A_n$ by $B(\cdot, e^{-\chi n}e^{2\epsilon n})$ in a Besicovitch manner, which means we can find a mutually disjoint sub-collection of measure at least $\frac{1}{C_d}e^{-\epsilon n}$, $\mathcal{F}$. Let $B(y,e^{-\chi n+2\epsilon n}), B(y',e^{-\chi n+2\epsilon n}) \in \mathcal{F}$, and let $z \in B(y,e^{-\chi n}) \cap A'_\delta$ and $z' \in B(y',e^{-\chi n}) \cap A'_\delta$. Let $w \in B(z,n,e^{-\epsilon n}) \cap \xi^u(x) \cap B(y,e^{-\chi n+\frac{\epsilon}{2}n})$ and $w' \in B(z',n,e^{-\epsilon n}) \cap \xi^u(x) \cap B(y',e^{-\chi n+\frac{\epsilon}{2}n})$.

If $B(z,n,e^{-\epsilon n}) \cap B(z',n,e^{-\epsilon n}) \neq \varnothing$, then $w \in B(w',n,4e^{-\epsilon n})$, which is impossible since both w and w' lie on $\xi^u(x)$ with distance at least $2\cdot(e^{-\chi n+2\epsilon n} - e^{-\chi n+\frac{\epsilon}{2}n}) \geq e^{-\chi n}$ between them.

Then, $\forall B, B' \in \mathcal{F}$, centered at y and y' respectively, $S_n(y) \cap S_n(y') = \varnothing$. However, $\#\mathcal{F} \geq \frac{e^{-\epsilon n}}{e^{-nd^u(\chi+3\epsilon)+\delta n}}$ (since $B(y,e^{-n(\chi-2\epsilon)}) \cap \xi^u(y) \subseteq B_{\xi^u(y)}(y,e^{-n(\chi-3\epsilon)})$). Thus,

$$1 \geq \#\mathcal{F}\cdot e^{-\chi d^u + n\sqrt{\delta}} \geq \frac{e^{-\epsilon n}}{e^{-nd^u(\chi+3\epsilon)+\delta n}}\cdot e^{-\chi d^u + n\sqrt{\delta}} \geq e^{\frac{1}{2}\sqrt{\delta}n},$$

a contradiction! □

It follows that for μ-a.e $x \in K'_\delta$, for all n large enough,

$$e^{-n\chi d^u + \sqrt{\delta}n} \geq \mu(\bigcup C^u \cap A_\delta) \geq e^{-3d\tau n}\cdot \#C^u \cdot \min_{B\in C^u}\mu(B\cap A_\delta) \geq e^{-3d\tau n}\cdot\#C^u\cdot e^{-n\mathcal{E}_\mu - \delta n}\cdot e^{-48d\epsilon n}.$$

Thus,

$$\#C^u \leq e^{n(\mathcal{E}_\mu - d^u\chi + 6\sqrt{\delta})}. \tag{8}$$

This concludes the bound

$$\mu(B(x,e^{-\chi n})) \leq e^\delta\cdot\mu(B(x,e^{-\chi n})\cap A'_\delta) \leq e^\delta\cdot\#C^u\cdot e^{-n(\mathcal{E}_\mu+\delta)} \leq e^\delta\cdot e^{n(\mathcal{E}_\mu - d^u\chi+6\sqrt{\delta})}\cdot e^{-n(\mathcal{E}_\mu+\delta)}.$$

Hence $\underline{d} \geq d^u - 7\sqrt{\delta}$ where $\delta > 0$ is arbitrary.

For the case of the negative exponents, construct similarly a cover C^s of $A'_\delta \cap B(x,e^{-\chi n})$ by exponential Bowen balls for f^{-1}, denoted by $B(\cdot,-n,e^{-\epsilon n})$. Then similarly one gets that $\#C^s \leq e^{n(\mathcal{E}_\mu - d^s\chi+6\sqrt{\delta})}$. We then define the cover $C := \{B^s \cap B^u : B^s \in C^s, B^u \in C^u, \text{ and } B^s\cap B^u \cap A'_\delta \neq \varnothing\}$. It follows immediately that

$$\#C \leq \#C^u\cdot\#C^s \leq e^{n(2\mathcal{E}_\mu - d^s\chi - d^u\chi + 12\sqrt{\delta})}. \tag{9}$$

Moreover, for every element $B \in C$, $B \subseteq B(x_B,-n,n,2e^{-\epsilon n})$ where $x_B \in A'_\delta$ by the triangle inequality. Therefore,

$$\mu(B(x,e^{-\chi n})) \leq e^\delta\cdot\#C\cdot e^{-n(2\mathcal{E}_\mu-\delta)} \leq e^\delta\cdot e^{-n(d^s+d^u)\chi+13\sqrt{\delta}n}. \tag{10}$$

Since $\delta > 0$ was arbitrary, we are done. □

[3]While formally Lemma 2.3 is only stated for the limit $\lim_{r\to 0}\limsup_{n\to\infty}\frac{-1}{n}\log\frac{\mu(B(x,n,e^{-nr})\cap A)}{\mu(B(x,n,e^{-nr}))} = 0$, the quantitative argument extends as is to the quantitative estimate of (7) whenever $\epsilon > 0$ is small w.r.t χ.

Remark: Note that the estimate from above of the measure of a ball in Theorem 4.1 is quite coarse: All elements of C^u and of C^s are of size $\sim e^{-\epsilon n}$ in the central direction, which is much longer than the diameter of the ball $e^{-\chi n}$. While this over-shooting may seem wasteful, one may may not expect an invariant measure to be concentrated in the central-direction, and in fact generally an invariant measure may even be atomic in the central direction, thus the estimate is tight for the general case. In some cases where we have more information regarding the central direction, we may say a bit more, as we show below in Corollary 4.2.

Corollary 4.2. *Under the assumptions of Theorem 4.1, if μ admits a measurable lamination by central leaves almost everywhere, then a.e,*

$$\liminf_{r\to 0}\frac{\log\mu(B(x,r))}{\log r} := \underline{d}(\mu) = \underline{d} \geq d^s + d^u + \underline{d}^c,$$

where $\underline{d}^c$ is the lower point-wise dimension of the conditional measures on the lamination by central leaves.

Proof. Let ξ^c be the measurable partition of μ into central leaves. Notice that $\underline{d}^c$ and $\underline{d}$ are invariant functions, and are constant a.e.

Let $\delta > 0$. We consider the set A''_δ, which is the set of density points of A'_δ which also satisfy dimension bounds, where A'_δ is as in the proof of Theorem 4.1. That is, for all $x \in A''_\delta$, and all $n \geq n'_\delta \geq n_\delta$,

$$\begin{aligned}\mu_{\xi^c(x)}(B(x, e^{-\chi n+n\epsilon}) \cap A'_\delta) &\leq e^{-\underline{d}^c\chi n+\delta n},\\ \mu_{\xi^c(x)}(B(x, e^{-2\epsilon n})) &\geq e^{-3d\epsilon n}.\end{aligned} \tag{11}$$

Furthermore, by the Borel-Cantelli lemma, there are infinitely many $n \in \mathbb{N}$ s.t $\mu(A_n) \geq e^{-\epsilon n}$, where

$$A_n := \{x \in A''_\delta : \mu(B(x, e^{-\chi n-\epsilon n})) \geq e^{-\chi\underline{d}n-\delta n}\}.$$

If for all $x \in A_n$, $\frac{\mu(A_n\cap B(x,e^{-\chi n-n\epsilon}))}{\mu(B(x,e^{-\chi n-\epsilon n}))} \leq e^{-d\epsilon n}$, then let $\mathcal{F}$ be a Besicovitch cover of A_n by balls $B(\cdot, e^{-\chi n-\epsilon n})$. It follows that,

$$e^{-\epsilon n} \leq \sum_{B\in\mathcal{F}}\mu(B\cap A_n) = \sum_{B\in\mathcal{F}}\frac{\mu(B\cap A_n)}{\mu(B)}\mu(B) \leq e^{-d\epsilon n}\sum_{B\in\mathcal{F}}\mu(B) \leq e^{-d\epsilon n}\cdot C_d,$$

$\Rightarrow$ a contradiction! Thus, by the definition of A_n, there exists $x \in A_n$ with

$$\mu(A_n \cap B(x, e^{-\chi n-\epsilon n})) \geq e^{-d\epsilon n}\cdot\mu(B(x, e^{-\chi n-\epsilon n})) \geq e^{-d\epsilon n-n\underline{d}\chi n-\delta n}. \tag{12}$$

Let $T_n(x) := B(x, e^{-\chi n+\epsilon n}) \cap B(x, -n, n, 2e^{-\epsilon n})$, then by (12) and by the bounds from Theorem 4.1, (9),

$$\begin{aligned}e^{-d\epsilon n-n\underline{d}\chi n-\delta n} &\leq \mu(B(x, e^{-\chi n-\epsilon n}) \cap A_n) \leq \mu(\bigcup C \cap B(x, e^{-\chi n-\epsilon n}) \cap A_n)\\ &\leq \#C\cdot\max_{B\in C:B\cap A_n\neq\varnothing}\mu(B\cap A_n\cap B(x, e^{-\chi n-\epsilon n}) \leq \#C\cdot\max_{B\in C:B\cap A_n\neq\varnothing}\mu(T_n(y_B)\cap A_n)\\ &\leq e^{-n\chi(d^u+d^s)+12\sqrt{\delta}n+2n\mathcal{E}_\mu}\cdot\max_{B\in C:B\cap A_n\neq\varnothing}\mu(T_n(y_B)\cap A_n),\end{aligned} \tag{13}$$

where y_B is a point in $B \cap A_n$.

We wish to bound the second term on the r.h.s of (13). We use the fact that $B(\cdot, -n, n, 2e^{-\epsilon n})$ is saturated by central leaves, and get that for all $B \in C$,

$$\begin{aligned}\mu(T_n(y_B)\cap A_n) &\leq \mu(\xi^c[T_n(y_B)\cap A_n]) = \int_{\xi^c[T_n(y_B)\cap A_n]}\mu_{\xi^c(z)}(T_n(y_B))d\mu(z)\\ &= \int_{\xi^c[T_n(y_B)\cap A_n]}\frac{\mu_{\xi^c(z)}(T_n(y_B))}{\mu_{\xi^c(z)}(B(y_B, e^{-2\epsilon n}))}\mu_{\xi^c(y)}(B(y_B, e^{-2\epsilon n}))d\mu(z)\\ (\because (11)) &\leq e^{-\chi\underline{d}^c n+\delta n+3d\epsilon n}\int_{\xi^c[T_n(y_B)\cap A_n]}\mu_{\xi^c(z)}(B(y_B, e^{-2\epsilon n}))d\mu(y)\\ &\leq e^{-\chi\underline{d}^c n+\delta n+3d\epsilon n}\mu(B(y_B, -n, n, 2e^{-\epsilon n})) \leq e^{-\chi\underline{d}^c n+\delta n+3d\epsilon n}e^{-2n\mathcal{E}_\mu+\delta n}.\end{aligned} \tag{14}$$

Plugging (14) back into (13), we get

$$e^{-d\epsilon n-n\underline{d}\chi n-\delta n} \leq e^{-n\chi(d^u+d^s)+12\sqrt{\delta}n+2n\mathcal{E}_\mu}\cdot e^{-\chi\underline{d}^c n+2\delta n+3d\epsilon n-2\mathcal{E}_\mu n}.$$

Therefore,

$$\underline{d} \geq d^s + d^u + \underline{d}^c - 19d\sqrt{\delta},$$

which concludes the proof since $\delta > 0$ is arbitrary. □

Remarks:

(1) Let $\overline{d}$ and $\overline{d}^c$ be the upper point-wise dimension of μ and of $\mu_{\xi^c(\cdot)}$, resp. One can show similarly to Corollary 4.2 that $\overline{d} \geq d^s + d^u + \overline{d}^c$, by adapting the definitions of the sets A''_δ and A_n, while the structure of the proof remains the same.

(2) In both Theorem 4.1 and Corollary 4.2, the non-ergodic case is treated similarly to the ergodic case, where A_δ restricts further to a subset where the values of $\underline{d}, \overline{d}, \underline{d}^c$, and $\overline{d}^c$ are almost constant.

Corollary 4.3. *Assume that μ admits a measurable lamination by central leaves $W^c(\cdot)$ almost everywhere. Let ξ^c be a measurable partition subordinated to the central lamination, and let $\lambda_{W^c(\cdot)}$ be the induced Riemannian volume on $W^c(\cdot)$. If the conditional measures of μ satisfy $\mu_{\xi^c(\cdot)} \ll \lambda_{W^c(\cdot)}$ a.e, then μ is exact and μ-a.e,*

$$d = d^s + d^u + \dim E^c,$$

where $\dim E^c$ is the dimension of the central Oseledec subspace.

Proof. By Corollary 4.2, and by [LY85b, Theorem F], μ-a.e,

$$\overline{d} \geq \underline{d} \geq d^s + d^u + \dim E^c \geq \overline{d}.$$

□

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