# HEIGHTS OF GENERALIZED HEEGNER CYCLES

by

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*To my parents, Sarah, and Roni.*

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# CHAPTER I

# Introduction

## 1.1 Background and motivation

The aim of this thesis is to relate the central derivatives of Rankin-Selberg $L$-functions to heights of algebraic cycles on varieties related to modular curves. Motivating our work is the formula of Gross and Zagier, which we now recall.

### 1.1.1 The Gross-Zagier formula

Let $f$ be a normalized newform of weight 2 and level $\Gamma_0(N)$. Let $K$ be an imaginary quadratic field with odd discriminant $D$, $H$ its Hilbert class field, and $\chi : \mathrm{Gal}(H/K) \to \bar{\mathbb{Q}}^\times$ a character. The theta series

$$\Theta_\chi = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \chi(\mathfrak{a}) q^{\mathrm{Nm}(\mathfrak{a})}$$

attached to $\chi$ is a weight 1 modular form, and we can form the Rankin-Selberg convolution $L(f, \chi, s) := L(f, \Theta_\chi, s)$. Gross and Zagier assume the *Heegner hypothesis*: every prime dividing $N$ splits in $K$. This forces the sign of the functional equation for $L(f, \chi, s)$ to be $-1$, and hence forces $L(f, \chi, s)$ to vanish at the central point $s = 1$.

On the other hand, the Heegner hypothesis guarantees that there exists a cyclic $N$-isogeny $\phi : A \to A'$ between two elliptic curves $A, A'$ both having complex multiplication by $\mathcal{O}_K$. Any such $\phi$ determines a point $y$ on the modular curve $X_0(N)$ parameterizing cyclic $N$-isogenies of elliptic curves. Since $\phi$ is defined over $H$, this point is $H$-rational, i.e. $y \in X_0(N)(H)$.

Using the Abel-Jacobi map

$$X_0(N) \to J_0(N) = \mathrm{Pic}^0(X_0(N))$$

$$y \mapsto c := [y] - [\infty],$$

we obtain a point $c \in J_0(N)(H)$ in the Mordell-Weil group of the Jacobian of $X_0(N)$ over $H$. As the actions of $\mathrm{Gal}(H/K)$ and the Hecke algebra $\mathbb{T}$ on $J_0(N)$ commute with each other, we may consider the $(f, \chi)$-isotypic component $c_{f,\chi} \in J_0(N) \otimes \mathbb{C}$ of $c$.

By the Mordell-Weil theorem, $J_0(N)(H)$ is a finitely generated abelian group. It is endowed with a symmetric bilinear pairing

$$\langle\, , \rangle_{\mathrm{NT}} : J_0(N)(H) \times J_0(N)(H) \to \mathbb{R}$$

called the Néron-Tate height pairing. The associated quadratic form on $J_0(N)(H) \otimes \mathbb{R}$ is positive definite, so $P \in J_0(N)(H)$ is torsion if and only if $\langle P, P \rangle_{\mathrm{NT}} = 0$. We extend this pairing to a Hermitian pairing on $J_0(N)(H) \otimes \mathbb{C}$ in order to compute the height of $c_{f,\chi}$.

**Theorem I.1** (Gross-Zagier [GZ])**.** *There is an explicit non-zero constant $\kappa = \kappa(f, K)$ such that*

$$L'(f, \chi, 1) = \kappa \cdot \langle c_{f,\chi}, c_{f,\chi} \rangle_{\mathrm{NT}}.$$

This formula gives a remarkable connection between the analytic realm of automorphic $L$-functions and the arithmetic of modular curves. It is also a key ingredient in the proof of many cases of the Birch and Swinnerton-Dyer conjecture for elliptic curves over $\mathbb{Q}$, a geometric conjecture whose statement makes no mention of modular forms at all.

To state this application of the Gross-Zagier formula, let us assume for simplicity that $f$ has rational Hecke eigenvalues. Then there is an elliptic curve $E_f/\mathbb{Q}$ which is quotient of $J_0(N)$ and such that $L(E_f, s) = L(f, s)$. If $\chi$ is the trivial character, then $L(f, \chi, s)$ is nothing other than $L(E_f/K, s)$, the $L$-function of the elliptic curve $E_f$ base changed to $K$. Moreover, we can think of $c_{f,\chi} = c_{f,1}$ in this case as a point in $E_f(K)$. Recall that our assumptions have forced $L(E_f/K, 1) = 0$. So the Gross-Zagier formula implies that if $L'(E_f/K, 1) \neq 0$ (i.e. if the analytic rank of $E_f$ is 1), then the rank of the group $E_f(K)$ is at least 1. This inequality is exactly as predicted by the Birch and Swinnerton-Dyer conjecture (BSD), which is the statement that the algebraic and analytic ranks agree:

$$\operatorname{rk} E_f(K) = \operatorname{ord}_{s=1} L(E_f/K, s).$$

In fact, Kolyvagin [Kol] proved that if $c_{f,1}$ is not torsion, then $\operatorname{rk} E_f(K) = 1$, and so the BSD conjecture for $E_f/K$ is verified in this case. Moreover, one can "descend" these results to prove BSD for $E_f/\mathbb{Q}$ as well (assuming the analytic rank is less than or equal to 1) . Since every elliptic curve $E/\mathbb{Q}$ is a quotient of $J_0(N)$ for some $N$, these arguments apply for all $E/\mathbb{Q}$.

### 1.1.2 The $p$-adic formula of Perrin-Riou

There are many variants and generalizations of the Gross-Zagier formula. One of the earliest variants was a $p$-adic version due to Perrin-Riou [PR1], in the case where $f$ is ordinary at $p$ (with respect to some chosen embedding $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$). Here, $p$ is a prime not dividing $N$ and which splits in $K$. She computes the derivative of a $p$-adic $L$-function $L_p(f, \chi, \lambda)$ instead of the usual complex Rankin-Selberg $L$-function. $L_p(f, \chi, -)$ is a $\mathbb{C}_p$-valued $p$-adic analytic function of characters $\lambda : \mathrm{Gal}(K_\infty/K) \to 1 + p\mathbb{Z}_p$, where $K_\infty$ is the unique $\mathbb{Z}_p^2$-extension of $K$. This $p$-adic $L$-function is characterized by an interpolation property of the form

$$L_p(f, \chi, \psi) \doteq L(f, \chi\psi, 1),$$

for all finite order characters $\psi$. Here, $\doteq$ means equality up to explicit (transcendental) constants, which must be divided out appropriately so that both sides of the equation are algebraic and the equality of elements of $\mathbb{C}_p$ and $\mathbb{C}$ can make sense.

Replacing the $\mathbb{C}$-valued Néron-Tate height pairing in her $p$-adic formula is a height pairing

$$\langle\, , \rangle_{\ell_K} : J_0(N)(H) \otimes \bar{\mathbb{Q}}_p \times J_0(N)(H) \otimes \bar{\mathbb{Q}}_p \to \bar{\mathbb{Q}}_p,$$

defined by Schneider and Mazur-Tate. This $p$-adic height pairing depends on a choice of "arithmetic logarithm"

$$\ell_K : \mathbb{A}_K^\times / K^\times \to \mathbb{Q}_p,$$

which we can alternatively view (via class field theory) as a homomorphism $\ell_K : \mathrm{Gal}(K_\infty/K) \to \mathbb{Q}_p$. In fact, we can write $\ell_K = p^{-n} \log_p \circ \lambda$ for some integer $n$ and some $\lambda : \mathrm{Gal}(K_\infty/K) \to 1 + p\mathbb{Z}_p$. Here $\log_p$ is Iwasawa's branch of the logarithm, so that $\log_p(p) = 0$. Then the derivative of $L_p(f, \chi, -)$ at the trivial character $\mathbb{1}$ in the direction of $\ell_K$ is defined as

$$L_p'(f, \chi, \ell_K, \mathbb{1}) = p^{-n} \frac{d}{ds} L_p(f, \chi, \lambda^s)\bigg|_{s=0}.$$

**Theorem I.2** (Perrin-Riou)**.**

$$L_p'(f, \chi, \ell_K, \mathbb{1}) \doteq \langle c_{f,\chi}, c_{f,\chi} \rangle_{\ell_K}$$

Kobayashi [Kob] later proved a similar $p$-adic formula when $f$ is non-ordinary at $p$. This case is more complicated because there are two different $p$-adic $L$-functions attached to $f$ and the height pairings now depend on a choice of splittings of the Hodge filtration (whereas there is a canonical choice in the ordinary case).

Perrin-Riou's formula implies cases of a $p$-adic version of the BSD conjecture, which states that (when $f$ has rational coefficients) the rank of $E_f(K)$ should equal the derivative $L'(f, 1, \ell_K, \mathbb{1})$ in the cyclotomic direction (i.e. $\ell_K = \log_p \circ \lambda$, with $\lambda$ the cyclotomic character). It is important here that $(p, N) = 1$, otherwise these two quantities are not necessarily equal, due to exceptional zero phenomena. Also note that $L_p(f, \chi, \mathbb{1}) = 0$ by the interpolation property.

Such $p$-adic formulas are interesting because they give new ways to prove statements about points on elliptic curves over $\mathbb{Q}$, or, more generally, algebraic cycles on varieties defined over number fields. Moreover, they have a certain flexibility that the archimedean formulas lack, in that they are amenable to methods of Iwasawa theory and techniques of $p$-adic variation. In fact, $p$-adic special value formulas are an important tool in recent proofs of "converse theorems" (e.g. [Sk] and [Zh]) concerning the usual (archimedean) BSD conjecture.

### 1.1.3 Higher weight formulas and Heegner cycles

In the 1990's, the formulas of Gross-Zagier and Perrin-Riou were generalized to eigenforms $f$ of weight $2r$, for any $r \geqslant 1$. In this case, the Rankin-Selberg $L$-function $L(f, \chi, s)$ again vanishes at its central point $s = r$. Already in [GZ, §V], it is attributed to Deligne that the derivative $L'(f, \chi, r)$ should be related to heights of *Heegner cycles*, which are certain algebraic cycles lying on the Kuga-Sato variety $W_{2r-2}$ of dimension $2r - 1$. This Kuga-Sato variety is a smooth compactification of the $(2r-2)$-th power

$$W^0_{2r-2} = \mathcal{E} \times_{Y(N)} \cdots \times_{Y(N)} \mathcal{E}$$

of the universal elliptic curve $\mathcal{E} \to Y(N)$ fibered over $Y(N)$ (the moduli space of elliptic curves with full level $N$ structure). By work of Deligne and Scholl, the Hecke operators can be used to construct a projector $\epsilon_f$ in the ring of algebraic correspondences of $W_{2r-2}$, which cuts out a motive $M_f$ (modulo homological equivalence) corresponding to the eigenform $f$. In particular, there is a subspace of $H^{2r-1}_{\text{ét}}(\bar{W}_{2r-2}, \mathbb{Q}_p(r))$ whose $L$-function equals $L(f, s)$, though it is in general necessary to extend the coefficient field in order to realize this subspace.

The Heegner cycle is a certain algebraic cycle lying in the fiber of

$$W_{2r-2} \to X(N)$$

above a point $\tilde{y} \in X(N)$ corresponding to an elliptic curve $A$ with $\mathrm{End}(A) \cong \mathcal{O}_K$; the fiber above $\tilde{y}$ is isomorphic to $A^{2r-2}$. Recall $D = \mathrm{Disc}(K)$, and let

$$\Gamma_{\sqrt{D}} = \{(P, \sqrt{D}(P)) : P \in A\} \subset A \times A$$

be the graph of the isogeny $\sqrt{D} : A \to A$. Then consider the cycle:

$$Y = \Gamma^{r-1}_{\sqrt{D}} \subset (A \times A)^{r-1} \subset W_{2r-2}.$$

Roughly speaking, the Heegner cycle $Y_{f,\chi}$ is the projection of $Y$ onto the $\chi$-isotypic part of $M_f/H$. The cohomology class of $Y_{f,\chi}$ in $H^{2r}_{\text{ét}}(\bar{W}_{2r-2}, \mathbb{Q}_p(r)) \otimes \bar{\mathbb{Q}}_p$ is trivial and hence $Y_{f,\chi}$ lies in the domain of the $p$-adic Abel-Jacobi map

$$\Phi : \mathrm{CH}^r(W_{2r-2})_0 \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p \to H^1(H, V) \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p,$$

where $V = H^{2r-1}_{\text{ét}}(\bar{W}_{2r-2}, \mathbb{Q}_p(r))$. The special value formulas for higher weight $f$ are due to Zhang [Z] (for the complex $L$-function) and Nekovář [N3] (for the $p$-adic $L$-function):

**Theorem I.3** (Zhang). *If $f$ has weight $2r$ and $\chi$ is a character of* $\mathrm{Gal}(H/K)$*, then*

$$L'(f, \chi, r) \doteq \langle Y_{f,\chi}, Y_{f,\chi} \rangle_{\mathrm{GS}}.$$

**Theorem I.4** (Nekovář)**.** *Let $p$ be a prime split in $K$ and not dividing $N$, and fix an embedding $\iota : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$. Suppose $f$ has weight $2r$ and is ordinary at $p$ with respect to $\iota$. For any character $\chi$ of* $\mathrm{Gal}(H/K)$ *and any choice of arithmetic logarithm $\ell_K$,*

$$L_p'(f, \chi, \ell_K, \mathbb{1}) \doteq \langle \Phi(Y_{f,\chi}), \Phi(Y_{f,\bar{\chi}}) \rangle_{\mathrm{Nek},\ell_K}$$

The $\mathbb{C}$-valued height pairing $\langle\, , \rangle_{\mathrm{GS}}$ is the one defined by Beilinson [Bei] and uses the arithmetic intersection theory of Gillet and Soulé [GS]. This pairing is a generalization of the Néron-Tate height pairing for polarized abelian varieties. Importantly, $\langle\, , \rangle_{\mathrm{GS}}$ is defined on the Chow group $\mathrm{CH}^r(W_{2r-2})_0$ of homologically trivial cycles.[1] This is in contrast to the $\bar{\mathbb{Q}}_p$-valued height pairing $\langle\, , \rangle_{\mathrm{Nek}}$ constructed by Nekovář (and generalizing the height pairings of Mazur-Tate and Schneider), which is defined on the Bloch-Kato subgroup $H^1_f(H, V) \subset H^1(H, V)$. It is known that the image of $\Phi$ lies in $H^1_f(H, V)$, in the case of Kuga-Sato varieties. This difference between the archimedean and $p$-adic heights makes it more difficult both to prove archimedean height formulas (as we will explain later) and also to apply them towards general conjectures on algebraic cycles, as we explain in the next section.

*Remark* I.5. There is important work of Yuan, Zhang, and Zhang [YZZ], which vastly generalizes the original Gross-Zagier formula in an orthogonal direction, namely by relaxing the Heegner hypothesis and other ramification conditions. In this general case, one relates $L'(f, \chi, s)$ (with $f$ having weight 2) to heights of special points on Shimura curves.

### 1.1.4 Conjectures of Beilinson-Bloch, Bloch-Kato, and Perrin-Riou

Assume for simplicity that $\chi$ is the trivial character $\mathbf{1}$ and $f$ has rational coefficients, and write $\epsilon_f$ for the algebraic correspondence on $W_{2r-2}$ which cuts out the motive $M_f$. The Beilinson-Bloch (BB) conjecture [Bei] is a vast generalization of the BSD conjecture (whose scope is limited to abelian varieties $A$ over number fields). The BB conjecture relates the rank of the Chow group of homologically trivial algebraic cycles on a smooth projective variety $X$ over a number field (or more generally, a Chow motive) to the order of vanishing of the $L$-functions attached to the étale cohomology of $X$. For the motive $M_f/K$, it predicts that

$$\dim_{\mathbb{Q}} \epsilon_f \mathrm{CH}^r(W_{2r-2}/K)_0 = \mathrm{ord}_{s=r} L(f, \chi, s).$$

Zhang's formula verifies one inequality in the BB conjecture when $M_f/K$ has analytic rank 1: if the order of vanishing equals 1, then the derivative is non-zero and so the height of the Heeger cycle is non-zero as well. Hence the cycle is non-torsion and the dimension on the left hand side is at least 1. One would like to use Kolyvagin's Euler system methods to show that the dimension is in fact equal to 1, just as in the weight two case. In fact, Nekovář [N1] was able to apply Kolyvagin's techniques in this case, but his result is that if $\Phi(Y_{f,1}) \neq 0$, then $\dim H^1_f(K, \epsilon_f V) = 1$. Unfortunately, the Abel-Jacobi map $\Phi$ is not known to be injective, so one cannot use Nekovář's Euler system result to prove BB in this case.

A related conjecture of Bloch-Kato [BK] predicts that the Abel-Jacobi map induces an isomorphism

$$\tilde{\Phi} : \epsilon_f \mathrm{CH}^r(W_{2r-2}/K)_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} H^1_f(K, \epsilon_f V),$$

and moreover that (in agreement with the BB conjecture):

$$\dim_{\mathbb{Q}_p} H^1_f(K, \epsilon_f V) = \mathrm{ord}_{s=r} L(f, \chi, s). \tag{1.1}$$

Again, since the injectivity of $\Phi$ is not known, Zhang's result cannot be used to unconditionally prove (1.1) when $L'(f, \chi, r) \neq 0$.

This is unfortunate, but we can take solace in the fact that these problems disappear in the $p$-adic realm. Perrin-Riou [PR3] has formulated a $p$-adic version of the Bloch-Kato conjecture (see also [Co, 2.7]), and in this case of good reduction the prediction is entirely similar:

$$\dim_{\mathbb{Q}_p} H^1_f(K, \epsilon_f V) = \mathrm{ord}_{\lambda=\mathbb{1}} L_p(f, \chi, \lambda),$$

[1] In fact it is defined only on a subgroup of $\mathrm{CH}^r(W_{2r-2})_0$ which is conjecturally equal to all of $\mathrm{CH}^r(W_{2r-2})_0$.

where the derivatives are taken in the cyclotomic direction, as before. Combining Theorem I.4 with Nekovář's results in [N1] immediately yields a proof of this conjecture when $L_p'(f, \mathbf{1}, \mathbb{1}) \neq 0$.

*Remark* I.6. There is another application of the original Gross-Zagier formula which fails to generalize to the higher weight case (in the current state of affairs). Namely, if the Heegner point $c_{f,\mathbf{1}}$ is non-torsion, then by the non-degeneracy of the Néron-Tate height pairing, we have $L'(f, \mathbf{1}, 1) \neq 0$. Using Kolyvagin once more, we conclude that BSD is true for $E_f$. This argument does not work in higher weight because the pairings $\langle\, ,\rangle_{\mathrm{GS}}$ and $\langle\, ,\rangle_{\mathrm{Nek}}$ are not known to be non-degenerate. In fact, the non-degeneracy of $p$-adic heights is not known even in weight 2, i.e. for abelian varieties, other than in the CM case [Be].

## 1.2 Main results

The goal of this thesis is to extend the results of [N3] and [Z] to a larger class of Rankin-Selberg $L$-functions, i.e. to a larger class of motives. Specifically, we will consider motives of the form $f \otimes \Theta_\chi$, where

$$\chi : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$$

is an unramified Hecke character of infinity type $(\ell, 0)$, with $0 < \ell = 2k < 2r$, and

$$\Theta_\chi = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \chi(\mathfrak{a}) q^{\mathbf{N}\mathfrak{a}}$$

is the associated theta series. The conditions on $\ell$ guarantee that the Hecke character $\chi_0 := \chi^{-1}\mathbf{N}^{r+k}$ of infinity type $(r-k, r+k)$ is central critical in the sense of [BDP1, §4], and that the central value $L(f, \chi, r+k) = L(f, \Theta_\chi, r+k)$ of the Rankin-Selberg $L$-function vanishes, as before. If we take $\ell = 0$, then $\chi$ comes from a character of $\mathrm{Gal}(H/K)$, so we are back in the situation considered in [N3] and [Z].

Our main result (Theorem I.7) extends Nekovář's formula to the case $\ell > 0$ by relating $p$-adic heights of *generalized* Heegner cycles to the derivative of a $p$-adic $L$-function attached to the pair $(f, \chi)$. We establish our assumptions and notation now and in the next subsections describe both the algebraic cycles and the $p$-adic $L$-function needed to state our $p$-adic formula.

For our $p$-adic formula, we let $p$ be an odd prime, $N \geqslant 3$ a positive integer prime to $p$, and $f = \sum a_n q^n$ a newform of weight $2r > 2$ on $X_0(N)$ with $a_1 = 1$. Fix embeddings $\bar{\mathbb{Q}} \to \mathbb{C}$ and $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$ once and for all, and suppose that $f$ is ordinary at $p$, i.e. the coefficient $a_p \in \bar{\mathbb{Q}}_p$ is a $p$-adic unit. We let $K$ be an imaginary quadratic field of odd discriminant $D$ such that each prime dividing $pN$ splits in $K$. As before, $H$ is the Hilbert class field of $K$.

### 1.2.1 Generalized Heegner cycles

Let $Y(N)/\mathbb{Q}$ be the modular curve parametrizing elliptic curves with full level $N$ structure, and let $\mathcal{E} \to Y(N)$ be the universal elliptic curve with level $N$ structure. Denote by $W = W_{2r-2}$, the canonical non-singular compactification of the $(2r-2)$-fold fiber product of $\mathcal{E}$ with itself over $Y(N)$ [Sc]. Finally, let $A/H$ be an elliptic curve with complex multiplication by the full ring of integers $\mathcal{O}_K$ and good reduction at primes above $p$. We assume further that $A$ is isogenous (over $H$) to each of its $\mathrm{Gal}(H/K)$-conjugates $A^\sigma$ and that $A^\tau \cong A$, where $\tau$ is complex conjugation. Such an $A$ exists since $K$ has odd discriminant [G, §11]. Set $X = W_H \times_H A^\ell$, where $W_H$ is the base change to $H$. The variety $X$ is fibered over the compactified modular curve $X(N)_H$, the typical geometric fiber being of the form $E^{2r-2} \times A^\ell$, for some elliptic curve $E$.

The $(2r+2k-1)$-dimensional variety $X$ contains a rich supply of *generalized* Heegner cycles supported in the fibers of $X$ above Heegner points on $X_0(N)$ (we view $X$ as fibered over $X_0(N)$ via $X(N) \to X_0(N)$). These cycles were first introduced by Bertolini, Darmon, and Prasanna in [BDP1]. In Chapter IV, we define certain cycles $\epsilon_B \epsilon Y$ and $\epsilon_B \bar{\epsilon} Y$ in $\mathrm{CH}^{r+k}(X)_K$ which sit in the fiber above a Heegner point on $X_0(N)(H)$, and which are variants of the generalized Heegner cycles which appear in [BDP2]. Here, $\mathrm{CH}^{r+k}(X)_K$ is the group of codimension $r+k$ cycles on $X$ with coefficients in $K$ modulo rational equivalence. In fact, for each ideal $\mathfrak{a}$ of $K$, we define cycles $\epsilon_B\epsilon Y^{\mathfrak{a}}$ and $\epsilon_B\bar{\epsilon}Y^{\mathfrak{a}}$ in $\mathrm{CH}^{r+k}(X)_K$, each one sitting in the fiber above a Heegner point. These cycles are replacements for the notion of $\mathrm{Gal}(H/K)$-conjugates of $\epsilon_B\epsilon Y$ and $\epsilon_B\bar{\epsilon}Y$ (recall that $\mathrm{Gal}(H/K) \cong \mathrm{Pic}(\mathcal{O}_K)$). The latter do not exist as cycles on $X$, as $X$ is not (generally) defined over $K$. In particular, we have $\epsilon_B\epsilon Y^{\mathcal{O}_K} = \epsilon_B\epsilon Y$.

The cycles $\epsilon_B\epsilon Y^{\mathfrak{a}}$ and $\epsilon_B\bar{\epsilon}Y^{\mathfrak{a}}$ are homologically trivial on $X$ (Corollary IV.5), so they lie in the domain of the $p$-adic Abel-Jacobi map

$$\Phi : \mathrm{CH}^{r+k}(X)_{0,K} \to H^1(H, V),$$

where $V$ is the $\mathrm{Gal}(\bar{H}/H)$-representation $H^{2r+2k-1}_{\text{ét}}(\bar{X}, \mathbb{Q}_p)(r+k)$. (See Chapter 5.2 for a definition of $\Phi$.) We will focus on a particular 4-dimensional $p$-adic representation $V_{f,A,\ell}$, which is a quotient of $V$. $V_{f,A,\ell}$ is an $F$-vector space, where $F/\mathbb{Q}_p$ is the field obtained by adjoining to $\mathbb{Q}_p$ the coefficents of $f$ and the coefficients of the Hecke character attached to $A$. As a Galois representation, $V_{f,A,\ell}$ is ordinary (Theorem VII.2) and is closely related to the $p$-adic realization of the motive $f \otimes \Theta_\chi$ (see Chapter IV). After projecting, one obtains a map

$$\Phi_f : \mathrm{CH}^{r+k}(X)_{0,K} \to H^1(H, V_{f,A,\ell}),$$

which we again call the Abel-Jacobi map. For any ideal $\mathfrak{a}$ of $K$, define $z_f^{\mathfrak{a}} = \Phi_f(\epsilon_B\epsilon Y^{\mathfrak{a}})$ and $\bar{z}_f^{\mathfrak{a}} = \Phi_f(\epsilon_B\bar{\epsilon}Y^{\mathfrak{a}})$. As before we write $z_f = z_f^{\mathcal{O}_K}$ and $\bar{z}_f = \bar{z}_f^{\mathcal{O}_K}$.

The image of $\Phi_f$ lies in the Bloch-Kato subgroup $H^1_f(H, V_{f,A,\ell}) \subset H^1(H, V_{f,A,\ell})$ (Theorem IV.6). If we fix a continuous homomorphism $\ell_K : \mathbb{A}_K^\times/K^\times \to \mathbb{Q}_p$, then [N2] provides a symmetric $F$-linear height pairing

$$\langle\ ,\ \rangle_{\ell_K} : H^1_f(H, V_{f,A,\ell}) \times H^1_f(H, V_{f,A,\ell}) \to F.$$

We can extend this height pairing $\bar{\mathbb{Q}}_p$-linearly to $H^1_f(H, V_{f,A,\ell}) \otimes \bar{\mathbb{Q}}_p$. The cohomology classes

$$z^{\mathcal{A}}_{f,\chi} := \chi(\mathfrak{a})^{-1} z_f^{\mathfrak{a}} \quad \text{and} \quad z^{\mathcal{A}}_{f,\bar{\chi}} := \bar{\chi}(\mathfrak{a})^{-1} \bar{z}_f^{\mathfrak{a}}$$

depend only on the class $\mathcal{A}$ of $\mathfrak{a}$ in the class group $\mathrm{Pic}(\mathcal{O}_K)$ (Lemma IV.10). Finally, set $h = \#\mathrm{Pic}(\mathcal{O}_K)$ and

$$z_{f,\chi} = \frac{1}{h} \sum_{\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)} z^{\mathcal{A}}_{f,\chi} \quad \text{and} \quad z_{f,\bar{\chi}} = \frac{1}{h} \sum_{\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)} z^{\mathcal{A}}_{f,\bar{\chi}},$$

both being elements of $H^1_f(H, V_{f,A,\ell}) \otimes \bar{\mathbb{Q}}_p$. Our main theorem relates $\langle z_{f,\chi}, z_{f,\bar{\chi}} \rangle_{\ell_K}$ to the derivative of a $p$-adic $L$-function which we now describe.

### 1.2.2 The $p$-adic $L$-function

Following [PR1], [N3], and the general construction of Hida, we construct a $p$-adic $L$-function attached to $f \otimes \Theta_\chi$. Recall, if $f = \sum a_n q^n \in M_j(\Gamma_0(M), \psi)$ and $g = \sum b_n q^n \in M_{j'}(\Gamma_0(M), \xi)$, then the Rankin-Selberg convolution is

$$L(f, g, s) = L_M(2s + 2 - j - j', \psi\xi) \sum_{n \geqslant 1} a_n b_n n^{-s},$$

where

$$L_M(s, \psi\xi) = \prod_{p \nmid M} \left(1 - (\psi\xi)(p) p^{-s}\right)^{-1}.$$

Let $K_\infty/K$ be the $\mathbb{Z}_p^2$-extension of $K$ and let $K_p$ be the maximal abelian extension of $K$ unramified away from $p$. In Chapter 2, we define a $p$-adic $L$-function $L_p(f \otimes \chi)(\lambda)$, which is a $\bar{\mathbb{Q}}_p$-valued function of continuous characters

$$\lambda : \mathrm{Gal}(K_\infty/K) \to 1 + p\mathbb{Z}_p.$$

The Iwasawa function $L_p(f \otimes \chi)$ is the restriction of an analytic function on

$$\mathrm{Hom}(\mathrm{Gal}(K_p/K), \mathbb{C}_p^\times),$$

which is characterized by the following interpolation property: if

$$\mathcal{W} : \mathrm{Gal}(K_p/K) \to \mathbb{C}_p^\times$$

is a finite order character of conductor $\mathfrak{f}$, with $\mathbf{N}\mathfrak{f} = p^\beta$, then

$$L_p(f \otimes \chi)(\mathcal{W}) = C_{f,k}\mathcal{W}(N)\overline{\chi\mathcal{W}}(\mathcal{D})\tau(\chi\mathcal{W})V_p(f,\chi,\mathcal{W})L(f,\Theta_{\overline{\chi\mathcal{W}}}, r+k)$$

with

$$C_{f,k} = \frac{2(r-k-1)!(r+k-1)!}{(4\pi)^{2r}\alpha_p(f)^\beta \langle f,f\rangle_N},$$

and where $\alpha_p(f)$ is the unit root of $x^2 - a_p(f)x + p^{2r-1}$, $\langle f,f\rangle_N$ is the Petersson inner product, $\mathcal{D} = \left(\sqrt{D}\right)$ is the different of $K$, $\Theta_{\overline{\chi\mathcal{W}}}$ is the theta series

$$\Theta_{\overline{\chi\mathcal{W}}} = \sum_{(\mathfrak{a},\mathfrak{f})=1} \overline{\chi\mathcal{W}}(\mathfrak{a})q^{\mathbf{N}\mathfrak{a}},$$

$\tau(\chi\mathcal{W})$ is the root number for $L(\Theta_{\chi\mathcal{W}}, s)$, and

$$V_p(f,\chi,\mathcal{W}) = \prod_{\mathfrak{p}|p}\left(1 - \frac{(\bar{\chi}\bar{\mathcal{W}})(\mathfrak{p})}{\alpha_p(f)}\mathbf{N}(\mathfrak{p})^{r-k-1}\right)\left(1 - \frac{(\chi\mathcal{W})(\mathfrak{p})}{\alpha_p(f)}\mathbf{N}(\mathfrak{p})^{r-k-1}\right).$$

Recall we have fixed a continuous homomorphism $\ell_K : \mathbb{A}_K^\times/K^\times \to \mathbb{Q}_p$. We define $L_p'(f\otimes\chi, \ell_K, \mathbb{1})$ as in 1.1.2. With these definitions, we can finally state our main result.

**Theorem I.7.** *If $\chi$ is an unramified Hecke character of $K$ of infinity type $(2k,0)$ with $0 < 2k < 2r$, then*

$$L_p'(f\otimes\chi, \ell_K, \mathbb{1}) = (-1)^k \prod_{\mathfrak{p}|p}\left(1 - \frac{\chi(\mathfrak{p})p^{r-k-1}}{\alpha_p(f)}\right)^2 \frac{h\left\langle z_{f,\chi}, z_{f,\bar{\chi}}\right\rangle_{\ell_K}}{u^2(4|D|)^{r-k-1}},$$

*where $h = h_K$ is the class number and $u = \frac{1}{2}\#\mathcal{O}_K^\times$.*

*Remark* I.8. When $k = 0$ the cycles and the $p$-adic $L$-function simplify to those constructed in [N3], and the main theorem becomes Nekovář's formula, at least up to a somewhat controversial sign. It appears that a sign was forgotten in [N3, II.6.2.3], causing the discrepancy with our formula and with Perrin-Riou's as well. Perrin-Riou's formula [PR1] covers the case $k = 0$ and $r = 1$.

In the last two chapters of this thesis, we also compute archimedean heights of generalized Heegner cycles and sketch a proof of a special value formula for the derivative $L'(f,\chi,r+k)$ of the complex $L$-function. This formula is analogous to the formula in Theorem I.7 and generalizes Theorem I.3 to Hecke characters of higher weight. We defer to a separate paper the technical aspects of the proof, and instead focus on the heart of the computation, which is the computation of the local heights at the infinite places. One of course needs to compute local heights at finite places as well, but these computations are essentially identical to our local $p$-adic height computations (at places away from $p$). We refer the reader to Chapter X for a description of our archimedean results.

### 1.2.3 Applications

Theorem I.7 implies special cases of Perrin-Riou's $p$-adic Bloch-Kato conjecture. The assumption that $A$ is isogenous to all its $\mathrm{Gal}(H/K)$-conjugates implies that the Hecke character

$$\psi_H : \mathbb{A}_H^\times \to \mathbb{C}^\times,$$

which is attached to $A$ by the theory of complex multiplication, factors as $\psi_H = \psi \circ \mathrm{Nm}_{H/K}$, where $\psi$ is a $(1,0)$-Hecke character of $K$. Assume for simplicity that $\chi = \psi^\ell$, and set $\chi_H = \psi_H^\ell$ and $G_H := \mathrm{Gal}(\bar{H}/H)$. Then the $G_H$-representation $V_{f,A,\ell}$ is the $p$-adic realization of a Chow motive $M(f)_H \otimes M(\chi_H)$. Here, $M(f)$ is the motive over $\mathbb{Q}$ attached to $f$ by Deligne, and $M(\chi_H)$ is a motive over $H$ (with coefficients in $K$) cutting out a two dimensional piece of the middle degree cohomology of $A^\ell$. In fact, the motive $M(\chi_H)$ descends to a motive $M(\chi)$ over $K$ with coefficients in $\mathbb{Q}(\chi)$. We write $V_{f,\chi}$ for the $p$-adic realization of $M(f)_K \otimes M(\chi)$, so that $V_{f,\chi}$ is a $G_K$-representation whose restriction to $G_H$ is isomorphic to $V_{f,A,\ell}$. In fact, $V_{f,\chi} \cong \chi \oplus \bar{\chi}$, where we now think of $\chi$ as a $\mathbb{Q}(\chi) \otimes \mathbb{Q}_p$-valued character of $G_K$. It follows that

$$L(V_{f,\chi}, s) = L(f, \chi, s)L(f, \bar{\chi}, s) = L(f, \chi, s)^2.$$

The Bloch-Kato conjecture for the motive $M(f)_K \otimes M(\chi)$ over $K$ reads

$$\dim H^1_f(K, V_{f,\chi}) = 2 \cdot \mathrm{ord}_{s=r+k} L(f, \chi, s).$$

Similarly, Perrin-Riou's $p$-adic conjecture [Co, 2.7] [PR3, 4.2.2] reads

$$\dim H^1_f(K, V_{f,\chi}) = 2 \cdot \mathrm{ord}_{\lambda=\mathbb{1}} L(f, \chi, \ell_K, \lambda), \tag{1.2}$$

where $\ell_K$ is the cyclotomic logarithm and the derivatives are taken in the cyclotomic direction. In Chapter VIII, we deduce the "analytic rank 1" case of Perrin-Riou's conjecture by combining our main formula with the results of Elias [E] on Euler systems for generalized Heegner cycles:

**Theorem I.9.** *If $L'_p(f \otimes \chi, \ell_K, \mathbb{1}) \neq 0$, then* (1.2) *is true, i.e. Perrin-Riou's p-adic Bloch-Kato conjecture holds for the motive $M(f)_K \otimes M(\chi)$.*

*Remark* I.10. Alternatively, we can think of $z_{f,\chi}$ (resp. $z_{f,\bar{\chi}}$) as giving a class in $H^1_f(K, V_f \otimes \chi)$ (resp. $H^1_f(K, V_f \otimes \bar{\chi})$), and note that $L(V_f \otimes \chi, s) = L(f, \chi, s) = L(V_f \otimes \bar{\chi}, s)$. The Bloch-Kato conjecture for the motive $f \otimes \chi$ over $K$ then reads

$$\dim H^1_f(K, V_f \otimes \chi) = \mathrm{ord}_{s=r+k} L(f, \chi, s),$$

and similarly for $\bar{\chi}$ and the $p$-adic $L$-functions.

We anticipate that Theorem I.7 can also be used to study the variation of generalized Heegner cycles in $p$-adic families, in the spirit of [Ca] and [Ho]. Theorem I.7 allows for variation in not just the weight of the modular form $f$, but in the weight of the Hecke character $\chi$ as well.

### 1.2.4 Related work

There has been much recent work on the connections between generalized Heegner cycles and $p$-adic $L$-functions. Generalized Heegner cycles were first studied in [BDP1], where their Abel-Jacobi classes were related to the special *value* (not the derivative) of a different Rankin-Selberg $p$-adic $L$-function. Brooks [Br] extended these results to Shimura curves over $\mathbb{Q}$ and recently Liu, Zhang, and Zhang proved a general formula for arbitrary totally real fields [LZZ]. In [D], Disegni computes $p$-adic heights of Heegner points on Shimura curves, generalizing the weight 2 formula of Perrin-Riou for modular curves. Kobayashi [Kob] extended Perrin-Riou's height formula to the supersingular case. Our work is the first (as far as we know) to study $p$-adic heights of generalized Heegner cycles.

### 1.2.5 Assumptions

We review all our assumptions and comment on the extent to which they may be relaxed.

- We have assumed $N \geqslant 3$ for the sake of exposition. For $N < 3$, the proof should be modified to account for the lack of a fine moduli space and extra automorphisms in the local intersection theory. These details are spelled out in [N3] and pose no new problems.

- We have assumed $D$ is odd, as is traditional in this area. If $D$ is even, various computations become more complicated, but are presumably not fundamentally more difficult.
- We have assumed $\chi$ is unramified. It should be straightforward to allow for unramified Hecke characters twisted by finite order ring class field characters. One does not necessarily expect a special value formula for more general $\chi$, i.e. if $\chi|_{\mathbb{A}_\mathbb{Q}^\times}$ is not a power of the norm, since then $\chi$ is not central critical with respect to $f$.
- One should be able to prove similar kinds of formulas when $f$ has odd weight, or more generally if the nebentypus of $f$ is non-trivial. But there will be restrictions on both the infinity type of $\chi$ and the Dirichlet character attached to $\chi$, again coming from the condition of central criticality. For example, if $f$ has odd weight, then $\chi$ will necessarily be an infinite order Hecke character.
- It would be worthwhile to relax the condition $(N, D) = 1$, as one could then consider $f$ with CM by $K$. The issue is that the $p$-adic $L$-function computations in this case become rather messy (this case was also avoided in [PR1] and [N3]).
- It would also be worthwhile to combine the methods here with the work of Disegni [D], i.e. to remove the Heegner hypothesis assumed in this paper. His adelic construction of the $p$-adic $L$-function is more amenable to generalization than our more classical approach, which is one reason why we have not pursued some of the strengthenings alluded to above.
- The assumption that $p$ splits in $K$ is important for the proof, but presumably can be removed *a fortiori*. This would follow from an argument similar to [Kob, proof of Theorem 5.9], but requires the archimedean height formula for generalized Heegner cycles.
- One might try to remove the assumption that $f$ is ordinary, using Kobayashi's approach [Kob] in the weight 2 case as a guide. However in higher weight there are some non-trivial technical issues to deal with in the computation of the local $p$-adic heights at places above $p$.
- Our assumption that $A^\tau \cong A$ implies that the lattice corresponding to $A$ is 2-torsion in the class group. This is convenient for proving the vanishing of the $p$-adic height in the anticyclotomic direction, and plays no other role in our proof (in particular, the proof in the interesting case where $\ell_K$ is cyclotomic does not use this assumption). One should be able to prove the theorem without this assumption by making use of the functoriality of the height pairing to relate heights on $X$ to heights on $X^\tau$, but we omit the details. Ultimately, the choice of auxiliary elliptic curve does not matter much and we should just choose $A$ as in Remark IV.1.

### 1.2.6 Sketch of proof

This rough sketch will assume $\ell_K$ is the cyclotomic character, because we show in IV.13 that both sides of Theorem I.7 vanish when $\ell_K$ is an anticyclotomic logarithm. We therefore drop $\ell_K$ from the notation. Again to ease notation, we assume $\mathrm{Pic}(\mathcal{O}_K)$ is trivial.

Following Hida, Perrin-Riou, and Nekovář, we construct a $p$-adic $L$-function $L_p(f, \chi, \lambda)$ roughly of the form

$$L_p(f, \chi, \lambda) = L_f\left(\int_{\mathrm{Gal}(K_\infty/K)} \lambda d\Psi\right).$$

Here $d\Psi$ is a $p$-adic measure, constructed from Eisenstein and theta measures (mimicking the Rankin-Selberg convolution), and valued in $p$-adic modular forms. The operator $L_f$ is the composition of Hida's ordinary projector $\lim_{j\to\infty} U_p^{c(j)p^j}$ with a $p$-adic analogue of taking the Peterson inner product with $f$. It follows that $L_p'(f, \chi, \mathbb{1}) = L_f(G)$ for some $p$-adic modular form $G$.

We then want to compare the two $p$-adic modular forms

$$G \quad \text{and} \quad F = \sum_g \langle \epsilon Y_g, \bar{\epsilon} Y_g \rangle_{\mathrm{Nek}} \cdot g$$

where the sum is over newforms of level dividing $N$. In fact, to prove the theorem we need to show that $L_f(G) = L_f(F)$. For $m \geqslant 1$ and prime to $N$, the $m$th coefficient of $F$ is

$$a_m(F) = \langle x, T_m \bar{x} \rangle_{\mathrm{Nek}},$$

where $x$ is the projection of $\Phi(\epsilon Y)$ onto $H^1_f(H, \oplus_g V_{g,A,\ell})$, i.e. we project onto the space of all modular forms, not just our chosen eigenform $f$ (and similarly for $\bar{x}$).

By an argument of Nekovář using Cebotarev's density theorem, it is enough to compare $m$th Fourier coefficients only for those integers $m$ such that there is no ideal in $\mathcal{O}_K$ of norm $m$ (this is the condition $r_{\mathcal{A}}(m) = 0$ in [GZ]). This condition amounts to saying that the cycles $Y$ and $T_mY$ do not intersect in the generic fiber. We can therefore decompose the global $p$-adic height into a sum of local heights, one for each *finite* place of $H$:

$$\begin{aligned} a_m(F) &= c_m(F) \qquad\qquad + \quad d_m(F) \\ &= \sum_{v \nmid p} \langle x, T_m \bar{x} \rangle_v \quad + \quad \sum_{v \mid p} \langle x, T_m \bar{x} \rangle_v \end{aligned}$$

Note that $p$-adic heights do not have a contribution from infinite places of $H$. Morally speaking, the local $p$-adic heights at places above $p$ replace the local archimedean heights at infinite places (both are very hard to compute in general).

After much computation (representing a large part of this thesis, and building off the work in [GZ], [PR1], and [N3]), we show that $a_m(G)$ and $c_m(F)$ are "essentially equal" (i.e. they are equal for the purposes of this sketch). Thus it suffices to show that

$$L_f\left(\sum d_m(F) q^m\right) = 0. \tag{1.3}$$

This part of the argument relies heavily on the fact that $p$ splits in $K$. The point is that the local *archimedean* heights at such primes $p$ are easily seen to vanish (c.f. [GZ, III]), and so it is not surprising that we still get the equality $a_m(G) = c_m(F)$ when we remove the $p$-adic height contributions from primes above $p$.

By a clever trick of Perrin-Riou using the norm-coherency of Heegner points, one reduces the proof of (1.3) to showing that for any place $v$ above $p$,

$$\lim_{j \to \infty} \left\langle x_f, b_{p^j} \right\rangle_v = 0,$$

where $b_{p^j}$ is the norm of a generalized Heegner cycle of conductor $p^j$ defined over the ring class field of conductor $p^j$, and $x_f$ is the $f$-isotypic component of $x$. In other words, we must show that certain height pairings become more and more divisible by $p$ as we move up the local ring class field tower. We do this in Chapter IX by fixing an approach suggested in [N3, II.5]. The key new inputs are local class field theory (via relative Lubin-Tate groups) and comparison isomorphisms in $p$-adic Hodge theory.

Roughly, we show that certain Galois representations ("mixed extensions") needed to compute the local height pairings are crystalline when $r_{\mathcal{A}}(m) = 0$. The key ingredient is Theorem IX.10 which relies on Faltings' proof of Fontaine's $C_{\mathrm{cris}}$ conjecture. This theorem (or rather, its proof) is quite general and should be useful for computing $p$-adic heights of algebraic cycles sitting on varieties fibered over curves. Returning to our context, it follows that the height pairing is a logarithm of a norm of unit in a ring class field. As the ring class field gets larger, the logarithm gets more divisible by $p$, and in the limit the height pairing goes to 0.

The remarkable aspect of the general approach outlined above (which is due to Perrin-Riou) is that it proves a formula for the global $p$-adic height without ever actually computing the local heights at $p$. This suggests that $p$-adic height formulas are more accessible than archimedean height formulas, for which explicit and often messy computations with Green's functions seem unavoidable (cf. the last chapter of this thesis). Of course, if this strategy is to work in greater generality (e.g. on higher dimensional Shimura varieties) then one still needs to prove that the contribution from local heights at $p$ vanishes. Perrin-Riou's proof in weight 2 relies heavily on the fact that the Galois representations at hand are Tate modules of abelian varieties or 1-motives. In higher weight, the proof uses more machinery, and required us to show that certain less accessible Galois representations are crystalline. Our hope is that the computations in Chapter IX will encourage further development of the necessary $p$-adic Hodge theoretic machinery needed for $p$-adic height pairings in situations even more general than ours (where we only consider algebraic cycles lying in fibers over a curve).

In the final two sections we present computations toward the archimedean special value formula for $f \otimes \chi$. Since the local archimedean heights at finite places are essentially already computed in earlier chapters, the crux of the matter is computing the local heights at infinity and comparing them to the analytic kernel (which we compute in Chapter X). The Green's functions and heights at infinity are computed in the final Chapter XI. The Green's functions we construct are eigenfunctions for the usual weight $\pm\ell$ Laplacian on the upper half plane, with a simple transformation property under the diagonal action of $\mathrm{SL}_2(\mathbb{R})$. These eigenfunctions might be of independent interest. At the end of the section, we sketch how to deduce an archimedean special value formula, at least assuming the modularity of certain generating series of height pairings.

### 1.2.7 Document outline

The proof of Theorem I.7 follows [N3] and [PR1] rather closely. We have therefore chosen not to dwell long on computations easily adapted to our situation.

We define the $p$-adic $L$-function $L_p(f \otimes \chi, \lambda)$ in Chapter II and show that it vanishes in the anticyclotomic direction. In Chapter III, we integrate the $p$-adic logarithm against the $p$-adic Rankin-Selberg measure to compute what is essentially the derivative of $L_p(f \otimes \chi)$ at the trivial character in the cyclotomic direction. In Chapter IV, we define the generalized Heegner cycles and describe Hecke operators and $p$-adic Abel-Jacobi maps attached to the variety $X$. After proving some properties of generalized Heegner cycles, we show that the RHS of Theorem I.7 vanishes when $\ell_K$ is anticyclotomic.

In Chapter V, we recall the definitions of Nekovář's local $p$-adic heights. In Chapter VI we compute the local cyclotomic heights of $z_f$ at places $v$ which are prime to $p$. In Chapter VII, we prove that $V_{f,A,\ell}$ is an ordinary representation. We complete the proof of the main theorem in Chapter VIII, modulo the results from Chapter IX. The latter is where we fix a technical issue in the proof in [N3, II.5], to complete a proof of the vanishing of the contribution coming from local heights at primes above $p$.

Chapters X and XI contain the archimedean computations described above.

# CHAPTER II

# Constructing the $p$-adic $L$-function

We fix once and for all an embedding $\iota : \bar{Q} \to \bar{\mathbb{Q}}_p$. Recall $f = \sum_{n\geqslant 1} a_n q^n \in S_{2r}(\Gamma_0(N))$ is a normalized newform which we assume to be ordinary, i.e. $\iota(a_p)$ is a $p$-adic unit. As in the introduction, $\chi : \mathbb{A}_K^\times \to \mathbb{C}^\times$ is an unramified Hecke character of infinity type $(2k, 0)$ with $0 < 2k = \ell < 2r$. This means

$$\chi(\alpha \cdot x \cdot z_\infty) = \chi(x) \cdot z_\infty^{-2k}, \qquad \text{for all } \alpha \in K^\times, z_\infty \in K_\infty^\times.$$

If $\mathfrak{a}$ is a prime ideal of $\mathcal{O}_K$, then we follow the usual convention and write $\chi(\mathfrak{a})$ for $\chi(\pi_\mathfrak{a})$, where $\pi_\mathfrak{a} \in K_\mathfrak{p}^\times \subset \mathbb{A}_K^\times$ is a uniformizer at $\mathfrak{a}$. Extending multiplicatively, we may think of $\chi$ as a character of the group of fractional ideals of $K$ satisfying $\chi(\mathfrak{a}) = \alpha^{2k}$ if $\mathfrak{a} = (\alpha)$ is a principal ideal. For more on Hecke characters, see [BDP1, §4.1].

All that follows will apply to $\chi$ of infinity type $(0, 2k)$ with suitable modifications. In this section, we follow [N3, I.3-5] and define a $p$-adic $L$-function attached to the pair $(f, \chi)$ which interpolates special values of certain Rankin-Selberg convolutions.

## 2.1 $p$-adic measures

We construct the $p$-adic $L$-function only in the setting needed for Theorem I.7; in the notation of [N3], this means that $\Omega = 1, N_1 = N_2 = c_1 = c_2 = c = 1, N_3 = N_3' = N, \Delta = \Delta_1 = \Delta_2 = |D|, \Delta_3 = 1$, and $\gamma = \gamma_3 = 0$. We begin by defining theta measures.

Fix an integer $m \geqslant 1$ and let $\mathcal{O}_m$ be the order of conductor $m$ in $K$. Let $\mathfrak{a}$ be proper $\mathcal{O}_m$-ideal whose class in $\mathrm{Pic}(\mathcal{O}_m)$ is denoted by $\mathcal{A}$. The quadratic form

$$Q_\mathfrak{a}(x) = \mathbf{N}(x)/\mathbf{N}(\mathfrak{a}),$$

takes integer values on $\mathfrak{a}$. Define the measure $\Theta_\mathcal{A}$ on $\mathbb{Z}_p^\times$ by

$$\Theta_\mathcal{A}(a (\mathrm{mod}\ p^\nu)) = \chi(\bar{\mathfrak{a}})^{-1} \sum_{\substack{x \in \mathfrak{a} \\ Q_\mathfrak{a}(x) \equiv a \ (\mathrm{mod}\ p^\nu)}} \bar{x}^\ell q^{Q_\mathfrak{a}(x)}. \tag{2.1}$$

To keep things from getting unwieldy we have omitted $\chi$ from the notation of the measure. If $\phi$ is a function on $\mathbb{Z}/p^\nu\mathbb{Z}$ with values in a $p$-adic ring $A$, then

$$\Theta_\mathcal{A}(\phi) = \chi(\bar{\mathfrak{a}})^{-1} \sum_{x \in \mathfrak{a}} \phi(Q_\mathfrak{a}(x)) \bar{x}^\ell q^{Q_\mathfrak{a}(x)} = \chi(\bar{\mathfrak{a}})^{-1} \sum_{n \geqslant 1} \phi(n) \rho_\mathfrak{a}(n, \ell) q^n, \tag{2.2}$$

where $\rho_\mathfrak{a}(n, \ell)$ is the sum $\sum \bar{x}^\ell$ over all $x \in \mathfrak{a}$ with $Q_\mathfrak{a}(x) = n$. We have

$$\rho_{\mathfrak{a}\cdot(\gamma)}(n, \ell) = \bar{\gamma}^\ell \rho_\mathfrak{a}(n, \ell),$$

for all $\gamma \in K^\times$, so that $\Theta_{\mathcal{A}}$ is independent of the choice of representative $\mathfrak{a}$ for the class $\mathcal{A}$. For $\mathfrak{a} \in \mathcal{A}$,

$$\chi(\bar{\mathfrak{a}})^{-1} \sum_{x\in\mathfrak{a}} \bar{x}^\ell q^{Q_{\mathfrak{a}}(x)} = w_m \sum_{\substack{\mathfrak{a}'\in\mathcal{A} \\ \mathfrak{a}'\subset\mathcal{O}_m}} \chi(\mathfrak{a}') q^{\mathbf{N}(\mathfrak{a}')} = w_m \sum_{n\geqslant 1} r_{\mathcal{A},\chi}(n) q^n, \tag{2.3}$$

since $\ell$ is a multiple of $w_m$ (recall $\chi$ is unramified). The coefficients $r_{\mathcal{A},\chi}(n)$ play the role of (and generalize) the numbers $r_{\mathcal{A}}(m)$ that appear in [GZ] and [N3].

**Proposition II.1.** *$\Theta_{\mathcal{A}}(\phi)$ is a cusp form in $M_{\ell+1}(\Gamma_1(M), A)$, with $M = \mathrm{lcm}(|D|m^2, p^{2\nu})$.*

*Proof.* It is classical [Og] that $\sum_{x\in\mathfrak{a}} \bar{x}^\ell q^{Q_{\mathfrak{a}}(x)}$ is a cusp form in $M_{\ell+1}(\Gamma_1(|D|m^2))$. It follows from [Hi, Proposition 1.1] that weighting this form by $\phi$ gives a modular form of the desired level. □

For a fixed integer $C$ prime to $N|D|p$, define the Eisenstein measures

$$E_1(\alpha(\mathrm{mod}\ p^\nu))(z) = E_1(z, \phi_{\alpha,p^\nu})$$

$$E_1^C(\alpha(\mathrm{mod}\ p^\nu))(z) = E_1(\alpha(\mathrm{mod}\ p^\nu))(z) - C E_1(C^{-1}\alpha(\mathrm{mod}\ p^\nu))(z),$$

where

$$E_1(\alpha(\mathrm{mod}\ p^\nu))(z) = \frac{1}{2}\tilde{L}(0,\delta_\alpha) + \sum_{\substack{j\cdot m>0 \\ j\equiv\alpha(\mathrm{mod}\ p^\nu)}} \mathrm{sgn}(j) q^{jm},$$

with notation as in [N3, I.3.6]. Similarly, we define the convolution measure on $\mathbb{Z}_p^\times$

$$\Phi_{\mathcal{A}}^C(a(\mathrm{mod}\ p^\nu)) = H\left[\sum_{\alpha\in(\mathbb{Z}/|D|p^\nu\mathbb{Z})^\times} \xi(\alpha)\Theta_{\mathcal{A}}(\alpha^2 a(\mathrm{mod}\ p^\nu))(z)\delta_1^{r-1-k}(E_1^C(\alpha(\mathrm{mod}\ |D|p^\nu))(Nz))\right],$$

which takes values in $\overline{M}_{2r}(\Gamma_0(N|D|p^\infty); \chi(\bar{\mathfrak{a}})^{-1}p^{-\delta}\mathbb{Z}_p)$, for some $\delta$ depending only on $r$ and $k$ [Hi, Lem. 5.1]. Here, $H$ is holomorphic projection, $\delta_1^{r-1-k}$ is Shimura's differential operator, and $\xi$ is the quadratic character $\left(\frac{D}{\cdot}\right)$. For the definitions of $H$ and $\delta_1^{r-k-1}$ for $p$-adic modular forms, see [N3, I.2-3] and [Hi, §5]. We are implicitly identifying $\mathbb{Z}_p$ with the ring of integers of $K_{\mathfrak{p}}$ for a prime $\mathfrak{p}$ above $p$ (which is split in $K$), so that $x^\ell \in \mathbb{Z}_p$ for all $x \in \mathfrak{a}$.

Another measure $\Psi_{\mathcal{A}}^C$ is defined by

$$\Psi_{\mathcal{A}}^C = \frac{1}{2w_m}\Phi_{\mathcal{A}}^C\Big|_{2r}\mathscr{T}(|D|)_{Np^\infty},$$

where

$$\mathscr{T}(|D)_{Np^\infty} : M_{2r}\left(\Gamma_0\left(N|D|p^\infty\right), \cdot\right) \to M_{2r}\left(\Gamma_0\left(Np^\infty\right), \cdot\right)$$

is the trace map, i.e. the adjoint to the operator $g \mapsto |D|^{r-1}g\Big|_{2r}\begin{pmatrix} |D| & 0 \\ 0 & 1\end{pmatrix}$.

Let $H_m/K$ be the ring class field of conductor $m$ for $K$. This is an abelian extension of $K$, unramified away from $m$, and corresponds via class field theory with the subgroup $K^\times\mathbb{A}_{K,\infty}^\times\hat{\mathcal{O}}_m^\times$ of $\mathbb{A}_K^\times/K^\times$. The global reciprocity map identifies $\mathrm{Gal}(H_m/K)$ with the group $\mathrm{Pic}(\mathcal{O}_m)$ of invertible $\mathcal{O}_m$-ideal classes. For ring class field characters $\rho : G(H_m/K) \to \overline{\mathbb{Q}}^\times$, define

$$\Phi_\rho^C = \sum_{[\mathcal{A}]\in\mathrm{Pic}(\mathcal{O}_m)} \rho([\mathcal{A}])^{-1}\Phi_{\mathcal{A}}^C,$$

and similarly for $\Psi_\rho^C$.

We define $\Psi^C_{f,\rho} = L_{f_0}(\Psi^C_\rho)$, where $L_{f_0}$ is Hida's projector attached to the $p$-stabilization

$$f_0 = f(z) - \frac{p^{2r-1}}{\alpha_p(f)} f(pz)$$

of $f$; recall $\alpha_p(f)$ is the unique root of $x^2 - a_p x + p^{2r-1}$ which is a $p$-adic unit. Intuitively, $L_{f_0}$ is the projection onto the space of ordinary forms, followed by projection onto "the $f_0$-part" (the $p$-adic version of taking a Peterson inner product with $f_0$). See [N3, I.2] for a proper definition and properties. Explicitly, if $g \in M_j(\Gamma_0(Np^\mu); \bar{\mathbb{Q}})$ with $\mu \geqslant 1$, then

$$L_{f_0(g)} = \left(\frac{p^{j/2-1}}{\alpha_p(f)}\right)^{\mu-1} \frac{\left\langle f_0^\tau\Big|_j \begin{pmatrix} 0 & -1 \\ Np^\mu & 0 \end{pmatrix}, g \right\rangle_{Np^\mu}}{\left\langle f_0^\tau\Big|_j \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix}, f_0 \right\rangle_{Np}}. \tag{2.4}$$

Once more, we define a measure $\Psi^C_f$, this time on $\mathrm{Gal}(H_{p^\infty}/K) \times \mathrm{Gal}(K(\mu_{p^\infty})/K)$, by

$$\Psi^C_f(\sigma \,(\mathrm{mod}\, p^n), \tau \,(\mathrm{mod}\, p^m)) = L_{f_0}(\Psi^C_{\mathcal{A}}(a \,(\mathrm{mod}\, p^m))),$$

where $\sigma$ corresponds to $\mathcal{A}$ and $\tau$ corresponds to $a \in (\mathbb{Z}/p^m\mathbb{Z})^*$ under the Artin map. Finally, as in [N3], we define modified measures $\tilde{\Psi}^C_{\mathcal{A}}, \tilde{\Psi}^C_\rho$, etc., by replacing $\mathscr{T}(|D|)$ with $\mathscr{T}(1)$ in the definition of $\Psi^C_{\mathcal{A}}$.

## 2.2 Integrating characters against the Rankin-Selberg measure

In this subsection, we integrate finite order characters of the $\mathbb{Z}_p^2$-extension of $K$ against the measures constructed in the previous section and show that they recover special values of Rankin-Selberg $L$-functions. This allows us to prove a functional equation for the (soon to be defined) $p$-adic $L$-function. We follow the computations in [N3, I.5] and [PR2, §4]. Let $\eta$ denote a character $(\mathbb{Z}/p^\nu\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$. Exactly as in [PR2, Lemma 7], we compute:

$$\int_{\mathbb{Z}_p^\times} \eta \, d\Phi^C_{\mathcal{A}} = \left(1 - C\xi(C)\bar{\eta}^2(C)\right) H[\Theta_{\mathcal{A}}(\eta)(z)\delta_1^{r-k-1}\left(E_1(Nz,\phi)\right)]. \tag{2.5}$$

Similarly, if $\rho$ is a ring class character with conductor a power of $p$,

$$\int_{\mathbb{Z}_p^\times} \eta \, d\Phi^C_\rho = w_m \left(1 - C\xi(C)\bar{\eta}^2(C)\right) H[\Theta_\chi(\mathcal{W}'')(z)\delta_1^{r-k-1}\left(E_1(Nz,\phi)\right)], \tag{2.6}$$

where $\mathcal{W}'' = \rho \cdot (\eta \circ \mathbf{N})$, the latter being thought of as a character modulo the ideal $\mathfrak{f} = \mathrm{lcm}(\mathrm{cond}\ \rho, \mathrm{cond}\ \eta, p)$. We denote by $\mathcal{W}$ the primitive character associated to $\mathcal{W}''$. By definition,

$$\Theta_\chi(\mathcal{W}'')(z) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O} \\ (\mathfrak{a},\mathfrak{f})=1}} \mathcal{W}''(\mathfrak{a})\chi(\mathfrak{a})q^{\mathbf{N}(\mathfrak{a})}.$$

This is a cusp form in $S_{\ell+1}\left(|D|\mathbf{N}^K_{\mathbb{Q}}(\mathfrak{f}), \left(\frac{D}{\cdot}\right)\eta^2\right)$, since $\chi$ is unramified (see [Og] for a more general result).

The computations of [N3, I.5.3-4] carry over to our situation, except the theta series transformation law now reads

$$\Theta_\chi(\mathcal{W}'')(z)\Big|_{\ell+1} \mathscr{F} = \left(\frac{D}{w}\right)\bar{\eta}^2(w)\Theta_\chi(W'')\Big|_{\ell+1} \begin{pmatrix} 0 & -1 \\ |D|p^\mu & 0 \end{pmatrix}, \tag{2.7}$$

where $\mathscr{F}$ is the involution

$$\begin{pmatrix} 0 & -1 \\ N|D|p^\mu & 0 \end{pmatrix} \begin{pmatrix} N & y \\ N|D|p^\mu t & N \end{pmatrix}$$

with $Nxw - |D|p^\mu ty = 1$. We thus obtain

$$\int_{\mathbb{Z}_p^\times} \eta\, d\Psi_{f,\rho}^C = \left(1 - C\xi(C)\bar{\eta}^2(C)\right) \frac{\left(\left(\frac{D}{\cdot}\right)\eta^2\right)(N)\lambda_N(f)|D|^{1/2}}{(4\pi i)\alpha_p(f)^{-1}} \left(\frac{|D|}{p}\right)^{r-1} \cdot \frac{\Lambda_\mu(\mathcal{W}'')}{\Lambda(f)}, \tag{2.8}$$

where

$$\Lambda(f) = \left\langle f_0^\tau \Big|_{2r} \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix}, f_0 \right\rangle_{Np}$$

and

$$\Lambda_\mu(\mathcal{W}'') = \frac{p^{\mu(r-1/2)}}{\alpha_p(f)^\mu} \left\langle f_0^\tau, \Theta_\chi\left(\mathcal{W}''\right)\Big|_{\ell+1} \begin{pmatrix} 0 & -1 \\ |D|p^\mu & 0 \end{pmatrix} \delta^{r-k-1}\left(E_1\left(z, \xi\bar{\eta}^2\right)\right) \right\rangle_{N|D|p^\mu},$$

and $\lambda_N(f)$ is the Atkin-Lehner eigenvalue of $f$. Define $\tau(\chi\mathcal{W})$ by the relation

$$\Theta_\chi(\mathcal{W})|_{\ell+1} \begin{pmatrix} 0 & -1 \\ |D|p^\beta & 0 \end{pmatrix} = (-1)^{k+1} i\tau(\chi\mathcal{W})\Theta_{\bar{\chi}}(\bar{\mathcal{W}}), \tag{2.9}$$

with $|D|p^\beta$ being the level $\Delta(\mathcal{W})$ of $\Theta_\chi(\mathcal{W})$. One knows ([M, Thm. 4.3.12]) that $\tau(\chi\mathcal{W}) \in \bar{\mathbb{Q}}^\times$, $|\tau(\chi\mathcal{W})| = 1$, and

$$\Lambda(\chi\mathcal{W}, s) = \tau(\chi\mathcal{W})\Lambda(\bar{\chi}\bar{\mathcal{W}}, \ell + 1 - s),$$

where

$$\Lambda(\chi\mathcal{W}, s) = \left(|D|p^\beta\right)^{s/2} (2\pi)^{-s}\Gamma(s)L(\Theta_\chi(\mathcal{W}), s).$$

Modifying the computations in [PR2, §4], we find that

$$\Lambda_\mu(\mathcal{W}'') = (-1)^{k+1} i\tau(\chi\mathcal{W}) \sum_{\substack{\mathfrak{a}|p \\ \mathbf{N}(\mathfrak{a})=p^s}} \mu(\mathfrak{a})\chi(\mathfrak{a})\mathcal{W}(\mathfrak{a})\Lambda_{\mu,s}, \tag{2.10}$$

with

$$\Lambda_{\mu,s} = \frac{p^{\mu\left(r-\frac{1}{2}\right)-s\left(k+\frac{1}{2}\right)}}{\alpha_p(f)^\mu} \left\langle f_0^\tau, \Theta_{\bar{\chi}}(\bar{\mathcal{W}})\Big|_{\ell+1} \begin{pmatrix} p^x & 0 \\ 0 & 1 \end{pmatrix} \delta^{r-k-1}(E_1(z, \xi\bar{\eta}^2)) \right\rangle_{N|D|p^\mu} \tag{2.11}$$

and $x = \mu - \beta - s$.

Following [PR2, §4.4], we compute:

(2.12)

$$\Lambda_\mu(\mathcal{W}'') = (-1)^r i\tau(\chi\mathcal{W})V_p(f, \chi, \mathcal{W}) \left(\frac{p^{r-1/2}}{\alpha_p(f)}\right)^\beta \frac{2(r+k-1)!(r-k-1)!}{(4\pi)^{2r-1}} L(f, \Theta_{\bar{\chi}}(\bar{\mathcal{W}}), r+k),$$

where

$$V_p(f, \chi, \mathcal{W}) = \prod_{\mathfrak{p}|p} \left(1 - \frac{(\bar{\chi}\bar{\mathcal{W}})(\mathfrak{p})}{\alpha_{\mathbf{N}(\mathfrak{p})}(f)} \mathbf{N}(\mathfrak{p})^{r-k-1}\right) \left(1 - \frac{(\chi\mathcal{W})(\mathfrak{p})}{\alpha_{\mathbf{N}(\mathfrak{p})}(f)} \mathbf{N}(\mathfrak{p})^{r-k-1}\right).$$

We have used the fact that

$$\left\langle f^\tau, g\delta_1^{r-k-1}(E_1(z, \phi)) \right\rangle_M = \frac{(1 - \epsilon(-1))(-1)^{r-k-1}(r+k-1)!(r-k-1)!}{(4\pi)^{2r-1}} L(f, g, r+k) \tag{2.13}$$

for any $g \in S_{2k+1}(M', \epsilon)$, and where $M = M'N$. Equation 2.13 follows from the usual unfolding trick and the fact [N3, I.1.5.3] that

$$\delta_1^{r-k-1}(E_1(z, \phi)) = \frac{(r-k-1)!}{(-4\pi)^{r-k-1}} E_{r-k}(z, \phi).$$

We have also used the following generalization of [PR2, Lemma 23].

**Lemma II.2.** *If $g$ is a modular form whose $L$-function admits a Euler product expansion $\prod_p G_p(p^{-s})$, then*

$$L(f_0, g, r+k) = G_p\left(p^{r-k-1}\alpha_p(f)^{-1}\right) L(f, g, r+k).$$

Finally, we also have [N3, II.5.7]

$$\alpha_p(f)^{-1} \cdot \Lambda(f) = \lambda_N(f) p^{1-r} H_p(f) \langle f, f \rangle_N,$$

with

$$H_p(f) = \left(1 - \frac{p^{2r-2}}{\alpha_p(f)^2}\right)\left(1 - \frac{p^{2r-1}}{\alpha_p(f)^2}\right).$$

Putting all these calculations together, we obtain the following interpolation result.

**Theorem II.3.** *For finite order characters $\mathcal{W} = \rho \cdot (\eta \circ \boldsymbol{N})$ as above,*

$$\left(1 - C\left(\frac{D}{C}\right)\bar{\mathcal{W}}(C)\right)^{-1} \int_{\mathbb{Z}_p^\times} \eta \, d\Psi_{f,\rho}^C = \frac{\mathcal{L}_p(f,\chi,\mathcal{W}) V_p(f,\chi,\mathcal{W}) \Delta(\mathcal{W})^{r-1/2}}{\alpha_p(f)^\beta H_p(f)},$$

*where*

$$\mathcal{L}_p(f,\chi,\mathcal{W}) = \left(\frac{D}{-N}\right) \mathcal{W}(N) \tau(\chi\mathcal{W}) C(r,k) \frac{L(f, \Theta_{\bar{\chi}}(\bar{\mathcal{W}}), r+k)}{\langle f, f \rangle_N},$$

*and*

$$C(r,k) = \frac{2(-1)^{r-1}(r-k-1)!(r+k-1)!}{(4\pi)^{2r}}.$$

The modified measures $\tilde{\Psi}_{f,\rho}^C$ satisfy

$$\int_{\mathbb{Z}_p^\times} \eta \, d\tilde{\Psi}_{f,\rho}^C = |D|^{1-r} \overline{(\chi\mathcal{W})}(\mathcal{D}) \int_{\mathbb{Z}_p^\times} \eta \, d\Psi_{f,\rho}^C,$$

where $\mathcal{D} = \left(\sqrt{D}\right)$ is the different of $K$.

## 2.3 Definition of the $p$-adic $L$-function

Recall we have fixed an integer $C$ prime to $N|D|p$.

**Definition II.4.** For any continuous character $\phi : G(H_{p^\infty}(\mu_{p^\infty})/K) \to \bar{\mathbb{Q}}_p^\times$ with conductor of $p$-power norm, we define

$$L_p(f \otimes \chi, \phi) = (-1)^{r-1} H_p(f) \left(\frac{D}{-N}\right)\left(1 - C\left(\frac{D}{C}\right)\phi(C)^{-1}\right)^{-1} \int_{G(H_{p^\infty}(\mu_{p^\infty})/K)} \phi \, d\tilde{\Psi}_f^C.$$

The $p$-adic $L$-function $L_p(f \otimes \chi)(\lambda) := L_p(f \otimes \chi, \lambda)$ is a function of characters

$$\lambda : G(H_{p^\infty}(\mu_{p^\infty})/K) \to (1 + p\mathbb{Z}_p).$$

$L_p(f \otimes \chi)$ is an Iwasawa function with values in $c^{-1}\mathcal{O}_{\widehat{\mathbb{Q}(f,\chi)}}$, where $\widehat{\mathbb{Q}(f,\chi)}$ is the $p$-adic closure (using our fixed embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$) of the field generated by the coefficients of $f$ and the values of $\chi$, and $c \in \widehat{\mathbb{Q}(f,\chi)}$ is non-zero.

We can construct analogous measures and an analogous $p$-adic $L$-function for $\bar{\chi}$, which is a Hecke character of infinity type $(0, \ell)$. There is a functional equation relating $L_p(f \otimes \chi)$ to $L_p(f \otimes \bar{\chi})$, which we now describe. First define

$$\Lambda_p(f \otimes \chi)(\lambda) = \lambda(\mathcal{D}N^{-1})\lambda(N)^{1/2} L_p(f \otimes \chi)(\lambda).$$

**Proposition II.5.** $\Lambda_p$ *satisfies the functional equation*

$$\Lambda_p\left(f \otimes \chi\right)(\lambda) = \left(\frac{D}{-N}\right) \Lambda_p\left(f \otimes \bar{\chi}\right)\left(\lambda^{-1}\right).$$

*Proof.* It suffices to prove this for all finite order characters $\mathcal{W}$. For such $\mathcal{W}$, the functional equation for the Rankin-Selberg convolution reads

$$L(f, \Theta_{\bar{\chi}}(\bar{\mathcal{W}}), r+k) = \frac{\left(\frac{D}{-N}\right)\bar{\mathcal{W}}(N)}{\tau(\chi\mathcal{W})^2} L(f, \Theta_{\chi}(\mathcal{W}), r+k), \tag{2.14}$$

so

$$\frac{\mathcal{L}_p(f, \chi, \mathcal{W})}{\mathcal{L}_p(f, \bar{\chi}, \bar{\mathcal{W}})} = \mathcal{W}(N)\left(\frac{D}{-N}\right).$$

We also have $V_p(f, \bar{\chi}, \bar{\mathcal{W}}) = V_p(f, \chi, \mathcal{W})$, so that

$$\frac{L_p(f \otimes \chi)(\mathcal{W})}{L_p(f \otimes \bar{\chi})(\bar{\mathcal{W}})} = \mathcal{W}(N)\left(\frac{D}{-N}\right)\bar{\mathcal{W}}(\mathcal{D})^2.$$

The proposition now follows from a simple computation. □

Recall the notation $\lambda^\tau(\mathfrak{a}) = \lambda(\mathfrak{a}^\tau)$, where $\tau \in \mathrm{Gal}(K/\mathbb{Q})$ is complex conjugation.

**Corollary II.6.** *Suppose* $\left(\frac{D}{N}\right) = 1$ *and* $\lambda$ *is anticyclotomic, i.e.* $\lambda\lambda^\tau = 1$. *Then* $L_p(f \otimes \chi)(\lambda) = 0$.

*Proof.* From the functional equation and the fact that

$$\Lambda_p(f \otimes \chi)(\lambda) = \Lambda_p(f \otimes \bar{\chi})(\lambda^\tau),$$

we obtain

$$\Lambda_p(f \otimes \chi)(\lambda) = -\Lambda_p(f \otimes \chi)(\lambda^{-\tau}).$$

Since $\lambda$ is anticyclotomic, this is equal to $-\Lambda_p(f \otimes \chi)(\lambda)$. □

# CHAPTER III

## Computing the $p$-adic $L$-function

This section is devoted to computing the Fourier coefficients of $\int_{\mathbb{Z}_p^\times} \lambda \, d\tilde{\Psi}_{\mathcal{A}}$, where $\lambda$ is a continuous function $\mathbb{Z}_p^\times \to \mathbb{Q}_p$. When $\ell_K : \mathbb{A}_K^\times / K^\times \to \mathbb{Q}_p$ is a cyclotomic logarithm, these computations will allow us to relate $L_p'(f \otimes \chi, \ell_K, \mathbb{1})$ to heights of generalized Heegner cycles. We follow the computations in [N3, I.6]; the main added subtlety is the transformation laws for theta series attached to Hecke characters.

Recall that for each ideal class $\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)$, we defined

$$\Phi_{\mathcal{A}}^C(a(\mathrm{mod}\ p^\nu)) =$$
$$H\left[\sum_{\alpha \in (\mathbb{Z}/|D|p^\nu\mathbb{Z})^\times} \xi(\alpha)\Theta_{\mathcal{A}}(\alpha^2 a(\mathrm{mod}\ p^\nu))(z)\delta_1^{r-1-k}(E_1^C(\alpha(\mathrm{mod}\ |D|p^\nu))(Nz))\right].$$

For each factorization $D = D_1 D_2$ (with the signs normalized so that $D_1$ is a discriminant), we choose integers $a, b, c, d$ and define

$$W_{D_1}^{(\nu)} = \begin{pmatrix} |D_1|a & b \\ N|D|p^\nu c & |D_1|d \end{pmatrix},$$

so that $W_{D_1}^{(\nu)}$ has determinant $|D_1|$.

**Lemma III.1.** *For $W_{D_1}^{(\nu)}$ as above and $\alpha \in (\mathbb{Z}/|D|p^\nu\mathbb{Z})^\times$,*

$$\Theta_{\mathcal{A}}\left(\alpha(\mathrm{mod}\ p^\nu)\right)(z)\Big|_{\ell+1} W_{D_1}^{(\nu)} = \frac{|D_1|^k}{\chi(\mathcal{D}_1)}\gamma\Theta_{\mathcal{A}\mathfrak{d}_1^{-1}}\left(|D_1|a^2\alpha(\mathrm{mod}\ p^\nu)\right)(z),$$

*where*

$$\gamma = \left(\frac{D_1}{cp^\nu N}\right)\left(\frac{D_2}{a\boldsymbol{N}(\mathfrak{a})}\right)\kappa(D_1)^{-1},$$

*and $\mathcal{D}_1$ is the ideal of norm $|D_1|$ in $\mathcal{O}_K$ and $\kappa(D_1) = 1$ if $D_1 > 0$, otherwise $\kappa(D_1) = i$.*

*Remark* III.2. Note that the factor $\frac{|D_1|^k}{\chi(\mathcal{D}_1)}$ is equal to $\pm 1$, since $\chi$ has infinity type $(2k, 0)$ and $\mathcal{D}_1$ is 2-torsion in the class group.

*Proof.* The proof proceeds as in [PR1, §3.2], but requires some extra Fourier analysis. We sketch the argument for the convenience of the reader. Fixing an ideal $\mathfrak{a}$ in the class of $\mathcal{A}$, we set $L = p^\nu\mathfrak{a}$ and let $L^*$ be the dual lattice with the respect to the quadratic form $Q_{\mathfrak{a}} = \mathrm{Nm}(x)/\mathrm{Nm}(\mathfrak{a})$. Denote by $S = S_{\mathfrak{a}}$ the symmetric bilinear form corresponding to $Q_{\mathfrak{a}}$, so $S_{\mathfrak{a}}(\alpha, \beta) = \frac{1}{\mathrm{Nm}(\mathfrak{a})}\mathrm{Tr}(\alpha\bar{\beta})$. For $u \in L^*$, define

$$\Theta_{\mathfrak{a},\chi}(u, L) = \chi(\bar{\mathfrak{a}})^{-1}\sum_{\substack{w-u \in L \\ w \in L^*}} \bar{w}^\ell q^{Q_{\mathfrak{a}}(w)}.$$

For any $c \in \mathbb{Z}$, one checks the following relations:

$$\Theta_{\mathfrak{a},\chi}(u, L) = \sum_{\substack{w-u\in L \\ w\in L^*/cL}} \Theta_{\mathfrak{a},\chi}(w, cL), \tag{3.1}$$

$$\Theta_{\mathfrak{a},\chi}(u, cL)(c^2 z) = c^{-\ell}\Theta_{\mathfrak{a},\chi}(cu, c^2 L)(z), \tag{3.2}$$

and for all $a \in \mathbb{Z}$ and $w \in L^*$,

$$\Theta_{\mathfrak{a},\chi}(w, cL)\left(z + \frac{a}{c}\right) = e\left(\frac{a}{c}Q_{\mathfrak{a}}(w)\right)\Theta_{\mathfrak{a},\chi}(w, cL). \tag{3.3}$$

We also have

$$z^{-(\ell+1)}\Theta_{\mathfrak{a},\chi}(w, cL)\left(\frac{-1}{z}\right) = -ic^{-2}[L^* : L]^{-1/2} \sum_{y\in(cL)^*/cL} e\left(S_{\mathfrak{a}}(w, y)\right)\Theta_{\mathfrak{a},\chi}(y, cL)(z). \tag{3.4}$$

This follows from the identity

$$z^{\ell+1}\sum_{x\in L} P(x + u)e\left(Q_{\mathfrak{a}}(x + y)z\right) = i[L^* : L]^{-1/2} \sum_{y\in L^*} P(y)e\left(\frac{-Q_{\mathfrak{a}}(y)}{z}\right)e\left(S_{\mathfrak{a}}(y, u)\right), \tag{3.5}$$

valid for any rank two integral quadratic space $(L, Q_{\mathfrak{a}}, S_{\mathfrak{a}})$ and any polynomial $P$ of degree $\ell$ which is spherical for $Q_{\mathfrak{a}}$. See [Wa] for a proof of this version of Poisson summation.

Now write
$$W_{D_1}^{(\nu)} = H\begin{pmatrix} |D_1| & 0 \\ 0 & 1 \end{pmatrix}$$
with $H \in \mathrm{SL}_2(\mathbb{Z})$. Exactly as in [PR1], we use the relations above to compute

$$\Theta_{\mathfrak{a},\chi}(\alpha(\mathrm{mod}\ p^\nu))\Big|_{\ell+1} H = \gamma|D_1|^{-1/2} \sum_{\substack{u\in\mathfrak{a}/L \\ Q_{\mathfrak{a}}(u)\equiv\alpha(\mathrm{mod}\ p^\nu)}} \sum_{\substack{w\in L^*/L \\ w+au\in\mathcal{D}_1^{-1}p^r\mathfrak{a}}} \Theta_{\mathfrak{a},\chi}(w, L)$$

so that

$$\begin{aligned}\Theta_{\mathfrak{a},\chi}(\alpha(\mathrm{mod}\ p^\nu))\Big|_{\ell+1} W_{D_1}^{(\nu)} &= \gamma|D_1|^k\chi(\bar{\mathfrak{a}})^{-1} \sum_{\substack{w\in\mathcal{D}_1^{-1}\mathfrak{a} \\ Q_{\mathfrak{a}\mathcal{D}_1^{-1}}(w)\equiv|D_1|a^2\alpha(\mathrm{mod}\ p^r)}} \bar{w}^\ell q^{Q_{\mathfrak{a}\mathcal{D}_1^{-1}}(w)} \\ &= \frac{|D_1|^k}{\chi(\mathcal{D}_1)}\gamma\Theta_{\mathfrak{a}\mathcal{D}_1^{-1},\chi}\left(|D_1|a^2\alpha(\mathrm{mod}\ p^\nu)\right)(z),\end{aligned}$$

as desired. □

For any function $\lambda$ on $(Z/p^\nu\mathbb{Z})^\times$, we define $h_{D_1}(\lambda)$ as in [N3, I.6.3], so that

$$\int_{\mathbb{Z}_p^\times} \lambda\, d\tilde{\Psi}_{\mathcal{A}} = \frac{1}{2w}\sum_{D=D_1\cdot D_2}\sum_{j\in\mathbb{Z}/|D_1|\mathbb{Z}} h_{D_1}(\lambda)\Big|_{2r}\begin{pmatrix} 1 & j \\ 0 & |D_1| \end{pmatrix}.$$

The Fourier coefficient computation in [N3, I.6.5] remains valid, except one needs to use the following proposition in place of [N3, I.1.9]:

**Proposition III.3.** *Let $f = \sum_{n\geqslant 1} a(n)q^n$ be a cusp form of weight $\ell+1 = 2k+1$, and $g = \sum_{n\geqslant 0} b(n)q^n$ a holomorphic modular form of weight one. Then $H(f\delta_1^{r-k-1}(g)) = \sum_{n\geqslant 1} c(n)q^n$ with*

$$c(n) = \frac{(-1)^{r-k-1}}{\binom{2r-2}{r-k-1}} n^{r-k-1} \sum_{i+j=n} a(i)b(j)H_{r-k-1,k}\left(\frac{i-j}{i+j}\right),$$

*where*

$$H_{m,k}(t) = \frac{1}{2^m\cdot(m+2k)!}\left(\frac{d}{dt}\right)^{m+2k}[(t^2-1)^m(t-1)^{2k}]$$

*Proof.* From [N3, I.1.2.4, I.1.3.2], we have

$$c(n) = \frac{(r-k-1)!}{(-4\pi)^{r-k-1}}\cdot\frac{(4\pi n)^{2r-1}}{(2r-2)!}\sum_{i+j=n} a(i)b(j)\int_0^\infty p_{r-k-1}(4\pi jy)e^{-4\pi ny}y^{r+k-1}dy,$$

where

$$p_m(x) = \sum_{a=0}^{m}\binom{m}{a}\frac{(-x)^a}{a!}.$$

The integral is evaluated using the following lemma.

**Lemma III.4.** *Let $m, k \geqslant 0$. Then*

$$\int_0^\infty p_m(4\pi jy)e^{-4\pi(i+j)y}y^{m+2k}dy = \frac{(m+2k)!}{(4\pi(i+j))^{m+2k+1}}H_{m,k}\left(\frac{i-j}{i+j}\right)$$

*Proof.* Evaluating the elementary integrals, we find that the left hand side is equal to

$$\frac{m!}{(4\pi(i+j))^{m+2k+1}}G_{m,k}\left(\frac{j}{i+j}\right).$$

where

$$G_{m,k}(t) = \sum_{a=0}^{m}(-1)^a\frac{(m+2k+a)!}{(a!)^2(m-a)!}t^a.$$

It therefore suffices to prove the identity

$$G_{m,k}(t) = \frac{(m+2k)!}{m!}H_{m,k}(1-2t). \tag{3.6}$$

This is proved by showing that both sides satisfy the same defining recurrence relation (and base cases). Indeed, one can check directly that for $m \geqslant 1$:

$$\begin{aligned}(m+1)^2(m+k)G_{m+1,k}(t) =& \\ &(2m+2k+1)[m^2+m+2km+k-(m+k)(2m+2k+2)t]G_{m,k}(t)\\ &-(m+k+1)(m+2k)^2G_{m-1,k}(t).\end{aligned} \tag{3.7}$$

That the right hand side of (3.6) satisfies the same recurrence relation amounts to the well known recurrence relation for the Jacobi polynomials

$$P_n^{(\alpha,\beta)}(t) = \frac{(-1)^n}{2^n n!}(1-t)^{-\alpha}(1+t)^{-\beta}\frac{d^n}{dt^n}\left[(1-t)^\alpha(1+t)^\beta(1-t^2)^n\right].$$

Indeed, we have

$$H_{m,k}(t) = 2^{2k}\cdot P_{m+2k}^{(0,-2k)}(t)(1+t)^{-2k},$$

and one checks that the recurrence relation

$$2(n+1)(n+\beta+1)(2n+\beta)P_{n+1}^{(0,\beta)}(t) = (2n+\beta+1)[(2n+\beta+2)(2n+\beta)t-\beta^2]P_n^{(0,\beta)}(t) - 2n(n+\beta)(2n+\beta+2)P_{n-1}^{(0,\beta)}(t)$$

translates (using $n = m+2k$ and $\beta = -2k$) into the recurrence (3.7) for the polynomials $\frac{(m+2k)!}{m!}H_{m,k}(1-2t)$. ☐

Finally, to prove the proposition, we simply plug in $m = r-k-1$ into the previous lemma and simplify our above expression for $c(n)$. ☐

Recall that for any ideal class $\mathcal{A}$, we have defined

$$r_{\mathcal{A},\chi}(j) = \sum_{\substack{\mathfrak{a}\in\mathcal{A} \\ \mathfrak{a}\subset\mathcal{O} \\ \mathbf{N}(\mathfrak{a})=j}} \chi(\mathfrak{a}).$$

Putting together Lemma III.1, Proposition 11.6, and the manipulation of symbols in [N3, I.6.5], we obtain

$$a_m\left(\int_{\mathbb{Z}_p^\times}\lambda d\tilde{\Psi}_{\mathcal{A}}\right) = \frac{(-1)^{r-k-1}}{\binom{2r-2}{r-k-1}} m^{r-k-1}\left(\frac{D}{-N}\right)\sum_{D=D_1D_2}\left(\frac{D_2}{N\mathfrak{a}}\right)\chi(\mathcal{D}_1)^{-1}\sum_{\substack{j+nN=|D_1|m \\ (p,j)=1}} \sum_{\substack{d|n \\ (p,d)=1}} r_{\mathcal{A}\mathcal{D}_1^{-1},\chi}(j)\left(\frac{D_2}{-dN}\right)\left(\frac{D_1}{|D_2|n/d}\right)\lambda\left(\frac{m|D_1|-nN}{|D_1|d^2}\right) \times H_{r-k-1,k}\left(1-\frac{2nN}{m|D_1|}\right).$$

**Lemma III.5.**

$$r_{\mathcal{A}\mathcal{D}_1^{-1},\chi}(j) = \chi(\mathcal{D}_2)^{-1} r_{\mathcal{A},\chi}(j|D_2|).$$

*Proof.* Since $\mathcal{D}_1$ is 2-torsion in the class group, the left hand side equals $r_{\mathcal{A}\mathcal{D}_1,\chi}(j)$. The lemma now follows from the definitions once one notes that $\mathfrak{b}\mapsto\mathfrak{b}\mathcal{D}_2$ is a bijection from integral ideals of norm $j$ in $\mathcal{A}\mathcal{D}_1$ to integral ideals of norm $j|D_2|$ in $\mathcal{A}\mathcal{D}$. ☐

Using the lemma and also the change of variables employed in [N3], we obtain our version of [N3, Proposition 6.6].

**Proposition III.6.** *If $p|m$, then*

$$a_m\left(\int_{\mathbb{Z}_p^\times}\lambda d\tilde{\Psi}_{\mathcal{A}}\right) = \frac{(-1)^{r-1}}{\binom{2r-2}{r-k-1}} m^{r-k-1}\left(\frac{D}{-N}\right)|D|^{-k}\sum_{\substack{1\leqslant n\leqslant \frac{m|D|}{N} \\ (p,n)=1}} r_{\mathcal{A},\chi}(m|D|-nN) \times H_{r-k-1,k}\left(1-\frac{2nN}{m|D|}\right)\sum_{d|n}\epsilon_{\mathcal{A}}(n,d)\lambda\left(\frac{m|D|-nN}{|D|}\cdot\frac{d^2}{n^2}\right).$$

*where $\epsilon_{\mathcal{A}}(n,d)=0$ if $(d,n/d,|D|)>1$, otherwise*

$$\epsilon_{\mathcal{A}}(n,d) = \left(\frac{D_1}{d}\right)\left(\frac{D_2}{-nN/d}\right)\left(\frac{D_2}{\boldsymbol{N}(\mathcal{A})}\right),$$

*where $(d,|D|)=|D_2|$ and $D=D_1D_2$.*

*Proof.* The proof is as in [N3]. We have also used the fact that $\chi(\mathcal{D}) = D^k$ to get the extra factor of $|D|^{-k}$ and the correct sign (recall that $D$ is negative!). $\square$

**Corollary III.7.** *If* $\left(\frac{D}{N}\right) = 1$ *and* $p|m$*, then*

$$a_m\left(\int_{\mathbb{Z}_p^\times} \log_p d\tilde{\Psi}_{\mathcal{A}}\right) = \frac{(-1)^r}{\binom{2r-2}{r-k-1}} m^{r-k-1}|D|^{-k} \sum_{\substack{1\leqslant n\leqslant \frac{m|D|}{N} \\ (p,n)=1}} r_{\mathcal{A},\chi}(m|D|-nN)\sigma_{\mathcal{A}}(n)H_{r-k-1,k}\left(1-\frac{2nN}{m|D|}\right),$$

*with*

$$\sigma_{\mathcal{A}}(n) = \sum_{d|n} \epsilon_{\mathcal{A}}(n,d)\log_p\left(\frac{n}{d^2}\right).$$

*Proof.* As in [PR1]. $\square$

# CHAPTER IV

## Generalized Heegner cycles

In the previous section we computed Fourier coefficients of $p$-adic modular forms closely related to the derivative of $L_p(f,\chi)$ at the trivial character and in the cyclotomic direction. We expect similar looking expressions to appear as the sum of local heights of certain cycles, with the sum varying over the finite places of $H$ which are prime to $p$.

These cycles should come from the motive attached to $f \otimes \Theta_\chi$. Since $\Theta_\chi$ has weight $2k+1$, work of Deligne and Scholl provides a motive inside the cohomology of a Kuga-Sato variety which is the fiber product of $2k-1$ copies of the universal elliptic curve over $X_1(|D|)$. Instead of using this motive, we work with a closely related motive, which we describe now.

We fix an elliptic curve $A/H$ with the following properties:

1. $\mathrm{End}_H(A) = \mathcal{O}_K$.

2. $A$ has good reduction at primes above $p$.

3. $A$ is isogenous to each of its $\mathrm{Gal}(H/K)$-conjugates.

4. $A^\tau \cong A$, where $\tau$ is complex conjugation.

*Remark* IV.1. Since $D$ is odd, we may even choose such an $A$ with the added feature that $\psi_A^2$ is an unramified Hecke character of type (2,0) (see [R]). In that case, $\psi_A^{2k}$ differs from $\chi$ by a character of $\mathrm{Gal}(H/K)$, so this is perhaps the most natural choice of $A$, given $\chi$. In general, $\psi_A^{2k}\chi^{-1}$ is a finite order Hecke character.

We will use a two-dimensional submotive of $A^{2k}$ whose $\ell$-adic realizations are isomorphic to those of the Deligne-Scholl motive for $\Theta_{\psi_A^{2k}}$ (see [BDP2]).

From Property (3), $A$ is isogenous to $A^\sigma$ over $H$ for each $\sigma \in G := \mathrm{Gal}(H/K)$. If $\sigma$ corresponds to an ideal class $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_K)$ via the Artin map, then one such isogeny $\phi_\mathfrak{a} : A \to A^\sigma$ is given by $A \to A/A[\mathfrak{a}]$, at least if $\mathfrak{a}$ is integral. A different choice of integral ideal $\mathfrak{a}' \in [\mathfrak{a}]$ gives an isomorphic elliptic curve over $H$, and the maps $\phi_\mathfrak{a}$ and $\phi_{\mathfrak{a}'}$ will differ by endomorphisms of $A$ and $A^\sigma$.

As in the introduction, let $Y(N)/\mathbb{Q}$ be the modular curve parametrizing elliptic curves with full level $N$ structure, and let $\mathcal{E} \to Y(N)$ be the universal elliptic curve with level $N$ structure. The canonical non-singular compactification of the $(2r-2)$-fold fiber product

$$\mathcal{E} \times_{Y(N)} \cdots \times_{Y(N)} \mathcal{E},$$

will be denoted by $W = W_{2r-2}$ [Sc]; $W$ is a variety over $\mathbb{Q}$. The map $W \to X(N)$ to the compactified modular curve has geometric fibers (over non-cuspidal points) of the form $E^{2r-2}$, for some elliptic curve $E$. We set

$$X = X_{r,N,k} = W_H \times A^{2k},$$

where $W_H$ is the base change to $H$. Recall the curve $X_0(N)/\mathbb{Q}$, the coarse moduli space of generalized elliptic curves with a cyclic subgroup of order $N$. $X_0(N)$ is the quotient of $X(N)$ by the action of the standard Borel subgroup $B \subset \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$.

The computations of the Fourier coefficients in the previous section suggest that we consider the following *generalized Heegner cycle* on $X$. Fix a Heegner point $y \in Y_0(N)(H)$ represented by a cyclic $N$-isogeny $A \to A'$, for some elliptic curve $A'/H$ with CM by $\mathcal{O}_K$. Such an isogeny exists since each prime dividing $N$ splits in $K$. Also let $\tilde{y}$ be a closed point of $Y(N)_H$ over $y$. The fiber $E_{\tilde{y}}$ of the universal elliptic curve $\mathcal{E} \to Y(N)$ above the point $\tilde{y}$ is isomorphic to $A_F$, where $F \supset H$ is the residue field of $\tilde{y}$. Let

$$\Delta \subset E_{\tilde{y}} \times A_F \cong A_F \times A_F$$

be the diagonal, and we write $\Gamma_{\sqrt{D}} \subset E_{\tilde{y}} \times E_{\tilde{y}}$ for the graph of $\sqrt{D} \in \mathrm{End}(E_{\tilde{y}}) \cong \mathcal{O}_K$. We define

$$Y = \Gamma_{\sqrt{D}}^{r-1-k} \times \Delta^{2k} \subset X_{\tilde{y}} \cong A_F^{2r-2} \times A_F^{2k},$$

so that $Y \in \mathrm{CH}^{k+r}(X_F)$. Here $X_{\tilde{y}}$ is the fiber of the natural projection $X \to X(N)$ above the point $\tilde{y}$.

Since $X$ is not defined over $\mathbb{Q}$, we need to find cycles to play the role of $\mathrm{Gal}(H/K)$-conjugates of $Y$. For each $\sigma \in \mathrm{Gal}(H/K)$ we have a corresponding ideal class $\mathcal{A}$. For each integral ideal $\mathfrak{a} \in \mathcal{A}$, define the cycle $Y^{\mathfrak{a}}$ as follows:

$$Y^{\mathfrak{a}} = \Gamma_{\sqrt{D}}^{r-k-1} \times \left(\Gamma_{\phi_{\mathfrak{a}}}^t\right)^{2k} \subset (A_F^{\mathfrak{a}} \times A_F^{\mathfrak{a}})^{r-k-1} \times (A_F^{\mathfrak{a}} \times A_F)^{2k} = X_{\tilde{y}^{\sigma}} \subset X_F.$$

Here, $\Gamma_{\phi_{\mathfrak{a}}}^t$ is the transpose of $\Gamma_{\phi_{\mathfrak{a}}}$, the graph of $\phi_{\mathfrak{a}} : A \to A^{\mathfrak{a}}$. The cycle $Y^{\mathfrak{a}} \in \mathrm{CH}^{k+r}(X_F)$ is *not* independent of the class of $\mathfrak{a}$ in $\mathrm{Pic}(\mathcal{O}_K)$, but certain expressions involving $Y^{\mathfrak{a}}$ *will* be independent of the class of $\mathfrak{a}$. Note that $Y = Y^{\mathcal{O}_K}$.

*Remark* IV.2. Alternatively, we could have worked with a variety over $K$ whose complex points are

$$W(\mathbb{C}) \times \coprod_{\sigma \in \mathrm{Gal}(H/K)} A^{\sigma}(\mathbb{C}),$$

and which does have an action of $\mathrm{Gal}(H/K)$. In some ways this is a more natural variety to work with (and we expect a similar height formula holds), but we found the height computations to be simpler on our $X$

## 4.1 Projectors

Next we define a projector $\epsilon \in \mathrm{Corr}^0(X, X)_K$ so that $\epsilon Y^{\mathfrak{a}}$ lies in the group $\mathrm{CH}^{r+k}(X_F)_{0,K}$ of homologically trivial $(r+k)$-cycles with coefficients in $K$. Here, $\mathrm{Corr}^0(X, X)_K$ is the ring of degree 0 correspondences with coefficients in $K$. For definitions and conventions concerning motives, correspondences, and projectors see [BDP2, §2].

The projector is defined as $\epsilon = \epsilon_X = \epsilon_W \epsilon_\ell$. Here, $\epsilon_W$ is the pullback to $X$ of the Deligne-Scholl projector $\tilde{\epsilon}_W \in \mathbb{Q}[\mathrm{Aut}(W)]$ which projects onto the subspace of $H^{2r-1}(W)$ coming from modular forms of weight $2r$ (see e.g. [BDP1, §2]). The second factor $\epsilon_\ell$ is the pullback to $X$ of the projector

$$\epsilon_\ell = \left(\frac{\sqrt{D} + [\sqrt{D}]}{2\sqrt{D}}\right)^{\otimes \ell} \circ \left(\frac{1 - [-1]}{2}\right)^{\otimes \ell} \in \mathrm{Corr}^0(A^\ell, A^\ell)_K,$$

denoted by the same symbol. The projector $\epsilon_1 \in \mathrm{Corr}^0(A, A)_K$ projects onto the 1-dimensional $\mathbb{Q}_p$-subspace $V_{\mathfrak{p}}A$ of $H_1(\bar{A}, \mathbb{Q}_p) \cong V_p A$. Here, $\mathfrak{p}$ is the prime of $K$ above $p$ which is determined by our chosen embedding $K \hookrightarrow \bar{\mathbb{Q}}_p$ and

$$V_{\mathfrak{p}}A = \left(\varprojlim_n A[\mathfrak{p}^n]\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is the $\mathfrak{p}$-adic Tate module of $A$. Hence, on the $p$-adic realization of the motive $M_{A^\ell,K}$, $\epsilon_\ell$ projects onto the 1-dimensional $\mathbb{Q}_p$-subspace of

$$H^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)^{\otimes 2k}(k) \subset H^{2k}_{\text{ét}}(\bar{A}^{2k}, \mathbb{Q}_p(k))$$

corresponding (after dualization and twist) to $(V_{\mathfrak{p}}A)^{\otimes 2k}$. See Chapter VII and [BDP2, §1.2] for more details.

We also make use of the projectors

$$\bar{\epsilon}_\ell = \left(\frac{\sqrt{D}-[\sqrt{D}]}{2\sqrt{D}}\right)^{\otimes \ell} \circ \left(\frac{1-[-1]}{2}\right)^{\otimes \ell} \in \mathrm{Corr}^0(A^\ell, A^\ell)_K$$

and $\kappa_\ell = \epsilon_\ell + \bar{\epsilon}_\ell$. The first projects onto $V_{\bar{\mathfrak{p}}}A^{\otimes \ell}$ and the latter onto $V_{\mathfrak{p}}A^{\otimes \ell} \oplus V_{\bar{\mathfrak{p}}}A^{\otimes \ell}$. Set $\bar{\epsilon} = \epsilon_W \bar{\epsilon}_\ell$ and $\epsilon' = \epsilon_W \kappa_\ell$.

*Remark* IV.3. For this remark, suppose that $\chi = \psi^\ell$, where $\psi$ is the $(1,0)$-Hecke character attached to $A$ by the theory of complex multiplication. Recall that this means the $G_H$-action on $H^1(\bar{A}, \mathbb{Q}_p)(1)$ is given by the $(K \otimes \mathbb{Q}_p)^\times$-values Galois character $\psi_H = \psi \circ \mathrm{Nm}_{H/K}$. If we write $\chi_H = \psi_H^\ell$, then the motive $M(\chi_H)$ over $H$ (with coefficients in $K$) from Section 1.2.3 is defined by the triple $(A^{2k}, \kappa_\ell, k)$.

We explain how to descend this to a motive over $K$ with coefficients in $\mathbb{Q}(\chi)$ (this a modification of a construction from an earlier draft of [BDP2]). Let $e_K$ and $\bar{e}_K$ be the idempotents in $K \otimes K$ corresponding to the first and second projections $K \otimes K \cong K \times K \to K$. For each $\sigma \in \mathrm{Gal}(H/K)$ choose an ideal $\mathfrak{a} \subset \mathcal{O}_K$ corresponding to $\sigma$ under the Artin map and define

$$\Gamma(\sigma) := e_K \cdot (\phi_{\mathfrak{a}} \times \cdots \times \phi_{\mathfrak{a}}) \otimes \chi(\mathfrak{a})^{-1} \in \mathrm{Hom}\left(A^\ell, (A^\ell)^\sigma\right) \otimes_{\mathbb{Q}} \mathbb{Q}(\chi)$$

$$\bar{\Gamma}(\sigma) := \bar{e}_K \cdot (\phi_{\mathfrak{a}} \times \cdots \times \phi_{\mathfrak{a}}) \otimes \bar{\chi}(\mathfrak{a})^{-1} \in \mathrm{Hom}\left(A^\ell, (A^\ell)^\sigma\right) \otimes_{\mathbb{Q}} \mathbb{Q}(\chi).$$

Since $\chi(\gamma\mathfrak{a}) = \gamma^\ell \chi(\mathfrak{a})$ and $\phi_{\gamma\mathfrak{a}} = \gamma\phi_{\mathfrak{a}}$, these definitions are independent of the choice of $\mathfrak{a}$. Moreover,

$$\Gamma(\sigma\tau) = \Gamma(\sigma)^\tau \circ \Gamma(\tau)$$

and similarly for $\bar{\Gamma}$. We set

$$\Lambda(\sigma) = \kappa_\ell \circ (\Gamma(\sigma) + \bar{\Gamma}(\sigma)) \circ \kappa_\ell^\sigma \in \mathrm{Corr}^0(A^\ell, (A^\sigma)^\ell)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}(\chi).$$

Then the collection $\{\Lambda(\sigma)\}_\sigma$ gives descent data for the motive $M(\chi_H) \otimes \mathbb{Q}(\chi)$, hence determines a motive $M(\chi)$ over $K$ with coefficients in $\mathbb{Q}(\chi)$. The $p$-adic realization of $M(\chi)$ is $\chi \oplus \bar{\chi}$ where $\chi$ is now thought of as a $\mathbb{Q}(\chi) \otimes \mathbb{Q}_p$-valued character of $G_K$.

Returning to the general situation, we define the sheaf $\mathcal{L} = j_* \mathscr{B}$ on $X(N)$, where

$$\mathscr{B} = \mathrm{Sym}^{2r-2}(R^1 f_* \mathbb{Q}_p)(r-1) \otimes \kappa_\ell H^{2k}_{\text{ét}}(\bar{A}^{2k}, \mathbb{Q}_p(k)),$$

and $j : Y(N) \hookrightarrow X(N)$ and $f : \mathcal{E} \to Y(N)$ are the natural maps.

From now on we drop the subscript 'ét' from all cohomology groups and set $\bar{Z} = Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \bar{k}$ for any variety defined over a field $k$. We also use the notation $V_K = V \otimes K$, for any abelian group $V$.

**Theorem IV.4.** *There is a canonical isomorphism*

$$H^1(\bar{X}(N), \mathcal{L}(1)) \xrightarrow{\sim} \epsilon' H^{2r+2k-1}(\bar{X}, \mathbb{Q}_p)(r+k) = \epsilon' H^*(\bar{X}, \mathbb{Q}_p)(r+k).$$

*Proof.* See [N3, II.2.4] and [BDP1, Prop. 2.4]. □

**Corollary IV.5.** *The cycles $\epsilon Y^{\mathfrak{a}}$ and $\bar{\epsilon} Y^{\mathfrak{a}}$ are homologically trivial on $X_F$, i.e. they lie in the domain of the $p$-adic Abel-Jacobi map*

$$\Phi : \mathrm{CH}^{r+k}(X_F)_{0,K} \to H^1(F, H^{2r+2k-1}(\bar{X}, \mathbb{Q}_p(r+k))).$$

*Proof.* By the theorem, $\epsilon' Y^{\mathfrak{a}}$ is in the kernel of the map

$$\mathrm{CH}^{r+k}(X_F)_K \to H^{2r+2k}(\bar{X}_F, \mathbb{Q}_p(r+k)),$$

i.e. it is homologically trivial. Moreover, $\epsilon = \epsilon\epsilon'$ and $\bar{\epsilon} = \bar{\epsilon}\epsilon'$. Since Abel-Jacobi maps commute with algebraic correspondences, it follows that $\epsilon Y^{\mathfrak{a}}$ and $\bar{\epsilon} Y^{\mathfrak{a}}$ are homologically trivial as well. □

## 4.2 Bloch-Kato Selmer groups

Let $F$ be a finite extension of $\mathbb{Q}_\ell$ ($\ell$ a prime, possibly equal to $p$) and let $V$ be a continuous $p$-adic representation of $G_F := \mathrm{Gal}(\bar{F}/F)$. Recall that $V$ is said to be *unramified* if the inertia subgroup $I_F \subset G_F$ acts trivially on $V$. If $\ell = p$, then $V$ is *crystalline* if

$$\dim_{F_0}(V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_F} = \dim_{\mathbb{Q}_p} V,$$

where $F_0$ is the maximal unramified extension of $\mathbb{Q}_p$ contained in $F$ and $B_{\mathrm{cris}}$ is Fontaine's ring of crystalline periods.

The Bloch-Kato subgroup $H^1_f(F,V) \subset H^1(F,V)$ is then defined to be the kernel of

$$H^1(F,V) \to H^1(F, V \otimes B_{\mathrm{cris}}).$$

For more details and some examples, see [BK] or [N2, 1.12 and 2.1.4]. If $\ell \neq p$ (resp. $\ell = p$) and $V$ is unramified (resp. crystalline), then $H^1_f(F,V) = \mathrm{Ext}^1(\mathbb{Q}_p, V)$ in the category of unramified (resp. crystalline) representations of $G_F$. In other words, $H^1_f(F,V)$ classifies isomorphism classes of extensions which are themselves unramified (resp. crystalline). If instead $F$ is a number field, then $H^1_f(F,V)$ is defined to be the set of classes in $H^1(F,V)$ which restrict to classes in $H^1_f(F_v,V)$ for all finite primes $v$ of $F$.

The Bloch-Kato Selmer group plays an important role in the general theory of $p$-adic heights of homologically trivial algebraic cycles on a smooth projective variety $X/F$ defined over a number field $F$. Indeed, Nekovář's $p$-adic height pairing is only defined on $H^1_f(F,V)$, and not on the Chow group $\mathrm{CH}^j(X)_0$ of homologically trivial cycles of codimension $j$. Here $V = H^{2j-1}(\bar{X}, \mathbb{Q}_p(j))$. This is compatible with the Bloch-Kato conjecture [BK], which asserts (among other, much deeper statements) that the image of the Abel-Jacobi map

$$\Phi : \mathrm{CH}^j(X)_0 \to H^1(F,V)$$

is contained in $H^1_f(F,V)$. The next couple of results follow [N3, II.2] and verify this aspect of the Bloch-Kato conjecture in our situation, allowing us to consider $p$-adic heights of generalized Heegner cycles. We also give a more concrete description of the Abel-Jacobi images of generalized Heegner cycles in terms of local systems on the modular curve.

Denote by $b(Y^{\mathfrak{a}})$ the cohomology class of $\epsilon(\bar{Y}^{\mathfrak{a}})$ in the fiber $\bar{X}_{\tilde{y}}$, so that $b(Y^{\mathfrak{a}})$ lies in

$$\epsilon' H^{2r+2k-2}\left(\bar{X}_{\tilde{y}^\sigma}, \mathbb{Q}_p(r+k-1)\right)^{G(\bar{F}/F)} \xrightarrow{\sim} H^0\left(\overline{\tilde{y}^\sigma}, \mathscr{B}\right)^{G(\bar{F}/F)},$$

where again

$$\mathscr{B} = \mathrm{Sym}^{2r-2}(R^1 f_* \mathbb{Q}_p)(r-1) \otimes \kappa_\ell H^{2k}\left(\bar{A}^{2k}, \mathbb{Q}_p(k)\right),$$

the sheaf on $Y(N)$. The isomorphism above follows from proper base change, Lemma 1.8 of [BDP1], and the Kunneth formula. Similarly, let $\bar{b}(Y^{\mathfrak{a}})$ be the class of $\bar{\epsilon}\bar{Y}^{\mathfrak{a}}$. For the next result, let $j : Y(N) \to X(N)$ be the inclusion.

**Theorem IV.6.** *Set* $V = H^{2r+2k-1}(\bar{X}, \mathbb{Q}_p(r+k))$.

1. *$V$ is a crystalline representation of* $\mathrm{Gal}(\bar{H}_v/H_v)$ *for all* $v|p$.

2. *The Abel-Jacobi images* $z^{\mathfrak{a}} = \Phi(\epsilon Y^{\mathfrak{a}}), \bar{z}^{\mathfrak{a}} = \Phi(\bar{\epsilon} Y^{\mathfrak{a}}) \in H^1(F,V)$ *lie in the subspace* $H^1_f(F,V)$.

3. *The element $z^{\mathfrak{a}}$, thought of as an extension of $p$-adic Galois representations, can be obtained as the pull back of*

$$0 \to H^1(\overline{X(N)}, j_*\mathscr{B})(1) \to H^1(\overline{X(N)} - \overline{\tilde{y}^\sigma}, j_*\mathscr{B})(1) \to H^0(\overline{\tilde{y}^\sigma}, \mathscr{B}) \to 0$$

*by the map* $\mathbb{Q}_p \to H^0\left(\overline{\tilde{y}^\sigma}, \mathscr{B}\right)$ *sending 1 to $b(Y^{\mathfrak{a}})$, and similarly for $\bar{z}^{\mathfrak{a}}$. In particular, $z^{\mathfrak{a}}$ and $\bar{z}^{\mathfrak{a}}$ only depend on $b(Y^{\mathfrak{a}})$ and $\bar{b}(Y^{\mathfrak{a}})$ respectively.*

*Proof.* (1) follows from Faltings' theorem [F] and the fact that $X$ has good reduction at primes above $p$. (2) is a general result due to Nekovář (Niziol also gave a proof), see [N4, Theorem 3.1]. To apply the result one needs to know the weight-monodromy conjecture for $X$ (also known as the purity conjecture). But this conjecture is known for $W$ and $A^\ell$, so it holds for $X$ as well [N4, 3.2]. We note that (2) is ultimately a local statement at each place $v$ of $H$, and for $v|p$, the approach taken in the proof of Theorem IX.10 below gives an alternate proof of this local statement. Statement (3) can be proved exactly as in [N3, II.2.4]. □

**Definition IV.7.** If $F/H$ is a field extension, then a *Tate vector* is an element in $H^0(\bar{y}_0, \mathscr{B})^{\mathrm{Gal}(\bar{F}/F)}$ for some $y_0 \in Y(N)(F)$. A *Tate cycle* is a formal finite sum of Tate vectors over $F$. The group of Tate cycles is denoted $Z(Y(N), F)$.

Let $\pi : X(N) \to X_0(N) = X(N)/B$ be the quotient map, and as in [N3], define $\epsilon_B = (\#B)^{-1} \sum_{g \in B} g$, which acts on $X(N)$ and its cohomology. Set $\mathscr{A} = (\pi_* \mathscr{B})^B$, $a(Y^{\mathfrak{a}}) = \epsilon_B b(Y^{\mathfrak{a}})$, and $\bar{a}(Y^{\mathfrak{a}}) = \epsilon_B \bar{b}(Y^{\mathfrak{a}})$. We define the group $Z(Y_0(N), F)$ of Tate cycles on $Y_0(N)$ exactly as for $Y(N)$, but with $\mathscr{B}$ replaced by $\mathscr{A}$. Let $j_0 : Y_0(N) \to X_0(N)$ be the inclusion. Note that $a(Y^{\mathfrak{a}})$ is an element of $Z(Y(N), H)$, not just $Z(Y(N), F)$.

**Proposition IV.8.** *The element*

$$\Phi(\epsilon_B \epsilon Y^{\mathfrak{a}}) \in H^1\left(H, H^1\left(\overline{X_0(N)}, (j_0)_* \mathscr{A}\right)(1)\right),$$

*thought of as an extension of p-adic Galois representations, can be obtained as the pull back of*

$$0 \to H^1\left(\overline{X_0(N)}, j_* \mathscr{A}\right)(1) \to H^1\left(\overline{X_0(N)} - \bar{y}^\sigma, j_* \mathscr{A}\right)(1) \to H^0(\bar{y}^\sigma, \mathscr{A}) \to 0$$

*by the map $\mathbb{Q}_p \to H^0\left(\overline{y^\sigma}, \mathscr{A}\right)$ sending 1 to $a(Y^{\mathfrak{a}})$. In particular, $\Phi(\epsilon_B \epsilon Y^{\mathfrak{a}})$ only depends on $a(Y^{\mathfrak{a}})$. Similarly, $\Phi(\epsilon_B \bar{\epsilon} Y^{\mathfrak{a}})$ depends only on $\bar{a}(Y^{\mathfrak{a}})$.*

In fact, for any field $F/H$ one can define a map

$$\Phi_T : Z(Y_0(N), F) \to H^1(F, H^1(\bar{X}_0(N), j_{0*}\mathscr{A})(1)),$$

by pulling back the appropriate exact sequence as above. We then have $\Phi(\epsilon_B \epsilon Y^{\mathfrak{a}}) = \Phi_T(a(Y^{\mathfrak{a}}))$ and $\Phi(\epsilon_B \bar{\epsilon} Y^{\mathfrak{a}}) = \Phi_T(\bar{a} Y^{\mathfrak{a}})$. For more detail, see [N3, II.2.6].

## 4.3 Hecke operators

The Hecke operators on $W_{2r-2}$ from [N3] pull back to give Hecke operators $T_m$ on $X$. The $T_m$ are correspondences on $X$; they act on Chow groups and cohomology groups and commute with Abel-Jacobi maps. To describe the action of the Hecke algebra $\mathbb{T}$ on Tate vectors, we need to say what $T_m$ does to an element of $H^0(\bar{y}_0, \mathscr{A})^{G(\bar{F}/F)}$ for an arbitrary point $y_0 \in X_0(N)(F)$, $F$ an extension of $H$. Such an element is represented by a triple $(E, C, b)$ where $E$ is an elliptic curve, $C$ is a subgroup of order $N$, and

$$b \in \mathrm{Sym}^w(H^1(\bar{E}, \mathbb{Q}_p))(r-1) \otimes \kappa_\ell H^{2k}(\bar{A}, \mathbb{Q}_p)(k).$$

As the Hecke operators are defined via base change from those on $W_{2r-2}$, we have:

$$T_m(E, C, b) = \sum_{\substack{\lambda : E \to E' \\ \deg(\lambda) = m}} (E', \lambda(C), (\lambda^w \times \mathrm{id})_*(b)),$$

where we are using the map $\lambda^w \times \mathrm{id} : E^w \times A^\ell \to E'^w \times A^\ell$.

Now set $V_{r,A,\ell} = \epsilon_B \epsilon' V = H^1(\overline{X_0(N)}, (j_0)_* \mathscr{A})(1)$, a subrepresentation of $V$. Then $z^{\mathfrak{a}} := \Phi(\epsilon_B \epsilon Y^{\mathfrak{a}})$ lands in the Bloch-Kato subspace $H^1_f(H, V_{r,A,\ell}) \subset H^1(H, V_{r,A,\ell})$, by Proposition IV.6.

For any newform $f \in S_{2r}(\Gamma_0(N))$, we let $V_{f,A,\ell}$ be the $f$-isotypic component of $V_{r,A,\ell}$ with respect to the action of $\mathbb{T}$. By this, we mean

$$V_{f,A,\ell} = V_f \otimes_{\mathbb{Q}_p(f)} \kappa_\ell H^{2k}(\bar{A}, \mathbb{Q}_p(k))_{\mathbb{Q}_p(f)},$$

where $V_f \subset V_{r,A,0}$ is the Galois representation (with coefficients in the $p$-adic field $\mathbb{Q}_p(f)$) attached to $f$ (see e.g. [N3, II.2.11]).

Consider the $f$-isotypic Abel-Jacobi map

$$\Phi_f : \mathrm{CH}^{r+k}(X)_{0,K} \to H^1_f(H, V_{f,A,\ell}),$$

and set $z_f^{\mathfrak{a}} = \Phi_f(\epsilon_B \epsilon Y^{\mathfrak{a}})$ and $\bar{z}_f^{\mathfrak{a}} = \Phi_f(\epsilon_B \bar{\epsilon} Y^{\mathfrak{a}})$.

As is shown in Chapter VII, the $p$-adic representation $V_{f,A,\ell}$ is ordinary and satisfies $V_{f,A,\ell} \cong V^*_{f,A,\ell}(1)$. The results of [N2] therefore give a symmetric pairing

$$\langle\, , \rangle_{\ell_K} : H^1_f(H, V_{f,A,\ell}) \times H^1_f(H, V_{f,A,\ell}) \to \mathbb{Q}_p(f),$$

depending on a choice of logarithm $\ell_H : \mathbb{A}_H^\times / H^\times \to \mathbb{Q}_p$ and the canonical splitting of the local Hodge filtrations at places $v$ of $H$ above $p$. We will always assume $\ell_H = \ell_K \circ \mathrm{Nm}_{H/K}$ for some $\ell_K : \mathbb{A}_K^\times / K^\times \to \mathbb{Q}_p$, which explains the notation above. We sometimes omit the dependence on $\ell_K$ in the notation, if a choice has been fixed. If $a, b \in Z(Y_0(N), F)$ are two Tate cycles, then we will write $\langle a, b\rangle_{\ell_K}$ for $\langle \Phi_T(a), \Phi_T(b)\rangle_{\ell_K}$.

## 4.4 Properties of generalized Heegner cycles

Here we collect some facts about generalized Heegner cycles and their corresponding cohomology classes. We first recall the intersection theory on products of elliptic curves; see [N3, II.3] for proofs.

Let $E, E', E''$ be elliptic curves over an algbraically closed field $k$ of characteristic not $p$, and set

$$H^i(Y) = H^i_{\text{ét}}(Y, \mathbb{Q}_p) = \left( \varprojlim_n H^i_{\text{ét}}(Y, \mathbb{Z}/p^n\mathbb{Z}) \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

for any variety $Y/k$. A pair $(\alpha, \beta)$ of isogenies $\alpha \in \mathrm{Hom}(E'', E)$ and $\beta \in \mathrm{Hom}(E'', E')$, determines a cycle

$$\Gamma_{\alpha,\beta} = (\alpha, \beta)_*(1) \in \mathrm{CH}^1(E \times E'),$$

where $(\alpha, \beta)_* : \mathrm{CH}^0(E'') \to \mathrm{CH}^1(E \times E')$ is the push forward. The image of $\Gamma_{\alpha,\beta}$ under the cycle class map $\mathrm{CH}^1(E \times E') \to H^2(E \times E')(1)$ will be denoted by $[\Gamma_{\alpha,\beta}]$. Also let $X_{\alpha,\beta}$ be the projection of $[\Gamma_{\alpha,\beta}]$ to $H^1(E) \otimes H^1(E')(1)$, i.e.

$$X_{\alpha,\beta} = [\Gamma_{\alpha,\beta}] - \deg(\alpha)h - \deg(\beta)v,$$

where $h$ is the horizontal class $[\Gamma_{1,0}]$ and $v$ is the vertical class $[\Gamma_{0,1}]$. If $\alpha \in \mathrm{Hom}(E, E')$, we write $\Gamma_\alpha$ and $X_\alpha$ for $\Gamma_{1,\alpha}$ and $X_{1,\alpha}$, respectively. If $\beta \in \mathrm{Hom}(E', E)$ we write $\Gamma^t_\beta$ and $X^t_\beta$ for $\Gamma_{\beta,1}$ and $X_{\beta,1}$, respectively. Finally, let

$$(\, , \,) : H^2(E \times E')(1) \times H^2(E \times E')(1) \to \mathbb{Q}_p,$$

be the non-degenerate cup product pairing.

**Proposition IV.9.** *With notation as above,*

1. *The map*
$$\mathrm{Hom}(E'', E) \times \mathrm{Hom}(E'', E') \to H^1(E) \otimes H^1(E')(1)$$
*given by* $(\alpha, \beta) \mapsto X_{\alpha,\beta}$ *is biadditive.*

2. *The map* $\mathrm{Hom}(E,E') \to H^1(E)\times H^1(E')(1)$ *given by* $\alpha \mapsto X_\alpha$ *is an injective group homomorphism.*

3. *If* $E = E'$, *then* $X_{\alpha,\beta} = X_{\beta\hat{\alpha}}$ *and* $(X_\alpha, X_\beta) = -\mathrm{Tr}(\alpha\hat{\beta})$ *for all* $\alpha,\beta \in \mathrm{End}(E)$.

*Here,* $\mathrm{Tr} : \mathrm{End}(E) \to \mathbb{Z}$ *is the map* $\alpha \mapsto \alpha + \hat{\alpha}$.

It is convenient to think of $H^1(E)$ as $V_pE^* = \mathrm{Hom}(V_pE, \mathbb{Q}_p)$, where $V_pE = T_pE \otimes \mathbb{Q}_p$ is the $p$-adic Tate module. The Weil pairing

$$V_pE \times V_pE \to \mathbb{Q}_p(1)$$

gives identifications $V_pE^*(1) \cong V_pE$ and $\bigwedge^2 V_pE \cong \mathbb{Q}_p(1)$. We then have the following diagram of isomorphisms

$$\begin{array}{ccccc}
(V_pE \otimes V_pE)(-1) & \longrightarrow & \left(\mathrm{Sym}^2 V_pE \oplus \bigwedge^2 V_pE\right)(-1) & \longrightarrow & \mathrm{Sym}^2 V_pE(-1) \oplus \mathbb{Q}_p \\
\downarrow & & & & \downarrow \delta \\
V_pE^* \otimes V_pE & \longrightarrow & \mathrm{End}(V_pE) & \longrightarrow & \mathrm{End}_0(V_pE) \oplus \mathbb{Q}_p
\end{array}$$

One checks that $\delta$ identifies $\mathrm{Sym}^2 V_pE(-1)$ with the space $\mathrm{End}_0(V_pE)$ of traceless endomorphisms of $V_pE$. Now suppose that $E$ has complex multiplication by $\mathcal{O}_K$ and that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in $K$. Then

$$V_pE = V_{\mathfrak{p}}E \oplus V_{\bar{\mathfrak{p}}}E,$$

where $V_{\mathfrak{p}} = \varprojlim E[\mathfrak{p}^n] \otimes \mathbb{Q}_p$ and $V_{\bar{\mathfrak{p}}} = \varprojlim E[\bar{\mathfrak{p}}^n] \otimes \mathbb{Q}_p$. Let $x^*$ and $y^*$ be a basis for $V_{\mathfrak{p}}E$ and $V_{\bar{\mathfrak{p}}}E$ respectively, and let $x, y$ be the dual basis of $H^1(E)$ arising from the Weil pairing. Since the Weil pairing is non-degenerate, we may assume that $e(x^*, y^*) = 1 \in \mathbb{Q}_p$.

If $\alpha \in \mathrm{End}(E)$, then the class $X_\alpha \in H^1(E) \otimes H^1(E)(1)$, when thought of as an element of $\mathrm{End}(V_pE)$ via the isomorphisms above, is simply the map $V\alpha : V_pE \to V_pE$ induced on Tate modules. Thus, $X_1 = \lambda(x \otimes y - y \otimes x)$ for some $\lambda \in \mathbb{Q}_p$. Recall that one can compute the intersection pairing on $H^1(E)^{\otimes 2}$ in terms of the cup product on $H^1(E)$:

$$(a \otimes b, c \otimes d) = -(a \cup c)(b \cup d).$$

Since $(X_1, X_1) = -2$, we conclude that $\lambda = 1$. Next we claim that

$$X_{\sqrt{D}} = \pm\sqrt{D}(x \otimes y + y \otimes x). \tag{4.1}$$

To prove this, it suffices to show that $V\sqrt{D}$ acts on $V_{\mathfrak{p}}$ by $\sqrt{D}$ and on $V_{\bar{\mathfrak{p}}}$ by $-\sqrt{D}$. Indeed, under the identifications

$$H^1(E) \otimes H^1(E)(1) \cong V_pE^* \otimes V_pE^*(1) \cong V_pE^* \otimes V_pE \cong \mathrm{End}(V_pE),$$

$x \otimes y$ corresponds to the element $f \in \mathrm{End}(V_p)$ such that $f(ax^* + by^*) = ax^*$ whereas $y \otimes x$ corresponds to $g \in \mathrm{End}(V_p)$ such that $g(ax^* + by^*) = -by^*$.

To understand how $V\sqrt{D}$ acts on $V_{\mathfrak{p}}$, write $\mathfrak{p}^n = p^n\mathbb{Z} + \frac{b+\sqrt{D}}{2}\mathbb{Z}$ for some $b, c \in \mathbb{Z}$ such that $b^2 - 4p^nc = D$, which is possible because $p$ splits in $K$. For $P \in E[\mathfrak{p}^n]$, one has $(b + \sqrt{D})(P) = 0$, so $\sqrt{D}(P) = -bP$. Since $b \equiv \pm\sqrt{D} \pmod{\mathfrak{p}^n}$, it follows upon taking a limit that $(V\sqrt{D})(x^*) = \pm\sqrt{D}x^*$. Since we can write $\bar{\mathfrak{p}}^n = p^n\mathbb{Z} + \frac{b-\sqrt{D}}{2}\mathbb{Z}$, we also have $(V\sqrt{D})(y^*) = \mp\sqrt{D}y^*$, and this proves the claim. Hence

$$X_\gamma = \gamma(x \otimes y) - \bar{\gamma}(y \otimes x) \in H^1(E) \otimes H^1(E)(1),$$

for all $\gamma \in \mathcal{O}_K \hookrightarrow \mathrm{End}(E)$.

Finally, note that the projector $\epsilon_1 \in \mathrm{Corr}^0(E,E)_K$ defined earlier acts on $H^1(E)$ as projection onto $V_{\mathfrak{p}}$.

**Proposition IV.10.** *Let $\mathfrak{a} \subset \mathcal{O}_K$ be an ideal and $\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)$ its ideal class. Then the elements*

$$z^{\mathcal{A}}_{f,\chi} = \chi(\mathfrak{a})^{-1} z^{\mathfrak{a}}_f \quad \text{and} \quad z^{\mathcal{A}}_{f,\bar{\chi}} = \bar{\chi}(\mathfrak{a})^{-1} \bar{z}^{\mathfrak{a}}_f$$

*in $H^1_f(H, V_{f,A,\ell})_{\bar{\mathbb{Q}}_p}$ depend only on $\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)$.*

*Proof.* To prove the proposition for $z^{\mathcal{A}}_{f,\chi}$, we wish to relate $z^{\mathfrak{a}}_f$ to $z^{\mathfrak{a}(\gamma)}_f$ for some $\gamma \in \mathcal{O}_K$ and some integral ideal $\mathfrak{a}$. The contribution to $z^{\mathfrak{a}}_f$ from one of the "generalized" components $\Gamma^t_{\phi_{\mathfrak{a}}} \subset A^{\mathfrak{a}} \times A$ is $\epsilon X_{\phi_{\mathfrak{a}},1}$, where $X_{\phi_{\mathfrak{a}},1} \in H^1(\bar{A}^{\mathfrak{a}}, \mathbb{Q}_p) \otimes H^1(\bar{A}, \mathbb{Q}_p)$ is the class of

$$\Gamma^t_{\phi_{\mathfrak{a}}} - \deg(\phi_{\mathfrak{a}})h - v \in \mathrm{CH}^1(A^{\mathfrak{a}} \times A),$$

as above. Let $x, y$ be a basis of $H^1(\bar{A}, \mathbb{Q}_p)$ such that

$$X_{\gamma,1} = \bar{\gamma}(x \otimes y) - \gamma(y \otimes x) \in H^1(\bar{A}, \mathbb{Q}_p) \otimes H^1(\bar{A}, \mathbb{Q}_p),$$

for all $\gamma \in \mathcal{O}_K$. Let $x_{\mathfrak{a}}, y_{\mathfrak{a}}$ be the basis of $H^1(\bar{A}^{\mathfrak{a}}, \mathbb{Q}_p)$ corresponding to $x, y$ under the isomorphism $\phi^*_{\mathfrak{a}} : H^1(\bar{A}^{\mathfrak{a}}, \mathbb{Q}_p) \to H^1(\bar{A}, \mathbb{Q}_p)$. One checks that

$$(\phi_{\mathfrak{a}} \times \mathrm{id})^*(X_{\phi_{\mathfrak{a}},1}) = \deg(\phi_{\mathfrak{a}}) X_{1,1}$$

and so

$$X_{\phi_{\mathfrak{a}},1} = \deg(\phi_{\mathfrak{a}}) \left( x_{\mathfrak{a}} \otimes y - y_{\mathfrak{a}} \otimes x \right).$$

Similarly,

$$X_{\phi_{\mathfrak{a}(\gamma)},1} = X_{\gamma\phi_{\mathfrak{a}},1} = \deg(\phi_{\mathfrak{a}}) \left( \bar{\gamma}(x_{\mathfrak{a}} \otimes y) - \gamma(y_{\mathfrak{a}} \otimes x) \right).$$

Since the projector $\epsilon$ kills $y$, we find that $\epsilon X_{\gamma\phi_{\mathfrak{a}},1} = \gamma \epsilon X_{\phi_{\mathfrak{a}},1}$. In the components which come purely from the Kuga-Sato variety $W_{2r-2}$, the two cycles $Y^{\mathfrak{a}}$ and $Y^{\mathfrak{a}(\gamma)}$ are identical – they both have the form $\epsilon \Gamma^{r-k-1}_{\sqrt{D}}$. Taking the tensor product of the $\ell$ "generalized" components and the $r - k - 1$ Kuga-Sato components, we conclude that

$$z^{\mathfrak{a}(\gamma)}_f = \gamma^{\ell} z^{\mathfrak{a}}_f,$$

as desired. The proof for $z^{\mathcal{A}}_{f,\bar{\chi}}$ is similar: since $\bar{z}^{\mathfrak{a}}_f$ is defined using $\bar{\epsilon}$ instead of $\epsilon$, the extra factor of $\bar{\gamma}^{\ell}$ which pops out is accounted for by the factor $\bar{\chi}(\mathfrak{a})^{-1}$. □

**Lemma IV.11.** *For any ideal classes $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathrm{Pic}(\mathcal{O}_K)$, we have*

$$\left\langle z^{\mathcal{A}}_{f,\chi}, z^{\mathcal{B}}_{f,\bar{\chi}} \right\rangle = \left\langle z^{\mathcal{AC}}_{f,\chi}, z^{\mathcal{BC}}_{f,\bar{\chi}} \right\rangle$$

*Proof.* It suffices to prove $\left\langle z^{\mathrm{id}}_{f,\chi}, z^{\mathcal{B}}_{f,\bar{\chi}} \right\rangle = \left\langle z^{\mathcal{A}}_{f,\chi}, z^{\mathcal{BA}}_{f,\bar{\chi}} \right\rangle$ for all $\mathcal{A}, \mathcal{B} \in \mathrm{Pic}(\mathcal{O}_K)$. Equivalently, we must show

$$\mathrm{Nm}(\mathfrak{a})^{\ell} \left\langle z^{\mathcal{O}_K}_f, \bar{z}^{\mathfrak{b}}_f \right\rangle = \left\langle z^{\mathfrak{a}}_f, \bar{z}^{\mathfrak{ba}}_f \right\rangle, \tag{4.2}$$

for all integral ideals $\mathfrak{a}$ and $\mathfrak{b}$. Let $\sigma \in \mathrm{Gal}(\bar{K}/K)$ restrict to an element of $\mathrm{Gal}(H/K)$ which corresponds to $\mathfrak{a}$ under the Artin map. Consider the homomorphisms of Chow groups

$$\sigma : \mathrm{CH}^*(\overline{W \times A^{\ell}})_K \to \mathrm{CH}^*(\overline{W \times (A^{\sigma})^{\ell}})_K$$

and

$$\xi = (\mathrm{id} \times \phi^{\ell}_{\mathfrak{a}})^* : \mathrm{CH}^*(\overline{W \times (A^{\sigma})^{\ell}})_K \to \mathrm{CH}^*(\overline{W \times A^{\ell}})_K.$$

After identifying $A^{\sigma}$ with $A^{\mathfrak{a}}$, one checks that $(\xi \circ \sigma)(Y^{\mathfrak{b}}) = Y^{\mathfrak{ab}}$. Indeed, since $\mathfrak{a}$ and $\mathfrak{b}$ are integral, the graph of $\phi^{\sigma}_{\mathfrak{b}} : A^{\sigma} \to (A^{\mathfrak{b}})^{\sigma}$ can be identified with the graph of the projection map $\phi : A/A[\mathfrak{a}] \to A/A[\mathfrak{ab}]$ (first note the two isogenies have the same kernel and then use the main theorem of complex multiplication). The latter is pulled back to $\Gamma_{\phi_{\mathfrak{ab}}}$ by $(\mathrm{id} \times \phi_{\mathfrak{a}})^*$. It follows that $(\xi \circ \sigma)(Y^{\mathfrak{b}}) = Y^{\mathfrak{ab}}$, and the identity therefore holds for the corresponding cohomology classes. On cohomology, $\sigma$ and $\xi$ are isomorphisms, so (4.2) follows from the functoriality of $p$-adic heights [N2, Theorem 4.11]. We are using the fact that $\left(\hat{\phi}^{\ell}_{\mathfrak{a}}\right)^*$ is adjoint to $\left(\phi^{\ell}_{\mathfrak{a}}\right)^*$ under the pairing given by Poincaré duality, and that $\deg \phi_{\mathfrak{a}} = \mathrm{Nm}(\mathfrak{a})$. □

The goal now is to compute $\langle z_{f,\chi}, z_{f,\bar{\chi}}\rangle$, where

$$z_{f,\chi} = \frac{1}{h}\sum_{\mathcal{A}\in \mathrm{Pic}(\mathcal{O}_K)} z_{f,\chi}^{\mathcal{A}} \quad \text{and} \quad z_{f,\bar{\chi}} = \frac{1}{h}\sum_{\mathcal{A}\in \mathrm{Pic}(\mathcal{O}_K)} z_{f,\bar{\chi}}^{\mathcal{A}}.$$

Here, we have extended the $p$-adic height $\bar{\mathbb{Q}}_p$-linearly.

Let $\tau \in \mathrm{Gal}(H/\mathbb{Q})$ be a lift of the generator of $\mathrm{Gal}(K/\mathbb{Q})$. As $A$ and $W$ are defined over $\mathbb{R}$, $\tau$ acts on $X = W \times A^\ell$ and its cohomology.

**Lemma IV.12.** *Let $\mathfrak{n} \subset \mathcal{O}_K$ be the ideal of norm $N$ corresponding to the Heegner point $y \in X_0(N)$, and let $(-1)^r\epsilon_f$ be the sign of the functional equation for $L(f,s)$. Then*

$$\tau(z_{f,\chi}^{\mathcal{A}}) = (-1)^{r-k-1}\epsilon_f\chi(\mathfrak{n})N^{-k}z_{f,\bar{\chi}}^{\mathcal{A}^{-1}[\bar{\mathfrak{n}}]}$$

*and*

$$\tau(z_{f,\bar{\chi}}^{\mathcal{A}}) = (-1)^{r-k-1}\epsilon_f\bar{\chi}(\mathfrak{n})N^{-k}z_{f,\chi}^{\mathcal{A}^{-1}[\bar{\mathfrak{n}}]}.$$

*Proof.* Let $W_j^0(N)$ be the Kuga-Sato variety over $X_0(N)$, i.e. the quotient of $W_j$ by the action of the Borel subgroup $B$. Recall the map $W_N : W_j^0 \to W_j^0$ which sends a point $P \in \bar{E}^j$ in the fiber above a diagram $\phi : E \to E/E[\mathfrak{n}]$ to the point $\phi^j(P)$ in the fiber above the diagram $\hat{\phi} : E/E[\mathfrak{n}] \to E/E[N]$. Meanwhile, complex conjugation sends the Heegner point $A^{\mathfrak{a}} \to A^{\mathfrak{a}}/A^{\mathfrak{a}}[\mathfrak{n}]$ to the Heegner point $A^{\bar{\mathfrak{a}}} \to A^{\bar{\mathfrak{a}}}/A^{\bar{\mathfrak{a}}}[\bar{\mathfrak{n}}]$. Thus on a generalized component of our cycle, we have

$$(W_N \times \mathrm{id})^*(X_{\phi_{\bar{\mathfrak{a}}\bar{\mathfrak{n}}},1}) = NX_{\phi_{\bar{\mathfrak{a}}},1} = N\tau(X_{\phi_{\mathfrak{a}},1}),$$

where these objects are thought of as Chow cycles on $X$ which are supported on the fiber of $X$ above $(\tilde{y})^{\sigma\tau}$. Since $\tau$ takes $V_{\mathfrak{p}}A$ to $V_{\bar{\mathfrak{p}}}A$, we even have

$$(W_N \times \mathrm{id})^*(\bar{\epsilon}_1 X_{\phi_{\bar{\mathfrak{a}}\bar{\mathfrak{n}}},1}) = N\bar{\epsilon}_1 X_{\phi_{\bar{\mathfrak{a}}},1} = N\tau(\epsilon_1 X_{\phi_{\mathfrak{a}},1}).$$

On the purely Kuga-Sato components, one computes [N1, 6.2]

$$W_N^*(X_{\sqrt{D}}) = NX_{\sqrt{D}} = -N\tau(X_{\sqrt{D}}),$$

where the $X_{\sqrt{D}}$ in the equation above are supported on $\tilde{y}^{\mathrm{Frob}(\bar{\mathfrak{a}}\bar{\mathfrak{n}})}$, $\tilde{y}^{\mathrm{Frob}(\bar{\mathfrak{a}})}$, and $\tilde{y}^{\mathrm{Frob}(\mathfrak{a})}$ respectively.

On the other hand, $(W_N \times \mathrm{id})^2 = [N] \times \mathrm{id}$, where $[N] : W_{2r-2}^0 \to W_{2r-2}^0$ is multiplication by $N$ in each fiber. On cycles and cohomology, $[N] \times \mathrm{id}$ acts as multiplication by $N^{2r-2}$. Since $W_N$ commutes with the Hecke operators, we see that $(W_N \times \mathrm{id})$ acts as multiplication by $\pm N^{r-1}$ on the $f$-isotypic part of cohomology, and this sign is well known to equal $\epsilon_f$. Putting things together, we obtain

$$\tau(z_f^{\mathfrak{a}}) = \frac{(-1)^{r-k-1}(W_N \times \mathrm{id})^*(\bar{z}_f^{\bar{\mathfrak{a}}\bar{\mathfrak{n}}})}{N^{2k+r-k-1}} = \frac{(-1)^{r-k-1}\epsilon_f\bar{z}_f^{\bar{\mathfrak{a}}\bar{\mathfrak{n}}}}{N^k},$$

from which the first identity in the lemma follows. The proof of the second identity is entirely analogous. □

**Theorem IV.13.** *If $\ell_K : \mathbb{A}_K^\times/K^\times \to \mathbb{Q}_p$ is anticyclotomic, i.e. $\ell_K \circ \tau|_K = -\ell_K$, then*

$$\langle z_{f,\chi}, z_{f,\bar{\chi}}\rangle_{\ell_K} = 0.$$

*In particular, Theorem I.7 holds for such $\ell_K$.*

*Proof.* From the previous lemma we have

$$\tau(z_{f,\chi}) = (-1)^{r-k-1}\epsilon_f\chi(\mathfrak{n})N^{-k}z_{f,\bar{\chi}}$$

and

$$\tau(z_{f,\bar{\chi}}) = (-1)^{r-k-1}\epsilon_f\bar{\chi}(\mathfrak{n})N^{-k}z_{f,\chi}.$$

Thus

$$\langle z_{f,\chi}, z_{f,\bar{\chi}}\rangle_{\ell_K} = \langle \tau(z_{f,\chi}), \tau(z_{f,\bar{\chi}})\rangle_{\ell_K\circ\tau} = \langle z_{f,\bar{\chi}}, z_{f,\chi}\rangle_{-\ell_K} = -\langle z_{f,\chi}, z_{f,\bar{\chi}}\rangle_{\ell_K},$$

which proves the vanishing. Theorem I.7 now follows from Corollary II.6. □

Since any logarithm $\ell_K$ can be decomposed into a sum of a cyclotomic and an anticyclotomic logarithm, it now suffices to prove Theorem I.7 for cyclotomic $\ell_K$, i.e. we may assume $\ell_K = \ell_K \circ \tau|_K$. By Lemma IV.11 we have

$$\langle z_{f,\chi}, z_{f,\bar{\chi}} \rangle = \left\langle z_{f,\chi}^{\mathcal{O}_K}, z_{f,\bar{\chi}} \right\rangle = \frac{1}{h} \sum_{\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)} \left\langle z_f, z_{f,\bar{\chi}}^{\mathcal{A}} \right\rangle. \tag{4.3}$$

The height $\langle\, , \rangle$ can be written as a sum of local heights:

$$\langle x, y \rangle = \sum_v \langle x, y \rangle_v,$$

where $v$ varies over the *finite* places of $H$. These local heights are defined in general in [N2] and computed explicitly for cyclotomic $\ell_K$ in [N3, Proposition II.2.16] in a situation similar to ours. In the next chapter we recall the definition of the local heights, and in the following chapter we compute the local heights $\langle z_f, z_{f,\bar{\chi}}^{\mathcal{A}} \rangle_v$ for finite places $v$ of $H$ not dividing $p$. The contribution from local heights at places $v|p$ will be treated in Chapter IX.

# CHAPTER V

# $p$-adic height pairings

## 5.1 Definition of local height pairings

In this section we recall the definition of the local $p$-adic heights pairings, at least in the cases that will concern us later. For more details, see [N2, §4] and [N3, II.1].

Let $F \supset H$ be a number field and set $V = H^{2r+2k-1}_{\text{ét}}(\bar{X}, \mathbb{Q}_p(r+k))$, thought of as a representation of $G_F := \mathrm{Gal}(\bar{F}/F)$. By Poincaré duality, we have $V \cong V^*(1)$. We will recall the definition of heights for the representation $V$; heights for the representation $V_{f,A,\ell}$ (which, after enlarging the coefficient field a bit, is a subrepresentation) are defined similarly.

To define Nekovář's global bilinear $p$-adic height pairing

$$\langle \, , \rangle : H^1_f(F, V) \times H^1_f(F, V) \to \mathbb{Q}_p$$

one needs two pieces of data:

- A continuous homomoprhism $\ell_F : \mathbb{A}_F^\times / F^\times \to \mathbb{Q}_p$, which we choose to view as collection of maps $\ell_v : F_v^\times \to \mathbb{Q}_p$ satisfying $\sum_v \ell_v(a) = 0$ for $a \in F^\times$.
- For each place $v|p$ of $F$, a $\mathbb{Q}_p$-linear splitting of the Hodge filtration

$$0 \to F^0 DR(V_v) \to DR(V_v) \to DR(V_v)/F^0 \to 0,$$

  where $V_v$ is the restriction of $V$ to a representation of $G_{F_v}$, and

$$DR(V_v) = \left(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}\right)^{G_{F_v}}.$$

By [F], we have an identification

$$DR(V_v) \cong H^{2r+2k-1}_{\mathrm{dR}}(X \otimes_F F_v / F_v),$$

under which the filtration $F^i DR(V_v)$ on $DR(V_v)$ (coming from the grading on $B_{\mathrm{dR}}$) is identified with the usual Hodge filtration (up to twist).

This pairing decomposes into a sum of local pairings, one for each *finite* place of $F$:

$$\langle \, , \rangle = \sum_v \langle \, , \rangle_v.$$

More precisely, suppose $a, b \in H^1_f(F, V)$ are Selmer classes. The class $a$ corresponds to an extension

$$0 \to V \to E_a \to \mathbb{Q}_p \to 0,$$

and dualizing the extension corresponding to $b$, we get

$$0 \to \mathbb{Q}_p(1) \to E_b^*(1) \to V \to 0.$$

As a consequence of the fact that $H^1_f(F_v, V)$ annihilates $H^1_f(F_v, V^*(1))$ via local duality, one may choose a "mixed extension" $E$ fitting in the commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & E_b^*(1) & \longrightarrow & V & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & E & \longrightarrow & E_a & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow{\scriptstyle h} & & \\
 & & & & \mathbb{Q}_p & = & \mathbb{Q}_p & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0, & &
\end{array}
$$

This choice of $E$ is not unique, but we fix one such $E$ (which we stress is a representation of $G_F$). The local height pairings to be defined depend on this choice of global mixed extension. However, the global pairing (i.e. the sum of the local pairings) will be independent of the choice of $E$.

**Proposition V.1.** *If $v$ does not divide $p$, then $H^1(F_v, V) = 0$.*

*Proof.* This is an easy application of local duality and the local Euler characteristic formula, once we know that $H^0(F_v, V) = 0$. If $X$ has good reduction at $v$, the latter follows from the Riemann hypothesis over finite fields, proved by Deligne. In general, this follows from the weight-monodromy conjecture of Deligne, see e.g. [J, Corollary 4.3]. As mentioned earlier, this conjecture is known for Kuga-Sato and abelian varieties and is stable under products, so it holds for our generalized Kuga-Sato $X$ as well. □

If $v$ is a place of $F$ not dividing $p$, then by the previous proposition, the restriction of the class $[E_a]$ to $H^1_f(F_v, V)$ is trivial. Hence, if we consider the diagram above in the category of $G_{F_v}$-representations, we may choose a splitting $s : \mathbb{Q}_p \to E_a$ of the map $h$. Pulling back the middle row of the diagram by $s$, gives a short exact sequence

$$0 \to \mathbb{Q}_p(1) \to U_v \to \mathbb{Q}_p \to 0$$

of $G_{F_v}$-representations, i.e. a class in $H^1(F_v, \mathbb{Q}_p(1)) \cong F_v^\times \hat{\otimes} \mathbb{Q}_p$. The local height at $v$ is then defined to be

$$\langle a, b \rangle_v = \ell_v([U_v]).$$

If $v$ is a place of $F$ above $p$, then we will define the local height at $v$ under the assumption that the mixed extension $E$ is crystalline at $v$. In general one does not expect to $E$ to be crystalline, even if (as in our situation) $V$, $E_a$ and $E_b$ are all crystalline. But this will turn out to be enough for our purposes. First we consider the exact sequence

$$0 \to H^1_f(F_v, \mathbb{Q}_p(1)) \xrightarrow{j} H^1_f(F_v, E_b^*(1)) \to H^1_f(F_v, V) \to 0,$$

obtained from the top row in the mixed extension diagram above. On the other hand, our choice of splitting of the Hodge filtration at $v$ gives rise to a splitting $r_v : H^1_f(F_v, E_b^*(1)) \to H^1_f(F_v, \mathbb{Q}_p(1))$ of the map $j$ above. We refer to [N2] for more details on this splitting; its definition is actually not important to us, only its existence. Since $E_v$ (the restriction of $E$ to a $G_{F_v}$-representation) is crystalline, it determines a class $[E_v]$ in $H^1_f(F_v, E_b^*(1))$, coming from the short exact sequence in the central column in the mixed extension diagram. The local height pairing at $v$ is then defined to be

$$\langle a, b \rangle_v = \ell_v(r_v([E_v])),$$

where we are thinking of $r_v$ as landing in

$$H^1_f(F_v, \mathbb{Q}_p(1)) \cong \mathcal{O}_{F_v}^{\times} \hat{\otimes} \mathbb{Q}_p \subset F_v^{\times} \hat{\otimes} \mathbb{Q}_p.$$

## 5.2 Mixed extensions attached to algebraic cycles

Now we consider the case where $a$ and $b$ are images of homologically trivial algebraic cycles under the $p$-adic Abel-Jacobi map. In this case, there is a natural choice for the global mixed extension $E$. For more details see [N2, §5].

We first recall a definition of the $p$-adic Abel-Jacobi map

$$\Phi : \mathrm{CH}^{2r+2k-1}(X)_0 \to H^1(F, V).$$

Suppose $a = \Phi([Y])$, for some homologically trivial cycle $Y$ representing a class in $\mathrm{CH}^{2r+2k-1}(X)_0$. We write $|\bar{Y}|$ for the (geometric) support of $Y$, and assume that $|\bar{Y}|$ is smooth for simplicity. Then the Gysin long exact sequence reads

$$0 \to V \to H^{2r+2k-1}_{\text{ét}}(\bar{X} - |\bar{Y}|, \mathbb{Q}_p(r+k)) \to H^{2r+2k}_{|\bar{Y}|}(\bar{X}, \mathbb{Q}_p(r+k)) \xrightarrow{\mathrm{cl}} H^{2r+2k}_{\text{ét}}(\bar{X}, \mathbb{Q}_p(r+k)).$$

Note that $H^{2r+2k}_{|\bar{Y}|}(\bar{X}, \mathbb{Q}_p(r+k)) \cong H^0(|\bar{Y}|, \mathbb{Q}_p)$ and the map cl is the usual cycle class map. Since $Y$ is homologically trivial, we can pull back this sequence along the map $\mathbb{Q}_p \to H^{2r+2k}_{|\bar{Y}|}(\bar{X}, \mathbb{Q}_p(r+k))$ which sends 1 to the class of $Y$. We get

$$\begin{array}{ccccccccc}
0 & \to & V & \to & E_a & \to & \mathbb{Q}_p & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & V & \to & H^{2k+2r-1}_{\text{ét}}(\bar{X} - |\bar{Y}|, \mathbb{Q}_p(r+k)) & \to & H^0(|\bar{Y}|, \mathbb{Q}_p) & \to & H^{2k+2r}_{\text{ét}}(\bar{X}, \mathbb{Q}_p(r+k)),
\end{array}$$

and $\Phi([Y]) = [E_a] = a$. Thus, we may realize $E_a$ as a subspace of

$$H^{2k+2r-1}_{\text{ét}}(\bar{X} - |\bar{Y}|, \mathbb{Q}_p(r+k)).$$

Dually, if $b = \Phi([Z])$, then we may realize $E_b$ as a quotient of

$$H^{2k+2r-1}_{\text{ét}}(\bar{X} \text{ rel } |\bar{Z}|, \mathbb{Q}_p(r+k)).$$

Here we are using relative cohomology, which in this case is simply cohomology with compact support along $|\bar{Z}|$.

Now assume that $Y$ and $Z$ are disjoint.

*Remark* V.2. If we start only with the classes $[Y]$ and $[Z]$, then we may use the moving lemma to arrange for $Y$ and $Z$ to be disjoint. However, in our height computations we do not want to use the moving lemma because we need a very explicit description of the cycles.

Using the Gysin sequence again (this time for the variety $\bar{X} - |\bar{Y}|$ relative to the subscheme $|\bar{Z}|$), one can check that we may choose the mixed extension $E$ to be a subquotient of cohomology with partial compact support along the boundary:

$$H^{2r+2k-1}_{\text{ét}}(\bar{X} - |\bar{Y}| \text{ rel } |Z|, \mathbb{Q}_p(r+k)).$$

Similarly, suppose the classes $a, b \in H^1_f(F, V)$ are images of Tate cycles $a', b' \in Z(Y_0(N), F)$ (recall these were introduced in Chapter 4.2), and suppose $a'$ and $b'$ are supported on finite disjoint sets of points $S, T \subset X_0(N)$. Then to compute $\langle \Phi_T(a'), \Phi_T(b') \rangle$, we may use use a mixed extension coming from the cohomology of $X_0(N)$ with coefficients in the local system $\mathscr{A}$ from the previous chapter. Specifically, we may choose $E$ to be a subquotient of the representation

$$H^1_{\text{ét}}(\bar{X}_0(N) - \bar{S} \text{ rel } \bar{T}, j_{0*}\mathscr{A})(1).$$

Finally, we remark that these constructions work integrally as well, i.e. we can work with $\mathbb{Z}_p$-coefficients instead of $\mathbb{Q}_p$-coefficients. One subtlety in the integral setting is that a cycle might be homologically trivial when we use $\mathbb{Q}_p$-coefficients only because its cycle class in $H^{2r+2k}_{\text{ét}}(\bar{X}, \mathbb{Z}_p(r+k))$ is torsion (but possibly non-zero). Therefore, naively copying the definition of the Abel-Jacobi map in the integral setting does not quite work. Instead, one can apply the Abel-Jacobi map to a multiple of the cycle (killing the torsion), and then proceed as before.

The point in working integrally is that then one can hope to show that $\langle a, b\rangle_v = \ell_v(r_v([E_v]))$ with $r_v([E_v])$ an element of $\mathcal{O}^\times_{F_v}\hat{\otimes}\mathbb{Z}_p$ (which is just the pro-$p$ part of $\mathcal{O}^\times_{F_v}$) instead of $\mathcal{O}^\times_{F_v}\hat{\otimes}\mathbb{Q}_p$. When we take into account the torsion subtlety from before, we get an element of $p^{-n}\mathcal{O}^\times_{F_v}\hat{\otimes}\mathbb{Z}_p$, for some $n$ which is independent of the cycle chosen. This will eventually allow us to measure how divisible by $p$ the quantity $\langle a, b\rangle_v$ is. For the definitions in the integral setting and for more details, see [N3, II.1.9].

# CHAPTER VI

# Local $p$-adic heights at primes away from $p$

Our ultimate goal is to compute $\left\langle z_f, z_{f,\bar{\chi}}^{\mathcal{A}} \right\rangle_{\ell_K}$ when $\ell_K : \mathbb{A}_K^\times/K^\times \to \mathbb{Q}_p$ is cyclotomic, i.e. $\ell_K \circ \tau = \ell_K$. Since such a homomorphism is unique up to scaling, we may assume that $\ell_K = \log_p \circ \lambda$, where $\lambda : G(K_\infty/K) \to 1 + p\mathbb{Z}_p$ is the cyclotomic character and $\log_p$ is Iwasawa's $p$-adic logarithm. We may write $\lambda = \tilde{\lambda} \circ \mathbf{N}$, where $\tilde{\lambda} : \mathbb{Z}_p^\times \to 1 + p\mathbb{Z}_p$ is given by $\tilde{\lambda}(x) = \langle x \rangle^{-1}$. Here, $\langle x \rangle = x\omega^{-1}(x)$, where $\omega$ is the Teichmuller character.

We maintain the following notations and assumptions for the rest of this section. Fix an ideal class $\mathcal{A}$ and an integer $m \geqslant 1$, and suppose that there are no integral ideals in $\mathcal{A}$ of norm $m$, i.e. $r_{\mathcal{A}}(m) = 0$. Choose an integral representative $\mathfrak{a} \in \mathcal{A}$ and let $\sigma \in \mathrm{Gal}(H/K)$ correspond to $\mathcal{A}$ under the Artin map. Write $x = a(Y)$ and $\bar{x}^{\mathfrak{a}} = \bar{a}(Y^{\mathfrak{a}})$ for the two Tate vectors supported at the points $y$ and $y^\sigma$ in $X_0(N)(H)$, as in Proposition IV.8. Let $v$ be a finite place of $H$ not dividing $p$ and set $F = H_v$. Write $\Lambda$ for the ring of integers in $F^{\mathrm{ur}}$, the maximal unramified extension of $F$, and let $\mathbb{F} = \bar{\mathbb{F}}_\ell$ be the residue field of $\Lambda$. Write $\underline{X}_0(N) \to \operatorname{Spec} \mathbb{Z}$ for the integral model of $X_0(N)$ constructed in [KM], and let $\underline{X}_0(N)_\Lambda$ be the base change to $\operatorname{Spec} \Lambda$. Finally, write $i : Y_0(N) \times_{\mathbb{Q}} F^{\mathrm{ur}} \hookrightarrow \underline{X}_0(N)_\Lambda$ for the inclusion.

Now suppose $a, b$ are elements of $Z(Y_0(N), F^{\mathrm{ur}})$ supported at points $y_a \neq y_b$ of $X_0(N)(F^{\mathrm{ur}})$ of good reduction. Let $\underline{y}_a$ and $\underline{y}_b$ be the Zariski closure of the points $y_a$ and $y_b$ in $\underline{X}_0(N)_\Lambda$ and let $\underline{a}$ and $\underline{b}$ be extensions of $a$ and $b$ to $H^0(\underline{y}_a, i_*\mathscr{A})$ and $H^0(\underline{y}_b, i_*\mathscr{A})$ respectively. If $\underline{y}_a$ and $\underline{y}_b$ have common special fiber $z$ (so $z$ corresponds to an elliptic curve $E/\bar{\mathbb{F}}$), then define

$$(a, b)_v = (\underline{y}_a \cdot \underline{y}_b)_z \cdot (\underline{a}_z, \underline{b}_z),$$

where $(\underline{y}_a \cdot \underline{y}_b)_z$ is the usual local intersection number on the arithmetic surface $\underline{X}_0(N)_\Lambda$ and $(\underline{a}_z, \underline{b}_z)$ is the intersection pairing on the cohomology of $E^{2r-2} \times A_{\mathbb{F}}^\ell$, where $A_{\mathbb{F}}$ is the reduction of $A_{\bar{F}}$.

*Remark* VI.1. Note that while $A$ may not have good reduction at $v$, it has potential good reduction. We can therefore identify $H^i_{\text{ét}}(A_{\bar{F}}, \mathbb{Q}_p)$ and $H^i_{\text{ét}}(A_{\mathbb{F}}, \mathbb{Q}_p)$ as vector spaces, but not as $\mathrm{Gal}(\bar{F}/F)$-representations. Since the ensuing intersection theoretic computations can be performed over an algebraic closure, this is enough for our purposes.

Our assumption that $r_{\mathcal{A}}(m) = 0$ implies that the Tate vectors $x$ and $T_m\bar{x}^{\mathfrak{a}}$ have disjoint support. The goal of this chapter is to compute $\langle x, T_m\bar{x}^{\mathfrak{a}} \rangle_v$ explicitly. Since the two Tate vectors have disjoint support, we may used the natural choice of mixed extension $E$ (discussed in Chapter 5.2) in order to perform these calculations. By [ST], we may assume that these cycles are supported at points of $\mathcal{X}_0(N)_\Lambda$ which are represented by elliptic curves with good reduction. The following proposition gives a way to compute the local heights purely in terms of Tate cycles. This technique of computing heights of cycles on higher dimensional varieties using a local system on a curve is the key to the entire computation. The idea goes back to work of Deligne, Beilinson, Brylinski, and Scholl, among others.

**Proposition VI.2.** *With notation and assumptions as above, we have*

$$\langle x, T_m\bar{x}^{\mathfrak{a}} \rangle_v = -\,(x, T_m\bar{x}^{\mathfrak{a}})_v \log_p(\boldsymbol{N}v), \tag{6.1}$$

*Proof.* The proof is exactly as in [N3, II.2.16 and II.4.5]. In our case, one uses that $H^2(\underline{X}_0(N), i_*\mathscr{A}(1)) = 0$. This follows from the fact that if

$$\mathscr{A}' = \left(\pi_* \mathrm{Sym}^{2r-2}(R^1 f_* \mathbb{Q}_p)(r-1)\right)^B,$$

then $\mathscr{A} = \mathscr{A}' \otimes W$, where $W$ is a trivial two-dimensional local system, and $H^2(\underline{X}_0(N), i_*\mathscr{A}') = 0$ [KM, 14.5.5.1]. □

Recall that over $\Lambda$, the sections $\underline{y}$ and $\underline{y}^\sigma$ correspond to cyclic isogenies of degree $N$. We will confuse the two notions, so that the notation $\mathrm{Hom}_\Lambda(\underline{y}^\sigma, \underline{y})$ makes sense. See [N3] and [C1] for details.

**Proposition VI.3.** *Suppose $v$ is a finite prime of $H$ not divisible by $p$. If $m \geqslant 1$ is prime to $N$ and satisfies $r_{\mathcal{A}}(m) = 0$, then*

$$(x, T_m \bar{x}^{\mathfrak{a}})_v = \frac{1}{2} m^{r-k-1} \sum_{n \geqslant 1} \sum_g \left(\bar{\epsilon}\left(X_{g\sqrt{D}g^{-1}}^{\otimes r-k-1} \otimes X_{g\phi_{\mathfrak{a}}}^{\otimes \ell}\right), \epsilon\left(X_{\sqrt{D}}^{\otimes r-k-1} \otimes X_1^{\otimes \ell}\right)\right),$$

*where the sum is over $g \in \mathrm{Hom}_{\Lambda/\pi^n}(\underline{y}^\sigma, \underline{y})$ of degree $m$. The intersection pairing on the right takes place in the cohomology of $E^{2r-2} \times A_{\mathbb{F}}^\ell$, where $E \cong A_{\mathbb{F}}$ is the elliptic curve over $\mathbb{F}$ corresponding to the special fiber $\underline{y}_s$ of $\underline{y}$.*

*Proof.* The proof builds on that of [N3, II.4.12], so we only mention what is new to our setting. We write $m$ as $m = m_0 q^t$ where $q$ is the rational prime below $v$ (this is what Nekovář calls $\ell$). In the notation of [N3], we need to compute the special fiber of $\underline{x}_g^{\mathfrak{a}}(j)$, where $g \in \mathrm{Hom}_\Lambda(\underline{y}^\sigma, \underline{y}_g^\sigma)$ is an isogeny of degree $m_0$. There is no harm in assuming $r = k+1$, because the description of the purely Kuga-Sato components of $\underline{x}_g^{\mathfrak{a}}(j)$ (i.e. coming from factors of the cycle $Y^{\mathfrak{a}}$ of the form $\Gamma_{\sqrt{D}} \subset E^{\mathfrak{a}} \times E^{\mathfrak{a}}$) is handled in [N3].

Assume now that $q$ is inert in $K$ and $t$ is even. In this case the special fiber $(\underline{y})_s$ is supersingular, and the special fiber $(\underline{x}_g^{\mathfrak{a}})_s$ of the Tate vector is represented by the pair

$$\left((\underline{y}_g^\sigma)_s, \bar{\epsilon}\left(X_{g\phi_{\mathfrak{a}},1}^{\otimes \ell}\right)\right).$$

This follows from the definition of the Hecke operators and the following fact: if $g : E \to E'$ is an isogeny and $\phi : A \to E$ is an isogeny, then

$$(g \times \mathrm{id})_* (\Gamma_\phi^t) = \Gamma_{g\phi}^t \in \mathrm{CH}^1(E' \times A).$$

Since any isogeny $h \in \mathrm{Hom}_{\Lambda/\pi^n}(\underline{y}_g^\sigma, \underline{y})$ of degree $q^t$ on the special fiber $\underline{y}_s \cong (\underline{y}_g^\sigma)_s$ is of the form $q^{t/2} h_0$, with $h_0$ of degree 1, we find that, assuming $\underline{y}$ and $\underline{y}_g^\sigma(j)$ intersect, $(\underline{x}_g^{\mathfrak{a}}(j))_s$ is represented by

$$\left((\underline{y}_g^\sigma)_s, \bar{\epsilon}\left(X_{q^{t/2}g\phi_{\mathfrak{a}},1}^{\otimes \ell}\right)\right) = \left(\underline{y}_s, \bar{\epsilon}\left(X_{h_0 q^{t/2} g\phi_{\mathfrak{a}},1}^{\otimes \ell}\right)\right) = \left(\underline{y}_s, \bar{\epsilon}\left(X_{hg\phi_{\mathfrak{a}},1}^{\otimes \ell}\right)\right) = \left(\underline{y}_s, \epsilon\left(X_{\overline{hg\phi_{\mathfrak{a}}}}^{\otimes \ell}\right)\right),$$

as desired. The proof when $t$ is odd or when $q$ is ramified is similar. If $q$ is split in $K$, then both sides of the equation are 0, as is shown in [GZ]. □

When $v$ lies over a non-split prime, $\mathrm{End}_{\Lambda/\pi}(\underline{y}) = \mathrm{End}(E)$ is an order $R$ in a quaternion algebra $B$ and we can make the double sum on the right hand side more explicit. To do this, we follow [GZ] and identify $\mathrm{Hom}_{\Lambda/\pi}(\underline{y}^\sigma, \underline{y})$ with $R\mathfrak{a}$ by sending a map $g$ to $b = g\phi_{\mathfrak{a}}$. The reduction of endomorphisms induces an embedding $K \hookrightarrow B$, which in turn determines a canonical decomposition $B = K \oplus Kj$. Thus every $b \in B$ can be written as $b = \alpha + \beta j$ with $\alpha, \beta \in K$. Recall also that the reduced norm on $B$ is additive with respect to this decomposition, i.e. $\mathbf{N}(b) = \mathbf{N}(\alpha) + \mathbf{N}(\beta j)$.

**Proposition VI.4.** *If $g\phi_{\mathfrak{a}} = b = \alpha + \beta j \in \mathrm{End}(E)$, then*

$$\left(\bar{\epsilon}(X_{g\sqrt{D}g^{-1}}^{r-k-1} \otimes X_{\bar{b}}^{\otimes \ell}), \epsilon(X_{\sqrt{D}}^{\otimes r-k-1} \otimes X_1^{\otimes \ell})\right) = \frac{(4D)^{r-k-1}}{\binom{2r-2}{r-k-1}} \bar{\alpha}^{2k} H_{r-k-1,k}\left(1 - \frac{2\boldsymbol{N}(\beta j)}{\boldsymbol{N}(b)}\right),$$

*where*

$$H_{m,k}(t) = \frac{1}{2^m \cdot (m+2k)!}\left(\frac{d}{dt}\right)^{m+2k}[(t^2-1)^m(t-1)^{2k}].$$

*Proof.* Recall from Chapter 4.4 that we have chosen a basis $x^*, y^*$ of $V_pE$, and a dual basis $x, y$ of $H^1(E)$ such that $x^* \in V_{\mathfrak{p}}E$, $y^* \in V_{\bar{\mathfrak{p}}}E$, and $(x^*, y^*) = 1$. We have already seen that $X_\alpha = \alpha x \otimes y - \bar{\alpha} y \otimes x$. Since $\gamma j = j\bar{\gamma}$ for all $\gamma \in K$, $Vj$ (the map induced by $j$ on $V_pE$) swaps $V_{\mathfrak{p}}E$ and $V_{\bar{\mathfrak{p}}}E$. So we can write

$$Vj = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$$

for some $u, v \in \mathbb{Q}_p$ such that $uv = \mathbf{N}(j) = -j^2$. It follows that

$$X_b = \alpha x \otimes y - \bar{\alpha} y \otimes x + \beta u y \otimes y - \bar{\beta} v x \otimes x.$$

Next note that $g\sqrt{D}g^{-1} = b\sqrt{D}b^{-1}$. We write $b\sqrt{D}b^{-1} = \gamma + \delta j$, so that $\gamma = \frac{\sqrt{D}}{\mathbf{N}(b)}(\mathbf{N}(\alpha) - \mathbf{N}(\beta j))$ and $\delta = \frac{-2\sqrt{D}}{\mathbf{N}(b)}\alpha\beta$. Thus $X_{g\sqrt{D}g^{-1}}$ already lies in $\mathrm{Sym}^2 H^1(E)$, and hence (working now in the symmetric algebra)

$$\bar{\epsilon} X_{g\sqrt{D}g^{-1}} = 2\gamma xy + \delta u y^2 - \bar{\delta} v x^2 = \frac{2\sqrt{D}}{\mathbf{N}(b)}(\bar{\alpha}x - \beta u y)(\alpha y + \bar{\beta} v x),$$

since $\bar{\epsilon}$ acts as Scholl's projector $\epsilon_W$ on the purely Kuga-Sato components.

The cohomology classes $X_{\bar{b}}$ in the statement of the proposition are on 'mixed' components, i.e. they live in $H^1(E) \otimes H^1(E')$, where $E$ comes from a Kuga-Sato component and $E'$ (which is abstractly isomorphic to E) comes from the factor $A^\ell$. Thus

$$X_{\bar{b}} = \bar{\alpha} x \otimes y' - \alpha y \otimes x' - \beta u y \otimes y' + \bar{\beta} v x \otimes x',$$

and $\bar{\epsilon} X_{\bar{b}} = (\bar{\alpha}x - \beta u y)y'$, since $\bar{\epsilon}$ acts trivially on $H^1(E)$ and kills the basis vector $x'$ in $H^1(E')$. Using these observations together with the compatibility of the projectors with the multiplication in the appropriate symmetric algebras, we compute

$$\begin{aligned}
&\left(\bar{\epsilon}(X^{r-k-1}_{g\sqrt{D}g^{-1}} \otimes X_{\bar{b}}^{\otimes \ell}), \epsilon(X_{\sqrt{D}}^{\otimes r-k-1} \otimes X_1^{\otimes \ell})\right) \\
&\quad = \left((2\gamma xy + \delta u y^2 - \bar{\delta} x^2)^{r-k-1}(\bar{\alpha}x - \beta u y)^{2k} \otimes y'^{2k}, (2\sqrt{D}xy)^{r-k-1} y^{2k} \otimes x'^{2k}\right) \\
&\quad = \left(\frac{4D}{\mathbf{N}(b)}\right)^{r-k-1} (y'^{2k}, x'^{2k}) \left((\bar{\alpha}x - \beta u y)^{r+k-1}(\alpha y + \bar{\beta} v x)^{r-k-1}, x^{r-k-1} y^{r+k-1}\right) \\
&\quad = \left(\frac{4D}{\mathbf{N}(b)}\right)^{r-k-1} (y'^{2k}, x'^{2k})(y^{r-k-1}x^{r+k-1}, x^{r-k-1}y^{r+k-1}) \cdot C \\
&\quad = \frac{(4D)^{r-k-1}}{\mathbf{N}(b)^{r-k-1}\binom{2r-2}{r-k-1}} \cdot C,
\end{aligned}$$

where $C$ is the coefficient of the monomial $y^{r-k-1}x^{r+k-1}$ in

$$(\bar{\alpha}x - \beta u y)^{r+k-1}(\alpha y + \bar{\beta} v x)^{r-k-1}.$$

The pairings in the second to last line are the natural ones on $\mathrm{Sym}^{2k} H^1(E')$ and $\mathrm{Sym}^{2r-2} H^1(E)$ induced from the pairings on the full tensor algebras. For example, $\mathrm{Sym}^{2r-2} H^1(E)$ has a natural pairing coming from the cup product $(\ ,\ )$ on $H^1(E)$:

$$(v_1 \otimes \cdots \otimes v_{2r-2}) \times (w_1 \otimes \cdots \otimes w_{2r-2}) \mapsto \frac{1}{(2r-2)!} \sum_{\sigma \in S_{2r-2}} \prod_{i=1}^{2r-2} (v_i, w_{\sigma(i)}).$$

In particular, $(x^a y^b, x^c y^d) = 0$ unless $a = d$ and $b = c$, and

$$(x^a y^b, y^a x^b) = \frac{a!b!}{(a+b)!} = \binom{a+b}{a}^{-1}.$$

We have also used that on $\mathrm{Sym}^{2r-2} H^1(E) \otimes \mathrm{Sym}^{2k} H^1(E')$ we have

$$(u \otimes v, w \otimes z) = (u, w)(v, z).$$

To compute the value of $C$, note that in general, the coefficient of $x^{m+2k}$ in

$$(ax+b)^{m+2k}(cx+d)^m$$

is equal to $a^{2k}(ad-bc)^m H_{m,k}\left(\frac{ad+bc}{ad-bc}\right)$. This is proved using the method of [Z, 3.3.3]. Applying this to the situation at hand, we find that

$$C = \bar{\alpha}^{2k} \mathbf{N}(b)^{r-k-1} H_{r-k-1,k}\left(1 - \frac{2\mathbf{N}(\beta j)}{\mathbf{N}(b)}\right).$$

Plugging this in, we obtain the desired expression for the pairing on the special fiber. □

For each prime $q$, define $\langle x, T_m \bar{x}^{\mathfrak{a}} \rangle_q = \sum_{v|q} \langle x, T_m \bar{x}^{\mathfrak{a}} \rangle_v$.

**Proposition VI.5.** *Assume that* $(m, N) = 1$, $r_{\mathcal{A}}(m) = 0$. *Then*

$$\chi(\bar{\mathfrak{a}})^{-1} \sum_{q \neq p} \langle x, T_m \bar{x}^{\mathfrak{a}} \rangle_q =$$
$$- u^2 \frac{(4|D|m)^{r-k-1}}{D^k \cdot \binom{2r-2}{r-k-1}} \sum_{0<n<\frac{m|D|}{N}} \sigma_{\mathcal{A}}(n) r_{\mathcal{A},\chi}(m|D| - nN) H_{r-k-1,k}\left(1 - \frac{2nN}{m|D|}\right),$$

*with* $\sigma_{\mathcal{A}}(n)$ *defined as in Corollary III.7.*

*Proof.* This type of sum arises from Proposition VI.3 exactly as in [N3, II.4.17] and [GZ], so we omit the details. The main new feature here is that each $b = \alpha + \beta j \in R\mathfrak{a}$ of degree $m$ is weighted by $\bar{\alpha}^{\ell}$, by the previous proposition. Thus the numbers $r_{\mathcal{A}}(j)$, with $j = m|D| - nN$, and which in [N3, II.4.17] are simply counting the number of such $b$, become non-trivial sums of the form

$$\sum_{\substack{\mathfrak{c} \subset \mathcal{O}_K \\ [\mathfrak{c}] = \mathcal{A}^{-1}\mathcal{D} \\ \mathrm{Nm}(\mathfrak{c}) = j}} \bar{\alpha}^{\ell}.$$

Here, $\alpha \in \mathfrak{d}^{-1}\mathfrak{a}$ and $\mathfrak{c} = (\alpha)\mathfrak{d}\mathfrak{a}^{-1}$ (see [GZ, p. 265]). Rewriting this sum, we obtain

$$\sum_{\substack{\mathfrak{c} \subset \mathcal{O}_K \\ [\mathfrak{c}] = \mathcal{A}^{-1}\mathcal{D} \\ \mathrm{Nm}(\mathfrak{c}) = j}} \bar{\chi}(\mathfrak{c}\mathfrak{a}\mathfrak{d}^{-1}) = \frac{\chi(\bar{\mathfrak{a}})}{\chi(\mathfrak{d})} \cdot \sum_{\substack{\mathfrak{c} \subset \mathcal{O}_K \\ [\mathfrak{c}] = \mathcal{A}^{-1}\mathcal{D} \\ \mathrm{Nm}(\mathfrak{c}) = j}} \chi(\bar{\mathfrak{c}}) = \frac{\chi(\bar{\mathfrak{a}})}{D^k} \cdot \sum_{\substack{\mathfrak{c} \subset \mathcal{O}_K \\ [\mathfrak{c}] = \mathcal{A} \\ \mathrm{Nm}(\mathfrak{c}) = j}} \chi(\mathfrak{c}) = \frac{\chi(\bar{\mathfrak{a}})}{D^k} r_{\mathcal{A},\chi}(j).$$

Multiplying by $\chi(\bar{\mathfrak{a}})^{-1}$, we get the desired result. □

We define

$$B_m^{\sigma} = m^{r-k-1} \sum_{\substack{n=1 \\ (p,n)=1}}^{\frac{m|D|}{N}} r_{\mathcal{A},\chi}(m|D| - nN)\sigma_{\mathcal{A}}(n) H_{r-k-1,k}\left(1 - \frac{2nN}{m|D|}\right)$$

$$C_m^\sigma = m^{r-k-1} \sum_{n=1}^{\frac{m|D|}{N}} r_{\mathcal{A},\chi}(m|D| - nN)\sigma_{\mathcal{A}}(n) H_{r-k-1,k}\left(1 - \frac{2nN}{m|D|}\right)$$

Up to a constant, the $B_m^\sigma$ appear as coefficients of the derivative of the $p$-adic $L$-function defined earlier (this will follow from Corollary III.7) and $C_m^\sigma$ contributes to the height of our generalized Heegner cycle, as we have just seen. Just as in [N3, I.6.7], we wish to relate the $B_m^\sigma$ to the $C_m^\sigma$.

Let $U_p$ be the operator defined by $C_m^\sigma \mapsto C_{mp}^\sigma$ and similarly for $B_m^\sigma$. For a prime $\mathfrak{p}$ of $K$ above $p$, we write $\sigma_{\mathfrak{p}}$ for $\mathrm{Frob}(\mathfrak{p}) \in \mathrm{Gal}(H/K)$. We will also let $\sigma_{\mathfrak{p}}$ be the operator $C_m^\sigma \mapsto C_m^{\sigma\sigma_{\mathfrak{p}}}$.

**Proposition VI.6.** *Suppose $p > 2$ is a prime which splits in $K$ and that $\chi$ is an unramified Hecke character of $K$ of infinity type $(\ell, 0)$ with $\ell = 2k$. Then*

$$\prod_{\mathfrak{p}|p} \left(U_p - p^{r-k-1}\chi(\bar{\mathfrak{p}})\sigma_{\mathfrak{p}}\right)^2 C_m^\sigma = \left(U_p^4 - p^{2r-2}U_p^2\right) B_m^\sigma.$$

*Proof.* The proof follows [PR1, Proposition 3.20], which is the case $r = 1$ and $\ell = k = 0$. We first generalize [PR1, Lemma 3.11] and write down relations between the various $r_{\mathcal{A},\chi}(-)$.

**Lemma VI.7.** *Set $r_{\mathcal{A},\chi}(t) = 0$ if $t \in \mathbb{Q}\backslash\mathbb{N}$. For all integers $m > 0$, we have*

1. $r_{\mathcal{A},\chi}(mp) + p^\ell r_{\mathcal{A},\chi}(m/p) = \chi(\bar{\mathfrak{p}}) r_{\mathcal{A}\mathfrak{p},\chi}(m) + \chi(\mathfrak{p}) r_{\mathcal{A}\bar{\mathfrak{p}},\chi}(m)$.
2. $r_{\mathcal{A},\chi}(mp^2) + p^{2\ell} r_{\mathcal{A},\chi}(m/p^2) = \chi(\bar{\mathfrak{p}}^2) r_{\mathcal{A}\mathfrak{p}^2,\chi}(m) + \chi(\mathfrak{p}^2) r_{\mathcal{A}\bar{\mathfrak{p}}^2,\chi}(m)$ *if* $p|m$.
3. $r_{\mathcal{A},\chi}(mp^2) - p^\ell r_{\mathcal{A},\chi}(m) = \chi(\bar{\mathfrak{p}}^2) r_{\mathcal{A}\mathfrak{p}^2,\chi}(m) + \chi(\mathfrak{p}^2) r_{\mathcal{A}\bar{\mathfrak{p}}^2,\chi}(m)$ *if* $(p, m) = 1$.
4. *If $n = n_0 p^t$ with $p \nmid n_0$, then $\sigma_{\mathcal{A}}(n) = (t+1)\sigma_{\mathcal{A},t}(n_0)$, where $\sigma_{\mathcal{A},t} = \sigma_{\mathcal{A}\mathfrak{p}^t} = \sigma_{\mathcal{A}\bar{\mathfrak{p}}^t}$.*
5. $\sigma_{\mathcal{A}\mathfrak{b}^2}(n) = \sigma_{\mathcal{A}}(n)$ *for any ideal* $\mathfrak{b}$.

*Proof.* Note that every integral ideal $\mathfrak{a}$ in $\mathcal{A}$ of norm $mp$ is either of the form $\mathfrak{a}'\mathfrak{p}$ with $\mathfrak{a}' \in \mathcal{A}\bar{\mathfrak{p}}$ of norm $m$ or it is of the form $\mathfrak{a}'\bar{\mathfrak{p}}$ with $\mathfrak{a}' \in \mathcal{A}\mathfrak{p}$ of norm $m$. Moreover, an ideal of norm $mp$ which can be written as such a product in two ways is necessarily the product of an integral ideal in $\mathcal{A}$ of norm $m/p$ with $(p)$. The first claim now follows from the fact that

$$r_{\mathcal{A},\chi}(t) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O} \\ \mathfrak{a}\in\mathcal{A} \\ \mathbf{N}(\mathfrak{a})=t}} \chi(\mathfrak{a}),$$

and that $\chi((p)) = p^\ell$. Parts (2) and (3) follow formally from (1). (4) is proven in [PR1] and (5) is clear from the definition. □

Going back to the proof of Proposition VI.6, the left hand side is equal to

$$\begin{aligned} C_{mp^4}^\sigma - 2p^{r-k-1}&\left(\chi(\bar{\mathfrak{p}})C_{mp^3}^{\sigma\sigma_{\mathfrak{p}}} + \chi(\mathfrak{p})C_{mp^3}^{\sigma\sigma_{\bar{\mathfrak{p}}}}\right) \\ &+ p^{2(r-k-1)}\left(\chi(\bar{\mathfrak{p}})^2 C_{mp^2}^{\sigma\sigma_{\mathfrak{p}}^2} + 4p^\ell C_{mp^2}^\sigma + \chi(\mathfrak{p})C_{mp^2}^{\sigma\sigma_{\bar{\mathfrak{p}}}^2}\right) \\ &- 2p^{3(r-k-1)+\ell}\left(\chi(\bar{\mathfrak{p}})C_{mp}^{\sigma\sigma_{\mathfrak{p}}} + \chi(\mathfrak{p})C_{mp}^{\sigma\sigma_{\bar{\mathfrak{p}}}}\right) + p^{4(r-1)}C_m^\sigma. \end{aligned}$$

In the following we write $v(p)$ for the $p$-adic valuation of an integer $n$, and $n = n_0 p^{v(p)}$. For the sake of brevity we also set $r_{\mathcal{A}}(u, v) = r_{\mathcal{A},\chi}(u|D| - vN)$ for integers $u$ and $v$ and $H(x) = H_{r-k-1,k}(x)$. Then by repeated usage of the previous lemma, the expression above is equal to

$$\sum_{n=1}^{m|D|/N} (v(n)+1)(mp^4)^{r-k-1} M(n),$$

where $M(n)$ equals

$$\begin{aligned}
&r_{\mathcal{A}}(mp^4,n)\sigma_{\mathcal{A},v(n)}(n_0)H\left(1-\frac{2nN}{mp^4|D|}\right)\\
&\quad -2\left[r_{\mathcal{A}}(mp^4,pn)+p^{\ell}r_{\mathcal{A}}\left(mp^2,n/p\right)\right]\sigma_{\mathcal{A},v(n)+1}(n_0)H\left(1-\frac{2nN}{mp^3|D|}\right)\\
&\quad +\left[r_{\mathcal{A}}(mp^4,p^2n)+\begin{cases} p^{2\ell}r_{\mathcal{A}}\left(m,n/p^2\right)+4p^{\ell}r_{\mathcal{A}}(mp^2,n) & \text{if } p|n\\ 3p^{\ell}r_{\mathcal{A}}(mp^2,n) & \text{if } p\nmid n\end{cases}\right]\times\\
&\qquad\qquad \times\sigma_{\mathcal{A},v(n)}(n_0)H\left(1-\frac{2nN}{mp^2|D|}\right)\\
&\quad -2p^{\ell}\left[r_{\mathcal{A}}(mp^2,pn)+p^{\ell}r_{\mathcal{A}}(m,n/p)\right]\sigma_{\mathcal{A},v(n)+1}(n_0)H\left(1-\frac{2nN}{mp|D|}\right)\\
&\quad +p^{2\ell}r_{\mathcal{A}}(m,n)\sigma_{\mathcal{A},v(n)}(n_0)H\left(1-\frac{2nN}{m|D|}\right).
\end{aligned}$$

Grouping in terms of the $n_0$ which arise in this sum, we can rewrite this as

$$\sum_{(n_0,p)=1}\sum_t \sigma_{\mathcal{A},t}(n_0)A_t,$$

where $A_t$ equals

$$\begin{aligned}
&(mp^4)^{r-k-1}r_{\mathcal{A}}(mp^4,p^tn_0)\left[t+1-2t+\begin{cases} t-1 & \text{if } t\geqslant 1\\ 0 & \text{if } t=0\end{cases}\right]H\left(1-\frac{2n_0p^tN}{mp^4|D|}\right)\\
&\quad +(mp^2)^{r-k-1}p^{2r-2}r_{\mathcal{A}}(mp^2,p^tn_0)\left[-2(t+2)+\begin{cases} 4(t+1)-2t & \text{if } t\geqslant 1\\ 3 & \text{if } t=0\end{cases}\right]\times\\
&\qquad\qquad \times H\left(1-\frac{2n_0p^tN}{mp^2|D|}\right)\\
&\quad +m^{r-k-1}p^{4r-4}r_{\mathcal{A}}(m,p^tn_0)\left[t+3-2(t+2)+t+1\right]H\left(1-\frac{2n_0p^tN}{m|D|}\right).
\end{aligned}$$

So $A_t=0$ unless $t=0$, and we conclude that the left hand side in Proposition VI.6 is equal to $(U_p^4-p^{2r-2}U_p^2)B_m^{\sigma}$, as desired. □

# CHAPTER VII

## Ordinary representations

The contributions to the $p$-adic height $\langle z_f, z^{\mathcal{A}}_{f,\bar{X}}\rangle$ coming from places $v|p$ will eventually be shown to vanish. The proof is as in [N3, II.5] (though see Chapter IX). An important input to Nekovář's approach is that the local $p$-adic Galois representation $V_f$ attached to $f$ is ordinary. We recall this notion and prove that the Galois representation $V_{f,A,\ell} = V_f \otimes \kappa_\ell H^\ell(\bar{A}^\ell, \mathbb{Q}_p)(k)$ is ordinary as well.

**Definition VII.1.** Let $F$ be a finite extension of $\mathbb{Q}_p$. A $p$-adic Galois representation $V$ of $G_F = \mathrm{Gal}(\bar{F}/F)$ is *ordinary* if it admits a decreasing filtration by subrepresentations

$$\cdots F^i V \supset F^{i+1} V \supset \cdots$$

such that $\bigcup F^i V = V$, $\bigcap F^i V = 0$, and for each $i$, $F^i V / F^{i+1} V = A_i(i)$, with $A_i$ unramified.

Recall we have defined $\epsilon' = \epsilon_W \kappa_\ell$, with

$$\kappa_\ell = \left[ \left( \frac{\sqrt{D} + [\sqrt{D}]}{2\sqrt{D}} \right)^{\otimes \ell} + \left( \frac{\sqrt{D} - [\sqrt{D}]}{2\sqrt{D}} \right)^{\otimes \ell} \right] \circ \left( \frac{1 - [-1]}{2} \right)^{\otimes \ell}.$$

**Theorem VII.2.** *Let $f \in S_{2r}(\Gamma_0(N))$ be an ordinary newform and let $V_f$ be the 2-dimensional $p$-adic Galois representation associated to $f$ by Deligne. Let $A/H$ be an elliptic curve with CM by $\mathcal{O}_K$ and assume $p$ splits in $K$ and $A$ has good reduction at primes above $p$. For any $\ell = 2k \geqslant 0$, set $W = \kappa_\ell H^\ell(\bar{A}^\ell, \mathbb{Q}_p)(k)$. Then for any place $v$ of $H$ above $p$, $V_{f,A,\ell} = V_f \otimes W$ is an ordinary $p$-adic Galois representations of* $\mathrm{Gal}(\bar{H}_v/H_v)$.

*Proof.* First we recall that $V_f$ is ordinary. Indeed, Wiles [Wi] proves that the action of the decomposition group $D_p$ on $V_f$ is given by

$$\begin{pmatrix} \epsilon_1 & * \\ 0 & \epsilon_2 \end{pmatrix}$$

with $\epsilon_2$ unramified. Since, $\det V_f$ is $\chi_{\mathrm{cyc}}^{2r-1}$, we have $\epsilon_1 = \epsilon_2^{-1} \chi_{\mathrm{cyc}}^{2r-1}$. Thus, the filtration

$$F^0 V_f = V_f \supset F^1 V_f = F^{2r-1} V_f = \epsilon_1 \supset F^{2r} V_f = 0,$$

shows that $V_f$ is an ordinary $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-representation and hence an ordinary $\mathrm{Gal}(\bar{H}_v/H_v)$-representation as well. Next we describe the ordinary filtration on (a Tate twist of) $W$.

**Proposition VII.3.** *Write $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ as ideals in $K$. Then the $p$-adic representation $M = \kappa_\ell H^\ell_{\text{ét}}(\bar{A}^\ell, \mathbb{Q}_p)(\ell)$ of* $\mathrm{Gal}(\bar{H}_v/H_v)$ *has an ordinary filtration*

$$F^0 M = M \supset F^1 M = F^\ell M \supset F^{\ell+1} M = 0.$$

*Proof.* The theory of complex multiplication associates to $A$ an algebraic Hecke character $\psi : \mathbb{A}_H^\times \to K^\times$ of type $\mathrm{Nm} : H^\times \to K^\times$ such that for any uniformizer $\pi_v$ at a place $v$ not dividing $p$ or the conductor of $A$, $\psi(\pi_v) \in K \cong \mathrm{End}(A)$ is a lift of the Frobenius morphism of the reduction $A_v$ at $v$. The composition

$$t_p : \mathbb{A}_H^\times \xrightarrow{\mathrm{Nm}} \mathbb{A}_K^\times \to (K \otimes \mathbb{Q}_p)^\times$$

agrees with $\psi$ on $H^\times$, giving a continuous map

$$\rho' = \psi t_p^{-1} : \mathbb{A}_H^\times / H^\times \to (K \otimes \mathbb{Q}_p)^\times.$$

Since the target is totally disconnected, this factors through a map

$$\rho : G_H^{\mathrm{ab}} \to (K \otimes \mathbb{Q}_p)^\times.$$

By construction of the Hecke character (and the Chebotarev density theorem), the action of $\mathrm{Gal}(\bar{H}/H)$ on the rank 1 $(K \otimes \mathbb{Q}_p)$-module $T_pA \otimes \mathbb{Q}_p$ is given by the character $\rho$. Since $p$ splits in $K$, we have

$$(K \otimes \mathbb{Q}_p)^\times \cong K_{\mathfrak{p}}^\times \oplus K_{\bar{\mathfrak{p}}}^\times = \mathbb{Q}_p^\times \oplus \mathbb{Q}_p^\times.$$

Now write $\rho = \rho_{\mathfrak{p}} \oplus \rho_{\bar{\mathfrak{p}}}$, where $\rho_{\mathfrak{p}}$ and $\rho_{\bar{\mathfrak{p}}}$ are the characters obtained by projecting $\rho$ onto $K_{\mathfrak{p}}^\times$ and $K_{\bar{\mathfrak{p}}}^\times$.

**Lemma VII.4.** *Let $\chi_{\mathrm{cyc}} : \mathrm{Gal}(\bar{H}_v/H_v) \to \mathbb{Q}_p^\times$ denote the cyclotomic character and consider $\rho_{\mathfrak{p}}$ and $\rho_{\bar{\mathfrak{p}}}$ as representations of $\mathrm{Gal}(\bar{H}_v/H_v)$. Then $\rho_{\mathfrak{p}}\rho_{\bar{\mathfrak{p}}} = \chi_{\mathrm{cyc}}$ and $\rho_{\bar{\mathfrak{p}}}$ is unramified.*

*Proof.* The non-degeneracy of the Weil pairing shows that $\bigwedge^2 T_pA \cong \mathbb{Z}_p(1)$. It then follows from the previous discussion that $\rho_{\mathfrak{p}}\rho_{\bar{\mathfrak{p}}} = \chi_{\mathrm{cyc}}$. That $\rho_{\bar{\mathfrak{p}}}$ is unramified follows from the fact that $t_{\bar{\mathfrak{p}}}(H_v) = 1$ and $v$ is prime to the conductor of $\psi$. Indeed, the conductor of $A$ is the square of the conductor of $\psi$ [G], and $A$ has good reduction at $p$. □

*Remark* VII.5. Let $\mathcal{A}/\mathcal{O}_H$ be the Néron model of $A/H$. Since $\mathcal{A}[\bar{\mathfrak{p}}^n]$ is étale, it follows that the $\bar{\mathfrak{p}}$-adic Tate module $V_{\bar{\mathfrak{p}}}A$ is unramified at $v$. We can therefore identify $\rho_{\mathfrak{p}} \cong V_{\mathfrak{p}}A$ and $\rho_{\bar{\mathfrak{p}}} = V_{\bar{\mathfrak{p}}}A$. One can also see this from the computation in equation 4.1.

**Lemma VII.6.** *As $\mathrm{Gal}(\bar{H}_v/H_v)$-representations,*

$$H^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)(1) \cong \rho_{\mathfrak{p}} \oplus \rho_{\bar{\mathfrak{p}}}$$

*and*

$$M = \kappa_\ell H^\ell_{\text{ét}}(\bar{A}^\ell, \mathbb{Q}_p)(\ell) \cong \rho_{\mathfrak{p}}^\ell \oplus \rho_{\bar{\mathfrak{p}}}^\ell.$$

*Proof.* The first claim follows from the fact that $T_pA \otimes \mathbb{Q}_p \cong H^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)(1)$. Fix an embedding $\iota : \mathrm{End}(A) \hookrightarrow K$, which by our choices, induces an embedding $\mathrm{End}(A) \hookrightarrow \mathbb{Q}_p$. By the definition of $\rho$, $\rho_{\mathfrak{p}}$ is the subspace of $H^1_{\text{ét}}(\bar{A}, \mathbb{Q}_p)(1)$ on which $\alpha \in \mathrm{End}(A)$ acts by $\iota(\alpha)$, whereas on $\rho_{\bar{\mathfrak{p}}}$, $\alpha$ acts as $\bar{\iota}(\alpha)$. The second statement now follows from the Kunneth formula and the definition of $\kappa_\ell$. □

Now set $F^0M = M$, $F^1M = F^\ell M = \psi^\ell$, and $F^{\ell+1}M = 0$. By the lemmas above, this gives an ordinary filtration of $M$ and proves the proposition. □

Now to prove the theorem. We have specified ordinary filtrations $F^iV_f$ and $F^iM$ above. A simple check shows that

$$F^i(V_f \otimes M) = \sum_{p+q=i} F^pV_f \otimes F^qM$$

is an ordinary filtration on $V_f \otimes M$. Since $V_{f,A,\ell} = V_f \otimes W = (V_f \otimes M)(-k)$ and Tate twisting preserves ordinarity, this proves $V_{f,A,\ell}$ is ordinary. □

*Remark* VII.7. Another way to obtain the ordinary filtration on $M$ is to use the fact that $M$ is isomorphic to the $p$-adic realization of the motive $M_{\theta_{\psi^\ell}}$ attached to the modular form $\theta_{\psi^\ell}$ of weight $\ell+1$. Since $A$ has ordinary reduction at $p$, $\theta_\psi$ is an ordinary modular form, and it follows that $\theta_{\psi^\ell}$ is ordinary as well. We may therefore apply Wiles' theorem again to obtain an ordinary filtration on $W$.

**Proposition VII.8.** *The* $\mathrm{Gal}(\bar{H}/H)$ *representation* $V_{f,A,\ell} = V_f \otimes W$ *satisfies* $V^*_{f,A,\ell}(1) \cong V_{f,A,\ell}$.

*Proof.* Recall that $V_f^*(1) \cong V_f$, so we need to show that $W^* \cong W$. This follows from the two lemmas above. □

# CHAPTER VIII

## Proof of Theorems I.7 and I.9

Let $\mathbb{T}$ be the Hecke algebra of level $N$, i.e. the $\mathbb{Q}$-algebra generated by the action of the Hecke operators $T_m$ $((m, N) = 1)$ on $S_2(\Gamma_0(N))$. In what follows, normalized primitive forms $f_\beta \in S_{2r}(\Gamma_0(N))$ (i.e. $f_\beta$ is a newform of some level dividing $N$) will be indexed by the corresponding $\mathbb{Q}$-algebra homomorphism $\beta : \mathbb{T} \to \bar{\mathbb{Q}}$. We let $\beta_0$ be the homomorphism corresponding to our chosen newform $f$. If $\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)$, then

$$F_{\mathcal{A}} := \sum_\beta \langle z_{\beta,\chi}, z^{\mathcal{A}}_{\beta,\bar{\chi}} \rangle f_\beta$$

is a cusp form in $S_{2r}(\Gamma_0(N); \mathbb{Q}_p(\chi))$. Indeed, for $(m, N) = 1$, we have

$$\chi(\bar{\mathfrak{a}}) a_m(F_{\mathcal{A}}) = \sum_\beta \langle z_\beta, \bar{z}^{\mathfrak{a}}_\beta \rangle \beta(T_m) = \langle z, T_m \bar{z}^{\mathfrak{a}} \rangle = \langle x, T_m \bar{x}^{\mathfrak{a}} \rangle \in \mathbb{Q}_p,$$

because the Hecke operators are self-adjoint with respect to the height pairing. If $r_{\mathcal{A}}(m) = 0$, then we have the decomposition

$$a_m(F_{\mathcal{A}}) = c^\sigma_m + d^\sigma_m$$

where

$$c^\sigma_m = \chi(\bar{\mathfrak{a}})^{-1} \sum_{v \nmid p} \langle x, T_m \bar{x}^{\mathfrak{a}} \rangle_v, \qquad d^\sigma_m = \chi(\bar{\mathfrak{a}})^{-1} \sum_{v | p} \langle x, T_m \bar{x}^{\mathfrak{a}} \rangle_v,$$

and the sums are over *finite* places of $H$.

Both sides of the equation in Theorem I.7 depend linearly on a choice of arithmetic logarithm $\ell_K : \mathbb{A}^\times_K / K^\times \to \mathbb{Q}_p$. By Theorem IV.13, it suffices to proves the main theorem for cyclotomic $\ell_K$, i.e. $\ell_K = \ell_K \circ \tau$. As cyclotomic logarithms are unique up to scalar we only need to consider the case $\ell_K = \ell_{\mathbb{Q}} \circ \mathbf{N}$. Thus, $\ell_K = \log_p \circ \lambda$, where $\lambda : G(K_\infty/K) \to 1 + p\mathbb{Z}_p$ is the cyclotomic character. As before, we write $\lambda = \tilde{\lambda} \circ \mathbf{N}$, where $\tilde{\lambda} : \mathbb{Z}_p^\times \to 1 + p\mathbb{Z}_p$ is given by $\tilde{\lambda}(x) = \langle x \rangle^{-1}$.

By definition,

$$L'_p(f \otimes \chi, \mathbb{1}) = \frac{d}{ds} L_p(f \otimes \chi, \lambda^s) \bigg|_{s=0}.$$

Also by definition,

$$\begin{aligned} L_p(f \otimes \chi, \lambda^s) &= (-1)^{r-1} H_p(f) \left(\frac{D}{-N}\right) \left(1 - C \left(\frac{D}{C}\right) \lambda^s(C)^{-1}\right)^{-1} \int_{G(H_{p^\infty}(\mu_{p^\infty})/K)} \lambda^s d\tilde{\Psi}^C_{f,1,1} \\ &= (-1)^r H_p(f) \left(1 - C \left(\frac{D}{C}\right) \tilde{\lambda}^{-2s}(C)\right)^{-1} \int_{G(H_{p^\infty}(\mu_{p^\infty})/K)} \lambda^s d\tilde{\Psi}^C_{f,1,1}, \end{aligned}$$

where $C$ is an arbitrary integer prime to $N|D|p$. The measure $\tilde{\Psi}^C_{f,1,1}$ is given by:

$$\tilde{\Psi}^C_{f,1,1}(\sigma (\mathrm{mod}\ p^n), \tau (\mathrm{mod}\ p^m)) = L_{f_0}(\tilde{\Psi}^C_{\mathcal{A},1}(a (\mathrm{mod}\ p^m)))$$

where $a$ corresponds to the restriction of $\tau$ under the Artin map and $\sigma$ corresponds to $[\mathcal{A}] \in \mathrm{Pic}(\mathcal{O}_{p^n})$. We therefore have

$$L_p(f\otimes\chi)(\lambda^s) = (-1)^r H_p(f)\left(1 - C\left(\frac{D}{C}\right)\langle C\rangle^{2s}\right)^{-1} L_{f_0}\left[\sum_{\mathcal{A}\in\mathrm{Pic}(\mathcal{O}_K)}\int_{\mathbb{Z}_p^\times}\langle x\rangle^{-s} d\tilde{\Psi}^C_{\mathcal{A},1}\right].$$

Using $\log\langle x\rangle = \log x$, we compute

$$\begin{aligned}\frac{d}{ds}\bigg|_{s=0}&\left(\left(1 - C\left(\frac{D}{C}\right)\langle C\rangle^{2s}\right)^{-1}\int_{\mathbb{Z}_p^\times}\langle x\rangle^{-s} d\tilde{\Psi}^C_{\mathcal{A}}\right)\\ &= \left(1 - C\left(\frac{D}{C}\right)\right)^{-1}\int_{\mathbb{Z}_p^\times}\log x\, d\tilde{\Psi}^C_{\mathcal{A}} + (*)\int_{\mathbb{Z}_p^\times} d\tilde{\Psi}^C_{\mathcal{A}}\\ &= \left(1 - C\left(\frac{D}{C}\right)\right)^{-1}\int_{\mathbb{Z}_p^\times}\log x\, d\tilde{\Psi}^C_{\mathcal{A}}\end{aligned}$$

The integral $\int_{\mathbb{Z}_p^\times} d\tilde{\Psi}^C_{\mathcal{A}}$ vanishes because by Corollary II.6, $L_p(f\otimes\chi)(\lambda) = 0$ for all anticyclotomic $\lambda$, in particular for $\lambda = 1$. Or more simply, it vanishes by the interpolation property of $L_p$ and the vanishing of $L(f,\chi,r+k)$.

If we set

$$G_\sigma = (-1)^r\int_{\mathbb{Z}_p^\times}\log_p\, d\tilde{\Psi}_{\mathcal{A}} \in \bar{M}_{2r}(\Gamma_0(Np^\infty);\mathbb{Q}_p(\chi)),$$

then using the identity

$$\int_{\mathbb{Z}_p^\times}\lambda(\alpha)\, d\tilde{\Psi}^C_{\mathcal{A}} = \int_{\mathbb{Z}_p^\times}\lambda(\alpha) - C\left(\frac{D}{C}\right)\lambda(C^{-2}\alpha)\, d\tilde{\Psi}_{\mathcal{A}},$$

we obtain

$$L'_p(f\otimes\chi,\mathbb{1}) = -H_p(f)\sum_{\sigma\in G(H/K)} L_{f_0}(G_\sigma).$$

Define the operator

$$\mathcal{F} = \prod_{\mathfrak{p}|p}\left(U_p - p^{r-k-1}\chi(\mathfrak{p})\sigma_{\bar{\mathfrak{p}}}\right)^2.$$

Putting together Corollary III.7 and Propositions VI.5 and VI.6, we obtain

**Proposition VIII.1.** *If $p|m$, $(m,N)=1$ and $r_{\mathcal{A}}(m) = 0$, then*

$$c^\sigma_m|\,\mathcal{F} = (-1)^{k+1}(4|D|)^{r-k-1}u^2 a_m(G_\sigma)\Big|\left(U_p^4 - p^{2r-2}U_p^2\right).$$

We define the $p$-adic modular form

$$H_\sigma = F_{\mathcal{A}}|\mathcal{F} + (-1)^k(4|D|)^{r-k-1}u^2 G_\sigma\Big|\left(U_p^4 - p^{2r-2}U_p^2\right).$$

By construction, when $p|m$, $(m,N)=1$ and $r_{\mathcal{A}}(m)=0$, we have

$$a_m(H_\sigma) = d^\sigma_m|\mathcal{F} = \chi(\bar{\mathfrak{a}})^{-1}\sum_{v|p}\langle x, T_m\bar{x}^{\mathfrak{a}}\rangle_v|\mathcal{F}.$$

**Proposition VIII.2.** *Define the operator*

$$\mathcal{F}' = (U_p - \sigma_{\mathfrak{p}})(U_p\sigma_{\mathfrak{p}} - p^{2r-2})(U_p - \sigma_{\bar{\mathfrak{p}}})(U_p\sigma_{\bar{\mathfrak{p}}} - p^{2r-2}).$$

*Then $L_{f_0}(H_\sigma|\mathcal{F}') = 0$.*

*Proof.* The proof should be exactly as in [N3, II.5.10], however the proof given there is not correct. In the next section we explain how to modify Nekovář's argument to prove the desired vanishing. For our purposes in this section, the important point is that this modified proof goes through if we replace the representation $V_{f,A,0} = V_f$ (i.e. the $\ell = 0$ case which Nekovář considers) with our representation $V_{f,A,\ell} = V_f \otimes W$, where $W$ corresponds to a trivial local system. Indeed, the proof works "on the curve" and essentially ignores the local system. The only inputs specific to the local system are two representation-theoretic conditions: it suffices to know that the representation $V_{f,A,\ell}$ is ordinary and crystalline. These follow from Theorems VII.2 and IV.6, respectively. □

It follows that

$$L_{f_0}\left(F_{\mathcal{A}}|\mathcal{F}\mathcal{F}'\right) = (-1)^{k+1}(4|D|)^{r-k-1}u^2 L_{f_0}\left(G_\sigma\middle|\left(U_p^4 - p^{2r-2}U_p^2\right)\mathcal{F}'\right).$$

Since $L_{f_0} \circ U_p = \alpha_p(f) L_{f_0}$, we can remove $\mathcal{F}'$ from the equation above; we may divide out the extra factors that arise as they are non-zero by the Weil conjectures. Summing this formula over $\sigma \in \mathrm{Gal}(H/K)$, we obtain

$$\begin{aligned} L_{f_0}(f) \prod_{\mathfrak{p}|p}\left(1 - \frac{\chi(\mathfrak{p})p^{r-k-1}}{\alpha_p(f)}\right)^2 &\sum_{\sigma \in \mathrm{Gal}(H/K)} \langle z_f, z_{f,\bar{\chi}}^{\mathcal{A}}\rangle \\ &= (-1)^k (4|D|)^{r-k-1} u^2 H_p(f)^{-1}\left(1 - \frac{p^{2r-2}}{\alpha_p(f)^2}\right) L_p'(f \otimes \chi, \mathbb{1}). \end{aligned}$$

Note that the operators $\sigma_{\mathfrak{p}}$ and $\sigma_{\bar{\mathfrak{p}}}$ (in the definition of $\mathcal{F}$) permute the various $\langle z_f, z_{f,\bar{\chi}}^{\mathcal{A}}\rangle$ as $\mathcal{A}$ ranges through the class group. So after summing over $\mathrm{Gal}(H/K)$, these operators have no effect and therefore do not show up in the Euler product in the left hand side.[1] By Hida's computation [N3, I.2.4.2]:

$$\left(1 - \frac{p^{2r-2}}{\alpha_p(f)^2}\right) = H_p(f) L_{f_0}(f),$$

so we obtain

$$L_p'(f \otimes \chi, \mathbb{1}) = (-1)^k \prod_{\mathfrak{p}|p}\left(1 - \frac{\chi(\mathfrak{p})p^{r-k-1}}{\alpha_p(f)}\right)^2 \frac{\sum_{\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)} \langle z_f, z_{f,\bar{\chi}}^{\mathcal{A}}\rangle}{(4|D|)^{r-k-1}u^2}.$$

By equation (4.3), this equals

$$(-1)^k \prod_{\mathfrak{p}|p}\left(1 - \frac{\chi(\mathfrak{p})p^{r-k-1}}{\alpha_p(f)}\right)^2 \frac{h\langle z_{f,\chi}, z_{f,\bar{\chi}}\rangle}{(4|D|)^{r-k-1}u^2}$$

and proves Theorem I.7.

*Proof of Theorem I.9.* We now assume $\chi = \psi^\ell$ as in Section 1.2.3. Recall that the cohomology classes $z_f$ and $\bar{z}_f$ live in $H^1_f(H, V_{f,A,\ell})$. Recall also $V_{f,A,\ell}$ is the 4-dimensional $p$-adic realization of the motive $M(f)_H \otimes M(\chi_H)$ over $H$ with coefficients in $\mathbb{Q}(f)$. Using Remark IV.3, we have a motive $M(f)_K \otimes M(\chi)$ over $K$ with coefficients in $\mathbb{Q}(f, \chi)$ descending $M(f)_H \otimes M(\chi_H) \otimes \mathbb{Q}(\chi)$. The $p$-adic realization of this motive over $K$ is what we called $V_{f,\chi}$.

Thus we may think of the classes $z_f$ and $\bar{z}_f$ in $H^1_f(H, V_{f,A,\ell}) \cong H^1(H, V_{f,\chi})$. Define

$$z_f^K = \mathrm{cor}_{H/K}(z_f) \quad \text{and} \quad \bar{z}_f^K = \mathrm{cor}_{H/K}(\bar{z}_f)$$

in $H^1_f(K, V_{f,\chi})$.

[1] This is unlike what happens in [N3]. The difference stems from the fact that we inserted the Hecke character into the definition of the measures defining the $p$-adic $L$-function.

**Lemma VIII.3.**

$$\mathrm{res}_{H/K}(z_f^K) = hz_{f,\chi} \quad \textit{and} \quad \mathrm{res}_{H/K}(\bar{z}_f^K) = hz_{f,\bar{\chi}}.$$

*Proof.* Note that there is a natural action of $\mathrm{Gal}(H/K)$ on $H^1(H, V_{f,\chi})$, since $V_{f,\chi}$ is a $G_K$-representation. Since $\mathrm{res} \circ \mathrm{cor} = \mathrm{Nm}$, it suffices to show that for each $\sigma \in \mathrm{Gal}(H/K)$, $z_f^\sigma = z_{f,\chi}^{\mathcal{A}}$ and $\bar{z}_f^\sigma = z_{f,\bar{\chi}}^{\mathcal{A}}$, where $\mathcal{A}$ corresponds to $\sigma$ under the Artin map. Recall that

$$z_{f,\chi}^{\mathcal{A}} = \chi(\mathfrak{a})^{-1}\Phi_f\left(\epsilon_B \epsilon Y^{\mathfrak{a}}\right) \quad \text{and} \quad z_{f,\bar{\chi}}^{\mathcal{A}} = \chi(\bar{\mathfrak{a}})^{-1}\Phi_f\left(\epsilon_B \bar{\epsilon} Y^{\mathfrak{a}}\right),$$

for any ideal $\mathfrak{a}$ in the class of $\mathcal{A}$.

To prove $z_f^\sigma = z_{f,\chi}^{\mathcal{A}}$, we first describe explicitly the action of $\mathrm{Gal}(\bar{K}/K)$ on the subspace $\epsilon V_{f,A,\ell} \subset V_{f,A,\ell}$ after identifying the spaces $V_{f,A,\ell}$ and $V_{f,\chi}$. For each $\sigma \in \mathrm{Gal}(\bar{K}/K)$, we have maps

$$\epsilon_\ell H^\ell(\bar{A}^\ell, \mathbb{Q}_p) \xrightarrow{\sigma^*} \epsilon_\ell^\sigma H^\ell(\overline{A^\sigma}^\ell, \mathbb{Q}_p) \xrightarrow{\chi(\mathfrak{a})^{-1}\phi_{\mathfrak{a}}^{\ell *}} \epsilon_\ell H^\ell(\bar{A}^\ell, \mathbb{Q}_p),$$

which induces an action of $G_K$ on $\epsilon V_{f,A,\ell} = V_f \otimes \epsilon H^\ell(\bar{A}^\ell, \mathbb{Q}_p(k))$. By definition of $M(\chi)$, this agrees with the action of $G_K$ on $V_{f,\chi}$. Now the argument in the proof of Lemma IV.11 shows that $z_f^\sigma = z_{f,\chi}^{\mathcal{A}}$. A similar argument works for $\bar{z}_f^\sigma$. □

By Lemma VIII.3, $\mathrm{res}_{H/K}(z_{f,\chi}^K) = hz_{f,\chi}$ and $\mathrm{res}_{H/K}(\bar{z}_f^K) = hz_{f,\bar{\chi}}$. It follows that

$$\left\langle z_f^K, \bar{z}_f^K \right\rangle_K = h\left\langle z_{f,\chi}, z_{f,\bar{\chi}}\right\rangle_H. \tag{8.1}$$

Now assume that $L_p'(f \otimes \chi, \ell_K, \mathbb{1}) \neq 0$. By Theorem I.7 and (8.1), the cohomology classes $z_f^K$ and $\bar{z}_f^K$ are non-zero, giving two independent elements of $H_f^1(K, V_{f,\chi})$. This proves one inequality in Perrin-Riou's conjecture (1.2). The other inequality follows from recent work of Elias [E] constructing an Euler system of generalized Heegner classes and extending the methods of Kolyvagin and Nekovář in [N1] to our setting. □

# CHAPTER IX

# Local $p$-adic heights at primes above $p$

The purpose of this chapter is to fix the proof of [N3, II.5.10] on which both Nekovář's Theorem A and our main theorem rely. In the first two subsections we gather some facts about relative Lubin-Tate groups and ring class field towers, and in 9.3 we explain how to modify the proof in [N3]. We have isolated and fixed only the two parts of [N3, II.5] with a serious mistake, instead of rewriting the entire argument of that section.

## 9.1 Relative Lubin-Tate groups

The reference for this material is [dS, §1].

Let $F/\mathbb{Q}_p$ be a finite extension and let $L$ be the unramified extension of $K$ of degree $\delta \geqslant 1$. Write $\mathfrak{m}_F$ and $\mathfrak{m}_L$ for the maximal ideals in $\mathcal{O}_F$ and $\mathcal{O}_L$ and write $q$ for the cardinality of $\mathcal{O}_F/\mathfrak{m}_F$. We let $\phi : L \to L$ be the Frobenius automorphism lifting $x \to x^q$ and normalize the valuation on $F$ so that a uniformizer has valuation 1. Let $\xi \in F$ be an element of valuation $\delta$ and let $f \in \mathcal{O}_L[[X]]$ be such that

$$f(X) = \varpi X + O(X^2) \quad \text{and} \quad f(X) \equiv X^q \bmod \mathfrak{m}_L,$$

where $\varpi \in \mathcal{O}_L$ satisfies $\mathrm{Nm}_{L/F}(\varpi) = \xi$. Note that $\varpi$ exists and is a uniformizer, since $\mathrm{Nm}_{L/F}(L^\times)$ is the set of elements in $F^\times$ with valuation in $\delta\mathbb{Z}$.

**Theorem IX.1** (de Shalit)**.** *There is a unique one dimensional formal group law $F_f \in \mathcal{O}_L[[X,Y]]$ for which $f$ is a lift of Frobenius, i.e. for which $f \in \mathrm{Hom}(F_f, F_f^\phi)$. $F_f$ comes equipped with an isomorphism $\mathcal{O}_F \cong \mathrm{End}(F_f)$ denoted $a \mapsto [a]_f$, and the isomorphism class of $F_f/\mathcal{O}_L$ depends only on $\xi$ and not on the choice of $f$.*

*Remark* IX.2. This extends the well known construction of Lubin and Tate in the case $\delta = 1$.

Now let $M$ be the valuation ideal of $\mathbb{C}_p$ and let $M_f$ the $M$-valued points of $F_f$. For each $n \geqslant 0$, the $\mathfrak{m}_F^n$-torsion points of $F_f$ are by definition

$$W_f^n = \{\omega \in M_f : [a]_f(\omega) = 0 \quad \text{for all} \quad a \in \mathfrak{m}_F^n\}$$

**Proposition IX.3.** *For each $n \geqslant 1$, set $L_\xi^n = L(W_f^n)$. Then*

1. *$L_\xi^n$ is a totally ramified extension of $L$ of degree $(q-1)q^{n-1}$ and is abelian over $F$.*
2. *There is a canonical isomorphism $(\mathcal{O}_F/\mathfrak{m}_F^n)^\times \cong \mathrm{Gal}(L_\xi^n/L)$ given by $u \mapsto \sigma_u$, where $\sigma_u(\omega) = [u^{-1}]_f(\omega)$ for $\omega \in W_f^n$.*
3. *Both the field $L_\xi^n$ and the isomorphism above are independent of the choice of $f$.*
4. *The map $u \mapsto \sigma_u$ is compatible with the local Artin map $r_F : F^\times \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$.*
5. *The field $L_\xi^n$ corresponds to the subgroup $\xi^{\mathbb{Z}} \cdot (1 + \mathfrak{m}_F^n) \subset F^\times$ via local class field theory.*

Writing $L_\xi = \bigcup_n L_\xi^n$, we see that $\mathrm{Gal}(L_\xi/L) \cong \mathcal{O}_F^\times$ and the group of universal norms in $F^\times$ coming from $L_\xi$ is $\xi^{\mathbb{Z}}$. Moreover, we have an isomorphism $\mathrm{Gal}(L_\xi/L) \to \mathcal{O}_F^\times$ who's inverse is $r_F|_{\mathcal{O}_F^\times}$ composed with the restriction $\mathrm{Gal}(F^{\mathrm{ab}}/F) \to \mathrm{Gal}(L_\xi/F)$.

## 9.2 Relative Lubin-Tate groups and ring class field towers

Now let $v$ be a place of $H$ above $p$ and above the prime $\mathfrak{p}$ of $K$. For each $j \geqslant 1$, write $H_{j,w}$ for the completion of the ring class field $H_{p^j}$ of conductor $p^j$ at the unique place $w = w(j)$ above $v$. In particular, $H_{0,v} = H_v$. If $\delta$ is the order of $\mathfrak{p}$ in $\mathrm{Pic}(\mathcal{O}_K)$, then $H_v$ is the unramified extension of $K_\mathfrak{p} \cong \mathbb{Q}_p$ of degree $\delta$. Since $p$ splits in $K$, $H_{j,w}/H_v$ is totally ramified of degree $(p-1)p^{j-1}/u$, where recall $u = \#\mathcal{O}_K^\times/2$. Moreover, $\mathrm{Gal}(H_{j,w}/H_v)$ is cyclic and $H_{j,w}$ is abelian over $\mathbb{Q}_p$. We call $H_\infty = \bigcup_j H_{j,w}$ the local ring class field tower; it contains the anticyclotomic $\mathbb{Z}_p$-extension of $K_\mathfrak{p}$. To ease notation and to recall the notation of the previous section, we write $L = H_v$.

**Proposition IX.4.** *Write $\mathfrak{p}^\delta = (\pi)$ for some $\pi \in \mathcal{O}_K$. Then $H_\infty$ is contained in the field $L_\xi$ attached to the Lubin-Tate group relative to the extension $L/\mathbb{Q}_p$ with parameter $\xi = \pi/\bar{\pi}$ in $K_\mathfrak{p} \cong \mathbb{Q}_p$. If $\mathcal{O}_K^\times = \{\pm 1\}$, then $H_\infty = L_\xi$.*

*Remark* IX.5. Note that there are other natural Lubin-Tate groups relative to $L/\mathbb{Q}_p$ coming from the class field theory of $K$, namely the formal groups of elliptic curves with complex multiplication by $\mathcal{O}_K$. These formal groups will have different parameters however, as can be seen from the discussion in [dS, II.1.10].

*Proof.* By (5) of Proposition IX.3, it is enough to prove that $H_\infty$ is the subfield of $\mathbb{Q}_p^{\mathrm{ab}}$ corresponding to the subgroup $(\pi/\bar{\pi})^{\mathbb{Z}} \cdot \mu_K^2$ under local class field theory. First we show that $(\pi/\bar{\pi})$ is norm from every $H_{j,w}$. Using the compatibility between local and global reciprocity maps, this will follow if the idele (with non-trivial entry in the $\mathfrak{p}$ slot)

$$(\ldots 1\,, 1\,, \pi/\bar{\pi}\,, 1,\ 1\,, \ldots) \in \mathbb{A}_K^\times$$

is in the kernel of the reciprocity map

$$r_j : \mathbb{A}_K^\times/K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K) \to \mathrm{Gal}(H_{p^j}/K),$$

for each $j$. Since the kernel of $r_j$ is $K^\times \mathbb{A}_{K,\infty}^\times \hat{\mathcal{O}}_{p^j}^\times$, it is enough to show that

$$(\ldots 1/\pi\,, 1/\pi\,, 1/\bar{\pi}\,, 1/\pi\,, 1/\pi\,, \ldots) \in \hat{\mathcal{O}}_{p^j}^\times.$$

This is clear at all primes away from $p$ since $\pi$ is a unit at those places. At $p$, it amounts to showing that $(1/\bar{\pi}, 1/\pi) \in K_\mathfrak{p} \times K_{\bar{\mathfrak{p}}}$ lands in the diagonal copy of $\mathbb{Z}_p$ under the identification $K_\mathfrak{p} \times K_{\bar{\mathfrak{p}}} \cong \mathbb{Q}_p \times \mathbb{Q}_p$, and this is also clear.

Since $L/\mathbb{Q}_p$ is unramified of degree $\delta$ and $\xi = \pi/\bar{\pi}$ has valuation $\delta$, it remains to prove that the only units in $\mathbb{Q}_p$ which are universal norms for the tower $H_\infty/\mathbb{Q}_p$ are those in $\mu_K^2$. But by the same argument as above, the only way $\alpha \in \mathbb{Z}_p^\times$ can be a norm from every $H_{j,w}$ is if $\alpha\zeta = \bar{\zeta}$ for some global unit $\zeta \in K$. But then $\zeta$ is a root of unity and $\alpha = \zeta^{-1}\bar{\zeta} = \zeta^{-2}$, so $\alpha$ is in $\mu_K^2$. Conversely, it's clear that each $\zeta \in \mu_K^2$ is a universal norm. $\square$

*Remark* IX.6. Since we are assuming $K$ has odd discriminant, the equality $H_\infty = L_\xi$ holds unless $K = \mathbb{Q}(\mu_3)$. For ease of exposition we will assume $K \neq \mathbb{Q}(\mu_3)$ for the rest of this chapter; the modifications needed for the case $K = \mathbb{Q}(\mu_3)$ are easy enough.

We will need one more technical fact about the relative Lubin-Tate group $F_f$ cutting out $H_\infty$. Let $\chi_\xi : \mathrm{Gal}(\bar{L}/L) \to \mathbb{Z}_p^\times$, be the character giving the Galois action on the torsion points of $F_f$. We let $\mathbb{Q}_p(\chi_\xi)$ denote the 1-dimensional $\mathbb{Q}_p$-vector space endowed with the action of $\mathrm{Gal}(\bar{L}/L)$ determined by $\chi_\xi$, and we denote by $D_{\mathrm{cris}}(\mathbb{Q}_p(\chi_\xi))$ the usual filtered $\phi$-module contravariantly attached to the $\mathrm{Gal}(\bar{L}/L)$-representation $\mathbb{Q}_p(\chi_\xi)$ by Fontaine.

**Proposition IX.7.** *The representation $\mathbb{Q}_p(\chi_\xi)$ is crystalline and the frobenius map on the 1-dimensional $L$-vector space $D_{\mathrm{cris}}(\mathbb{Q}_p(\chi_\xi))$ is given by multiplication by $\xi$.*

*Proof.* This is presumably well known, but with a lack of reference we will verify this fact using [C2, Prop. B.4]. There it is shown that $\mathbb{Q}_p(\chi_\xi)$ is crystalline if and only if there exists a homomorphism of tori $\chi' : L^\times \to \mathbb{Q}_p^\times$ which agrees with the restriction of $\chi_\xi \circ r_L$ to $\mathcal{O}_L^\times$. In that case, frobenius on $D_{\mathrm{cris}}(\mathbb{Q}_p(\chi_\xi))$ is given by multiplication by $\chi_\xi(r_L(\varpi)) \cdot \chi'(\varpi)^{-1}$, where $\varpi$ is any uniformizer of $L$.[1] Combining (2) and (4) of Proposition IX.3 with the commutativity of the following diagram

$$\begin{array}{ccc} L^\times & \xrightarrow{r_L} & \mathrm{Gal}(L^{\mathrm{ab}}/L) \\ {\scriptstyle \mathrm{Nm}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathbb{Q}_p^\times & \xrightarrow{r_{\mathbb{Q}_p}} & \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p), \end{array}$$

we see that $\chi' = \mathrm{Nm}^{-1}$ gives such a homomorphism, so the crystallinity follows. Note that by construction $\chi_\xi : \mathrm{Gal}(L^{\mathrm{ab}}/L) \to \mathbb{Z}_p^\times$ factors through a character

$$\tilde{\chi}_\xi : \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/L) \to \mathbb{Z}_p^\times.$$

So if we choose $\varpi$ to be such that $\mathrm{Nm}_{L/\mathbb{Q}_p}(\varpi) = \xi$, then

$$\begin{aligned} \chi_\xi(r_L(\varpi)) &= \tilde{\chi}_\xi(r_{\mathbb{Q}_p}(\mathrm{Nm}(\varpi))) \\ &= \tilde{\chi}_\xi(r_{\mathbb{Q}_p}(\xi)) = 1. \end{aligned}$$

Thus, the frobenius is given by multiplication by $\chi'(\varpi)^{-1} = \mathrm{Nm}_{L/\mathbb{Q}_p}(\varpi) = \xi$. □

## 9.3 Local heights at $p$ in ring class field towers

The proofs of both [N3, II.5.6] and [N3, II.5.10] mistakenly assert that $H_{j,w}$ contains the $j$-th layer of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$ (as opposed to the anticyclotomic $\mathbb{Z}_p$-extension). This issue first arises in the proofs of [N3, II.5.9] and [N3, II.5.10]. We explain now how to adjust the proof of [N3, II.5.10]; similar adjustments may be used to fix the proof of [N3, II.5.9]. The adjustments we make are still in the spirit of Nekovář's original argument, but we will use some deep results from $p$-adic Hodge theory to carry the argument through.

Recall the setting of [N3, II.5.10]: $x$ is the Tate vector corresponding to our (generalized) Heegner cycle $\epsilon_B \epsilon Y$, and $V = H^1_{\text{ét}}(\bar{X}_0(N), j_{0*}\mathcal{A})(1)$. We have the Tate cycle

$$x_f = \sum_{m \in S} c_{f,m} T_m x \in Z(Y_0(N), H) \otimes_{\mathbb{Q}_p} L,$$

a certain linear combination (with coefficients $c_{f,m}$ living in a large enough field $L$) of $T_m x$ such that

$$\Phi_T(x_f) = z_f \in H^1_f(H, V) \otimes_{\mathbb{Q}_p} L.$$

Moreover, each $m \in S$ satisfies $(m, pN) = 1$ and $r(m) = 0$, where $r(m)$ is the number of ideals in $K$ of norm $m$. To fix the proof of [N3, II.5.10], we prove the following vanishing result for local heights at primes $v$ of $H$ above $p$.

**Theorem IX.8.** *For each $j \geqslant 1$, let $h_j^\sigma \in Z_f(Y_0(N), H_{j,w})$ be a Tate vector supported on a point $y_j \in Y_0(N)$ corresponding to an elliptic curve $E_j$ such that* $\mathrm{End}(E_j)$ *is the order in $\mathcal{O}_K$ of index $p^j$. Then*

$$\lim_{j \to \infty} \langle x_f, N_{H_{j,w}/H_v}(h_j^\sigma) \rangle_v = 0.$$

[1] Note that we are using the contravariant $D_{\mathrm{cris}}$, whereas [C2] uses the covariant version.

*Proof.* Recall that $E_j$ is a quotient of an elliptic curve $E$ with CM by $\mathcal{O}_K$ by a (cyclic) subgroup of order $p^j$ which does not contain either the canonical subgroup $E[\mathfrak{p}]$ or its dual $E[\bar{\mathfrak{p}}]$. By the compatibility of local heights with norms [N3, II.1.9.1], we have

$$\left\langle x_f, N_{H_{j,w}/H_v}(h_j^\sigma)\right\rangle_{v,\ell_v} = \left\langle x_f, h_j^\sigma\right\rangle_{w,\ell_w}, \tag{9.1}$$

where $\ell_w = \ell_v \circ N_{H_{j,w}/H_v}$. Recall that we are assuming now that $\ell_K = \log_p \circ \lambda$, where $\lambda : \mathrm{Gal}(K_\infty/K) \to 1 + p\mathbb{Z}_p$ is the cyclotomic character. Thus the local component $\ell_v : H_v^\times \to \mathbb{Q}_p$ of $\ell_K$ is $\ell_v = \log_p \circ N_{H_v/\mathbb{Q}_p}$, and

$$\ell_w = \log_p \circ N_{H_{j,w}/\mathbb{Q}_p}.$$

We have seen that the ring class field tower $H_\infty$ is cut out by a relative Lubin-Tate group. In fact, it follows from the results in the previous sections that $H_{j,w} = L_\xi^j$, where $L = H_v$ and $\xi = \pi/\bar{\pi}$ as before. Let $E$ be the mixed extension used to compute the height pairing of $x_f$ and $h_j^\sigma$, chosen as in Chapter 5.2, and let $E_w$ be its restriction to the decomposition group at $w$. Assume that

$$E_w \text{ is a crystalline representation of } \mathrm{Gal}(\bar{H}_{j,w}/H_{j,w}).$$

Then by definition of the local height (see Chapter 5.1), we have

$$\begin{aligned}\left\langle x_f, h_j^\sigma\right\rangle_{w,\ell_w} &= \ell_w(r_w([E_w])) \\ &= \log_p\left(N_{H_{j,w}/\mathbb{Q}_p}(r_w([E_w]))\right).\end{aligned}$$

where $r_w([E_w])$ is an element of $\widehat{\mathcal{O}_{H_{j,w}}^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. In fact, the ordinarity of $f$ allows Nekovář to "bound denominators"; i.e. he shows

$$p^{-d_j}\left\langle x_f, h_j^\sigma\right\rangle_{w,\ell_w} \in \log_p\left(N_{H_{j,w}/\mathbb{Q}_p}\left(\widehat{\mathcal{O}_{H_{j,w}}^\times}\right)\right).$$

for some integer $d_j$. Indeed, see the proofs in [N3, II.1.10, II.5.10] and note that $H_f^1(H_{v,w}, \mathbb{Z}_p(1)) = \widehat{\mathcal{O}_{H_{j,w}}^\times}$. Moreover, the $d_j$ are uniformly bounded as $j$ varies (Nekovář's proof of this fact does not quite work, but we fix this issue in Proposition IX.14). Let us write $d = \sup_j d_j$. By Proposition IX.3, we have

$$p^{-d}\left\langle x_f, h_j^\sigma\right\rangle_{w,\ell_w} \in \log_p(1 + p^j\mathbb{Z}_p) \subset p^j\mathbb{Z}_p.$$

The theorem would then follow upon taking the limit as $j \to \infty$.

It therefore remains to show that $E_w$ is crystalline. First we need a lemma.

**Lemma IX.9.** *Let $m \in S$ and $j$ be as above. Then the supports of $T_m x$ and $b_{p^j}^\sigma$ are disjoint on the generic and special fibers of the integral model $\mathcal{X}$ of $X_0(N)$.*

*Proof.* Let $z \in Y_0(N)(\bar{\mathbb{Q}}_p)$ be in the support of $T_m x$ and let $y$ be the Heegner point supporting the Tate cycle $x$. Thinking of these points as elliptic curves via the moduli interpretation, there is an isogeny $\phi : y \to z$ of degree prime to $p$ since $(p, m) = 1$. Recall $p$ splits in $K$, so that $y$ has ordinary reduction $y_s$ at $v$. Since $\mathrm{End}(y) \cong \mathcal{O}_K \cong \mathrm{End}(y_s)$, $y$ is a Serre-Tate canonical lift of $y_s$. As $\phi$ induces an isomorphism of $p$-divisible groups, $z$ is also a canonical lift of its reduction. On the other hand, the curve $E_j$ supporting $h_j^\sigma$ has CM by a non-maximal order of $p$-power index in $\mathcal{O}_K$ and is therefore not a canonical lift of its reduction. Indeed, the reduction of $E_j$ is an elliptic curve with CM by the full ring $\mathcal{O}_K$ as it obtained by successive quotients of $y_s$ by either the kernel of Frobenius or Verschiebung. This shows that $T_m x$ and $b_{p^j}^\sigma$ have disjoint support in the generic fiber.

By [GZ, III.4.3], the divisors $T_{mn}y$ and $y^\tau$ are disjoint in the generic fiber, for any $\tau \in \mathrm{Gal}(H/K)$. Since all points in the support of these divisors are canonical lifts, the divisors must not intersect in the special fiber either. But we saw above that the special fiber of $E_j$ is a Galois conjugate of the reduction of $y$, so $E_j$ and $T_m y$ are disjoint on the special fiber as well. □

Next we note that $T_m x$ is a sum $\sum d_i$, where each $d_i$ is supported on a single closed point $S$ of $Y_0(N)/H_{j,w}$. Using norm compatibility once more and base changing to an extension $\mathbb{F}/H_{j,w}$ which splits $S$, we may assume that $S \in Y_0(N)(\mathbb{F})$.

It then suffices to show that the mixed extension $E'_w$ corresponding to $d_i$ and $h_j^\sigma$ is crystalline. Recall from Chapter V that this mixed extension is a subquotient of

$$H^1(\bar{X}_0(N) - \bar{S} \text{ rel } \bar{T}, j_{0*}\mathcal{A})(1),$$

where $T = y_j$ is the point supporting $h_j^\sigma$. So it is enough to show that this cohomology group is itself crystalline. Finally, this follows from combining the previous lemma with the following result. □

**Theorem IX.10.** *Suppose $\mathbb{F}$ is a finite extension of $\mathbb{Q}_p$ and let $S, T \in Y_0(N)(\mathbb{F})$ be points with non-cuspidal reduction and which do not intersect in the special fiber. Then $H^1(\bar{X}_0(N) - \bar{S} \text{ rel } \bar{T}, j_{0*}\mathcal{A})(1)$ is a crystalline representation of $G_\mathbb{F}$.*

*Remark* IX.11. Suppose $F$ is a $p$-adic field and $X/\text{Spec } \mathcal{O}_F$ is a smooth projective variety of relative dimension $2k-1$. If $Y, Z \subset X$ are two (smooth) subvarieties of codimension $k$ which do not intersect on the special fiber, then one expects that $H^{2k-1}(\bar{X}_F - \bar{Y}_F \text{ rel } \bar{Z}_F, \mathbb{Q}_p(k))$ is a crystalline representation of $G_F$. The theorem above proves this for cycles sitting in fibers of a map $X \to C$ to a curve, but the method of proof does not seem to apply in the general case.

*Proof.* Write $\mathbb{V} = H^1(\bar{X}_0(N) - \bar{S} \text{ rel } T, j_{0*}\mathcal{A})(1)$. The sketch of the proof is as follows. Faltings' comparison isomorphism [F] identifies $D_{\text{cris}}(\mathbb{V})$ with the crystalline analogue of $\mathbb{V}$, which we will refer to (in this sketch) as $H^1_{\text{cris}}(X - S \text{ rel } T, j_{0*}\mathcal{A})$. The dimension of $\mathbb{V}$ is determined by the standard exact sequences

$$0 \to H^0(\bar{T}, j_{0*}\mathcal{A})(1) \to \mathbb{V} \to H^1(\bar{X} - \bar{S}, j_{0*}\mathcal{A})(1) \to 0 \tag{9.2}$$

$$0 \to H^1(\bar{X}, j_{0*}\mathcal{A})(1) \to H^1(\bar{X} - \bar{S}, j_{0*}\mathcal{A})(1) \to H^0(\bar{S}, j_{0*}\mathcal{A}) \to 0$$

Similar exact sequences should hold in the crystalline theory (i.e. with $H^1$ replaced by $H^1_{\text{cris}}$ everywhere) since $S$ and $T$ reduce to distinct points on the special fiber. Using the known crystallinity of $H^1(\bar{X}, j_{0*}\mathcal{A})(1)$, $H^0(\bar{T}, j_{0*}\mathcal{A})(1)$, and $H^0(\bar{S}, j_{0*}\mathcal{A})$ (the latter two because the fibers of $X \to X(N)$ above $S$ and $T$ have good reduction), we conclude that

$$\dim_{\mathbb{Q}_p} \mathbb{V} = \dim_{F_0} H^1_{\text{cris}}(X - S \text{ rel } T, j_{0*}\mathcal{A}),$$

i.e. that $\mathbb{V}$ is crystalline. To turn this sketch into a proof, we need to say explicitly what $H^1_{\text{cris}}(X - S \text{ rel } T, j_{0*}\mathcal{A})$ is. Note that the usual crystalline cohomology is not a good candidate because it is not usually finite dimensional unless the variety is smooth and projective.

Let us describe in more detail the comparison isomorphism which we invoked above. The main result of [F] concerns the cohomology of a smooth projective variety with trivial coefficients. In our setting, however, we deal with cohomology of an affine variety with partial support along the boundary and with non-trivial coefficients. The proof of the comparison isomorphism in this more complicated situation is sketched briefly in [F] as well, but we follow the exposition [Ol], where the modifications we need are explained explicitly and in detail.

Let $R$ be the ring of integers of $\mathbb{F}$ and set $V = \text{Spec }(R)$. Let $X/V$ be a smooth projective curve and let $S, T \in X(V)$ be two rational sections which we think of as divisors on $X$. We assume that $S$ and $T$ do not intersect, even on the closed fiber. Set $D = S \cup T$ and $X^o = X - D$. The divisor $D$ defines a log structure $M_X$ on $X$ and we let $(Y, M_Y)$ be the closed fiber of $(X, M_X)$. We use the log-convergent topos $((Y, M_Y)/V)_{\text{conv}}$ to define the 'crystalline' analogue of $\mathbb{V}$. There is an isocrystal $J_S$ on $((Y, M_Y)/V)_{\text{conv}}$ which is étale locally defined by the ideal sheaf of $S$; see [Ol, §13] for its precise definition and for more regarding the convergent topos.

**Theorem IX.12** (Faltings, Olsson). *Let $L$ be a crystalline sheaf on $X^o_\mathbb{F}$ associated to a filtered isocrystal $(F, \varphi_F, Fil_\mathcal{F})$. Then there is an isomorphism*

$$B_{\text{cris}}(\bar{V}) \otimes_\mathbb{F} H^1(((Y, M_Y)/V)_{\text{conv}}, F \otimes J_S) \to B_{\text{cris}}(\bar{V}) \otimes_{\mathbb{Q}_p} H^1(\bar{X} - \bar{S} \text{ rel } \bar{T}, L). \tag{9.3}$$

As $L = j_{0*}\mathcal{A}$ is crystalline [F, 6.3], we may apply this theorem in our situation. Taking Galois invariants, we conclude that $D_{\mathrm{cris}}(\mathbb{V}) = H^1(((Y, M_Y)/V)_{\mathrm{conv}}, F \otimes J_S)$. To complete the proof of Theorem IX.10, it would be enough to know that the convergent cohomology group $D_{\mathrm{cris}}(\mathbb{V})$ sits in exact sequences analogous to the standard Gysin sequences (9.2). These sequences hold in any cohomology theory satisfying the Bloch-Ogus axioms, but unfortunately convergent cohomology is not known to satisfy these axioms. On the other hand, rigid cohomology does satisfy the Bloch-Ogus axioms [P]. So we apply Shiho's log convergent-rigid comparison isomorphism [Sh, 2.4.4] to identify $D_{\mathrm{cris}}(\mathbb{V})$ with $H^1_{\mathrm{rig}}(Y - S_s \text{ rel } T_s, j^\dagger\mathcal{E})$, for a certain overconvergent isocrystal $j^\dagger\mathcal{E}$ which is the analogue of $j_{0*}\mathcal{A}$ on the special fiber. Here $S_s$ and $T_s$ are the points on the special fiber. We have similar identifications with rigid cohomology for each term appearing in the sequences (9.2), and the corresponding short exact sequences of rigid cohomology groups are exact. The crystallinity of $\mathbb{V}$ now follows from dimension counting. □

*Remark* IX.13. Theorem IX.8 has two components: first one must bound denominators and then one shows that the heights go to 0 $p$-adically. In the argument above, the ordinarity of $f$ was the crucial input needed to bound denominators. We briefly explain the modifications need to fix the proof of [N3, II.5.9], where one pairs Heegner cycles of $p$-power conductor with cycles in the kernel of the local Abel-Jacobi map (the analogue of principal divisors in weight 2). The fact that these cycles are Abel-Jacobi trivial allows us to make a "bounded denominators" argument even without an ordinarity assumption; see [N3, II.1.9]. To kill the $p$-adic height, we further note that the particular AJ-trivial cycles in the proof of II.5.9 are again linear combinations of various $T_n x$, with $r(n) = 0$. This lets us invoke Lemme IX.9 and Theorem IX.10, as before.

As we alluded to in the proof of Theorem IX.8, the proof of [N3, II.5.11] again assumes (incorrectly) that $H_\infty$ contains the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$. To fix the proof there, it is enough to prove the following proposition.

**Proposition IX.14.** *Let $V$ be the Galois representation $H^1_{\mathrm{\acute{e}t}}(\bar{X}_0(N), j_{0*}\mathcal{A})(1)$ attached to weight $2r$ cusp forms. Writing $H_\infty$ for $\bigcup_j H_{j,w}$, we have $H^0(H_\infty, V) = 0$.*

*Proof.* We follow Nekovář's approach, but instead of using the cyclotomic character we use the character $\chi_\xi$ coming from the relative Lubin-Tate group attached to $H_\infty$, defined above. By Proposition IX.7, the $G_{\mathbb{Q}_p}$-representation $\mathbb{Q}_p(\chi_\xi)$ is crystalline and the frobenius on $D_{\mathrm{cris}}(\mathbb{Q}_p(\chi_\xi))$ is given by multiplication by $\xi$, where $\xi$ is defined in Proposition IX.4.

Since $V$ is Hodge-Tate, there is an inclusion of $\mathrm{Gal}(H_\infty/H_v)$-representations

$$H^0(H_\infty, V) \subset \oplus_{j\in\mathbb{Z}} H^0(H_v, V(\chi_\xi^j))(\chi_\xi^{-j}).$$

Indeed, $H^0(H_\infty, V)$ has an action by $\mathrm{Gal}(H_\infty/H)$ which we can break up into isotypic parts indexed by characters $\chi_\xi^s$, with $s \in \mathbb{Z}_p$. But of these characters, the only ones which are Hodge-Tate are those with $s \in \mathbb{Z}$, so we obtain the inclusion above.

So it suffices to show that for each $j$, $H^0(H_v, V(\chi_\xi^j))(\chi_\xi^{-j}) = 0$. Tensoring the inclusion $\mathbb{Q}_p \to B_{\mathrm{cris}}^{f=1}$ by $V(\chi_\xi^j)$, taking invariants, and then twisting the resulting filtered frobenius modules by $\chi_\xi^{-j}$, we obtain

$$H^0(H_v, V(\chi_\xi^j))(\chi_\xi^{-j}) \subset D_{\mathrm{cris}}(V)^{f=\xi^{-j}}$$

As an element of $\mathbb{C}$, $\xi$ has absolute value 1. Since $V$ appears in the odd degree cohomology of the Kuga-Sato variety, [KM] implies that $D_{\mathrm{cris}}(V)^{f=\xi^{-j}}$ vanishes and the proposition follows. □

Finally, for completeness, we explain how Proposition IX.14 is used in the proof of Proposition VIII.2. Let $X$ be the (generalized) Kuga-Sato variety over $H_v$ and let $T$ be the image of the map

$$H^{2r+2k-1}(\bar{X}, \mathbb{Z}_p(r+k)) \to V = H^{2r+2k-1}(\bar{X}, \mathbb{Q}_p(r+k)).$$

Proposition IX.14 is used to infer the following fact, whose proof was left to the reader in [N3].

**Proposition IX.15.** *The numbers $\#H^1(H_{j,w}, T)_{\mathrm{tors}}$ are bounded as $j \to \infty$.*

*Proof.* From the short exact sequence

$$0 \to T \to V \to V/T \to 0,$$

we have

$$(V/T)^{G_j} \to H^1(G_j, T) \to H^1(G_j, V) \to 0,$$

where $G_j = \mathrm{Gal}(\bar{H}_{j,w}/H_{j,w})$. As $H^1(G_j, V)$ is torsion-free, we see that $(V/T)^{G_j}$ maps surjectively onto $H^1(G_j, T)_{\mathrm{tors}}$. An element of order $p^a$ in $(V/T)^{G_j}$ is of the form $p^{-a}t$ for some $t \in T$ not divisible by $p$ in $T$. We then have $\sigma t - t \in p^a T$ for all $\sigma \in G_j$. As $V/T \cong (\mathbb{Q}_p/\mathbb{Z}_p)^n$ for some integer $n$, it suffices to show that $a$ is bounded as we vary over all elements of $(V/T)^{G_j}$ and all $j$.

Suppose these $a$ are not bounded. Then we can find a sequence $t_i \in T$ such that $t_i \notin pT$ and such that $\sigma t_i - t_i \in p^{a(i)}T$ for all $\sigma \in G_\infty := \mathrm{Gal}(\bar{H}/H_\infty)$. Here, $a(i)$ is a non-decreasing sequence going to infinity with $i$. Since $T$ is compact we may replace $t_i$ with a convergent subsequence, and define $t = \lim t_i$. We claim that $t \in H^0(H_\infty, V)$. Indeed, for any $i$ we have

$$\sigma t - t = \sigma(t - t_i) - (t - t_i) + \sigma t_i - t_i.$$

For any $n > 0$, we can choose $i$ large enough so that $(t - t_i) \in p^n T$ and $\sigma t_i - t_i \in p^n T$, showing that $\sigma t = t$. By Proposition IX.14, $t = 0$, which contradicts the fact that $t = \lim t_i$ and $t_i \notin pT$. $\square$

## CHAPTER X

# Complex $L$-functions

In the previous section we completed the proof of the $p$-adic Gross-Zagier formula for a weight $2r$ ordinary modular form $f$ together with an unramified Hecke character $\chi$ of type $(2k, 0)$ with $k < r$. Of course, one expects an archimedean version of this formula, directly generalizing the original Gross-Zagier formula and Zhang's higher weight formula [Z], both of which concern the case $k = 0$. Zhang's archimedean formula relates the central derivative of the complex $L$-function $L(f, \chi, s)$ to archimedean heights of Heegner cycles. The $\mathbb{C}$-valued height pairing he uses is Beilinson's height pairing on homologically trivial algebraic cycles [Bei], which can be computed using the arithmetic intersection theory of Gillet and Soulé [GS].

In the remaining two sections, we sketch a proof of the archimedean version of Theorem I.7. In this first section, we compute the Fourier coefficients of the modular form which represents the linear functional $f \mapsto L'_{\mathcal{A}}(f, \chi, s)$ on the space of newforms of weight $2k$. One wishes to relate these coefficients to height pairings roughly of the form $\langle \epsilon' Y, T_m \epsilon' Y^{\mathfrak{a}} \rangle_{\mathrm{GS}}$ (see Chapter IV for the definition of the projector $\epsilon'$). These pairings decompose into local heights at both finite and infinite places. The local heights at finite places more or less agree with the our $p$-adic local height computations at places away from $p$ (Proposition VI.5). Indeed they are both computed by arithmetic intersection theory. Moreover, these contributions are seen to match up with the first term in the expression for the Fourier coefficient computation (see Proposition X.8). It therefore remains to compute local heights of generalized Heegner cycles at archimedean places, which is what we do in the next and last section. The local heights at infinity will ultimately match up with the remaining terms in the Fourier coefficient expression.

Our computations build off the work of [GZ] on the analytic side and the approach of [Br] (and, to a lesser degree, [Z]) for the height computations. We therefore switch our notation to match with those papers. So for the rest of this document, we let $f \in S_{2k}(\Gamma_0(N))$ be a newform and $\chi$ an unramified Hecke character of $K$ with infinite type $(2t, 0)$ and $0 < t < k$. We also set $\ell = 2t$ for occasional notational convenience.

Let $A/H$ be an elliptic curve (chosen as in Chapter IV) over the Hilbert class field $H$ of $K$ with CM by $\mathcal{O}_K$. For convenience, we choose an embedding $H \to \mathbb{C}$ so that the base change $A_{\mathbb{C}}$ is isomorphic to $\mathbb{C}/\mathcal{O}_K$. As before, we assume all primes dividing $N$ split in $K$ and that the discriminant $D$ of $K$ is odd. Let $X = W_{2k-2} \times A^{2t}$ be the generalized Kuga-Sato variety, defined over $H$ and fibered over the modular curve $X(N)$ parameterizing elliptic curves with full level $N$ structure.

For each ideal $\mathfrak{a} \subset \mathcal{O}_K$, we have constructed in Chapter IV generalized Heegner cycles $\epsilon_B \epsilon Y^{\mathfrak{a}} \subset X$ and $\epsilon_B \bar{\epsilon} Y^{\mathfrak{a}}$ (with coefficients in $K$) sitting in fibers above Heegner points in $X_0(N)(H)$. These are homologically trivial cycles of codimension $k + t$ in $X$. For any ideal class $\mathcal{A}$, define

$$Z_{\mathcal{A}} = \chi(\mathfrak{a})^{-1} \epsilon_B \bar{\epsilon} Y^{\mathfrak{a}} \quad \text{and} \quad \bar{Z}_{\mathcal{A}} = \chi(\bar{\mathfrak{a}})^{-1} \epsilon_B \bar{\epsilon} Y^{\mathfrak{a}},$$

where $\mathfrak{a}$ is any choice of integral ideal in the class $\mathcal{A}$. We have extended coefficients of our Chow groups to $K(\chi)$. Like in [Z], the cycle $Z_{\mathcal{A}}$ is a formal sum $\sum_Q Y_0$ of identical copies of a certain symmetrized algebraic cycle $Y_0$ in the fiber of the variety $X$ over the point $Q$ in $X(N)$. The sum is over points $Q$ in the preimage of a Heegner point on $X_0(N)$. For $\mathcal{A} = [\mathcal{O}_K]$, we just write $Z$ and $\bar{Z}$. We also define

$$H_m(\mathcal{A}) = \langle Z + \bar{Z}, T_m(Z_{\mathcal{A}} + \bar{Z}_{\mathcal{A}})\rangle_{\mathrm{GS}}.$$

On the other hand, for each ideal class $\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)$ we consider the Dirichlet series

$$L_{\mathcal{A}}(f, \chi, s) = \sum_{(n,ND)=1} \left(\frac{D}{n}\right) n^{-2s+2k+2t-1} \sum_{m \geqslant 1} a_f(m) r_{\mathcal{A},\chi}(m) m^{-s}.$$

Here, $r_{\mathcal{A},\chi}(m) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})$, the sum being over integral ideals in $\mathcal{A}$ of norm $m$. As usual, we write $r_{\mathcal{A}}(m)$ for $r_{\mathcal{A},1}(m)$, the number of integral ideals in $\mathcal{A}$ of norm $m$.

Following the methods of Gross and Zagier [GZ], we show that $L_{\mathcal{A}}(f, \chi, s)$ has analytical continuation to all of $\mathbb{C}$ and satisfies a functional equation when $s$ is replaced by $2k + 2t - s$. By our assumption on the primes dividing $N$ and on the weights of $f$ and $\chi$, $L_{\mathcal{A}}(f, \chi, s)$ vanishes at the central point $s = k+t$. Moreover we will show that there exists a $g_{\mathcal{A}} = \sum_{m \geqslant 1} a_m(\mathcal{A}) q^m \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ representing the linear functional

$$f \mapsto \frac{(2k-2)!\sqrt{|D|}D^t}{2^{4k-1}\pi^{2k}} L'_{\mathcal{A}}(f, \chi, r+k),$$

on $S_{2k}^{\mathrm{new}}(\Gamma_0(N))$.

Our goal is to sketch a proof of the following result:

**Theorem X.1.** *Set $u = \#\mathcal{O}_K^{\times}/2$ as before. Then for $m \geqslant 1$ such that $(m, N) = 1$ and $r_{\mathcal{A}}(m) = 0$, we have*

$$H_m(\mathcal{A}) + H_m(\bar{\mathcal{A}}) = \frac{u^2(4|D|)^{k-t-1}}{\binom{2k-2}{k-t-1}} \left(a_m(\mathcal{A}) + a_m(\bar{\mathcal{A}})\right).$$

*Remark* X.2. Assume for simplicity that the class number of $K$ is 1. Then we would ultimately like to prove

$$\langle Y_{f,\chi}, Y_{f,\chi}\rangle_{\mathrm{GS}} \doteq L'(f, \chi, r+k), \tag{10.1}$$

where $Y_{f,\chi}$ is the $f$-isotypic component of $Z + \bar{Z} = \epsilon_B \epsilon' Y$. If one knew the modularity of the generating series $\sum \langle Z + \bar{Z}, T_m(Z + \bar{Z})\rangle_{\mathrm{GS}}\, q^m$, then (10.1) would follow from Theorem X.1 via a standard argument. Even without knowing the modularity, we can still deduce the formula (10.1) with some extra work (much like what is done in [Z]); we will explain this in a separate paper. See also the end of Section 11.3.

*Remark* X.3. We call the proof of Theorem X.1 a "sketch" because we will leave out some details of the proof. For example we will at some point assume $t < k-1$. The extremal case $t = k-1$ (where there are the same number of Kuga-Sato factors as powers of $A$ in the variety $X$) is a more delicate computation from an analytic point of view, and we wish to avoid technical issues of convergence in this sketch.

## 10.1 Functional equational and preliminary special value formulas

In this section we prove the functional equation and analytic continuation of $L_{\mathcal{A}}(f, \chi, s)$ and compute the coefficients $a_m(\mathcal{A})$ from the introduction. These computations follow [GZ, §IV] closely and we retain the notation there. Let $\epsilon(n) = \left(\frac{D}{n}\right)$ be the quadratic character attached to $K$ and set $L^{(N)}(s, \epsilon) = \sum_{(n,N)=1} \epsilon(n) n^{-s}$, so that

$$L_{\mathcal{A}}(f, \chi, s) = L^{(N)}(2s - 2k - 2t + 1, \epsilon) \sum_{n \geqslant 1} a_f(n) r_{\mathcal{A},\chi}(n) n^{-s}.$$

If we set $\ell = 2t$, then the theta series

$$\theta_{\mathcal{A}}(z) = \sum_{n \geqslant 1} r_{\mathcal{A},\chi}(n) q^n = \frac{1}{w\chi(\bar{\mathfrak{a}})} \sum_{x \in \mathfrak{a}} \bar{x}^{\ell} q^{Q_{\mathfrak{a}}(x)}.$$

is in $S_{\ell+1}(\Gamma_0(D),\epsilon)$. Here $w$ is the number of units in $\mathcal{O}_K$, $\mathfrak{a}$ is any ideal in $\mathcal{A}$ and $Q_{\mathfrak{a}}(x) = \mathrm{Nm}(x)/\mathrm{Nm}(\mathfrak{a})$. By the Rankin-Selberg method, we have

$$\begin{aligned}\frac{\Gamma(s+2k-1)}{(4\pi)^{s+2k-1}}\sum_{n\geqslant 1}\frac{a_f(n)r_{\mathcal{A},\chi}(n)}{n^{s+2k-1}} &= \int_0^\infty \sum_{n\geqslant 1} a_f(n)r_{\mathcal{A},\chi}(n)e^{-4\pi ny}y^{s+2k-2}dy\\ &= \int_0^\infty\int_0^1 f(x+iy)\overline{\theta_{\bar{\mathcal{A}}}(z)(x+iy)}dxy^{s+2k-2}dy\\ &= \iint\limits_{\Gamma_\infty\backslash\mathcal{H}} f(z)\overline{\theta_{\bar{\mathcal{A}}}(z)}y^{s+2k}\frac{dxdy}{y^2}\end{aligned}$$

Here, $\bar{\mathcal{A}}$ is the class $\mathcal{A}^{-1}$ of $\bar{\mathfrak{a}}$. It follows that

$$\frac{\Gamma(s+2k-1)}{(4\pi)^{s+2k-1}}L_{\mathcal{A}}(f,\chi,s+2k-1) = (f,\theta_{\bar{\mathcal{A}}}E_{\bar{s}})_{\Gamma_0(M)} := \iint\limits_{Y_0(M)} f(z)\overline{\theta_{\bar{\mathcal{A}}}(z)E_{\bar{s}}(z)}y^{2k}\frac{dxdy}{y^2},$$

where $M = N|D|$ and $E_s(z)$ is the weight $(2k-2t-1)$ Eisenstein series

$$\begin{aligned}E_s(z) &= E_{M,\epsilon,2k-2t-1,s}(z)\\ &= L^{(N)}(2s+2k-2t-1,\epsilon)\sum_{\Gamma_\infty\backslash\Gamma_0(M)}\frac{\epsilon(d)}{(cz+d)^{2k-2t-1}}\frac{y^s}{|cz+d|^{2s}}\\ &= \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\ M|c\\ (d,M)=1}}\frac{\epsilon(d)}{(cz+d)^{2k-2t-1}}\frac{y^s}{|cz+d|^{2s}}.\end{aligned}$$

Note that the weight of the Eisenstein series is at least 1, with equality if and only if $t = k-1$. In this case we need to be careful about convergence, as in [GZ].

We define $E_s^{(1)}(z)$ just as $E_s(z)$ but with $M$ replaced by 1, i.e.

$$E_s^{(1)}(z) = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\ D|c}}\frac{\epsilon(d)}{(cz+d)^{2k-2t-1}}\frac{y^s}{|cz+d|^{2s}}.$$

Then

$$\tilde{\Phi}_s(z) = \mathrm{Tr}_N^{ND}(\theta_{\bar{\mathcal{A}}}(z)E_s^{(1)}(Nz))$$

is a non-homolorphic modular form of weight $2k$ and level $N$ such that

$$(4\pi)^{-s-2k+1}N^s\Gamma(s+2k-1)L_{\mathcal{A}}(f,\chi,s+2k-1) = (f,\tilde{\Phi}_{\bar{s}}) \qquad (10.2)$$

To make notation simpler, we define $\ell_1 = 2k-2t$.

**Proposition X.4.** *Suppose* $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *with* $(c,D) = |D_2|$ *and* $D_1\cdot D_2 = D$ *and* $\delta_i = |D_i|$. *Then*

$$E_s^{(1)}|_{\ell_1-1}\,\gamma = \epsilon_{D_1}(c)\epsilon_{D_2}(d\delta_1)\delta_1^{-s-\ell_1+1}E_s^{(D_1)}\left(\frac{z+c^*d}{\delta_1}\right).$$

$$\theta_{\mathcal{A}}|_{\ell+1}\,\gamma = \frac{\epsilon_{D_1}(c/\delta_2)\epsilon_{D_2}(d)\chi_{D_1\cdot D_2}(\mathcal{A})}{\kappa(D_1)\delta_1^{\frac{1}{2}}\chi(\mathfrak{d}_1)}\theta_{\mathcal{A}\mathcal{D}_1}\left(\frac{z+c^*d}{\delta_1}\right).$$

*Here,* $\kappa(D_1)$ *is 1 or* $i$, *depending on whether* $D_1 > 0$ *or* $D_1 < 0$, *and* $\mathcal{D}$ *is the ideal class of* $\mathfrak{d}_1$.

*Proof.* The first formula follows from (2.2) in [GZ]. To prove the theta series transformation law we follow the arguments in [GZ, IV.2.3] and assume that $c = \delta_2$. Setting $\zeta = -1/c(cz+d)$, we have

$$\theta_{\mathcal{A}}\left(\frac{az+b}{cz+d}\right) = \theta_{\mathcal{A}}\left(\frac{a}{c}+\zeta\right) = \frac{1}{\chi(\bar{\mathfrak{a}})w}\sum_{\lambda\in\mathfrak{a}}\bar{x}^{\ell} e\left(Q_{\mathfrak{a}}(\lambda)(\frac{a}{c}+\zeta)\right)$$
$$= \frac{1}{\chi(\bar{\mathfrak{a}})w}\sum_{\lambda\in\mathfrak{a}/\mathfrak{a}\mathfrak{d}_2} e_c(aQ_{\mathfrak{a}}(\lambda))\sum_{\mu\in\mathfrak{a}\mathfrak{d}_2}(\overline{\lambda+\mu})^{\ell} e(Q_{\mathfrak{a}}(\lambda+\mu)\zeta).$$

Poisson summation for any fractional ideal $\mathfrak{b}$ reads [T]

$$\sum_{\mu\in\mathfrak{b}}(\overline{\lambda+\mu})^{\ell} e(N(\lambda+\mu)z) = \frac{i}{z^{2k+1}}\frac{\delta^{-1/2}}{N(\mathfrak{b})}\sum_{\nu\in\mathfrak{b}^{-1}\mathfrak{d}^{-1}}\nu^{\ell} e\left(-\frac{N(\nu)}{z}\right)e(\mathrm{Tr}(\lambda\nu)).$$

Setting $A = \mathrm{Nm}(\mathfrak{a})$, we therefore have

$$\theta_{\mathcal{A}}|_{\ell+1}\gamma = \theta_{\mathcal{A}}\left(\frac{az+b}{cz+d}\right)(cz+d)^{-\ell-1}$$
$$= \frac{i(-cA)^{\ell+1}}{w\chi(\bar{\mathfrak{a}})\delta^{1/2}A\delta_2}\sum_{\lambda\in\mathfrak{a}/\mathfrak{a}\mathfrak{d}_2} e_c(aQ_{\mathfrak{a}}(\lambda))\sum_{\nu\in\mathfrak{a}^{-1}\mathfrak{d}_2^{-1}\mathfrak{d}^{-1}}\nu^{\ell} e(AN(\nu)c(cz+d))e(\mathrm{Tr}(\lambda\nu))$$
$$= \frac{-i}{w\chi(\bar{\mathfrak{a}})\delta^{1/2}}\sum_{\nu\in\mathfrak{a}^{-1}\mathfrak{d}_1^{-1}} C(\nu)(\nu A)^{\ell} e\left(AN(\nu)\left(z+\frac{d}{c}\right)\right),$$

with

$$C(\nu) = \sum_{\lambda\in\mathfrak{a}/\mathfrak{a}\mathfrak{d}_2} e_c\left(aQ_{\mathfrak{a}}(\lambda)\right)e_c(\mathrm{Tr}(\lambda\nu)).$$

Evaluating $C(\nu)$ as in [GZ], we conclude

$$\theta_{\mathcal{A}}|_{\ell+1}\gamma = \frac{-i\kappa(D_2)}{\delta_1^{1/2}w\chi(\bar{\mathfrak{a}})}\epsilon_{D_2}(d)\chi_{D_1\cdot D_2}(\mathcal{A})\sum_{\nu\in\mathfrak{a}^{-1}\mathfrak{d}_1^{-1}}(\nu A)^{\ell} e\left(AN(\nu)\left(z+\frac{d}{c}\right)\right)$$
$$= \frac{-i\kappa(D_2)}{\delta_1^{1/2}w}\epsilon_{D_2}(d)\chi_{D_1\cdot D_2}(\mathcal{A})\chi(\mathfrak{a})\sum_{\nu\in\bar{\mathfrak{a}}^{-1}\mathfrak{d}_1^{-1}}\bar{\nu}^{\ell} e\left(\frac{N(\nu)N(\mathfrak{d}_1)}{N(\bar{\mathfrak{a}}^{-1})}\left(\frac{z+c^*d}{\delta_1}\right)\right)$$
$$= \frac{\epsilon_{D_2}(d)\chi_{D_1\cdot D_2}(\mathcal{A})}{\delta_1^{1/2}\chi(\mathfrak{d}_1)\kappa(D_1)}\theta_{\mathcal{A}\mathcal{D}_1}\left(\frac{z+c^*d}{\delta_1}\right).$$

In the last line we have used the fact that $\bar{\mathfrak{d}}_1 = \mathfrak{d}_1$; in particular $\mathcal{D}_1 = \mathcal{D}_1^{-1}$. □

It follows from this proposition that

$$E_s^{(1)}(Nz)\theta_{\bar{\mathcal{A}}}(z)|_{2k}\gamma = \frac{\epsilon_1(N)\chi_{D_1\cdot D_2}(\bar{\mathcal{A}})}{\kappa(D_1)\chi(\mathfrak{d}_1)\delta_1^{s+\ell_1-\frac{1}{2}}}E_s^{(D_1)}\left(N\frac{z+c^*d}{\delta_1}\right)\theta_{\bar{\mathcal{A}}\mathcal{D}_1}\left(\frac{z+c^*d}{\delta_1}\right).$$

Next note that when $\delta_2$ divides $n$, the $n$th coefficient of $\theta_{\bar{\mathcal{A}}\mathcal{D}_1}(\delta_2 z) = \theta_{\bar{\mathcal{A}}\mathcal{D}_2}(\delta_2 z)$ is equal to the $n$th coefficient of $\chi(\mathfrak{d}_2)^{-1}\theta_{\bar{\mathcal{A}}}(z)$. Thus, following [GZ, p. 276], we have

$$\tilde{\Phi}_s(z) = D^{-t}\mathcal{E}_s(Nz)\theta_{\bar{\mathcal{A}}}(z)|_{U_{|D|}},$$

where

$$\mathcal{E}_s(z) = \sum_{D_1\cdot D_2}\frac{\epsilon_{D_1}(N)\chi_{D_1\cdot D_2}(\bar{\mathcal{A}})}{\kappa(D_1)|D_1|^{s+\ell_1-\frac{3}{2}}}E_s^{(D_1)}(|D_2|z),$$

is as is [GZ], except with weight $\ell_1 - 1$ instead of $2k-1$.

The fourier coefficients of $\mathcal{E}_s(z)$ are computed in [GZ, IV.3], giving us the following result.

**Corollary X.5.** *For each $r \in Z$ in the range $0 \leqslant r \leqslant k-t-1$, we have*

$$\tilde{\Phi}_{-r}(z) = D^{-t} \sum_{m=0}^{\infty} \sum_{0 \leqslant n \leqslant \frac{m\delta}{N}} e_{n,r}(y) r_{\bar{\mathcal{A}},\chi}(m\delta - nN) e^{2\pi i m z}$$

*where*

$$e_{0,k-t-1}(y) = \left[ L(1,\epsilon) - \epsilon(N) \frac{\sqrt{\pi}}{\delta} L(0,\epsilon) \right] (Ny)^{1-k+t}$$

$$e_{n,r}(y) = (-1)^{k-t-r} \epsilon(N) \frac{2\pi}{\sqrt{\delta}} (Ny)^{r-\ell_1+2} p_{k-t,r} \left( \frac{4\pi N n y}{\delta} \right) \sum_{d|n} \epsilon_{\mathcal{A}}(n,d) d^{2r-\ell_1+2}$$

*for $n > 0$.*

For each $\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)$, the completed $L$-function is defined to be

$$L^*_{\mathcal{A}}(f,\chi,s) = (2\pi)^{-2s} N^s \delta^s \Gamma(s) \Gamma(s-2t) L_{\mathcal{A}}(f,\chi,s).$$

**Theorem X.6.** *$L^*_{\mathcal{A}}(f,\chi,s)$ satisfies the functional equation*

$$L^*_{\mathcal{A}}(f,\chi,s) = -\epsilon(N) L^*_{\mathcal{A}}(f,\chi,2k+2t-s)$$

*Proof.* Let $\mathcal{E}_s(z) = \sum_{n\in\mathbb{Z}} e_s(n,y) e(nx)$ be the Fourier expansion of $\mathcal{E}_s(z)$, and set

$$e^*_s(n,y) = \pi^{-s} \delta^s \Gamma(s+2k-2t-1) e_s(n,y).$$

Then the functional equation follows from (10.2) and the formula

$$e^*_s(n,y) = -\epsilon(N) e^*_{2-2k+2t-s}(n,y),$$

which is proved in [GZ, §IV.4]. Indeed, we compute

$$L^*_{\mathcal{A}}(f,\chi,s) = \pi^{-s} N^{2k-1} \delta^s \Gamma(s) (f, \tilde{\Phi}_{\bar{s}-2k+1})$$

and so

$$\begin{aligned} L^*_{\mathcal{A}}(f,\chi,2k+2t-s) &= \pi^{s-2k-2t} N^{2k-1} \delta^{2k+2t-s} \Gamma(2k-s) (f, \tilde{\Phi}_{1+2t-\bar{s}}) \\ &= -\epsilon(N) \pi^{-s} N^{2k-1} \delta^s \Gamma(s-2t) (f, \tilde{\Phi}_{\bar{s}-2k+1}) \\ &= -\epsilon(N) L^*_{\mathcal{A}}(f,\chi,s). \end{aligned}$$

□

**Proposition X.7.** *There is a non-holomorphic modular form $\tilde{\Phi} \in \tilde{M}_{2k}(\Gamma_0(N))$ such that*

$$L'_{\mathcal{A}}(f,\chi,k+t) = \frac{2^{2k+2t+1} \pi^{k+t+1}}{(k+t-1)! \sqrt{|D|} D^t} (f, \tilde{\Phi}),$$

*and the Fourier expansion of $\tilde{\Phi}$ is*

$$\begin{aligned} \tilde{\Phi}(z) = \sum_{m=\infty}^{\infty} \Bigg[ &- \sum_{0<n\leqslant \frac{m\delta}{N}} \sigma'_{\mathcal{A}}(n) r_{\bar{\mathcal{A}},\chi}(m\delta - Nn) p_{k-t-1}\left( \frac{r\pi n N y}{\delta} \right) \\ &+ \frac{h}{u} r_{\bar{\mathcal{A}},\chi}(m) \left( \log y + \frac{\Gamma'}{\Gamma}(u) + \log N\delta - \log \pi + 2\frac{L'}{L}(1,\epsilon) \right) \\ &- \sum_{n=1}^{\infty} \sigma_{\mathcal{A}}(n) r_{\bar{\mathcal{A}},\chi}(m\delta + nN) q_{k-t-1}\left( \frac{4\pi n N y}{\delta} \right) \Bigg] y^{1-k+t} e^{2\pi i m z}. \end{aligned}$$

*Here,*

$$p_{m-1}(z) = \sum_{j=0}^{m-1} \binom{m-1}{j} \frac{(-z)^j}{j!}, \quad q_{m-1}(z) = \int_1^{\infty} \frac{(x-1)^{m-1}}{x^m} e^{-xz} dx$$

Next we wish to prove a version of the previous proposition, but with $\tilde{\Phi}$ replaced by a holomorphic modular form of weight $2k$. When $k-t>1$, $\tilde{\Phi}$ satisfies the growth conditions needed to apply holomorphic projection [GZ, IV.5.1].

**Proposition X.8.** *Suppose $k-t>1$. Then*

$$L'_{\mathcal{A}}(f,\chi,k+t) = \frac{2^{4k-1}\pi^{2k}}{(2k-2)!\sqrt{|D|}D^t}(f, \sum a_m(\mathcal{A})q^m),$$

*where $\sum a_m(\mathcal{A})q^m$ is a holomorphic cusp form of weight $2k$ and level $N$ with coefficients:*

$$\begin{aligned} a_m(\mathcal{A}) = m^{k-t-1}\Bigg[ &- \sum_{0<n\leqslant \frac{m\delta}{N}} \sigma'_{\mathcal{A}}(n) r_{\bar{\mathcal{A}},\chi}(m\delta - Nn) H_{k-t-1,t}\left(1 - \frac{2nN}{m|D|}\right) \\ &+ \frac{h}{u} r_{\bar{\mathcal{A}},\chi}(m)\left(\frac{\Gamma'}{\Gamma}(k+t) + \frac{\Gamma'}{\Gamma}(k-t) + \log\frac{N|D|}{4\pi^2 m} + 2\frac{L'}{L}(1,\epsilon)\right) \\ &- \sum_{n=1}^{\infty} \sigma_{\mathcal{A}}(n) r_{\bar{\mathcal{A}},\chi}(m\delta + nN) Q_{k,t}\left(1 + \frac{2nN}{m|D|}\right)\Bigg]. \end{aligned}$$

*Here we have defined*

$$H_{m,t}(z) = \frac{1}{2^m \cdot (m+2t)!}\left(\frac{d}{dz}\right)^{m+2t}[(z^2-1)^m(z-1)^{2t}]$$

$$Q_{k,t}(z) = \int_{-\infty}^{\infty} \frac{2^{2t}dw}{(z+\sqrt{z^2-1}\cosh w)^{k-t}(z+1+\sqrt{z^2-1}e^w)^{2t}}$$

*Proof.* The proof is as in [GZ, Theorem IV.5.8], so we will not go through the details. Instead of using the identity in the second equation on [GZ, p. 293], one uses Lemma III.4. □

*Remark* X.9. Extra care needs to be taken when performing holomorphic projection in the case $t=k-1$. This is the source of serious complications in the weight 2 case (i.e. $k=1$, $t=0$) of [GZ, IV.6]. We will not go into the details here and will assume $t<k-1$ for the remainder of the paper.

# CHAPTER XI

# Archimedean Heights

In this chapter we compute the local heights of generalized Heegner cycles at the infinite places of $H$. In the last section, we relate these heights to the Fourier coefficients computed in the previous section and finish the proof of Theorem X.1. We also deduce an archimedean version of Theorem I.7, under the assumption that a certain geometrically defined $q$-expansion is a modular form.

## 11.1 Generalities on height pairings

Let $X$ be a smooth projective variety of dimension $n$ over a number field $F$. Beilinson and Gillet-Soulé define a global height pairing

$$\langle\, ,\, \rangle_{\mathrm{GS}} : \mathrm{CH}^j(X)_0 \times \mathrm{CH}^{n+1-j}(X)_0 \to \mathbb{R}$$

between homologically trivial algebraic cycles (modulo rational equivalence) of arithmetically complementary codimensions. This pairing decomposes into a sum of local heights

$$\langle\, ,\, \rangle_{\mathrm{GS}} = \sum_v \langle\, ,\, \rangle_v,$$

where the sum is over all places of $F$, including the archimedean ones.[1] The local heights are only defined for algebraic cycles with disjoint support, so one may need to use the moving lemma for the decomposition above to make sense. There are several ways to define these local height pairings; we refer to [Bei], [GS], [M], and [Z] for details.

*Remark* XI.1. The global height pairing is actually defined on a group *a priori* smaller than $\mathrm{CH}^j(X)_0$, though conjecturally they should coincide [Bei, Remark 4.0.1]. It will not matter for our purposes, as generalized Heegner cycles are contained in both groups.

In our situation, $X = W_{2k-2} \times A^\ell$ is fibered over the curve $X(N)$, and the generalized Heegner cycles, whose height we wish to compute, are finite formal sums $\sum Z_i$ of cycles $Z_i$ supported in the fiber of $g : X \to X(N)$ over points $x_i \in X(N)$. These $Z_i$ are of codimension $k+t$ on the $2k+2t-1$ dimensional variety $X$, so they are of middle arithmetic dimension and can be paired against each other. Brylinski [Br] gives a formula for the local height pairings $\langle\, ,\, \rangle_v$ of such fibral cycles, in terms of local systems. For the finite places $v$, he uses the $p$-adic local systems $R^{2k+2t-2}g_*\mathbb{Q}_p(k+t-1)$ for a prime $p$ such that $v \nmid p$ (and with appropriate adjustments at the cusp). His formula is exactly the same as our formula in Proposition 6.1, for the $p$-adic heights at places $v$ not above $p$ (but with log replacing $\log_p$). Indeed he proves that the local height can be described in terms of intersection theory on the arithmetic surface $X(N)_{\mathbb{Z}}$ and geometric intersection on the special fiber of $g^{-1}(x_i)$.[2]

[1] Our notation for local heights unfortunately does not distinguish between $p$-adic and archimedean versions. This should not cause too much confusion, especially because there is no archimedean component to the $p$-adic height. Moreover, at finite places $v$ not above $p$, the $p$-adic and archimedean heights are in some sense "the same" (see below).

[2] In fact, Nekovář's proof of Proposition 6.1 is based in part off the proof in [Br].

For archimedean $v$, Brylinski gives an analogous formula for the local height $\langle\,,\,\rangle_v$ in terms of the local system $R^{2k+2t-2}g_*\mathbb{Q}(k+t-1)$ over $X(N)_{\mathbb{C}}$ (from now on, we view all varieties and maps over $\mathbb{C}$). More specifically, the cycles $Z_i$ determine Hodge classes $v_i$ in $H^{2k+2t-2}(X_i,\mathbb{Q})$, where $X_i$ is the fiber of $g$ above $x_i$. This Betti cohomology group is the fiber at $x_i$ of $R^{2k+2t-2}g_*\mathbb{Q}(k+t-1)$. The latter local system is in fact a polarized variation of Hodge structures of weight 0; the polarization comes from the cup product in each fiber (note that algebraic cycles in the same fiber are now of complementary codimension in the geometric sense).

In fact, Brylinski defines height pairings attached to any polarized variation of Hodge structures over a smooth complex curve $C^*$. He even allows degenerating variation of Hodge structures, which we will need to handle the cusps on $X(N)$. To state this properly, let $V$ be a $\mathbb{Q}$-local system on a smooth curve $C^*$, underlying a polarized variation of Hodge structures $(\mathscr{V},F^p\mathscr{V})$ of weight 0. There is a canonical way to extend $\mathscr{V}$ to a vector bundle $\bar{\mathscr{V}}$ on the compactification $C$ of $C^*$ (i.e. it is characterized by certain properties) [Br, §1].

**Definition XI.2.** A Hodge vector $v_x$ at $x\in C^*$ is an element $v_x\in V_x$ which belongs to $F^0\mathscr{V}_x$ (so of type $(0,0)$). The group of Hodge vectors at $x$ is written $\mathrm{Hdg}(V)_x$.

**Definition XI.3.** A Hodge cycle is an element of $\mathrm{Hdg}(V):=\oplus_{x\in C^*}\mathrm{Hdg}(V)_x$.

*Remark* XI.4. Brylinski defines Hodge vectors for any $x\in C$, but we will not bother, as our generalized Heegner cycles avoid the cusps.

To define Brylinski's height pairing, we need the notion of a Green's kernel attached to $V$. Let us write $\bar{\mathscr{V}}_{0,\mathbb{R}}$ for the $C^\infty$ vector bundle of sections of $\bar{\mathscr{V}}$ which are real and of type $(0,0)$. Also write $p_1,p_2$ for the projections $C\times C\to C$.

**Proposition XI.5** ([Br])**.** *If $V$ has no non-zero global sections, then there exists a unique $C^\infty$-section $G$ of $\mathrm{Hom}(p_1^{-1}\bar{\mathscr{V}}_{0,\mathbb{R}},p_2^{-1}\bar{\mathscr{V}}_{0,\mathbb{R}})$ over the complement of the diagonal $\Delta_C$ in $C\times C$ such that*

1. *$\square_2 G=0$, where $\square_2$ is the Laplacian $\square$ attached to $V$ in the second variable.*
2. *$G(x,y)-\log|z(x)-z(y)|$ is bounded near any point $(a,a)$ of $\Delta_C$, if $z$ is a local coordinate on $C$ near $a$.*

The section $G$ is called the *Green's kernel* for $V$ on $C$. Brylinski then defines a height pairing $\langle v_1,v_2\rangle_{\mathrm{Br}}$ for any pair $v_1=\sum_x v_{1,x}$ and $v_2=\sum_x v_{2,x}\in\mathrm{Hdg}(V)$ of Hodge cycles with disjoint support as:

$$\langle v_1,v_2\rangle_{\mathrm{Br}}=\sum_{x,y}(G(x,y)(v_{1,x}),v_{2,y}).$$

Here, the pairing $(\,,\,)$ is the given polarization in the fiber of $\bar{\mathscr{V}}_{0,\mathbb{R}}$ at $y$.

The following result states that we may use Brylinski's pairing to compute Beilinson's archimedean local height when the two cycles are in distinct fibers of a map to a curve:

**Theorem XI.6.** *Suppose $X$ is a complex variety of dimension $2n+1$ and we are given a projective morphism $\tilde{g}:X\to C$ which restricts to a smooth map $g:\tilde{g}^{-1}(C^*)\to C^*$. Suppose $Z_1$ and $Z_2$ are homologically trivial cycles on $X$ of codimension $n+1$, supported in disjoint non-cuspidal fibers of $g$. Also suppose $X,C,Z_i$ are defined over a number field $F$ with a given embedding $v:F\to\mathbb{C}$. Then*

$$\langle Z_1,Z_2\rangle_v=\langle v_1,v_2\rangle_{\mathrm{Br}},$$

*where $v_1$ and $v_2$ are the corresponding Hodge cycles in the stalks of $V=R^{2n}g_*\mathbb{Q}(n)$. Here, the Brylinski pairing is with respect to the polarization on $V$ which is the cup product pairing on each fiber.*

*Proof.* A proof is sketched at the very end of [Br]. □

## 11.2 Local heights at infinity for generalized Heegner cycles

Let us return to our situation, where $X = W_{2k-2} \times A^{2t}$ and $\tilde{g} : X \to X(N)$ is the usual map. Write $X^0 = \tilde{g}^{-1}(Y(N))$ and write $g : X^0 \to Y(N)$ for the restriction. We wish to pair the generalized Heegner cycles defined in Chapter IV, so we consider the local system

$$W = \mathrm{Sym}^{2k-2} R^1 f_* \mathbb{Q}(k-1) \otimes \kappa_\ell H^{2t}(A^{2t}, \mathbb{Q})(t),$$

where $f : \mathcal{E} \to Y(N)$ is the universal elliptic curve and where $\kappa_\ell$ is the projector defined in Chapter IV. The factor $\kappa_\ell H^{2t}(A^{2t}, \mathbb{Q})(t)$ is a constant local system, with Hodge structure of weight 0 and type $(t, -t) + (-t, t)$. As $W$ is a summand of $R^{2k+2t-2} g_* \mathbb{Q}(k+t-1)$, it inherits the structure of polarized variation of Hodge structures and we may use Theorem XI.6 to compute local heights of generalized Heegner cycles using the Green's kernel attached to $W$.

This was done by Brylinski himself for classical Heegner cycles, i.e. when $t = 0$ [Br, §3]. We will build on his computations, so we begin by recalling notation. Let $E = \mathbb{R}^2$ be the standard representation of $G = \mathrm{GL}_2(\mathbb{R})^+$, with basis $u_1$ and $u_2$. Let $K = \mathbb{C}^\times$ be the stabilizer of $i$ in $G$ acting on the upper half plane $\mathfrak{h}$. Then $\mathscr{E} = G \times^K E$, is a $G$-equivariant vector bundle on $\mathfrak{h}$. The holomorphic subbundle $\mathscr{F}^1\mathscr{E}$ is generated by the holomorphic section $zu_1 + u_2$, where $z$ is a coordinate on $\mathfrak{h}$. $E$ is polarized by the skew symmetric form $( \, , \, ) : E \times E \to \mathbb{R}(-1)$ such that $(u_1, u_2) = -1/2\pi i$. The associated hermitian form satisfies

$$\langle zu_1 + u_2, zu_1 + u_2 \rangle = 2y$$

$$\langle zu_1 + u_2, \bar{z}u_1 + u_2 \rangle = 0$$

$$\langle \bar{z}u_1 + u_2, \bar{z}u_1 + u_2 \rangle = 2y,$$

where as usual $z = x + iy$. Recall that $zu_1 + u_2 \in \mathscr{E}_z^{1,0}$ and $\bar{z}u_1 + u_2 \in \mathscr{E}_z^{0,1}$.

The usual local system for weight $2k$ modular forms on $Y(N)$ is obtained by considering the representation $V_p = \mathrm{Sym}^{2p}(E)(p)$ of $G$, where $p = k-1$. The corresponding polarized variation of Hodge structures on $\mathfrak{h}$ is $\mathscr{V}_p = \mathrm{Sym}^{2p}(\mathscr{E})(p)$, and is pure of weight 0. The sections

$$v_n = \frac{\binom{2p}{n}^{1/2}}{(2y)^p (2\pi i)^p} \cdot (zu_1 + u_2)^n (\bar{z}u_1 + u_2)^{2p-n}$$

for $0 \leqslant n \leqslant 2p$ form a $C^\infty$-basis of $\mathscr{V}_p$ and each $v_n$ is of pure type $(-p+n, p-n)$. Moreover, this basis is orthonormal with respect to the Hermitian pairing on $\mathscr{V}_p$ obtained from the pairing $\langle \, , \, \rangle$ on $E$ in the usual way.

Recall that to define the cycle $\epsilon_B \epsilon Y$ in Chapter IV, we chose a Heegner point corresponding to the elliptic curve $A$. We fix a value $\tau_0 \in \mathfrak{h}$ corresponding to this chosen Heegner point on $X_0(N)$. Denote by $\mathbb{R}^2$ the trivial two dimensional representation of $G$ with basis $\{e_1, e_2\}$. Then we can realize the constant Hodge structure $\kappa_\ell H^{2t}(A^{2t}, \mathbb{Q})(t)$ as coming from the $G$-representation $\mathbb{R}^2$ and we can suggestively write a basis of sections as

(11.1)
$$\mu_0 = \frac{1}{(2y_0)^t (2\pi i)^t} (\bar{\tau}_0 e_1 + e_2)^{2t}$$

$$\mu_{2t} = \frac{1}{(2y_0)^t (2\pi i)^t} (\tau_0 e_1 + e_2)^{2t},$$

of type $(-t, t)$ and $(t, -t)$ respectively. Moreover, this basis is again orthonormal with respect to the Hermitian pairing coming from the polarization on $W$.

Let $\mathscr{W} = \mathscr{W}_{p,t}$ be the polarized variation of structure associated to $V_p \otimes \mathbb{R}^2$, so that $W$ is its underlying local system. The sections

$$w_{n,j} = v_n \otimes \mu_j$$

with $0 \leqslant n \leqslant 2p$ and $j = 0, 2t$, form a basis of $\mathscr{W}$. Each $w_{n,j}$ is pure of type $(-p-t+n+j, p+t-n-j)$. The $C^\infty$-subbundle $\mathscr{W}_{0,\mathbb{R}}$ of real $(0,0)$-vectors has rank 2, with basis $w_{p+t,0}$ and $w_{p-t,2t}$. One sees from the definitions of the projectors (see Chapter IV and also Lemma XI.19 below) that the Abel-Jacobi image of the cycle $\epsilon Y$ can be identified with a section in the fiber $\mathscr{W}_{\tau_0}$ which is a multiple of $w_{p-t,2t}$. Similarly, the Abel-Jacobi image of $\bar{\epsilon} Y$ is a multiple of $w_{p+t,0}$.

The vector bundle $\mathscr{W}_{p,t}$ is endowed with a Gauss-Manin connection $D$. To compute heights of generalized Heegner cycles, we must first find a Green's kernel, i.e. a sufficiently nice function $G$ on $\mathfrak{h} \times \mathfrak{h}$ which is harmonic with respect to the Laplacian $\Box_D = 2\Box_{D'}$ acting on the second variable. The following lemma identifies the restriction of the Laplacian operator $\Box_{D'}$ to the vectors of type $(0,0)$.

**Lemma XI.7.** *Let $F$ be a $C^\infty$ function on $\mathfrak{h}$. Set $w^- = y^{-t} \cdot w_{p-t,2t}$ and $w^+ = y^t \cdot w_{p+t,0}$. Then*

$$\Box_D \left(F \cdot w^\pm\right) = \frac{1}{2}\left[\Delta + (k \pm t - 1)(k \mp t) \pm 4ity\frac{\partial}{\partial \bar{z}}\right] F \cdot w^\pm,$$

*where the $\pm$ signs should be taken consistently, and where $\Delta = -4y^2 \frac{\partial^2}{\partial z \partial \bar{z}}$ is the usual Laplacian.*[3]

*Proof.* First consider $w^-$. Recall the decomposition [Br, §3]

$$\Box_{D'} = \partial\partial^* + \partial^*\partial + \bar{\nabla}'\left(\bar{\nabla}'\right)^* + \left(\bar{\nabla}'\right)^* \bar{\nabla}'.$$

As $\partial^*$ and $\left(\bar{\nabla}'\right)^*$ kill 0-forms, we have

$$\Box_{D'}(Fw^-) = (\partial^*\partial + \left(\bar{\nabla}'\right)^* \bar{\nabla}')(Fw^-).$$

If $\kappa = (4\pi i)^p \binom{2p}{n}^{-1/2}$, then

$$\begin{aligned}\kappa\partial(Fw^-) = {} & \frac{\partial F}{\partial z} y^{-p-t}(zu_1 + u_2)^{p-t}(\bar{z}u_1 + u_2)^{p+t} dz \otimes \mu_{2t} \\ & - \frac{p+t}{2i} F y^{-p-t-1}(zu_1 + u_2)^{p-t}(\bar{z}u_1 + u_2)^{p+t} dz \otimes \mu_{2t} \\ & + (p-t) F y^{-p-t} u_1 (zu_1 + u_2)^{p-t-1}(\bar{z}u_1 + u_2)^{p+t} dz \otimes \mu_{2t}.\end{aligned}$$

Since $u_1 = (2iy)^{-1}(zu_1 + u_2 - (\bar{z}u_1 + u_2))$, we obtain by taking the $(0,0)$ component:

$$\partial(Fw^-) = \frac{\partial F}{\partial z} dz \otimes w^- - \frac{t}{iy} F dz \otimes w^-.$$

Recall that by definition $\partial^* = - * \bar{\partial} *$, where $*$ is the Hodge $*$-operator. We recall how to compute the Hodge $*$-operator with respect to a metric $g$, and for a general basis $e_1, \cdots, e_n$. If $g_{ij} = \langle e_i, e_j \rangle$, then

$$* : e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto \sum_{J'} (-1)^{\sigma(I',J')} \frac{\det(g_{i_a i'_b})}{\sqrt{\det g}} e_{j'_1} \wedge \cdots \wedge e_{j'_{r-k}},$$

the sum varying over sets $J'$ of complementary length to the initial indexing set $I = \{i_1, \cdots, i_k\}$, and $I' = [n] - J'$. To compute the Hodge $*$-operator in our situation, we use the usual Poincaré metric on $\mathfrak{h}$ for which $\langle dx, dx \rangle = y^2 = \langle dy, dy \rangle$. So we have $*dx = dy$, $*dy = -dx$, and $*(dx \wedge dy) = y^2$.

[3] Note that there is a sign error in the statement of [Br, Lemma 3.2], which is the special case $t = 0$.

We can now compute

$$\begin{aligned}\partial^*\left(\frac{\partial F}{\partial z}dz\otimes w^-\right) &= -*\bar\partial*\frac{\partial F}{\partial z}dz\otimes w^- \\ &= -*\bar\partial\frac{\partial F}{\partial z}(-idz)\otimes w^- \\ &= i*\frac{\partial^2 F}{\partial\bar z\partial z}d\bar z\wedge dz\otimes w^- \\ &= 2i^2*\frac{\partial^2 F}{\partial\bar z\partial z}dx\wedge dy\otimes w^- \\ &= -2y^2\frac{\partial^2 F}{\partial\bar z\partial z}\cdot w^-.\end{aligned}$$

We have tacitly used the fact that $\langle w_{p-t,2t}, w_{p-t,2t}\rangle = 1$. Similarly,

$$\begin{aligned}\partial^*\left(\frac{t}{iy}Fdz\cdot w^-\right) &= -*\bar\partial*\frac{t}{iy}Fdz\otimes w^- \\ &= *\bar\partial\frac{t}{y}Fdz\otimes w^- \\ &= *t\left(\frac{\partial F}{\partial\bar z}y^{-1}-\frac{iF}{2y^2}\right)d\bar z dz\otimes w^- \\ &= \left(2ity\frac{\partial F}{\partial\bar z}+tF\right)\cdot w^-.\end{aligned}$$

On the other hand, we also have:

$$\begin{aligned}((\bar\nabla')^*\bar\nabla')(Fw^-) &= (\bar\nabla')^*\left(\kappa F\frac{p+t}{2i}y^{-p-t-1}(zu_1+u_2)^{p-t+1}(\bar z u_1+u_2)^{p+t-1}d\bar z\otimes\mu_{2t}\right) \\ &= -*\nabla'*\left(\kappa F\frac{p+t}{2i}y^{-p-t-1}(zu_1+u_2)^{p-t+1}(\bar z u_1+u_2)^{p+t-1}d\bar z\otimes\mu_{2t}\right) \\ &= -*\nabla'\frac{\kappa F(p+t)}{2}y^{-p-t-1}(zu_1+u_2)^{p-t+1}(\bar z u_1+u_2)^{p+t-1}d\bar z\otimes\mu_{2t} \\ &= -*\frac{\kappa F(p+t)(p-t+1)}{2(2iy)}y^{-p-t-1}(zu_1+u_2)^{p-t}(\bar z u_1+u_2)^{p+t}dzd\bar z\otimes\mu_{2t} \\ &= \frac{1}{2}\kappa F(p+t)(p-t+1)y^{-p-t}(zu_1+u_2)^{p-t}(\bar z u_1+u_2)^{p+t}\otimes\mu_{2t} \\ &= \frac{1}{2}F(p+t)(p-t+1)\cdot w^-.\end{aligned}$$

The incorrect sign in [Br, Lemma 3.2] presumably comes from a misapplication of the formula $\nabla'(\alpha s) = (-1)^p\alpha\wedge\nabla(s)$, if $\alpha$ is a $p$-form and $s$ is a section of the vector bundle. Putting everything together proves the lemma for $w^-$. The result for $w^+$ follows by a similar computation, replacing $t$ with $-t$ in the appropriate places. $\square$

Now let $z = x+iy$ and $z' = x'+iy'$ be parameters on $\mathfrak{h}\times\mathfrak{h}$ and define

$$g(z,z') = -Q_{k,t}\left(1+\frac{|z-z'|^2}{2yy'}\right),$$

where $Q_{k,t}(z)$ is as in Proposition X.8. We also define

$$\mu^-(z,z') = g(z,z')\left(\frac{\bar z - z'}{2iy}\right)^\ell.$$

$$\mu^+(z,z') = g(z,z')\left(\frac{z-\bar{z}'}{2iy'}\right)^\ell.$$

As $g(z,z')$ is a function of the hyperbolic distance between $z$ and $z'$, it is invariant under $\mathrm{SL}_2(\mathbb{R})$. A quick computation then shows that

$$\mu^-(\gamma z,\gamma z') = \mu^-(z,z')j(\gamma,z)^\ell j(\gamma,z')^{-\ell}$$

$$\mu^+(\gamma z,\gamma z') = \mu^+(z,z')j(\gamma,z)^{-\ell} j(\gamma,z')^{\ell}$$

for $\gamma \in \mathrm{SL}_2(\mathbb{R})$ and where $j(\gamma,z) = cz+d$, as usual. On the other hand, the sections $w^\pm$ are themselves not $\mathrm{SL}_2(\mathbb{R})$-invariant, but instead satisfy

(11.2)
$$w^-(\gamma z) = j(\gamma,z)^\ell(\gamma\cdot w^-(z))$$

$$w^+(\gamma z) = j(\gamma,z)^{-\ell}(\gamma\cdot \bar{w}^+(z)).$$

It follows that the section

$$\mu^-(z,z')\cdot[w^-(z)\mapsto w^-(z')]$$

$$\mu^+(z,z')\cdot[w^+(z)\mapsto w^+(z')]$$

of the vector bundle

$$\mathrm{Hom}(p_1^{-1}\bar{\mathscr{W}}_{0,\mathbb{R}}, p_2^{-1}\bar{\mathscr{W}}_{0,\mathbb{R}})$$

on $\mathfrak{h}^*\times\mathfrak{h}^*$ is invariant under the diagonal action of $\mathrm{SL}_2(\mathbb{R})$. Here, $\bar{\mathscr{W}}$ is the canonical extension of $\mathscr{W} = \mathscr{W}_{p,t}$ to the compactification $\mathfrak{h}^*$ and the $p_i$ are the projection maps. Therefore

$$G^\pm_{k,t,N}(z,z') := \sum_{\gamma\in\Gamma(N)} \mu^\pm(z,\gamma z')\cdot[w^\pm(z)\mapsto w^\pm(\gamma z')]$$

each descend to a section of the descended bundle $\mathrm{Hom}(p_1^{-1}\bar{\mathscr{W}}_{0,\mathbb{R}}, p_2^{-1}\bar{\mathscr{W}}_{0,\mathbb{R}})$ on $X(N)\times X(N)$ (assuming they converge). We define

$$G_{k,t,N}(z,z') = G^-_{k,t,N}(z,z') + G^+_{k,t,N}(z,z').$$

We wish to show that $G(z,z') := G_{k,t,N}(z,z')$ is a *Green's kernel* for the variation of Hodge structure $\mathscr{W}_{p,t}$ on $X(N)$ and hence can be used to compute the local height pairing of Hodge cycles in different fibers. Specifically, if $w_1$ and $w_2$ are two sections of $\bar{\mathscr{W}}_{0,\mathbb{R}}$ at points $z_1$ and $z_2$ of $X(N)$, then the local height pairing $\langle w_1,w_2\rangle_v$ (at an infinite place $v$ of $H$) is given by

(11.3)
$$\langle G(z_1,z_2)(w_1),w_2\rangle_{z_2},$$

where $\langle\,,\,\rangle_{z_2}$ is the Hermitian pairing from before on the fiber above $z_2$.[4] Recall from Proposition XI.5 that the Green's kernel is characterized as the section of the rank 4 vector bundle

$$\mathrm{Hom}(p_1^{-1}\bar{\mathscr{W}}_{0,\mathbb{R}}, p_2^{-1}\bar{\mathscr{W}}_{0,\mathbb{R}})$$

on $X(N)\times X(N)$, which is killed by the Laplacian $\square_D$ acting on the second variable and which has logarithmic poles along the diagonal. Thus, by Lemma XI.7, we want to show that for both choices of sign, $G^\pm(z,z')$, as a function in the second variable, is an eigenfunction for the weight $\pm\ell$ Laplacian

$$\Delta_{\pm\ell} = \Delta \pm 2i\ell y\frac{\partial}{\partial\bar{z}},$$

with eigenvalue $-(k\pm t-1)(k\mp t)$.

[4]The Hermitian pairing agrees with the pairing $(\,,\,)$ when both are restricted to real vectors of type $(0,0)$. We consider the Hermitian extension of this pairing because the projectors $\epsilon$ and $\bar{\epsilon}$ do not preserve the space of real vectors of type $(0,0)$, and also to handle scalars such as $\chi(\mathfrak{a})$.

*Remark* XI.8. Unlike $\Delta$, the operators $\Delta_{\pm\ell}$ are not $\mathrm{SL}_2(\mathbb{R})$ invariant. But to check that $G^{\pm}(z,z')$ is an eigenfunction for $\Delta_{\pm\ell}$, it still suffices to check that $\mu^{\pm}(z,z')$ is an eigenfunction with eigenvalue independent of $z$. Indeed, simply use the fact that we may also write $G^{\pm}(z,z') = \sum_\gamma \mu^{\pm}(\gamma z, z') \cdot [w^{\pm}(\gamma z) \mapsto w^{\pm}(z')]$.

We will need a few facts from the theory of special functions; a general reference is [BE1, BE2]. First we recall the usual hypergeometric function (for $|z| < 1$ and $c > 0$)

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where

$$(a)_n = \begin{cases} 1 & n = 0 \\ a(a+1)\cdots(a+n-1) & n > 0 \end{cases}$$

is the rising Pochhammer symbol. The hypergeometric function satisfied various transformation laws which will be useful for us. For example, there is Euler's transformation law

$$F(a,b,c;z) = (1-z)^{c-a-b} F(c-a, c-b, c; z). \tag{11.4}$$

and the Pfaff transformation

$$F(a,b,c;z) = (1-z)^{-b} F\left(b, c-a, c; \frac{z}{z-1}\right). \tag{11.5}$$

Our interest in hypergeometric functions stems from the fact that they are solutions to second order differential equations. A special family of hypergeometric functions called Jacobi functions of the second kind (depending on parameters $n$, $\alpha$ and $\beta$) are defined as follows:

$$Q_n^{(\alpha,\beta)}(x) = \frac{2^{n+\alpha+\beta}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)(x-1)^{n+\alpha+1}(x+1)^{\beta}} F\left(n+1, n+\alpha+1, 2n+\alpha+\beta+2; \frac{2}{1-x}\right).$$

The function $Q_n^{(\alpha,\beta)}$ is a solution to the differential equation

$$(1-x^2)y'' + [\beta - \alpha - (\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0, \tag{11.6}$$

and has the following integral representation [BE2, p. 172]:

$$Q_n^{(\alpha,\beta)}(x) = \frac{2^{-n-1}}{(x-1)^{\alpha}(x+1)^{\beta}} \int_{-1}^{1} \frac{(1+u)^{n+\beta}(1-u)^{n+\alpha} du}{(x-u)^{n+1}}. \tag{11.7}$$

In the special case where $\alpha = \beta = 0$, the function $Q_{k-1}^{(0,0)}$ is a Legendre function of the second kind, from which the Green's kernel for intersections of classical Heegner cycles of "weight" $2k$ is constructed [GZ, Z]. It is very natural then that our candidate Green's kernel $G_{k,t,N}(z,z')$ for the intersection of *generalized* Heegner cycles of "weight" $(2k, 2t)$ is in fact built out of the Jacobi functions $Q_{k-t-1}^{(0,2t)}$, as the next lemma shows.

**Lemma XI.9.** *We have* $Q_{k,t}(z) = 2Q_{k-t-1}^{(0,2t)}(z)$ *and so*

$$Q_{k,t}(z) = \frac{\Gamma(k-t)\Gamma(k+t)}{\Gamma(2k)} \left(\frac{2}{z-1}\right)^{k-t} \left(\frac{2}{z+1}\right)^{2t} F\left(k-t, k-t, 2k, \frac{2}{1-z}\right).$$

*Proof.* It is not hard (cf. [GZ, p. 293]) to rewrite $Q_{k,t}(z)$ as

$$Q_{k,t}(z) = 2^{k+t} \int_0^{\infty} \frac{v^{k-t-1} dv}{(v+1)^{k-t} \left(v(z-1)+z+1\right)^{k+t}}$$

and via the change of variable $v = \frac{1+u}{1-u}$, we eventually get

$$2^{t-k+1} \int_{-1}^{1} \frac{(1+u)^{k-t-1}(1-u)^{k+t-1}du}{(z-u)^{k+t}}.$$

Using the integral representation (11.7) for $Q_n^{(\alpha,\beta)}$, this is equal to

$$\frac{2^{\ell+1}}{(1+z)^{\ell}} Q_{k+t-1}^{(0,-\ell)}(z).$$

Applying Euler's transformation law (11.4) to the hypergeometric definition of $Q_{k+t-1}^{(0,-2t)}$, we find that the latter is equal to $2Q_{k-t-1}^{(0,2t)}(z)$, as desired. □

**Corollary XI.10.** *$Q_{k,t}(z)$ satisfies the differential equation*

$$(1-z^2)Q''(z) + [\ell - (\ell+2)z]Q'(z) + (k-t-1)(k+t)Q(z) = 0.$$

*Proof.* Use (11.6). □

**Proposition XI.11.** *For each fixed $z \in \mathfrak{h}$, the function $\mu^{\pm}(z,z')$ satisfies (as a function of $z'$)*

$$\Delta_{\pm\ell}(\mu(z,z')) = \lambda^{\pm} \cdot \mu^{\pm}(z,z'),$$

*with $\lambda^{\pm} = -(k \pm t - 1)(k \mp t)$.*

*Proof.* This is a long computation, which ultimately boils down to Corollary XI.10. For the reader wishing to verify this on his or her own, we record the useful formulas: if $s = 1 + \frac{|z-z'|^2}{2yy'}$, then

$$\frac{\partial s}{\partial \bar{z}'} = \frac{z'-z}{2yy'} + \frac{s-1}{2iy'},$$

$$4y'^2 \frac{\partial^2 s}{\partial z' \partial \bar{z}'} = 2s,$$

and

$$4y'^2 \left(\frac{\partial s}{\partial \bar{z}'}\right)\left(\frac{\partial s}{\partial z'}\right) = s^2 - 1.$$

□

Next we address the convergence of the functions $G^{\pm}(z,z')$ defined earlier.

**Proposition XI.12.** *For all integers $0 < t < k-1$, the sums*

$$G_{k,t,N}^{\pm}(z,z') = \sum_{\gamma\in\Gamma} \mu^{\pm}(z,\gamma z') \cdot [w^{\pm}(z) \mapsto w^{\pm}(\gamma z')]$$

*converges uniformly on compact subsets of $\mathfrak{h}^2 - \{(z,z') : z \in \Gamma z'\}$.*

*Proof.* The proof is similar to [H, Prop. 6.2]. □

It follows from Proposition XI.11, that $G^{\pm}(z,z')$ is an eigenfunction for the weight $\pm\ell$ Laplacian $\Delta_{-\ell}$ with eigenvalue $-(k \pm t - 1)(k \mp t)$. To prove that $G(z,z')$ is the Green's kernel attached to $\mathcal{W}_{p,t}$, it remains to understand its behavior along the diagonal of $Y(N) \times Y(N)$ and also at the cusps. In this direction, we have the following lemmas.

**Lemma XI.13.** *As the real parameter $s \to 1$ from above,*

$$Q_{k,t}(s) = -\log(s-1) + O(1).$$

*Proof.* We have the well known asymptotic

$$F(a,b,a+b,s) = -\frac{1}{B(a,b)}\log(1-s) + O(1),$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, and where $s$ is approaching 1 from below. Then as $s \to 1$ from above,

$$\begin{aligned} Q(s) &= \frac{2^{k+t}B(k-t,k+t)}{(s-1)^{k-t}(1+s)^{2t}} F\left(k-t,k-t,2k,\frac{2}{1-s}\right) \\ &= \frac{2^{k+t}B(k-t,k+t)}{(s-1)^{k-t}(s+1)^{2t}} \cdot \left(\frac{s-1}{s+1}\right)^{k-t} F\left(k-t,k+t,2k,\frac{2}{1+s}\right) \\ &= -\log\left(\frac{s-1}{s+1}\right) + O(1) \\ &= -\log(s-1) + O(1), \end{aligned}$$

where we have used (11.5) in the second equality. □

**Corollary XI.14.** $G^{\pm}_{k,t,N}(z,z') = \log|z-z'|^2 + O(1)$ *as* $z'$ *approaches* $z$.

*Proof.* We should clarify what this even means, as $G_{k,t,N}(z,z')$ is a section of the vector bundle $\mathrm{Hom}(p_1^{-1}\bar{\mathscr{W}}_{0,\mathbb{R}}, p_2^{-1}\bar{\mathscr{W}}_{0,\mathbb{R}})$, not a scalar quantity. But by choosing the sections $w^{\pm}$, we have trivialized this bundle, so the statement of the corollary should be taken to mean

$$(\mu^{\pm}(z,z') - \log|z-z'|^2) \cdot [w^{\pm}(z) \mapsto w^{\pm}(z')]$$

is bounded as $z \to z'$. This follows from the previous lemma and the fact that $\lim_{z' \to z}(\bar{z} - z') = -2iy$. □

Next we analyze the behavior of the Green's kernel at the cusps.

**Proposition XI.15.** *For* $z$ *in a neighborhood of a cusp* $\gamma\infty$*, the function*

$$\mathrm{Im}(\gamma z)^{p-t} G(z,z')$$

*is* $C^{\infty}$.

*Proof.* We omit the proof, which is rather technical. One can proceed as in [H, §6]. □

It follows from Propositions XI.11, XI.14, and XI.15 that $G(z,z')$ satisfies all the properties characterizing the Green's kernel. By the formalism in [Br, §2], we conclude:

**Theorem XI.16.** *Write* $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$*. Then for* $z_1 \notin \Gamma(N)z_2$*, the local archimedean height pairing at infinity is given by*

$$\langle w^-(z_1), w^-(z_2)\rangle_{\mathrm{Br}} = \frac{1}{(2iy_1y_2)^{\ell}} \sum_{\gamma \in \Gamma(N)} g(z_1, \gamma z_2)(\bar{z}_1 - \gamma z_2)^{\ell} j(\gamma, z_2)^{\ell}.$$

$$\langle w^+(z_1), w^+(z_2)\rangle_{\mathrm{Br}} = \frac{1}{(2i)^{\ell}} \sum_{\gamma \in \Gamma(N)} g(z_1, \gamma z_2)(z_1 - \gamma \bar{z}_2)^{\ell} j(\gamma, \bar{z}_2)^{\ell},$$

*where*

$$g(z,z') = -Q_{k,t}\left(1 + \frac{|z-z'|^2}{2yy'}\right).$$

*Proof.* Now that we have identified the Green's kernel, this follows from equation (11.3) and the fact that $\langle w^-(z_2), w^-(z_2)\rangle_{z_2} = y_2^{-2t} = y_2^{-\ell}$, while $\langle w^+(z_2), w^+(z_2)\rangle_{z_2} = y_2^{\ell}$. □

To compute heights of generalized Heegner cycles in the corresponding local system on $X_0(N)$ instead of $X(N)$, we will follow [Z] and identify our generalized Heegner cycles $Z_{\mathcal{A}}$ and $\bar{Z}_{\mathcal{A}}$ at a point $P \in X_0(N)$ with the sum of the same cohomology class over the preimages $Q \in X(N)$ of $P$ (this is the purpose of the projector $\epsilon_B$ in the definition of the classes $Z_{\mathcal{A}}$ and $\bar{Z}_{\mathcal{A}}$ at the beginning of Chapter X). In fact, we can define the local height pairing in the same way for $X_0(N)$ simply by summing over $\Gamma_0(N)$ instead of $\Gamma(N)$ above. Also, as all of our vector bundles come equipped with a $\mathrm{GL}_2(\mathbb{R})$-action, it makes sense to apply Hecke operators to sections such as $w^{\pm}$. This action agrees with the geometric action of Hecke operators on algebraic cycles, defined in Chapter IV.

**Proposition XI.17.** *Assume $m \geqslant 1$ and Let $z_1, z_2$ be points of $\mathfrak{h}$ such that the $z_1$ and $T_m z_2$ have disjoint support on $X_0(N)$. Then*

$$\langle w^-(z_1), T_m w^-(z_2)\rangle_{\mathrm{Br}} = \frac{m^{p-t}}{(2iy_1y_2)^{\ell}} \sum_{\substack{\gamma \in R_N \\ \det\gamma = m}} g(z_1, \gamma z_2)(\bar{z}_1 - \gamma z_2)^{\ell} j(\gamma, z_2)^{\ell},$$

$$\langle w^+(z_1), T_m w^+(z_2)\rangle_{\mathrm{Br}} = \frac{m^{p-t}}{(2i)^{\ell}} \sum_{\substack{\gamma \in R_N \\ \det\gamma = m}} g(z_1, \gamma z_2)(z_1 - \gamma \bar{z}_2)^{\ell} j(\gamma, \bar{z}_2)^{\ell},$$

*where* $R_N = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ *and $p = k-1$.*

*Proof.* We prove only the first formula as the proof of the second is similar. Let the variable $\gamma_2$ range through a set of representatives of $\Gamma(N)\backslash R_N$ of determinant $m$, and let $\gamma_1$ range through a set of representatives of $\Gamma$. Then $\gamma = \gamma_1\gamma_2$ ranges through the set of elements in $R_N$ of determinant $m$. Then we compute:

$$\begin{aligned}
\big\langle w^-(z_1),\ T_m w^-(z_2)\big\rangle_{\mathrm{Br}} &= \sum_{\gamma_2} \big\langle w^-(z_1), \gamma_2 \cdot w^-(z_2)\big\rangle_{\mathrm{Br}} \\
&= \sum_{\gamma_2} \left\langle w^-(z_1), \frac{m^{p+t}}{j(\gamma_2, z_2)^{\ell}} w^-(\gamma z_2)\right\rangle_{\mathrm{Br}} \\
&= m^{p+t} \sum_{\gamma_2} j(\gamma_2, \bar{z}_2)^{-\ell} \langle w(z_1), w(\gamma_2 z_2)\rangle_{\mathrm{Br}} \\
&= m^{p+t} \sum_{\gamma_2} j(\gamma_2, \bar{z}_2)^{-\ell} \frac{1}{(2iy_1 \mathrm{Im}(\gamma_2 z_2))^{\ell}} \sum_{\gamma_1} g(z_1, \gamma z_2)(\bar{z}_1 - \gamma z_2)^{\ell} j(\gamma_1, \gamma_2 z_2)^{\ell} \\
&= \frac{m^{p+t}}{(2iy_1y_2)^{\ell} m^{\ell}} \sum_{\gamma_2} j(\gamma_2, z_2)^{\ell} \frac{1}{(2iy_1y_2)^{\ell}} \sum_{\gamma_1} g(z_1, \gamma z_2)(\bar{z}_1 - \gamma z_2)^{\ell} j(\gamma_1, \gamma_2 z_2)^{\ell} \\
&= \frac{m^{p-t}}{(2iy_1y_2)^{\ell}} \sum_{\gamma} g(z_1, \gamma z_2)(\bar{z}_1 - \gamma z_2)^{\ell} j(\gamma, z_2)^{\ell}.
\end{aligned}$$

□

Now we specialize to the case where $z_1 = \tau_1$ and $z_2 = \tau_2$ correspond to Heegner points on $X_0(N)$ with CM by $\mathcal{O}_K$. Recall from [GZ, §2] that $\mathbb{Z} + \tau_i\mathbb{Z} = \mathfrak{a}_i^{-1}$ for some $\mathcal{O}_K$-ideal $\mathfrak{a}_i \subset \mathcal{O}_K$. If $A_i > 0$ and $B_i$ are integers such that $\tau_i = \frac{-B_i + \sqrt{D}}{2A_i}$, then $A_i = \mathrm{Nm}(\mathfrak{a}_i)$. We have already fixed a Heegner point $\tau_0 \in \mathfrak{h}$ corresponding to our choice of CM elliptic curve $A$ isomorphic to $\mathbb{C}/\mathcal{O}_K$ over $\mathbb{C}$. To simplify computations, we choose $\tau_0 = \frac{-1+\sqrt{D}}{2}$, so that the corresponding lattice $\mathfrak{a}_0$ is $\mathcal{O}_K$ (one can check that the results do not depend on this choice).

Note that if $\tau \in \mathfrak{h}$ is a Heegner point, and $\mathfrak{a}_\tau$ the corresponding lattice, then $\mathfrak{a}_{\gamma\tau} = j(\gamma, \tau)\mathfrak{a}_\tau$, for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. By (11.2), $\chi(\mathfrak{a}_\tau)^{-1} w^-(\tau)$ and $\chi(\mathfrak{a}_\tau) w^+(\tau)$ descend to well-defined cohomology classes in the fiber above the corresponding point of $X(N)$ and hence a class in the local system on $X(N)$ and $X_0(N)$ as well.

**Proposition XI.18.** *Let $\tau_1, \tau_2 \in X_0(N)$ be two Heegner points as above and suppose $\tau_1$ and $T_m\tau_2$ are disjoint on $X_0(N)$. Then*

$$\left\langle \frac{w^-(\tau_1)}{\chi(\mathfrak{a}_1)}, T_m \frac{w^-(\tau_2)}{\chi(\mathfrak{a}_2)} \right\rangle_{\mathrm{Br}} = \frac{(-1)^t 2^\ell m^{p-t}}{|D|^\ell} \chi(\bar{\mathfrak{a}}_1 \mathfrak{a}_2) \sum_{\gamma \in R_N^m} g(\tau_1, \gamma\tau_2) \alpha(\gamma, \tau_1, \tau_2)^\ell,$$

$$\left\langle \frac{w^+(\tau_1)}{y_1^{2t}\chi(\mathfrak{a}_1)}, T_m \frac{w^+(\tau_2)}{y_2^{2t}\chi(\mathfrak{a}_2)} \right\rangle_{\mathrm{Br}} = \frac{(-1)^t 2^\ell m^{p-t}}{|D|^\ell} \chi(\mathfrak{a}_1 \bar{\mathfrak{a}}_2) \sum_{\gamma \in R_N^m} g(\tau_1, \gamma\tau_2) \alpha(\gamma, \bar{\tau}_1, \bar{\tau}_2)^\ell,$$

*where $\alpha(\gamma, \tau_1, \tau_2) = c\bar{\tau}_1\tau_2 + d\bar{\tau}_1 - a\tau_2 - b$ and $R_N^m = \{\gamma \in R_N : \det\gamma = m\}$.*

*Proof.* This follows from the previous proposition. Note that $y_1 = \sqrt{|D|}/2A_1$ and $y_2 = \sqrt{D}/2A_2$. Also $\chi(\bar{\mathfrak{a}}_i)\chi(\mathfrak{a}_i) = \chi(\mathrm{Nm}(\mathfrak{a}_i)) = \mathrm{Nm}(\mathfrak{a}_i)^\ell = A_i^\ell$. □

**Lemma XI.19.** *For $\tau = x + iy \in \mathfrak{h}$, let $A_\tau$ be the elliptic curve $\mathbb{C}/\mathfrak{a}_\tau^{-1}$. Then the class of the generalized Heegner cycle $\epsilon Y^{\mathfrak{a}_\tau}$ in $\epsilon H_{\mathrm{dR}}^{2r+2k}(A_\tau^{2k-2} \times A^{2t}, \mathbb{C})$ is*

$$\pm \frac{2^{p-t}\sqrt{|D|}^{p-t}}{\binom{2p}{p-t}^{1/2}} y_0^t \cdot w^-(\tau).$$

*Proof.* Set $\mathfrak{a} = \mathfrak{a}_\tau$. The generalized Heegner cycle is constructed from graphs of isogenies. We may work factor by factor and compute the cycle class of the (adjusted) graph of the isogeny $\sqrt{D}$ on $A_\tau \times A_\tau$ and the graph of $\phi = \hat{\phi}_{\mathfrak{a}} : A_\tau = A^{\mathfrak{a}} \to A$ on $A_\tau \times A$. This is the de Rham analogue of the $p$-adic computations we did in Section 4.4, so we will use the notation $X_{\sqrt{D}}$ and $X_{1,\phi} = X_\phi$ for the adjusted graphs (recall that $X_\phi$ is the projection of the graph of the $\phi$ onto the orthogonal complement of the horizontal and vertical fibers, i.e. onto $H^1 \otimes H^1$).

Recall that if $C \subset A^{\mathfrak{a}} \times A$ is an algebraic cycle, then its de Rham cohomology class is represented by a differential form $\omega$ satisfying

$$\int_C \eta = \int_{A^{\mathfrak{a}} \times A} \eta \wedge \omega,$$

for all $\eta \in H_{\mathrm{dR}}^2(A^{\mathfrak{a}} \times A, \mathbb{C})$. In the case of $X_\phi$, one computes that its cycle class is represented by the differential form

$$\frac{\mathrm{Nm}(\mathfrak{a})}{2iy_0}(dz_1 d\bar{z}_2 - d\bar{z}_1 dz_2),$$

where $dz_1$ is the pullback of the usual holomorphic differential form on $\mathbb{C}/\mathfrak{a}^{-1} = A_\tau$ and $dz_2$ is the differential form on $A = \mathbb{C}/\mathcal{O}_K$. The factor of $\mathrm{Nm}(\mathfrak{a})$ comes from the fact that the dual of $\phi_{\mathfrak{a}} : \mathbb{C}/\mathcal{O}_K \to \mathbb{C}/\mathfrak{a}^{-1}$ is given by the map $\mathbb{C}/\mathfrak{a}^{-1} \to \mathbb{C}/\mathcal{O}_K$ which is multiplication by $\mathrm{Nm}(\mathfrak{a})$ on the underlying complex vector spaces. Note also $\int_A dz d\bar{z} = -2iy_0$. The effect of the projector $\epsilon$ is to kill the $d\bar{z}_2$ terms.

Similarly, one finds that the class of $X_{\sqrt{D}}$ on $A_\tau \times A_\tau$ is

$$\pm \frac{\sqrt{|D|}}{2y}(dz_1 d\bar{z}_2 + d\bar{z}_1 dz_2).$$

Note that the effect of the projector $\epsilon$ on these purely Kuga-Sato components was to force the cycle to lie in $\mathrm{Sym}^2 H^1(A_\tau)$, which is why we look at $X_{\sqrt{D}}$ and not simply the graph of $\sqrt{D}$.

The differential form $dz$ on $\mathbb{C}/\mathfrak{a}^{-1} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ is given by $\tau u_1 + u_2$ and $d\bar{z}$ is $\bar{\tau}u_1 + u_2$. So each of the $2t$ factors of the form $\epsilon X_\phi$ contribute

$$-\mathrm{Nm}(\mathfrak{a})(2iy_0)^{-1}(\bar{\tau}u_1 + u_2)(\tau_0 e_1 + e_2)$$

to the class of $\epsilon Y^{\mathfrak{a}}$ and each of the $p - t$ factors of $X_{\sqrt{D}}$ contribute

$$\pm \frac{\sqrt{|D|}}{y}(\tau u_1 + u_2)(\bar{\tau}u_1 + u_2).$$

Now just compare these computations with the definition of $w^-(\tau) = y^{-t} w_{p-t,2t}(\tau)$. □

For any ideal class $\mathcal{A}$, recall from the previous section that we write $Z_{\mathcal{A}}$ and $\bar{Z}_{\mathcal{A}}$ for the generalized Heegner cycles $\chi(\mathfrak{a})^{-1}\epsilon_B\epsilon Y^{\mathfrak{a}}$ and $\chi(\bar{\mathfrak{a}})^{-1}\epsilon_B\bar{\epsilon}Y^{\mathfrak{a}}$. When $\mathcal{A} = [\mathcal{O}_K]$, we just write $Z$ or $\bar{Z}$. Since $y_0 = \sqrt{D}/2$, the previous lemma and proposition show that

$$\langle Z_{\mathcal{A}_1}, T_m Z_{\mathcal{A}_2}\rangle_{\mathrm{Br}} = \frac{(4m|D|)^{p-t}}{D^t\binom{2p}{p-t}}\chi(\bar{\mathfrak{a}}_1\mathfrak{a}_2)\sum_{\gamma\in R_N^m} g(\tau_1,\gamma\tau_2)\alpha(\gamma,\tau_1,\tau_2)^{\ell}, \tag{11.8}$$

$$\left\langle \bar{Z}_{\mathcal{A}_1}, T_m \bar{Z}_{\mathcal{A}_2}\right\rangle_{\mathrm{Br}} = \frac{(4m|D|)^{p-t}}{D^t\binom{2p}{p-t}}\chi(\mathfrak{a}_1\bar{\mathfrak{a}}_2)\sum_{\gamma\in R_N^m} g(\tau_1,\gamma\tau_2)\alpha(\gamma,\bar{\tau}_1,\bar{\tau}_2)^{\ell},$$

for $\tau_1$ and $\tau_2$ such that $[\mathfrak{a}_{\tau_i}] = \mathcal{A}_i$.

Now let $\langle\,,\,\rangle_\infty = \sum_{v|\infty}\langle\,,\,\rangle_v$ be the sum of the local heights on $X = W\times A^{\ell}$ over all the infinite places $v$ of $H$. In other words, $\langle\,,\,\rangle_v$ is the pairing $\langle\,,\,\rangle_{\mathrm{Br}}$ applied to the base change $X_v/\mathbb{C}$ of $X$ to $\mathbb{C}$ using the embedding $H\to\mathbb{C}$ corresponding to $v$ (we are using Theorem XI.6 here). We extend these height pairings to algebraic cycles with coefficients in $\mathbb{Q}(\chi)$, the field generated by the values of $\chi$.

The next result gives the final expression for the local heights at infinity.

**Proposition XI.20.** *Let $\mathcal{A}$ be an ideal class in $K$ and assume $r_{\mathcal{A}}(m) = 0$. Then*

$$\langle Z, T_m Z_{\mathcal{A}}\rangle_\infty = -\frac{(4m|D|)^{p-t}}{D^t\binom{2p}{p-t}}u^2\sum_{n=1}^{\infty}\sigma_{\mathcal{A}}(n)r_{\bar{\mathcal{A}},\chi}(m|D|+nN)Q_{k,t}\left(1+\frac{2nN}{m|D|}\right),$$

$$\left\langle \bar{Z}, T_m\bar{Z}_{\mathcal{A}}\right\rangle_\infty = -\frac{(4m|D|)^{p-t}}{D^t\binom{2p}{p-t}}u^2\sum_{n=1}^{\infty}\sigma_{\bar{\mathcal{A}}}(n)r_{\mathcal{A},\chi}(m|D|+nN)Q_{k,t}\left(1+\frac{2nN}{m|D|}\right),$$

*and*

$$\left\langle Z, T_m\bar{Z}_{\mathcal{A}}\right\rangle_\infty = 0 = \left\langle \bar{Z}, T_m Z_{\mathcal{A}}\right\rangle_\infty$$

*Remark* XI.21. Notice that $r_{\bar{\mathcal{A}},\chi}$ appears in the first formula, whereas $r_{\mathcal{A},\chi}$ appears in the second.

*Proof.* As usual, the assumption $r_{\mathcal{A}}(m) = 0$ implies that $Z$ and $T_m Z_{\mathcal{A}}$ are supported on fibers above a disjoint set of points of $X_0(N)$. The final equation follows from the fact that $w^-$ and $w^+$ are orthogonal to each other. We next prove the first formula and omit a proof of the second, as it is similar.

One difficulty here is that $X = W\times A^{\ell}$ is defined only over $H$, so the complex varieties $X_v$ will be non-isomorphic as $v$ ranges over the archimedean places of $H$. We could redo the computations on the new variety (where we would have to replace the base point $\tau_0$ by a Galois conjugate). Alternatively, one can show (similar to the proof of Lemma IV.11) that

$$\langle Z, T_m Z_{\mathcal{A}}\rangle_v = \langle Z_{\mathcal{B}}, T_m Z_{\mathcal{B}\mathcal{A}}\rangle_{\mathrm{Br}}, \tag{11.9}$$

where $\mathcal{B}$ is the ideal class corresponding to $v$ (i.e. $\mathcal{B}$ is the class of the lattice attached to the base change $A_v$ of $A$ to $\mathbb{C}$). Since the archimedean places of $H$ are in bijection with the class group, the proposition now follows from (11.8), (11.9), and the proof of [GZ, II.3.17, IV.4.6]. Note that our $\alpha(\gamma,\tau_1,\tau_2) = c\bar{\tau}_1\tau_2 + d\bar{\tau}_1 - a\tau_2 - b$ is not the same as the $\alpha$ defined in [GZ, II.3.6], but they have the same norm; their $\alpha$ is just $\alpha(\gamma,\tau_2,\tau_1)$. Further note that $\alpha(\gamma,\tau_1,\tau_2)$ is an element of $\bar{\mathfrak{a}}^{-1}$, where $\mathfrak{a} = \mathfrak{a}_1\bar{\mathfrak{a}}_2$, as in [GZ], and the value of $g(\tau_1,\gamma\tau_2)$ depends only on the norm of $\alpha(\gamma,\tau_1,\tau_2)$. Since $[\mathfrak{a}] = \mathcal{A}$, we have

$$r_{\bar{\mathcal{A}},\chi}(j) = \chi(\bar{\mathfrak{a}})\sum_{\substack{x\in\bar{\mathfrak{a}}^{-1}\\ \mathrm{Nm}(x)=j/\mathrm{Nm}(\mathfrak{a})}} x^{\ell},$$

which explains the appearance of $r_{\bar{\mathcal{A}},\chi}$ in the sum. ∎

## 11.3 Sketch of proof of Theorem X.1

Following [GZ] and [Z], we might expect for any $m$ prime to $N$ a formula of the form

$$\langle Z + \bar{Z},\, T_m(Z_{\mathcal{A}} + \bar{Z}_{\mathcal{A}})\,\rangle_{\mathrm{GS}} \doteq a_m(\mathcal{A}), \tag{11.10}$$

where $a_m(\mathcal{A})$ is the Fourier coefficient from Proposition X.8. This formula would ideally hold for any $m$ with $(m, N) = 1$ and $\mathcal{A}$ such that $r_{\mathcal{A}}(m) = 0$. The constant implicit in the notation $\doteq$ should be independent of $m$ and $\mathcal{A}$.

We break up the height pairing above into four different terms, each of which can be decomposed into a sum of local heights. For example, we have

$$\langle Z, T_m Z_{\mathcal{A}}\rangle_{\mathrm{GS}} = \langle Z, T_m Z_{\mathcal{A}}\rangle_{\mathrm{fin}} + \langle Z, T_m Z_{\mathcal{A}}\rangle_{\infty},$$

where we have decomposed into the sum of local heights at finite places and the sum of local heights at infinity.

By Proposition X.8

$$\begin{aligned} a_m(\mathcal{A}) = -m^{k-t-1} &\sum_{0<n\leqslant \frac{m\delta}{N}} \sigma'_{\mathcal{A}}(n) r_{\bar{A},\chi}(m\delta - Nn) H_{k-t-1,t}\left(1 - \frac{2nN}{m|D|}\right) \\ &- m^{k-t-1} \sum_{n=1}^{\infty} \sigma_{\mathcal{A}}(n) r_{\bar{A},\chi}(m\delta + nN) Q_{k,t}\left(1 + \frac{2nN}{m|D|}\right). \end{aligned}$$

Write

$$a_m(\mathcal{A}) = a_m^1(\mathcal{A}) + a_m^2(\mathcal{A}),$$

where $a_m^1(\mathcal{A})$ is the sum with $H_{k-t-1,t}$ and $a_m^2(\mathcal{A})$ is the sum with $Q_{k,t}$.

By Proposition XI.20, we have

$$\langle Z, T_m Z_{\mathcal{A}}\rangle_{\infty} = \frac{u^2(4|D|)^{p-t}}{D^t \cdot \binom{2p}{p-t}} a_m^2(\mathcal{A}). \tag{11.11}$$

Similarly, we have

$$\langle \bar{Z}, T_m \bar{Z}_{\mathcal{A}}\rangle_{\infty} = \frac{u^2(4|D|)^{p-t}}{D^t \cdot \binom{2p}{p-t}} a_m^2(\bar{\mathcal{A}}). \tag{11.12}$$

Note that it is $\bar{\mathcal{A}}$ in the right hand side and not $\mathcal{A}$.

On the other hand, $\langle Z, T_m \bar{Z}_{\mathcal{A}}\rangle_{\mathrm{fin}}$ can be computed exactly as in Chapter VI, since local heights at places above $\ell$ not equal to $p$ (for the $p$-adic heights) or $\infty$ (for the archimedean heights) are determined by intersection on the special fiber. One sees from a calculation entirely similar to Proposition VI.5 that

$$\langle Z, T_m \bar{Z}_{\mathcal{A}}\rangle_{\mathrm{fin}} = \frac{u^2(4|D|)^{p-t}}{D^t \cdot \binom{2p}{p-t}} a_m^1(\bar{\mathcal{A}}).$$

(Note that there is a notation clash here: the function $\sigma_{\mathcal{A}}(n)$ in VI.5 is the $p$-adic analogue of what is called $\sigma'_{\mathcal{A}}(n)$ above.) Similarly,

$$\langle \bar{Z}, T_m Z_{\mathcal{A}}\rangle_{\mathrm{fin}} = \frac{u^2(4|D|)^{p-t}}{D^t \cdot \binom{2p}{p-t}} a_m^1(\mathcal{A}).$$

One can either compute this directly as before, or note that

$$\begin{aligned} \left\langle \bar{Z}, T_m Z_{\mathcal{A}}\right\rangle_{\mathrm{fin}} &= \left\langle T_m Z, \bar{Z}_{\bar{\mathcal{A}}}\right\rangle_{\mathrm{fin}} \\ &= \left\langle Z, T_m \bar{Z}_{\bar{\mathcal{A}}}\right\rangle_{\mathrm{fin}} \\ &\doteq a_m^1(\mathcal{A}). \end{aligned}$$

The first equality here is proved much as in Lemma IV.11.

Finally, the terms $\langle Z, T_m\bar{Z}_{\mathcal{A}}\rangle_\infty$, $\langle \bar{Z}, T_m Z_{\mathcal{A}}\rangle_\infty$, $\langle Z, T_m Z_{\mathcal{A}}\rangle_{\text{fin}}$, and $\langle \bar{Z}, T_m\bar{Z}_{\mathcal{A}}\rangle_{\text{fin}}$ are immediately seen to vanish by orthogonality. For the heights at infinity, this is Proposition XI.20. For the heights at the finite primes, this is ultimately because the cup product pairing on $H^1(A)$ is alternating (see the proof of Proposition VI.5).

Recall we defined $H_m(\mathcal{A}) = \langle Z + \bar{Z}, T_m(Z_{\mathcal{A}} + \bar{Z}_{\mathcal{A}})\rangle_{\text{GS}}$. From the four equations above, we cannot expect (11.10) to hold unless $\mathcal{A} = \bar{\mathcal{A}}$. In general, we can only hope for:

$$H_m(\mathcal{A}) + H_m(\bar{\mathcal{A}}) \doteq a_m(\mathcal{A}) + a_m(\bar{\mathcal{A}}), \tag{11.13}$$

which is the desired equality in Theorem X.1. And indeed, this follows from the four formulas above applied to both $\mathcal{A}$ and $\bar{\mathcal{A}}$.

Equation (11.13) is already remarkable in that it relates archimedean heights of generalized Heegner cycles to Fourier coefficients of a certain modular form encoding the central derivatives of Rankin-Selberg $L$-functions. As was indicated earlier, if we assume that the generating series

$$F_{\mathcal{A}} + F_{\bar{\mathcal{A}}} := \sum_{m\geqslant 1} (H_m(\mathcal{A}) + H_m(\bar{\mathcal{A}}))q^m$$

is a modular form of weight $2k$ and level $N$, then one deduces that $F_{\mathcal{A}} + F_{\bar{\mathcal{A}}}$ equals $g_{\mathcal{A}} + g_{\bar{\mathcal{A}}}$, up to addition of an old form (recall, $g_{\mathcal{A}} = \sum a_m(\mathcal{A})q^m$). This follows by a lemma of Nekovář [N3, II.5.7], which says that knowing the $m$th Fourier coefficients of a modular form for all $m$ (prime to $N$) such that $r_{\mathcal{A}}(m) = 0$ in fact determines the modular form up to the addition of an old form. From the equality of these modular forms, one deduces (similar to the argument in [GZ], though there is a bit more work to do in our setting) a formula relating $L'(f, \chi, k+t) = \sum_{\mathcal{A}} L'_{\mathcal{A}}(f, \chi, k+t)$ to the height of the $f$-isotypic component of the algebraic cycle $\sum_{\mathcal{A}}(Z_{\mathcal{A}} + \bar{Z}_{\mathcal{A}})$.

Unfortunately, it is not known (at least to us) that the generating series above is modular. To unconditionally prove the desired formula

$$L'(f, \chi, k+t) \doteq \left\langle \sum_{\mathcal{A}} (Z_{\mathcal{A}} + \bar{Z}_{\mathcal{A}}), \sum_{\mathcal{A}} (Z_{\mathcal{A}} + \bar{Z}_{\mathcal{A}}) \right\rangle_{\text{GS}}, \tag{11.14}$$

one can instead follow the approach of [Z] in the case $t = 0$. This requires proving the equality (11.13) in the more delicate situation where $Z$ and $T_m Z_{\mathcal{A}}$ have intersecting supports (i.e., the case $r_{\mathcal{A}}(m) \neq 0$, but $m$ still prime to $N$). One then needs to compare the Hecke action on the space of generalized Heegner cycles with the usual Hecke action on the space of modular forms. Zhang's approach for computing self-intersections fits nicely into the framework of Brylinski that we have used here, so we can use this approach in our situation as well. We plan to explain this in detail and complete the proof of (11.14) in a separate paper.

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