Vanishing criteria for Ceresa cycles and examples

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VaNTAGe seminar

August 27, 2024

Motivation

X an algebraic variety with subvarieties $Y, Y' \subset X$.

Basic questions:

- Can we algebraically "deform" Y to Y' inside X?
- How many such equivalence classes of subvarieties on X are there?

Equivalence relations on algebraic cycles

X smooth, projective, n-dimensional, algebraic variety over a field k. Define

$$Z_i(X) = \left\{\sum n_j Y_j \quad : n_j \in \mathbb{Z}\right\},$$

the free abelian group on closed integral subschemes $Y_j \subset X$ of dimension i.

Define $Z^i(X) = Z_{n-i}(X) =$ codimension i algebraic cycles.

Equivalence relations on $Z_i(X)$: (very roughly)

- For every $W \subset X$ of dim $i+1$ and every rational map $W \xrightarrow{f} \mathbb{P}^1$, we declare the fibers to be rationally equivalent.
- Algebraic equivalence is similar, but with maps $W \xrightarrow{f} C$, for arbitrary curves C.

Example: $Z^1(X) = \mathrm{Div}(X)$ and rational equivalence is linear equivalence.

Chow groups

Definitions:

- $\mathrm{CH}_i(X) = Z_i(X)/\sim_{\mathrm{rat}}$
- $\mathrm{CH}^i(X) = \mathrm{CH}_{n-i}(X)$ as before.
- $\mathrm{CH}^i(X)_{\mathrm{alg}} \subset \mathrm{CH}^i(X)$ subgroup of algebraically trivial cycles.

Properties:

- $\mathrm{CH}(X) := \bigoplus_{i=0}^n \mathrm{CH}^i(X)$ forms a ring under intersection pairing.
- Functoriality: (suitably behaved) morphisms induce pullback and push-forward.

Question: How big is $\mathrm{CH}^i(X)$? How do we tell whether two cycles are equivalent?

Cycle class map

Let $\mathrm{H}^*(X) = \mathrm{H}^*(X(\mathbb{C}), \mathbb{Z})$ (if $k = \mathbb{C}$) or your favorite Weil cohomology theory.

The cycle class map $\mathrm{CH}^i(X) \xrightarrow{\mathrm{cyc}_i} \mathrm{H}^{2i}(X)(i)$ sends $[Y]$ to its homology class.

Set $\mathrm{CH}^i(X)_{\mathrm{hom}} := \ker(\mathrm{cyc}_i)$.

- $\mathrm{Im}(\mathrm{cyc}_i)$ is finitely generated. Its rank is predicted by the Hodge/Tate conjecture.

What about $\mathrm{CH}^i(X)_{\mathrm{hom}}$?

- If $i = 1$ and X has a k-point, then $\mathrm{CH}^1(X)_{\mathrm{hom}} = A(k)$ for some abelian variety A.
- If $i > 1$ an $k = \mathbb{C}$, it is typically "infinite-dimensional" in some sense (Mumford).
- If k is a number field, it is conjecturally finitely generated (Bass).
- Beilinson-Bloch conjecture: $\mathrm{rk}\,\mathrm{CH}^i(X)_{\mathrm{hom}} = \mathrm{ord}_{s=i}\, L(\mathrm{H}^{2i-1}_{et}(X), s)$.

Let's explain why $\mathrm{H}^{2i-1}(X)$ is relevant.

Abel-Jacobi maps

Classical case: For a curve $C/\mathbb{C}$, let $V = \mathrm{H}^0(C, \Omega_C)$ and $\Lambda = \mathrm{H}_1(C, \mathbb{Z})$. The Jacobian of C is $J := V^*/\Lambda$, a g-dimensional complex torus. Have

$$\mathrm{Pic}^0(C) = \mathrm{CH}^1(C)_{\mathrm{hom}} \xrightarrow{\mathrm{AJ}} J$$
$$p - q \mapsto \left[\omega \mapsto \int_p^q \omega\right]$$

Hodge theory: $\mathrm{H}^1(C, \mathbb{C}) = \mathrm{H}^{1,0} \oplus \mathrm{H}^{0,1} = V \oplus \bar{V}$, i.e. $V = \mathrm{Fil}^1\mathrm{H}^1(C, \mathbb{C})$.

General case: Griffiths defines the **intermediate Jacobian**

$$J^i(X) := \frac{\mathrm{H}^{2i-1}(X, \mathbb{C})}{\mathrm{Fil}^i + \mathrm{H}^{2i-1}(X, \mathbb{Z})} \simeq \frac{\mathrm{Fil}^{n-i+1}\mathrm{H}^{2n-2i+1}(X, \mathbb{C})^*}{\mathrm{H}_{2n-2i+1}(X, \mathbb{Z})}$$

and the "higher" Abel-Jacobi map

$$\mathrm{CH}^i(X)_{\mathrm{hom}} \xrightarrow{\mathrm{AJ}} J^i(X)$$
$$Z \mapsto \left[\omega \mapsto \int_{\partial^{-1}Z} \omega\right]$$

Beilinson-Bloch conjecture

Analogously, for k a number field, we have the ℓ-adic Abel-Jacobi map

$$\mathrm{CH}^i(X)_{\mathrm{hom}} \xrightarrow{\mathrm{AJ}_\ell} H^1(\mathrm{Gal}_k, \mathrm{H}^{2i-1}_{et}(\overline{X}, \mathbb{Z}_\ell(i))),$$

whose kernel is conjecturally torsion (Beilinson-Bloch).

Another probe at our disposal is the Beilinson-Bloch height:

$$\langle\, , \,\rangle_{\mathrm{BB}} : \mathrm{CH}^i(X)_{\mathrm{hom}} \times \mathrm{CH}^{n+1-i}(X)_{\mathrm{hom}} \longrightarrow \mathbb{R},$$

generalizing the Néron-Tate height on abelian varieties.

Griffiths group

Consider the chain of subgroups

$$\{0\} \subset \mathrm{CH}^i_{\mathrm{alg}} \subset \mathrm{CH}^i(X)_{\mathrm{hom}} \subset \mathrm{CH}^i(X)$$

- Fact: $\mathrm{CH}^1_{\mathrm{alg}}(X) = \mathrm{CH}^1(X)_{\mathrm{hom}}$.
- Griffiths: not true for $i \geq 2$! The difference of two lines $[L] - [L'] \in \mathrm{CH}^2(X)_{\mathrm{hom}}$ on a very general quintic threefold $X \subset \mathbb{P}^4$ over $\mathbb{C}$ is not algebraically trivial.
- Define the Griffiths group

$$\mathrm{Gr}^i(X) := \mathrm{CH}^i(X)_{\mathrm{hom}}/\mathrm{CH}^i(X)_{\mathrm{alg}}.$$

- $\mathrm{Gr}^i(X)$ is countable but can have infinite rank (Clemens).

Ceresa cycle

Let C be a curve of genus $g \geq 2$ over k, with a degree one divisor $e \in \mathrm{Div}(C)$.

Let $C \overset{\iota}{\hookrightarrow} J$ be the Abel-Jacobi embedding $x \mapsto x - e$.

The Ceresa cycle (based at e) is

$$\kappa_e(C) = [\iota(C)] - (-1)^*[\iota(C)] \in \mathrm{CH}_1(J)_{\mathrm{hom}}.$$

Homologically trivial since $\mathrm{H}^{2g-2}(J) = \bigwedge^{2g-2}\mathrm{H}^1(J)$ and $(-1)^*$ acts as -1 on $\mathrm{H}^1(J)$.

Its image $\kappa_{\mathrm{Gr}}(C) \in \mathrm{Gr}_1(J)$ is independent of the choice of e.

Theorem (Ceresa)

For very general C over $\mathbb{C}$ of genus $g \geq 3$, $\kappa_{\mathrm{Gr}}(C)$ has infinite order.

For proof of something stronger, see Dick's talk.

Clemens' question

So most curves have infinite order Ceresa cycle, even modulo $\sim_{\mathrm{alg}}$. On the other hand:

Example: If C is hyperelliptic and e is a Weierstrass point, then $\kappa_e(C) = 0 \in Z_1(J)$.

Proof: Let $\tau \in \mathrm{Aut}(C)$ be the hyperelliptic involution. Then

$$(-1)^*(x - e) = \tau^*(x - e) = \tau(x) - e,$$

so that $(-1)^*(\iota(C)) = \iota(C)$.

Question (Clemens '87)

$\kappa_{\mathrm{Gr}}(C) = 0$ *if and only if* C *is hyperelliptic?*

I suspect this is not true, but the spirit of the question is:

The Ceresa class vanishes only if there is a good geometric reason.

Vanishing in the Chow group: choosing the base point

Let $K_C \in \mathrm{Pic}(C)$ be the canonical class.

Lemma

If $\kappa_e(C)$ is torsion then $(2g-2)e \equiv K_C$ in $\mathrm{Pic}(C) \otimes \mathbb{Q}$.

Upshot: We should and do assume $(2g-2)e \equiv K_C$.
(Let's also assume $\mathrm{char}(k) = 0$ from this point on...)

There are $(2g-2)^{2g}$ choices of such $\kappa_e(C)$, differing only by torsion.

Thus, there is a canonical Ceresa class in $\kappa(C) \in \mathrm{CH}_1(J) \otimes \mathbb{Q}$.

Note: $\kappa_e(C)$ is torsion if and only if $\kappa(C) = 0$.

Possible counterexamples to Clemens' question

Recently, non-hyperelliptic curves with $\mathrm{AJ}(\kappa(C)) = 0$ were found. Let $G = \mathrm{Aut}(C)$.

- (Bisogno-Li-Litt-Srinivasan) Fricke-Macbeath curve: $g = 7$ and $G \simeq \mathrm{PSL}_2(\mathbb{F}_8)$.
- (Beauville) The genus 3 curve $C : y^3 = x^4 + x$ with $G \simeq C_9$.

Proof.

AJ is G-equivariant and $\kappa(C)$ is canonical and hence G-invariant. One computes that the target $J^{g-1}(J)^G \otimes \mathbb{Q} = 0$, hence $\mathrm{AJ}(\kappa(C)) = 0$. □

This proof was then upgraded to a statement about $\kappa(C)$ itself:

Theorem (Qiu-Zhang)

If $\mathrm{H}^1(C)^{\otimes 3}$ has no G-invariants, then $\kappa(C) = 0$.

Chow-vanishing criterion

It turns out, it is enough to check a smaller G-representation.

Theorem (Laga-S)

If $\mathrm{H}^3(J)^G = 0$ *then* $\kappa(C) = 0$.

- Note that $\mathrm{H}^3(J) = \bigwedge^3 \mathrm{H}^1(C)$.
- Even $\mathrm{H}^3(J)^G_{\mathrm{prim}} = 0$ is enough, but if $g(C/G) = 0$, this is equivalent.

Examples:

- $y^3 = x^4 + 1$ (Qiu-Zhang, $g = 3$, $\#G = 48$)
- $y^3 = x^4 + x$ (Beauville-Schoen, $g = 3$, $G \simeq C_9$)
- $y^3 = x^5 + 1$ (Lilienfeldt-S, $g = 4$, $G \simeq C_{15}$)
- $y^3 = (x^3 + t)^2(tx^3 - 1)$ (Qiu-Zhang, $g = 4$, $G \simeq S_3^2$)
- 2-dimensional family of Humbert-Edge curves (Laterveer, $g = 5$, $G \simeq C_2^4$)
- If $\kappa(C) = 0$ and $C \to D$, then $\kappa(D) = 0$.

Griffiths-vanishing criterion

Recall $V = \mathrm{H}^0(C, \Omega_C)$, so $\mathrm{H}^1(C)_{\mathbb{C}} \simeq V \oplus \bar{V}$.

Theorem (Laga-S)

Assume Hodge conjecture for abelian varieties. If $(\wedge^3 V)^G = 0$ *then* $\kappa_{\mathrm{Gr}}(C)$ *is torsion.*

- The hypotheses imply $\mathrm{H}^3(J)^G \simeq \mathrm{H}^1(A)(-1)$. Need Hodge for $J \times A$.

Examples:

- Picard curves: $y^3 = x^4 + ax^2 + bx + c$. HC proved by Schoen.
- Genus four D_5-curves: $y^5 = x^3(x-1)^2(x-t)$. HC easy.
- Torelli locus in $\mathrm{U}(2,2)$-Shimura variety: $y^3 = x^2(x-1)^2(x^3 + ax^2 + bx + c)$. HC?
- Families in genus $5, 6, 8$.

Proof sketches

Exploit the structure of the Chow motive $h(J)$, especially the Chow-Kunneth and Beauville decompositions:

$$h(J) = \bigoplus_{i=0}^{2g} h^i(J) \quad \text{and} \quad \mathrm{CH}^{g-1}(J) = \bigoplus_{s=0}^{g-1} \mathrm{CH}^{g-1}_{(s)}(J).$$

Proof of Chow vanishing.

$\kappa(C) = 0$ if and only if $[C]_1 = 0$ and $[C]_1 \in \mathrm{CH}(h^{2g-3}(J)^G) \simeq \mathrm{CH}(h^3(J)^G)$. But $h^3(J)^G$ has trivial cohomology. By Kimura[a], $h^3(J)^G = 0$, hence $\kappa(C) = 0$. □

[a] S.-I. Kimura. Chow groups are finite dimensional, in some sense. Math. Ann., 2005.

Proof of Griffiths vanishing.

$h^3(J)^G$ need not be 0, but its Hodge structure is weight 1. So Hodge conjecture implies $h^3(J)^G \simeq h^1(A)(-1)$ and hence $\kappa(C)$ is algebraically trivial. □

Can $\kappa(C)$ vanish for non-group theoretic reasons? Yes!

For $f(x) = x^4 + ax^2 + bx + c$, let $C_f : y^3 = f(x)$ be the corresponding genus 3 curve.

Recall the I- and J-invariants of f:

- $I(f) = a^2 + 12c$
- $J(f) = 72ac - 2a^3 - 27b^2$

Classical observation: $J(f)^2 = 4I(f)^3 - 27\mathrm{Disc}(f)$.

In other words, $P_f := (I(f), J(f))$ lies on the elliptic curve $E_f: y^2 = 4x^3 - 27\mathrm{Disc}(f)$.

Theorem (Laga-S)

$\kappa(C_f) = 0$ *if and only if* $P_f \in E_f(\mathbb{C})$ *is torsion.*

In fact: $\langle \kappa(C_f), \kappa(C_f) \rangle_{\mathrm{BB}} = N^2 \langle P_f, P_f \rangle_{\mathrm{NT}}$ for some constant N.

Proof sketch

More generally, suppose $\mathcal{C} \to S$ is a family of curves with fiber-wise algebraically trivial Ceresa cycle.

Then the holomorphic normal function $S \xrightarrow{\sigma} \mathcal{J}^{g-1}(\mathcal{J})$ to the family of intermediate Jacobians should factor through an algebraic section of an abelian scheme $\mathcal{A} \to S$.

Suppose the Mordell-Weil group $\mathcal{A}(S)$ is free of rank 1 (absent further information, this is what we should expect!). In our case, we compute $\mathcal{A}(S)$ via Shioda-Tate.

If $\sigma \neq 0$, then it must be a multiple of this generator. To prove this, simply specialize and verify for a single example! (See Padma's talk.)

The Ceresa vanishing locus in $\mathcal{M}_g$

We can consider the torsion locus of either $\kappa(C)$ or $\kappa_{\mathrm{Gr}}(C)$ in $\mathcal{M}_g$. A priori, this is just some countable union of proper closed subvarieties.

Theorem (Gao-Zhang, '24)

For $g \geq 3$, there is an open dense subset $U_g \subset \mathcal{M}_g$ on which the height of the Ceresa/modified diagonal cycle satisfies a Northcott property for $\overline{\mathbb{Q}}$-points.

See also work of Hain and Kerr-Tayou. Our previous theorem implies:

Corollary

The 2-dimensional locus of Picard curves in $\mathcal{M}_3$ is contained in the complement of U_3

This is despite the fact that the normal function is not constant on this locus!

Proof

- The Picard locus is (essentially) the universal elliptic curve $\mathcal{E}$.
  - The map sends $y^3 = f(x)$ to $y^2 = f(x)$.
- The elliptic fibration $\mathcal{E}$ has base $S = \mathbb{P}(2,3)$.
- S can be thought of as the elliptic curve $\hat{E}\colon y^2 = x^3 + 1$.
- Our result: the torsion Ceresa cycles are the fibers of $\mathcal{E}$ above $\hat{E}_{\mathrm{tors}}$.
- In the coarse space, the fibers are of the form $E/\langle \pm 1\rangle \simeq \mathbb{P}^1$.
- So Picard locus contains infinitely many rational curves on which $\kappa(C) = 0$.
- Hence the Picard locus cannot be in U_3.

Many open questions about the subset of $\mathcal{M}_g$ with $\kappa(C) = 0$.

For example, what is its Zariski closure? All of $\mathcal{M}_g$?

Thank you!


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Köszönöm
Dankjewel
Grazie
Misaotra
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Takk
Terima Kasih
Tak
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Grazie
Dankie
Köszönöm
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Merci Gracias
Thank You
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Some open questions related to Ceresa vanishing

Let $G = \mathrm{Aut}(C)$.

- Are there non-hyperelliptic curves of arbitrary large genus with $\mathrm{H}^3(J)^{\mathrm{Aut}(C)} = 0$?
- With $\left(\bigwedge^3 \mathrm{H}^0(C, \Omega_C)\right)^G = 0$?
- Which genus 3 curves C have a cover with these properties?
- When $\mathrm{H}^3(J)^G(1)$ is pure of weight 1, what is the corresponding abelian variety?

Chow motives cheat sheet

The category of pure Chow k-motives is obtained from the category of smooth projective k-varieties by taking $\mathrm{Hom}(X,Y) = \mathrm{CH}(X \times Y)$, i.e. morphisms are algebraic correspondences. Such a hom induces $\mathrm{CH}(X) \to \mathrm{CH}(Y)$ and $\mathrm{H}^*(X) \to \mathrm{H}^*(Y)$ via "pullback-intersect-pushforward".

- Chow motives are tuples $(X, e, \ldots)$ with $e \in \mathrm{End}(X) = \mathrm{CH}(X \times X)$ idempotent.
- The object corresponding to X itself, denoted $h(X)$, is $(X, \Delta, \ldots)$.
- One expects a canonical Chow-Kunneth decomposition $h(X) = \bigoplus_{i=0}^{2\dim(X)} h^i(X)$, analogous to the decomposition of cohomology.
- An abelian variety has such a decomposition, which is even multiplicative.
- A curve C has a multiplicative Chow-Kunneth decomposition iff $\kappa(C) = 0$.

Does $\kappa(C)$ ever vanish for non-group theoretic reasons?

Yes!

Example: Let $C_{a,b}\colon y^3 = x^4 + ax^2 + b$, a plane quartic with $C_6 \hookrightarrow \mathrm{Aut}(C)$.

Alternate description: double cover of $y^2 = x^3 + k$, branched along ∞ and μ_3-orbit.

This gives an interesting involution $C_{a,b} \leftrightarrow C_{8a,16(a^2-4b)} = \hat{C}_{a,b}$ called bigonal duality.

Theorem (Laga-S)

$\kappa(C_{a,b}) = 0$ *if and only if* $\hat{C}_{a,b}$ *is branched along torsion points.*

This geometric criteria does not hold for general bielliptic plane quartics.

Does it generalize to other bielliptic curves with extra structure?

Aside on the modified diagonal cycle

Gross and Schoen defined a modified diagonal cycle

$$\Delta_e = \Delta_{123} - \Delta_{12e} - \Delta_{1e3} - \Delta_{e23} + \Delta_{1ee} + \Delta_{e2e} + \Delta_{ee3} \in \mathrm{CH}^2(C \times C \times C) \otimes \mathbb{Q}$$

which is closely related to the Ceresa cycle: $\Delta_e = 0$ if and only if $\kappa(C) = 0$.

Pros:

- Easier to work with when doing explicit computations
- In middle arithmetic dimension for all g

Cons:

- C^3 is not an abelian variety
- Harder to see the full picture, e.g. Poincare's formula $\Theta^{g-1} = (g-1)![C]$ holds modulo algebraic equivalence if and only if $\kappa(C) = 0$.