

## Relative Trace Formula

$$G \hookrightarrow G \times G$$

$$[G] = \frac{G(A)}{G(F)}$$

$$H \times H \hookrightarrow G \times G$$

$$[H] \times [H] \hookrightarrow [G] \times [G]$$

Define for  $f \in C_c^\infty(G(A))$

$$I(f) = \int_{[H]^2} k_f(h_1, h_2) dh_1 dh_2$$

$$= \sum_{\pi \in \mathcal{A}(G)} \sum_{\varphi \in \text{Basis}(\pi)} \int_{[H]} (R(f)\varphi)(h_1) dh_1 \int_{[H]} \overline{\varphi(h_2)} dh_2$$

On the other hand:

$$I(f) = \iint_{[H]^2} \sum_{\gamma \in G(F)} f(h_1^{-1} \gamma h_2) dh_1 dh_2$$

$$= \sum_{\sigma \in \frac{G(F)}{H(F)}} \text{Vol}\left(\frac{H_\sigma(A)}{H_\sigma(F)}\right) \int_{H_\sigma(A)} f(h_1^{-1} \gamma h_2) dh_1 dh_2$$

$$H_\gamma = \left\{ (h_1, h_2) \mid h_1^{-1} \circ h_2 = \gamma \right\}$$

$\uparrow$   
 $H^2$

$$|P_T(\varphi)|^2 \doteq L(\pi_k, \frac{1}{2})$$

$$\begin{array}{c} T \hookrightarrow \mathbb{B}^\times /_{F^\times} \quad \forall \in \pi^B \\ \parallel \quad K^\times /_{F^\times} \end{array}$$

$L(\pi, \frac{1}{2}) \quad L(\pi \otimes \eta, \frac{1}{2})$   
 $A \hookrightarrow GL_2$

Jacquet's RTF I

$F$  = number field

$K/F$  quadratic field

$G = PGL_2/F$

$A = \left\{ \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \right\} \hookrightarrow G$

For  $f \in C_c^\infty(G(A))$ , define

$$I(f) = \lim_{c \rightarrow \infty} \iint K_f(a, b) \eta(\det b) da db$$

$[A^c] [A^c]$

$G(A)$

$$\begin{array}{ccc} \gamma: \mathbb{B}^\times /_{F^\times} & \rightarrow & \{ \pm 1 \} \\ \parallel & & \uparrow \\ \gamma_k & \downarrow & \gamma_{11 \mp 1, a^b} \end{array}$$

$\mathcal{G}_{\mathbb{A}^1(F)}$

where  $A^c(A_F) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in A(A) \mid c^{-1} < |a| < c \right\}$

## Spectral Decomposition

$$K_f(x, y) = K_{f, \text{cusp}} + K_{f, \text{res}} + K_{f, \text{Eis}}$$

$$I(f) = I_{\text{cusp}}(f) + I_{\text{res}}(f) + I_{\text{Eis}}(f)$$

$$I_{\text{cusp}}(f) = \sum_{\pi \in \mathcal{A}_{\text{cusp}}(G)} I_{\pi}(f)$$

$$I_{\pi}(f) = \sum_{\psi \in \text{Basis}(\pi)} \underbrace{\int (R(f)(\psi))(a) da}_{[A]} \underbrace{\int \overline{\psi(L)} \gamma(b) db}_{[A]} L(\pi, \frac{1}{2}) L(\pi \otimes \gamma, \frac{1}{2})$$


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$B$  quart alg /  $F$

$$\mathcal{G}_B = PB^\times = \frac{B^\times}{F^\times}$$

$$K \hookrightarrow B$$

$$T \hookrightarrow G_B$$

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$$K^{\times}/F^{\times}$$

For  $f^B \in C_c^\infty(G_B(A))$  define

$$\underline{I}^B(f^B) = \iint_{[T][T]} K_{f^B}(t_1, t_2) dt_1 dt_2$$

$$\underline{I}_{\text{cusp}}^B(f^B) = \sum_{\pi^B \in \mathcal{A}_{\text{cusp}}(G_B)} I_{\pi^B}(f^B)$$

where  $I_{\pi^B}(f^B) =$

$$\sum_{\varphi \in \text{Basis}(\pi^B)} P_T(R(f^B)\varphi) \overline{P_T(\varphi)}$$

Goal:

$$I_{\text{cusp}}(f) = \sum_B I_{\text{cusp}}^B(f^B)$$

(fundamental lemma)

for "q matching"

$$f \mapsto f^B$$

$$I_{\pi}(f) = \sum_{B \in \mathcal{B}(\pi, k)} I_{\pi^B}(f^B)$$



$\dots R \dots$

$$\mathcal{I}_{\pi}(f) = \mathcal{I}_{\pi_B}(f^\sigma)$$

for  $B$  dictated by  
Sato-Tunnel.

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Geometric side of  $\mathrm{RTF}_S$  Start with  $\mathcal{I}^B, \bar{T}$

$$K_{f^B}(x, y) = \sum_{\delta \in G^B(F)} f^B(x^{-1}\delta y)$$

$$= \sum_{\delta \in G_B(F)/T(F)} \sum_{\gamma \in T(F) \setminus T(F)} f^B(x^{-1}\delta y)$$

$$\circlearrowleft \rightarrow T_\delta(F) \rightarrow T(F) \times T(F) \rightarrow T(F) \setminus T(F) \rightarrow$$

$$t_1, t_2 \mapsto t_1^{-1}\delta t_2$$

$$T_\delta = \{(t_1, t_2) \in T^2 \mid t_1^{-1}\delta t_2 = \delta\}$$

$$= \sum_{\delta \in G_B(F), (\tau_1, \tau_2) \in T(F)^2} f^B(x^{-1}\tau_1^{-1}\delta \tau_2 y)$$

$$\cup \in T(F) \quad T(F) \quad T_S(F)$$

$$S_0 \quad I^B(f^B) = \int \int \chi_{f^B}(t_1, t_2) dt_1 dt_2$$

$$= \sum_{\gamma \in \frac{G_B(F)}{T(F)}} \text{vol} \left( T_S(F) \backslash T_S(A) \right) \int f^B(t_1^{-1} \gamma t_2) dt_1 dt_2$$

$$T \times T \hookrightarrow G_B$$

$$(t, t') \mapsto t^{-1} g t'$$

$$I^B(f^B) = \sum_{\gamma \in \frac{G_B(F)}{T(F)}} \text{vol}(\ ) \quad \text{Orb}_S(f^B)$$

reg. \$\delta\$   \$\text{vol}(\ ) = 1\$

Similarly,

$$\underline{I}(f) = \sum_{\gamma \in \frac{G(F)}{A(F)}} \cancel{\text{vol}(\ )} \quad \underline{\text{Orb}}_S(f)$$

where  $\text{Orb}_G(f) = \iint_{A(A)^2} f(a^{-1} \gamma b) \gamma(b) da db$

for  $\gamma$  that are regular.

Consider

$$\begin{array}{ccc} A(F) \backslash G(F) / A(F) & \xrightarrow{\text{inv}} & \mathbb{P}^1(F) - \{1\} \\ \left( \begin{matrix} a & b \\ c & d \end{matrix} \right) & \longmapsto & [bc : ad] \end{array}$$

$$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$$

Def:  $\gamma \in A(F) \backslash G(F) / A(F)$  is regular

if  $\text{Inv}(\gamma) \neq 0, \infty$ .

$G_{rs} = \text{regular } \cancel{\text{semi-infinite}} \text{ elements of } G$ .

$$\begin{array}{ccc} \text{Lemma} & A(F) \backslash G_{rs}(F) / A(F) & \xrightarrow{\text{bij}} \mathbb{P}^1(F) - \{0, 1, \infty\} \\ & \left[ \begin{matrix} u \\ 1 \end{matrix} \right] & \longleftarrow \quad \left[ \begin{matrix} u & 1 \end{matrix} \right] \end{array}$$

... for orbits:

There are 6 non-regular

$$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \\ 1 & \end{bmatrix}$$

$$\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & \\ & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & \\ -1 & \end{bmatrix}$$

$K \hookrightarrow B$ . Can write

$$B = K + K_j$$

with  $j \in B^\times$  such that

$$1) \quad \lambda j = j \bar{\lambda} \quad \forall \lambda \in K$$

$$2) \quad j^2 \in F.$$

$$K \longrightarrow K \hookrightarrow B$$

$$\lambda \longmapsto \bar{\lambda} \rightsquigarrow \bar{\lambda}$$

$$\Rightarrow \bar{\lambda} = j^{-1} \lambda j \quad \text{for some } j.$$

If  $b \in B = K + K_j$

$$b = b_+ + b_-$$

then  $Nm(b) = Nm(b_+) + Nm(b_-)$

$$B^\times \xrightarrow{Nm} F^\times$$

$$b \longmapsto b\bar{b}$$

$$\bar{b} = \text{Tr } b - b$$

$$\begin{array}{ccc} G_B(F) / & \xrightarrow{\text{inv}_B} & P'(F) - \{1\} \\ T(F) \backslash & & \\ \parallel & & \\ K^\times \backslash & \xrightarrow{B^\times} & F^\times / \\ & & \parallel \\ & & j^2 Nm(\alpha) \end{array}$$

$$b = b_+ + b_- \mapsto \begin{cases} Nm(b_-) : -Nm(b_+) \\ \parallel \\ Nm(\alpha j) = -j^2 Nm(\alpha) \end{cases}$$

$$b_\alpha = b_+\alpha + \underbrace{b_-\alpha}_{\parallel}$$

$$\alpha' j \alpha = \alpha' \bar{j} \bar{\alpha}$$

$b \in G_B(F)$  is regular if  $\text{inv}_B(b) \neq 0, \infty$

Prop: There is a bijection:

$$\begin{array}{ccc} A(F) \backslash G_{rs}(F) / & \hookleftarrow & \coprod_B G_{B,rs}(F) / \\ A(F) & & T(F) \end{array}$$

$\exists k \in \mathbb{R}$



$$R^1(F) - \{0, 1, \infty\}$$

Rmk:

$$\left\{ B : \exists \quad k \hookrightarrow B \right\} \longleftrightarrow \frac{F^\times}{Nm(k^\times)}$$

$\sim$

!!

$$B(k)$$

The non-regular elements of  $\frac{G_B(F)}{T(F)}$

are 1, j.

## Transfer

Thm (Global transfer) let  $f \in C_c^\infty(G(A))$ ,

For each  $B \in \mathcal{B}(k)$ , there exists

$f^B \in C_c^\infty(G_B(A))$  such that

for all  $\delta \in G_{B,rs}(F)$ , if

$inv_B(\delta) = inv(\gamma)$ , for  $\gamma \in G(F)$

$$\text{then } \mathcal{O}_S(f^B) = \mathcal{O}_S(f).$$

Rank: suffices to prove this for

$$f = \bigotimes_v f_v \quad \text{with} \quad f_v \in C_c^\infty(G(F_v))$$

$$\text{and with } f_v = \prod_{\mathcal{O}_v} G(\mathcal{O}_v) \quad \text{for all}$$

$$\text{but finitely many } v. \quad \mathcal{O}_v \subseteq F_v$$

$$\mathcal{O}_S(f) = \iint_{A(A) A(A)} f(a^{-1} \gamma b) \underline{\gamma(b)} da db$$

$$= \prod_v \int_{A(F_v)} \int_{A(F_v)} f_v(a^{-1} \gamma_v b) \underline{\gamma_v(b)} da db$$

$$= \prod_v \mathcal{O}_S(f_v)$$

Reduce to  $\Rightarrow$  2 local theorems:

$\pi$  (local transfer) Let  $K/F$

Thm      1. Local fields  
be a quadratic extension of local fields.

let  $B_1 = \text{Mat}_2(F)$

$B_2 = D = \text{unique division quat. alg}/F$ .

For  $f \in C_c^\infty(G(F))$ , ~~then~~  $\int \frac{f^{B_1}}{\gamma} \frac{f^{B_2}}{\gamma} \in C_c^\infty(D^\times)$

such that for all  $\delta \in G^B(F)$   
 $(B = B_1 \text{ or } B_2)$

if  $\text{inv}_B(\delta) = \text{inv}(\gamma)$ ,  $\gamma \in G(F)$

then  $\mathcal{O}_\delta(f^B) = \mathcal{O}_\gamma(f)$

Thm (Fundamental Lemma)

Let  $K/F$  be an unramified quadratic extension of non-arch local fields.  $\mathcal{O} \subseteq F$  ring of integers

Then for  $f = \mathbb{1}_{G(\mathcal{O})} \in C_c^\infty(G(F))$

We can take in the local transfer,

$$C_{B_1} - 11$$

$$x = \underbrace{\mathbb{1}_{G(O)}}_{\sim}$$

$$f^B_2 = 0$$

Rank: a) We choose measures so that

$$A(O_v) \text{ and } T(O_v)$$

have volume 1.

Let's check that  $O_\gamma(f_v) = 1$  for a.e.  $v$ .

We may assume

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{rs}(F)$$

$$a, b, c, d \in O_v^\times \quad \text{and} \quad \det \gamma \in O_v^\times$$

Then for a.e.  $v$ ,

$$O_\gamma(f_v) = O_\gamma(\mathbb{1}_{\underline{G(O_v)}})$$

$$= \iint_{\frac{1}{\underline{GL}_2(O_v)} \setminus (\bar{F}_v^\times \times \bar{F}_v^{\times 2})} (t^{-1} \gamma t') \gamma(t')$$

$$\text{where } t = \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \quad t' = \begin{pmatrix} t'_1 & \\ & t'_2 \end{pmatrix}$$

$$F_v^\times \xrightarrow{\Delta} F_v^{\times 2} \times F_v^{\times 2}$$

$$a \mapsto (a, a, a, a)$$

$$\left( \begin{array}{c} \frac{1}{\pi^{a_1}} \\ & \ddots \\ & & \frac{1}{\pi^{a_2}} \end{array} \right) \left( \begin{array}{c} \frac{1}{\pi^{b_1}} \\ & \ddots \\ & & \frac{1}{\pi^{b_2}} \end{array} \right)$$

where  $v(\pi) = 1$ .

$$= \sum_{\substack{a_1, a_2, b_1, b_2 \in \mathbb{Z} \\ \text{modulo } (a_1, a_2, a_1, a_2)}} 1$$

$$\frac{1}{GL_2(\mathcal{O}_v)} \begin{pmatrix} \pi^{b_1 - a_1} & \pi^{b_2 - a_1 + b} \\ \pi^{b_1 - a_2} & \pi^{b_2 - a_2 + b} \end{pmatrix} (-1)^{b_1 - b_2}$$

$$\eta_v : \mathbb{F}_v^\times \rightarrow \{ \pm 1 \}^{v(\alpha)}$$

$$\alpha \mapsto (-1)^{v(\alpha)}$$

$$\text{Need } v(\det(t^{-1} \alpha t)) = 0$$

$$\Rightarrow v(\det t) = v(\det t')$$

$$\Rightarrow a_1 + a_2 = b_1 + b_2$$

Also need

$$\begin{aligned} b_1 &\geq a_1 \\ b_2 &\geq a_2 \\ b_1 &\geq a_2 \\ b_2 &\geq a_1 \end{aligned}$$

$$\Rightarrow a_1 = a_2 \\ \quad \quad \quad b_1 = b_2$$

$$\sum = 1$$

Proof FL : Note:  $\eta(x) = (-1)^{v(x)}$

since  $K/F$  unramified

$$\text{Recall: } \left( \begin{smallmatrix} \cdot & x \\ 1 & \cdot \end{smallmatrix} \right) \longleftrightarrow [x : 1]$$

$$\tilde{A}(F) \backslash GL_2^{\text{rs}}(F) / \tilde{A}(F) \longleftrightarrow P^1(F) - \{0, 1, \infty\}$$

$$\longleftrightarrow \coprod_{B \text{ quat}} K^\times \backslash B_{\text{rs}}^\times / K^\times$$

$K \hookrightarrow B$

$$Nm(K^\times) \cup \varepsilon Nm(K^\times) \longleftrightarrow \underline{K^\times \backslash GL_2^{\text{rs}}(F) / K^\times} \cup \underline{K^\times \backslash D_5^\times / K^\times}$$

for  $\varepsilon \in F^\times$ , not a norm

$$x \xrightarrow{\quad} v(x) \pmod{2}$$

$$\frac{F^\times}{Nm(K^\times)} \cong \mathbb{Z}/2\mathbb{Z}$$



$$\mathcal{B}(K)$$

So want to show that if  $\gamma = \left( \begin{smallmatrix} \cdot & x \\ 1 & \cdot \end{smallmatrix} \right)$  then

$$\mathcal{O}_\gamma \left( \mathbb{I}_{G(O)} \right) = \begin{cases} \mathcal{O}_\gamma \left( \mathbb{I}_{G(O)} \right) & \text{if } v(x) \text{ eve} \\ 0 & \text{if } v(x) \text{ odd} \end{cases}$$

$\text{inv}(\gamma) = \text{inv}_{B_i}(S)$

Let's compute  $\cup_{\sigma} (\mathbb{H} G(\mathbb{Q}))$

$$\gamma = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

$$= \sum_{a_1, a_2, b_1} \prod_{GL_2(\mathbb{O})} \begin{pmatrix} \pi^{b_1 - a_1} & \pi^{-a_1} x \\ \pi^{b_1 - a_2} & \pi^{-a_2} \end{pmatrix} (-1)^{b_1}$$

Need  $v(\det(\gamma)) = 0$

so  $a_1 + a_2 = b_1 + s$

$$b_1 = a_1 + a_2 - s$$

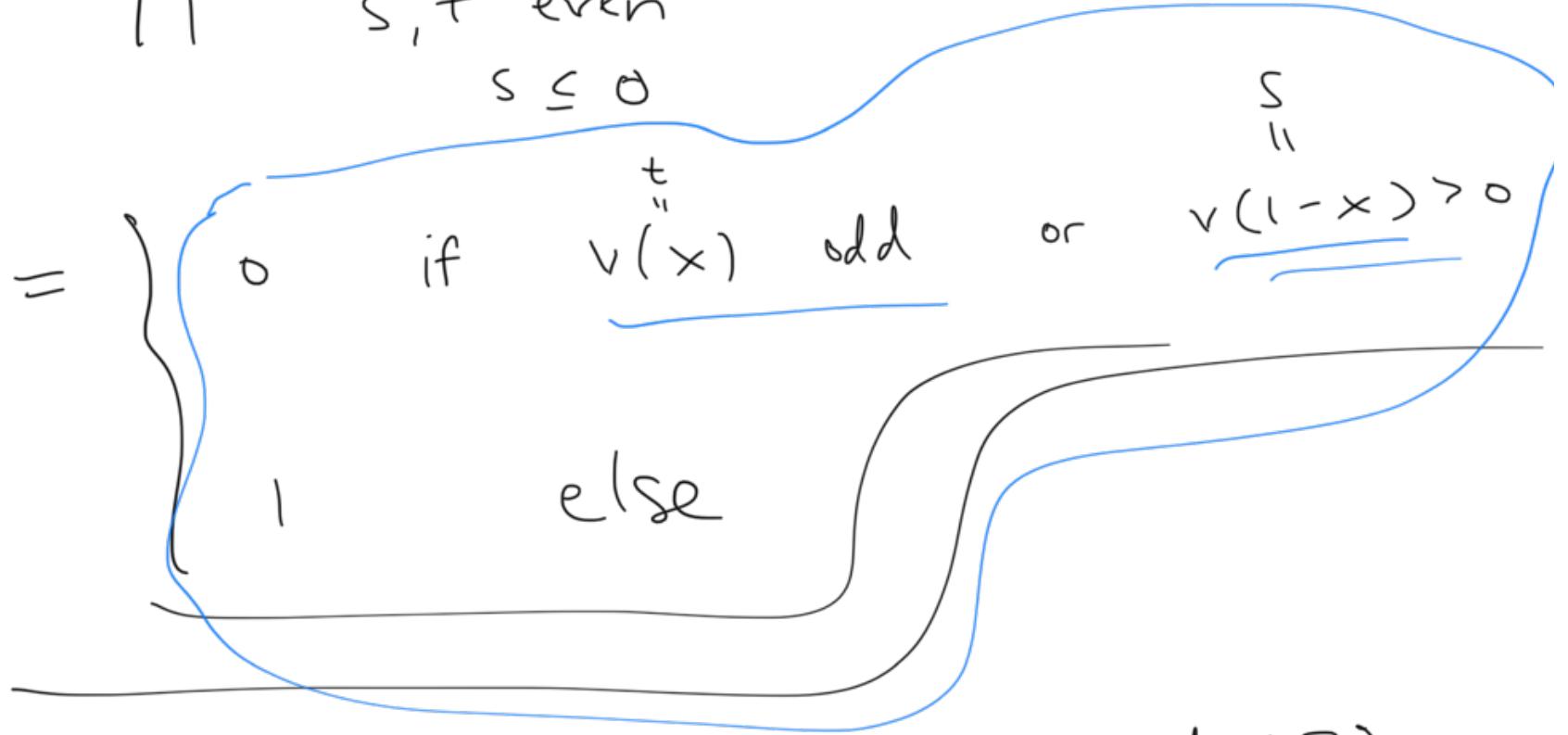
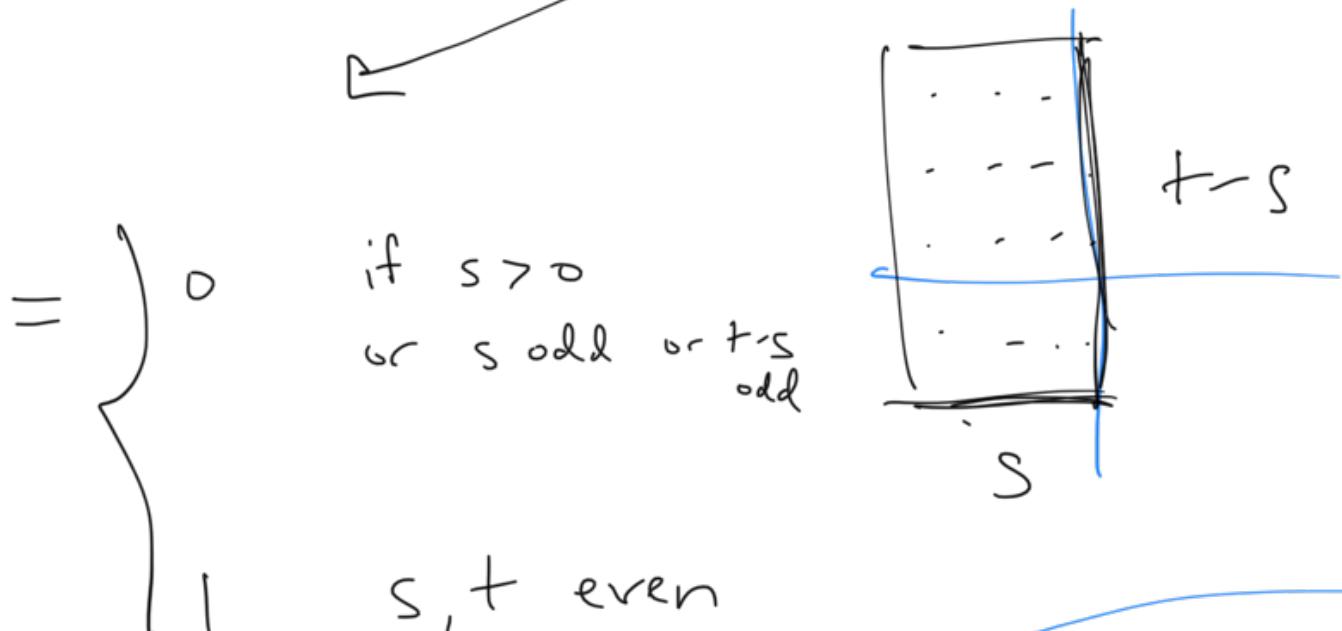
write  
 $s := v(1-x)$   
 $= v(\det \gamma)$   
 $t := v(x)$

$$\sum_{a_1, a_2} \prod_{GL_2(\mathbb{O})} \begin{pmatrix} \pi^{a_2 - s} & \pi^{t - a_1} \\ \pi^{a_1 - s} & \pi^{-a_2} \end{pmatrix} (-1)^{a_1 + a_2 - s}$$

Need  $a_2 \geq s$ ,  $t \geq a_1$   
 $a_1 \geq s$ ,  $a_2 \leq 0$

$$= \sum_{\substack{s \leq a_2 \leq 0 \\ s \leq a_1 \leq t}} (-1)^{a_1 + a_2 - s} = (-1)^s \sum_{\substack{s \leq a_2 \leq 0 \\ s \leq a_1 \leq t}} (-1)^{a_1 + a_2}$$

$$= 0 \quad \text{if } s > 0 \\ \text{hence } + = 0$$



$$\text{let } \gamma = \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} \leftrightarrow \delta \in \mathcal{B}_1 = \text{Mat}_2(\mathbb{F})$$

$$= K + Kj$$

$$= \underline{1} + \underline{j}\underline{\alpha} \\ \underline{\alpha} \in K^\times$$

$$\text{w/ } \underline{v(x)} \text{ even}$$

$$K \hookrightarrow \text{Mat}_2(\mathbb{F})$$

$$K = \mathbb{F}(\sqrt{d})$$

$$\frac{d}{11} \mapsto$$

$$\begin{pmatrix} a & b \\ bd & a \end{pmatrix}$$

$$a + b\sqrt{d}$$

$$d \in \mathcal{O}_F^\times$$

$$j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\alpha_j = j \bar{\alpha}.$$

$$\bar{z} = a - b\sqrt{d}.$$

$$S = \boxed{1 + j\bar{\alpha}} = \begin{pmatrix} 1+a & b \\ -bd & 1-a \end{pmatrix}$$

Since  $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \leftrightarrow S$  we have  $\det z = 1-x$

$$\boxed{x = \det \alpha = a^2 - b^2 d}$$

$$O_S(\mathbb{1}_{G(0)}) = \int_{T^2(F)} \mathbb{1}_{G(0)} (t^{-1} \sigma t) dt dt'$$

$$= \int_{K^\times} \int_{F^\times} \mathbb{1}_{GL_2(O)} (t^{-1} \sigma t) dt dt'$$

Note:  
 $K^\times \subseteq \mathcal{Z} \cdot GL_2(O)$

$$= \int_{GL_2(O)} (z \sigma) dz$$

$$Z(F) = F^\times$$

$$= \sum_{k \in \mathcal{U}} \mathbb{1}_{GL_2(O)} \left( \frac{\pi^k}{\underline{\underline{\sigma}}} \right)$$

$$= \prod_{GL_2(O)} \left( \pi^{-\frac{1}{2}(\nu(1-x))} S \right)$$

Since  $\det S = 1-x$

$$= \begin{cases} \prod_{GL_2(O)} \left( \pi^{-\frac{1}{2}\nu(1-x)} S \right) & \text{if } \nu(1-x) \text{ even} \\ 0 & \text{if } \nu(1-x) \text{ odd} \end{cases}$$

Case 1 :  $\nu(x) < 0$

$$-2S = \nu(1-x) = \underbrace{\nu(x)}_{\text{even}} \quad \boxed{x = \underline{a^2} - \underline{b^2} d}$$

$$\prod_{GL_2(O)} \begin{pmatrix} \pi^s(1+a) & \pi^s b \\ -\pi^s bd & \pi^s(1-a) \end{pmatrix}$$

$\nu(a) \geq -S$   
 $\nu(b) \geq -S$

 $\Rightarrow 1$

Case 2 :  $\nu(x) \geq 0 \Rightarrow$  all entries  
 $\nu(1-x) = 0$  are integral

$\Rightarrow$  get 1.

Case 3:  $v(1-x) > 0 \Rightarrow$  get 0.  
 $v(x) = 0$

Comparing both computations

$\Rightarrow$  this proves the FL.

Rmk's:

a) Can extend the fundamental lemma to include not just matching for  $\prod_{v \in S} G(O_v)$  but for any element of

$$\mathcal{H}\left(\left(PGL_2(F_v), PGL_2(O_v)\right)\right)$$

$$\therefore = C_c^\infty\left(G(O_v) \backslash G(F_v) / G(O_v)\right)$$

b) See Jacquet for proof of local transfer

## Unramified representations

$v$  place of  $F$ .  $G = \mathrm{PGL}_2$

$\pi_v$  irred (smooth) rep of  $G(F_v)$ .

$\pi_v$  unramified  $\Leftrightarrow \pi_v^{G(O_v)} \neq 0$ .

If so,  $\dim \pi_v^{G(O_v)} = 1$ .

$\mathcal{H}_v = \mathcal{H}(G(F_v), G(O_v))$ , algebra  
under

$$(f_1 * f_2)(h) = \int_{G(F_v)} f_1(hg^{-1}) f_2(g) dg$$

$\mathcal{H}_v \hookrightarrow \pi_v$  via:

$$f \cdot \omega = \int_{G(F_v)} f(g) \pi_v(g) \omega dg$$

$\mathcal{H}_v \hookrightarrow \pi_v^{G(O_v)}$

$$\mathcal{H}_v \longrightarrow \mathbb{C}$$

$$f \longmapsto \lambda_{\pi_v}(f)$$

$$\left\{ \begin{array}{l} \text{irred unramified} \\ \text{reps of } G(F_v) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{characters} \\ \mathcal{H}_v \xrightarrow{\lambda} \mathbb{C} \end{array} \right\}$$