Ceresa cycles

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Oberwolfach: Explicit methods in number theory

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Algebraic cycles and equivalence

X smooth, projective algebraic variety over a field k.

$Z^i(X)$ = free abelian group of algebraic cycles of codim i on X.

Have various equivalence relations

$$Z^i(X)_{\mathrm{rat}} \subset Z^i(X)_{\mathrm{alg}} \subset Z^i(X)_{\mathrm{hom}} \subset Z^i(X).$$

Taking quotients modulo rational equivalence gives:

$$0 \subset \mathrm{CH}^i(X)_{\mathrm{alg}} \subset \mathrm{CH}^i(X)_{\mathrm{hom}} \subset \mathrm{CH}^i(X)$$

Chow groups: how big are they (conjecturally)?

$$0 \subset \mathrm{CH}^i(X)_{\mathrm{alg}} \subset \mathrm{CH}^i(X)_{\mathrm{hom}} \subset \mathrm{CH}^i(X)$$

- Mumford: $\mathrm{CH}^i(X)_{\mathrm{alg}}$ can be big, hard to understand, when $i \geq 2$.
- When $i = 1$, we have $\mathrm{CH}^1(X)_{\mathrm{alg}} = \mathrm{CH}^1(X)_{\mathrm{hom}}$
- Griffiths: the group

$$\mathrm{Gr}^i(X) := \mathrm{CH}^i(X)_{\mathrm{hom}} / \mathrm{CH}^i(X)_{\mathrm{alg}}$$

need not be 0 for $i \geq 2$!

“Simplest” examples of non-zero Griffiths groups

- (Griffiths) Difference of two lines $[L] - [L']$ on a quintic threefold $X \subset \mathbb{P}^4$.
- For $J = \mathrm{Jac}(C)$ genus $g \geq 2$ Jacobian, define

$$\kappa_{\mathrm{Gr}}(C) = [\iota(C)] - [-\iota(C)] \in \mathrm{Gr}^{g-1}(J)$$

where we have used some $e \in \mathrm{Pic}^1(C)$ to embed $C \overset{\iota}{\hookrightarrow} J$.

Theorem (Ceresa ’83)

For very general C over $\mathbb{C}$ of genus $g \geq 3$, the class $\kappa_{\mathrm{Gr}}(C)$ has infinite order.

Main tool: Abel-Jacobi map

$$\mathrm{CH}^i(X)_{\mathrm{hom}} \xrightarrow{\mathrm{AJ}} J^i(X) := \frac{\mathrm{H}^{2i-1}(X, \mathbb{C})}{\mathrm{Fil}^i + \mathrm{H}^{2i-1}(X, \mathbb{Z})}$$

Clemens' question

"Most" curves have infinite order Ceresa cycle, but which ones exactly?

Example: If C is hyperelliptic then $\kappa_{\mathrm{Gr}}(C) = 0$.

Question (Clemens '87)

$\kappa_{\mathrm{Gr}}(C) = 0$ *if and only if* C *is hyperelliptic?*

I suspect this is not true....but maybe in the Chow group?

The Ceresa class vanishes only if there is a good reason.

Vanishing in the Chow group: choosing the base point

Let $\kappa_e(C)$ be the class of the Ceresa cycle in $\mathrm{CH}^{g-1}(J)$. Depends on e now!

Let $K_C \in \mathrm{Pic}(C)$ be the canonical class.

Lemma

If $\kappa_e(C)$ is torsion then $(2g-2)e \equiv K_C$ in $\mathrm{Pic}(C) \otimes \mathbb{Q}$.

Upshot: We should and do assume $(2g-2)e \equiv K_C$.

We then get a canonical Ceresa class in $\kappa(C) \in \mathrm{CH}_1(J) \otimes \mathbb{Q}$.

Note: $\kappa_e(C)$ is torsion if and only if $\kappa(C) = 0$.

Theorem (Gao-Zhang, '24)

If $g \geq 3$, there is an open dense set $U_g \subset \mathcal{M}_g$ such that the Beilinson-Bloch height $\langle \kappa(C), \mathcal{F}\kappa(C) \rangle$ satisfies a Northcott property on $U_g(\bar{\mathbb{Q}})$.

Possible counterexamples to Clemens' question?

Recent examples with $\mathrm{AJ}(\kappa(C)) = 0$. Let $G = \mathrm{Aut}(C)$.

- (Bisogno-Li-Litt-Srinivasan) Fricke-Macbeath curve: $g = 7$ and $G \simeq \mathrm{PSL}_2(\mathbb{F}_8)$.
- (Beauville) The genus 3 curve $C : y^3 = x^4 + x$ with $G \simeq C_9$.

Proof.

AJ is G-equivariant and $\kappa(C)$ is canonical and hence G-invariant. One computes that the target $J^{g-1}(J)^G \otimes \mathbb{Q} = 0$, hence $\mathrm{AJ}(\kappa(C)) = 0$. □

Beilinson-Bloch conjecture implies that $\kappa(C) = 0$. Can we prove it unconditionally?

Chow-vanishing criterion

Let $\mathrm{H}^*(X) = \mathrm{H}^*(X(\mathbb{C}), \mathbb{Q})$. So $\mathrm{H}^3(J) = \bigwedge^3 \mathrm{H}^1(C)$.

Theorem (Laga-S)

If $\mathrm{H}^3(J)^G = 0$ *then* $\kappa(C) = 0$.

- Strengthens a result of Qiu-Zhang.

Examples:

- $y^3 = x^4 + 1$ (Qiu-Zhang, $g = 3$, $G = \mathrm{SmallGroup}(48, 33)$)
- $y^3 = x^4 + x$ (Beauville-Schoen, $g = 3$, $G = C_9$)
- $y^3 = x^5 + 1$ (Lilienfeldt-S, $g = 4$, $G = C_{15}$)
- 1-parameter family $C_t : y^3 = (x^3 + t)^2(tx^3 - 1)$ (Qiu-Zhang, $g = 4$, $G = S_3^2$)
- Humbert-Edge curves (Laterveer, $g = 5$, $G = C_2^4$)
- $\kappa(C) = 0$ and $C \to D \implies \kappa(D) = 0$. E.g. Fricke-Macbeath quotient of $g = 3$.

Griffiths-vanishing criterion

Take now $k = \mathbb{C}$. Set $V = \mathrm{H}^0(C, \Omega_C)$.

Theorem (Laga-S)

Assume Hodge conjecture for abelian varieties. If $(\bigwedge^3 V)^G = 0$ *then* $\kappa_{\mathrm{Gr}}(C)$ *is torsion.*

- The hypotheses imply $\mathrm{H}^3(J)^G \simeq \mathrm{H}^1(A)(-1)$. Need Hodge for $J \times A$.

Examples

- Picard curves $y^3 = x^4 + ax^2 + bx + c$
  - $g = 3$, $G = C_3$. HC proved by Schoen.
  - "The" $\mathrm{U}(2,1)$ Picard modular surface
- $y^5 = x(x-1)^2(x-t)^3$
  - $g = 4$, $G = D_5$. HC known.
- Three dimensional family $y^3 = x(x-1)(x^3 + ax^2 + bx + c)^2$ in genus 4.
  - Torelli locus in $\mathrm{U}(2,2)$-Shimura variety. HC not known?
- At least a few more families, e.g. $y^9 = x(x-1)^3(x-t)^6$ in genus 6. HC?
  - This covers a genus 3 family with $\mathrm{Aut}(C) = 0$!

Proof sketches

Exploit Chow-Kunneth and Beauville decompositions:

$$h(J) = \bigoplus_{i=0}^{2g} h^i(J) \quad \text{and} \quad \mathrm{CH}^{g-1}(J) = \bigoplus_{s=0}^{g-1} \mathrm{CH}^{g-1}_{(s)}(J).$$

Proof of Chow vanishing.

$\kappa(C) = 0$ if and only if $[C]_1 = 0$ and $[C]_1 \in \mathrm{CH}(h^{2g-3}(J)^G) \simeq \mathrm{CH}(h^3(J)^G)$.
But $h^3(J)^G$ has trivial cohomology. By Kimura[a], $h^3(J)^G = 0$, hence $\kappa(C) = 0$. □

[a] S.-I. Kimura. Chow groups are finite dimensional, in some sense. Math. Ann., 2005.

Proof of Griffiths vanishing.

$h^3(J)^G$ need not be 0, but its Hodge structure is weight 1. So Hodge conjecture implies $h^3(J)^G \simeq h^1(A)(-1)$ and hence $\kappa(C)$ is algebraically trivial. □

Can $\kappa(C)$ vanish for non-group theoretic reasons? Yes!

For $f(x) = x^4 + ax^2 + bx + c$, let C_f be the genus 3 curve $y^3 = f(x)$.

Recall the I- and J-invariants of f:

- $I(f) = a^2 + 12c$
- $J(f) = 72ac - 2a^3 - 27b^2$

Classical observation: $J(f)^2 = 4I(f)^3 - 27\mathrm{Disc}(f)$.

In other words, $P_f := (I(f), J(f))$ lies on the elliptic curve $E_f\colon y^2 = 4x^3 - 27\mathrm{Disc}(f)$.

Theorem (Laga-S)

$\kappa(C_f) = 0$ *if and only if* $P_f \in E_f(\mathbb{C})$ *is torsion.*

In fact, the ratio of the orders of $\kappa_\infty(C_f)$ and P_f is uniformly bounded.

Corollaries

From this we conclude:

- Countably many rational curves in $\mathcal{M}_3(\bar{\mathbb{Q}})$ on which $\kappa_e(C)$ is identically torsion.
  - Here is one over $\mathbb{Q}$: $C_t : y^3 = x^4 - 12x^2 + tx - 12$.
  - Conjecture: for all t, the class $\kappa_\infty(C)$ has order 3.
- The Picard locus lies in the complement of $U_3 \subset \mathcal{M}_3$.
- The torsion order of $\kappa_\infty(C)$ in $S(\bar{\mathbb{Q}})$ is unbounded.
- Over number fields of fixed degree, the torsion order is bounded.

Proof sketch

Have a family $\mathcal{C} \to S$ of curves with fiber-wise algebraically trivial Ceresa cycle.

The holomorphic normal function $S \xrightarrow{\sigma} \mathcal{J}^{g-1}(\mathcal{J})$ to the family of intermediate Jacobians should factor through an algebraic section of an abelian scheme $\mathcal{A} \to S$.

Identify the abelian scheme (which is E_f in our case) and compute $\mathcal{A}(S)$.

In our case, $\mathcal{A}(S)$ is free of rank is 1 via Shioda-Tate.

Now show $\sigma \neq 0$ by specializing and verifying on a single example!

Thank you!


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