MANIN-DRINFELD CYCLES AND DERIVATIVES OF L-FUNCTIONS

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Abstract. We introduce a Manin-Drinfeld cycle in the moduli space of PGL_2-shtukas with r legs, arising from the diagonal torus. We show that its intersection pairing with Yun and Zhang's Heegner-Drinfeld cycle is equal to the product of the r-th central derivative of an automorphic L-function $L(\pi, s)$ and Waldspurger's toric period integral. When $L(\pi, \frac{1}{2}) \neq 0$, this gives a new geometric interpretation for the Taylor series expansion. When $L(\pi, \frac{1}{2}) = 0$, the pairing vanishes, suggesting "higher order" analogues of the vanishing of cusps in the modular Jacobian.

Our proof sheds new light on the algebraic correspondence introduced by Yun-Zhang, which is the geometric incarnation of "differentiating the L-function". We realize it as the Lie algebra action of $e + f \in \mathfrak{sl}_2$ on $(\mathbb{Q}_\ell^2)^{\otimes 2d}$. The comparison of relative trace formulas needed to prove our formula is then a consequence of Schur-Weyl duality.

1. Introduction

Let K/F be a quadratic extension of global function fields, corresponding to a double cover $\nu \colon Y \to X$ of smooth, projective, geometrically connected curves over $\boldsymbol{k} = \mathbb{F}_q$. We consider cuspidal automorphic representations π on $G = \mathrm{PGL}_{2,F}$. Let T be the torus $K^\times/F^\times$ over F. For simplicity, we assume both π and K/F are everywhere unramified.

1.1. **Summary.** In [8], Yun and Zhang relate the rth central derivative of the base change L-function $\mathscr{L}(\pi_K, s)$ to the self-intersection of a certain algebraic cycle. Let $\mathrm{Sht}_G^r \to X^r$ be the $2r$-dimensional moduli stack over $\boldsymbol{k}$ parameterizing G-shtukas with r legs. Let $\mathrm{Sht}_G^{r\prime} \to Y^r$ denote its base change to Y^r. Yun and Zhang define Heegner-Drinfeld cycles $[\mathrm{Sht}_T^{r\prime}]_\pi \in \mathrm{Ch}_c^r(\mathrm{Sht}_G^{r\prime})$, which are higher analogues of CM divisors on (Drinfeld) modular curves. Their special value formula takes the shape (equality up to explicit non-zero constants):

$$\mathscr{L}^{(r)}(\pi_K, 1/2) \doteq \left\langle [\mathrm{Sht}_T^{r\prime}]_\pi, [\mathrm{Sht}_T^{r\prime}]_\pi \right\rangle_{\mathrm{Sht}_G^{r\prime}}. \tag{1.1}$$

It encompasses the function field versions of Waldspurger's formula [7], when $r = 0$, and the Gross-Zagier formula, when $r = 1$.

In [9, Cor. 1.4], they give a self-intersection formula for the Heegner-Drinfeld cycle $[\mathrm{Sht}_T^r]_\pi \in \mathrm{Ch}_c^r(\mathrm{Sht}_G^r)$ obtained by push-forward to X^r. It reads:

$$\mathscr{L}^{(r)}(\pi, 1/2)\mathscr{L}(\pi \otimes \eta, 1/2) \doteq \langle [\mathrm{Sht}_T^r]_\pi, [\mathrm{Sht}_T^r]_\pi \rangle_{\mathrm{Sht}_G^r}. \tag{1.2}$$

Here, η is the quadratic character associated to K/F.

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In this paper, we define *Manin-Drinfeld cycles* $[\mathrm{Sht}^r_A]_\pi \in \mathrm{Ch}^r_c(\mathrm{Sht}^r_G)$ coming from A-shtukas, where $A \hookrightarrow G$ is the diagonal torus. Whereas the Heegner-Drinfeld cycles play the role of CM divisors on the modular curve, the Manin-Drinfeld cycles play the role of cuspidal divisors. Our main result (Theorem 1) has the following shape:

$$\mathscr{L}^{(r)}(\pi, 1/2) \int_{[T]} \phi(t)dt \doteq \langle [\mathrm{Sht}^r_A]_\pi, [\mathrm{Sht}^r_T]_\pi \rangle_{\mathrm{Sht}^r_G}\,, \tag{1.3}$$

for an appropriate spherical vector $\phi \in \pi$. When $r = 0$, this formula amounts to the statement that $\mathscr{L}(\pi, s)$ can be written as a Mellin transform. When $r = 1$, it is closely related to the function field version of the Manin-Drinfeld theorem [2, 6], that the cusps on the modular Jacobian are torsion.

For $r \geq 2$, our formula has no (known) analogue over number fields. If $\mathscr{L}(\pi_K, 1/2) \neq 0$, it shows that the Manin-Drinfeld cycles are non-vanishing and our formula gives a new expression for the Taylor series expansion of $\mathscr{L}(\pi, s)$. If $\mathscr{L}(\pi_K, 1/2) = 0$, then the Manin-Drinfeld cycle pairs trivially against the Heegner-Drinfeld cycle, raising the question of whether the cycles themselves vanish. See §1.2 for further speculation.

Yun and Zhang's proof of (1.1) proceeds by *geometrizing* Jacquet's relative trace formula (RTF) comparison approach to Waldspurger's formula [4]. One side of this comparison involves computing traces of Frobenius on $\beta_*\mathbb{Q}_\ell$, where $\beta\colon M_d \to A_d$ is a version of the Hitchin fibration (one for each integer $d \geq 0$). One of the key insights in [8] is that one can extend this approach to the case $r > 0$, using a certain natural correspondence

$$\begin{tikzcd} & \mathrm{YZ}_d \arrow[dl] \arrow[dr] & \\ M_d \arrow[dr, "\beta"'] & & M_d, \arrow[dl, "\beta"] \\ & A_d & \end{tikzcd}$$

which we call the *Yun-Zhang* correspondence. The induced operator

$$[\mathrm{YZ}_d]\colon \beta_*\mathbb{Q}_\ell \to \beta_*\mathbb{Q}_\ell$$

plays the role of "differentiation" on the geometric side of the RTF comparison. When $r > 0$, Yun and Zhang show that the traces of the operator $[\mathrm{YZ}_d]^r \circ \mathrm{Frob}$ match up with the rth derivative of certain traces on the L-function side.

Our proof of (1.3) makes use of the RTF approach of [8] and its extension developed in our work with Howard [3]. In fact, one can view (1.3) as a degenerate version of our formula in [3] for the intersection of two different Heegner-Drinfeld cycles. In the degenerate case considered here (where one torus is split), the RTF comparison is a tautology when $r = 0$. But in order to prove (1.3) for $r > 0$, we are forced to dig deeper into the representation theory of the Hitchin fibration. The key insight is a representation-theoretic interpretation of the operator $[\mathrm{YZ}_d]$. In our setting, the local system $\beta_*\mathbb{Q}_\ell$ comes from the representation

$(\mathbb{Q}_\ell^2)^{\otimes 2d}$ of the symmetric group S_{2d}. This is also a representation of $\mathfrak{sl}_2$, and the Yun-Zhang operator is given by the action of the element $e+f = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2$. Combined with Schur-Weyl duality, this allows us to compare the two RTF's in §4 and prove (1.3).

1.2. **Precise statement of results.** Let $Y_0 = X \coprod X \to X$ be the split double cover. The F-algebra of rational functions on Y_0 is $K_0 = F \oplus F$.

There are natural closed immersions

$$\begin{array}{ccc} \widetilde{T} = \underline{\mathrm{Aut}}_{\nu_*\mathcal{O}_Y}(\nu_*\mathcal{O}_Y) & \qquad & \widetilde{A} = \underline{\mathrm{Aut}}_{\nu_{0*}\mathcal{O}_{Y_0}}(\nu_{0*}\mathcal{O}_{Y_0}) \\ \big\downarrow & & \big\downarrow \\ \widetilde{G_1} = \underline{\mathrm{Aut}}_{\mathcal{O}_X}(\nu_*\mathcal{O}_Y) & & \widetilde{G} = \underline{\mathrm{Aut}}_{\mathcal{O}_X}(\nu_{0*}\mathcal{O}_{Y_0}) \end{array}$$

of group schemes over X. Let $T \subset G_1$ and $A \subset G$ be the quotients by the central $\mathbb{G}_m$. Then G_1 is Zariski-locally isomorphic to PGL_2 and $G = \mathrm{PGL}_2$ over X. The group scheme T is a non-split torus, while A is the split diagonal torus in PGL_2. On F-points, we have $T(F) = K^\times/F^\times$ and $A(F) \simeq F^\times$.

Let $\mathbb{A}$ be the adele ring of F, and $\mathbb{O}$ the subring of integral elements. Define $U = G(\mathbb{O})$ and $U_1 = G_1(\mathbb{O})$. The pair $U_1 \subset G_1(\mathbb{A})$ is non-canonically isomorphic to $U \subset G(\mathbb{A})$. There is, however, a canonical isomorphism of spaces of unramified cuspidal automorphic forms

$$\mathcal{A}_{\mathrm{cusp}}(G_1)^{U_1} \cong \mathcal{A}_{\mathrm{cusp}}(G)^U.$$

These are finite dimensional $\mathbb{C}$-vector spaces, and the space on the right carries a natural action of the Hecke algebra $\mathscr{H}$ of $\mathbb{Q}$-valued compactly supported U-bi-invariant functions on $G(\mathbb{A})$.

We adopt the usual notation

$$[T] = T(F)\backslash T(\mathbb{A}) \quad \text{and} \quad [A] = A(F)\backslash A(\mathbb{A}).$$

For any $\phi \in \mathcal{A}_{\mathrm{cusp}}(G)^U$, consider the toric period integrals

$$\mathscr{P}_A(\phi, s) = \int_{[A]} \phi(a)|a|^{2s} da,$$

and

$$\mathscr{P}_T(\phi) = \int_{[T]} \phi(t) dt.$$

The Haar measures are chosen so that the volume of $A(\mathbb{O})$ and $T(\mathbb{O})$ is 1.

To precisely define the geometric side of (1.3), recall from [8] the stack Sht_T^r of T-shtukas with r modifications, and the $2r$-dimensional $\boldsymbol{k}$-stack Sht_G^r of PGL_2-shtukas with r modifications. The stack Sht_T^r is proper over $\boldsymbol{k}$ of dimension r, and admits a finite morphism $\theta_T^r \colon \mathrm{Sht}_T^r \to \mathrm{Sht}_G^r$. Pushing forward the fundamental class along θ_T^r gives a class $[\mathrm{Sht}_T^r] \in \mathrm{Ch}_c^r(\mathrm{Sht}_G^r)$ in the Chow group of compactly supported cycles.

Remark 1.1. The definitions above require a choice of $\mu = (\mu_i) \in \{\pm 1\}^r$ satisfying $\sum_{i=1}^r \mu_i = 0$; in particular we assume that r is even. We suppress the choice of μ in the introduction.

Analogously, we define in Section 3 a stack Sht_A^r parameterizing A-shtukas with r modifications. It is not of finite type over $\boldsymbol{k}$, but can be written as a union

$$\mathrm{Sht}_A^r = \bigcup_{d\geq 0} \mathrm{Sht}_A^{r,\leq d}$$

of stacks $\mathrm{Sht}_A^{r,\leq d}$ which are proper over $\boldsymbol{k}$, and which admit finite maps

$$\mathrm{Sht}_A^{r,\leq d} \to \mathrm{Sht}_G^r.$$

Define $[\mathrm{Sht}_A^{r,\leq d}] \in \mathrm{Ch}_{c,r}(\mathrm{Sht}_G^r)$ as the push-forward of the fundamental class.

Fix $d \geq 0$, and denote by $\widetilde{W}_A^d, \widetilde{W}_T \subset \mathrm{Ch}_{c,r}(\mathrm{Sht}_G^r)$ the $\mathscr{H}$-submodule generated by the classes $[\mathrm{Sht}_A^{r,\leq d}]$, $[\mathrm{Sht}_T^r]$ respectively. Restricting the intersection pairing on the Chow group defines a pairing $\langle\cdot,\cdot\rangle : \widetilde{W}_A^d \times \widetilde{W}_T \to \mathbb{Q}$. If we define

$$\begin{aligned} W_A^d &= \widetilde{W}_A / \{c \in \widetilde{W}_A^d : \langle c, \widetilde{W}_T\rangle = 0\} \\ W_T^d &= \widetilde{W}_T / \{c \in \widetilde{W}_T : \langle c, \widetilde{W}_A^d\rangle = 0\}, \end{aligned}$$

this pairing descends to $W_A^d \times W_T^d$, and we extend it to an $\mathbb{R}$-bilinear pairing

$$\langle\ ,\ \rangle\colon W_A^d(\mathbb{R}) \times W_T^d(\mathbb{R}) \to \mathbb{R}.$$

We show in §4 that for $d \gg 0$, the space W_T^d is independent of d, and for each $* \in \{A, T\}$, there a decomposition into isotypic components

$$W_*^d(\mathbb{R}) = W_{*,\mathrm{Eis}}^d \oplus \left(\bigoplus_\pi W_{*,\pi}^d\right),$$

where the sum is over all unramified cuspidal π, and $\mathscr{H}$ acts on $W_{*,\pi}^d$ via $\lambda_\pi : \mathscr{H} \to \mathbb{R}$.

Let

$$[\mathrm{Sht}_A^{r,\leq d}]_\pi \in W_{A,\pi}^d, \quad \text{ and } \quad [\mathrm{Sht}_T^r]_\pi \in W_{T,\pi}^d,$$

be the projections of $[\mathrm{Sht}_A^{r,\leq d}] \in W_A^d(\mathbb{R})$ and $[\mathrm{Sht}_T^r] \in W_T^d(\mathbb{R})$. These are the *Manin-Drinfeld* and *Heegner-Drinfeld* classes, respectively.

Our main result is the following intersection formula. We write $\mathscr{P}_A^{(r)}(\phi, s)$ for the r-th derivative of $\mathscr{P}_A(\phi, s)$. Throughout, we tacitly assume $d \gg 0$, since the Manin-Drinfeld cycles are independent of d, for d large enough.

Theorem 1. *Let $\phi \in \pi^U$ be non-zero and let $r \geq 0$ be even. Then*

$$\frac{\mathscr{P}_A^{(r)}(\phi, 0)\mathscr{P}_T(\bar{\phi})}{(\log q)^r \langle\phi,\phi\rangle_{\mathrm{Pet}}} = \left\langle [\mathrm{Sht}_A^{r,\leq d}]_\pi, [\mathrm{Sht}_T^r]_\pi \right\rangle_{\mathrm{Sht}_G^r}.$$

Remark 1.2. The left side is independent of the choice of ϕ, as the Peterson inner product is Hermitian.

When $r = 0$, the formula is a tautology, after one unwinds the definition of the right hand side. To understand the formula when $r > 0$, we interpret it in terms of L-functions. Let g be the genus of X. Then the normalized L-function

$$\mathscr{L}(\pi, s) := q^{2(g-1)(s-\frac{1}{2})} L(\pi, s)$$

satisfies $\mathscr{L}(\pi, 1-s) = \mathscr{L}(\pi, s)$. Moreover, for a suitably scaled $\phi \in \pi^U$, we have

$$\mathscr{L}(\pi, 2s + 1/2) = \mathscr{P}_A(\phi, s).$$

The kind of information we learn from Theorem 1 depends on whether the base change L-function $\mathscr{L}(\pi_K, \frac{1}{2})$ vanishes at $s = 1/2$ or not. Recall

$$\mathscr{L}(\pi_K, s) = \mathscr{L}(\pi, s)\mathscr{L}(\pi \otimes \eta, s).$$

By Waldspurger's formula [8, Rem. 1.3], the central value $\mathscr{L}(\pi_K, \frac{1}{2})$ vanishes if and only if $\int_{[T]} \bar{\phi}\, dt$ vanishes.

Thus, if $\mathscr{L}(\pi_K, \frac{1}{2}) \neq 0$, Theorem 1 gives a geometric interpretation for the non-leading Taylor series coefficients of $\mathscr{L}(\pi, s)$, after dividing by the non-zero toric period integral. To formulate this better, we consider the ratio with the leading term:

Theorem 2. *If $\mathscr{L}(\pi_K, \frac{1}{2}) \neq 0$, then for even $r \geq 0$, we have $[\mathrm{Sht}_A^{r, \leq d}]_\pi \neq 0$, and*

$$\frac{\mathscr{L}^{(r)}(\pi, 1/2)}{\mathscr{L}(\pi, 1/2)} = 2^{-r} (\log q)^r \frac{\Big\langle [\mathrm{Sht}_A^{r, \leq d}]_\pi, [\mathrm{Sht}_T^r]_\pi \Big\rangle_{\mathrm{Sht}_G^r}}{\Big\langle [\mathrm{Sht}_A^{0, \leq d}]_\pi, [\mathrm{Sht}_T^0]_\pi \Big\rangle_{\mathrm{Sht}_G^0}}.$$

Remark 1.3. The non-vanishing follows from the positivity of $\mathscr{L}^{(r)}(\pi, 1/2)$ [8, Thm. B.2].

On the other hand, the precise version of (1.2) implies

$$\frac{\mathscr{L}^{(r)}(\pi, 1/2)}{\mathscr{L}(\pi, 1/2)} = (\log q)^r \frac{\langle [\mathrm{Sht}_T^r]_\pi, [\mathrm{Sht}_T^r]_\pi \rangle_{\mathrm{Sht}_G^r}}{\langle [\mathrm{Sht}_T^0]_\pi, [\mathrm{Sht}_T^0]_\pi \rangle_{\mathrm{Sht}_G^0}}.$$

This *suggests* that when $\mathscr{L}(\pi_K, \frac{1}{2}) \neq 0$, we have $[\mathrm{Sht}_A^{r, \leq d}]_\pi = 2^r c_{\pi, K} [\mathrm{Sht}_T^r]_\pi$, where $c_{\pi,K}$ is an explicit non-zero ratio of period integrals.

If $\mathscr{L}(\pi_K, \frac{1}{2}) = 0$, the story is different, as Theorem 1 then says nothing about $\mathscr{L}^{(r)}(\pi, 1/2)$. But we still learn interesting information about algebraic cycles:

Theorem 3. *If $\mathscr{L}(\pi_K, \frac{1}{2}) = 0$, then for even $r \geq 0$, we have*

$$\Big\langle [\mathrm{Sht}_A^{r, \leq d}]_\pi, [\mathrm{Sht}_T^r]_\pi \Big\rangle_{\mathrm{Sht}_G^r} = 0.$$

Let us give one consequence of this theorem. Let $r(\pi)$ be the order of vanishing of $\mathscr{L}(\pi, s)$ at $s = 1/2$. If $r \geq r(\pi) \geq 0$ and $r(\pi \otimes \eta) = 0$, then $[\mathrm{Sht}_T^r]_\pi \neq 0$ by (1.2). We infer:

Corollary 1.4. *If $r \geq r(\pi) > 0$ and $r(\pi \otimes \eta) = 0$, then either*

(i) $[\mathrm{Sht}_A^{r, \leq d}]_\pi = 0$, *or*
(ii) any lifts of $[\mathrm{Sht}_A^{r, \leq d}]_\pi$ *and* $[\mathrm{Sht}_T^r]_\pi$ *to the group* $\mathrm{Ch}_{c,r}(\mathrm{Sht}_G^r)_{\mathbb{R}}$ *are linearly independent.*

Remark 1.5. One gets a similar statement, without any mention of lifts, for the cycle classes $[\mathrm{Sht}^{r,\leq d}_A]_{\pi,\lambda}$ and $[\mathrm{Sht}^r_T]_{\pi,\lambda}$ in $H^{2r}_c(\mathrm{Sht}^r_G \otimes_{\boldsymbol{k}} \bar{\boldsymbol{k}}, \mathbb{Q}_\ell(r))_{\pi,\lambda}$, as in [8, 1.5].

Of course, we would like to know which option (i) or (ii) actually holds. The author's expectation is that $[\mathrm{Sht}^r_A]_\pi \neq 0$ precisely when $r \geq 2r(\pi)$. In work in progress, we hope to prove a special value formula for the *self-intersection* of the Manin-Drinfeld cycle $[\mathrm{Sht}^r_A]_\pi$ in terms of the rth derivative of the *square* $\mathscr{L}(\pi, s)^2$, which would indeed imply $[\mathrm{Sht}^r_A]_\pi \neq 0$, for $r \geq 2r(\pi)$. At the same time, we plan to generalize our formula to allow for mildly ramified representations π and ramified quadratic extensions K/F, as in [9]; see also [5].

1.3. **Outline.** We follow the RTF approach of [8, 3], giving references to those papers when the proofs are not substantially different. In Section 2, we define an analytic distribution on $\mathcal{H}$, and relate it to L-functions on one hand, and weighted traces of Frobenius along the Hitchin fibration $N_d \to A_d$, on the other. The only genuinely new computation is Lemma 2.5. The Manin-Drinfeld cycles are defined in Section 3. We use intersection pairings to define a geometric distribution, and relate it to traces of Frobenius along *the same* Hitchin fibration. In Section 4, we work out the representation theory of the Hitchin fibration and use it to show that the analytic and geometric distributions agree. This is the most interesting part of the computation, and the main theorem follows quickly from there.

1.4. **Notation.** $|X|$ is the set of closed points of X. The absolute value

$$|\cdot| = \prod_{x \in |X|} |\cdot|_x \colon \mathbb{A}^\times \to \mathbb{Q}^\times$$

sends the uniformizer $\pi_x \in F_x$ with residue field $\boldsymbol{k}_x$ to $q^{-[\boldsymbol{k}_x \colon \boldsymbol{k}]}$. If H is an algebraic group, Haar measure on $H(\mathbb{A})$ is normalized so that $H(\mathbb{O})$ has volume 1.

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2. Analytic distribution

2.1. **Automorphic forms.** We recall some notation from [3]. Denote by $\mathcal{A}(G)$ the space of automorphic forms [1, §5] on $G(\mathbb{A})$, and by $\mathcal{A}_{\mathrm{cusp}}(G) \subset \mathcal{A}(G)$ the subspace of cuspidal automorphic forms. The subspace of unramified (U-invariant) cuspforms is finite-dimensional, and admits a decomposition

$$\mathcal{A}_{\mathrm{cusp}}(G)^U = \bigoplus_{\mathrm{unr.\,cusp.}\,\pi} \pi^U$$

as a direct sum of lines, where the sum is over the unramified cuspidal automorphic representations $\pi \subset \mathcal{A}_{\mathrm{cusp}}(G)$.

Denote by $\mathscr{H}$ the Hecke algebra of compactly supported functions $f : U\backslash G(\mathbb{A})/U \to \mathbb{Q}$. The $\mathscr{H}$-module of compactly supported unramified $\mathbb{Q}$-valued automorphic forms is denoted

$$\mathscr{A} = C^\infty_c(G(F)\backslash G(\mathbb{A})/U, \mathbb{Q}).$$

We let $\mathscr{A}_{\mathbb{C}} = \mathscr{A} \otimes \mathbb{C}$ denote the corresponding complex space, so that

$$\mathcal{A}_{\mathrm{cusp}}(G)^U \subset \mathscr{A}_{\mathbb{C}} \subset \mathcal{A}(G)^U. \tag{2.1}$$

Following [8, §4.1], we view the Satake transform as a $\mathbb{Q}$-algebra surjection $a_{\mathrm{Eis}} : \mathscr{H} \to \mathbb{Q}[\mathrm{Pic}_X(\boldsymbol{k})]^{\iota_{\mathrm{Pic}}}$, for a particular involution ι_{Pic} of $\mathbb{Q}[\mathrm{Pic}_X(\boldsymbol{k})]$. The Eisenstein ideal is

$$\mathcal{I}^{\mathrm{Eis}} := \ker\left(a_{\mathrm{Eis}} : \mathscr{H} \to \mathbb{Q}[\mathrm{Pic}_X(\boldsymbol{k})]^{\iota_{\mathrm{Pic}}}\right). \tag{2.2}$$

As in [8, §7.3], define $\mathbb{Q}$-algebras

$$\mathscr{H}_{\mathrm{aut}} = \mathrm{Image}\big(\mathscr{H} \to \mathrm{End}_{\mathbb{Q}}(\mathscr{A}) \times \mathbb{Q}[\mathrm{Pic}_X(\boldsymbol{k})]^{\iota_{\mathrm{Pic}}}\big)$$
$$\mathscr{H}_{\mathrm{cusp}} = \mathrm{Image}\left(\mathscr{H} \to \mathrm{End}_{\mathbb{C}}(\mathcal{A}_{\mathrm{cusp}}(G)^U)\right).$$

The quotient map $\mathscr{H} \to \mathscr{H}_{\mathrm{cusp}}$ factors through $\mathscr{H}_{\mathrm{aut}}$, and the resulting map

$$\mathscr{H}_{\mathrm{aut}} \to \mathscr{H}_{\mathrm{cusp}} \times \mathbb{Q}[\mathrm{Pic}_X(\boldsymbol{k})]^{\iota_{\mathrm{Pic}}} \tag{2.3}$$

is an isomorphism [8, Lemma 7.16].

For each unramified cuspidal automorphic representation $\pi \subset \mathcal{A}_{\mathrm{cusp}}(G)$, denote by

$$\lambda_\pi : \mathscr{H} \to \mathbb{C}$$

the character through which the Hecke algebra acts on the line π^U. As in [8, §7.5.1], the $\mathbb{Q}$-algebra $\mathscr{H}_{\mathrm{cusp}}$ is isomorphic to a finite product of number fields, and the product of all characters λ_π induces an isomorphism

$$\mathscr{H}_{\mathrm{cusp}} \otimes \mathbb{C} \cong \bigoplus_{\mathrm{unr.\,cusp.}\,\pi} \mathbb{C}.$$

The above the discussion holds word-for-word if G is replaced by G_1.

Lemma 2.1. [3, Lem. 3.3] *There is a canonical bijection*

$$G(F)\backslash G(\mathbb{A})/U \to G_1(F)\backslash G_1(\mathbb{A})/U_1.$$

It induces an isomorphism of $\mathbb{C}$-vector spaces

$$\mathcal{A}(G)^U \cong \mathcal{A}(G_1)^{U_1}$$

respecting the subspaces of cusp forms.

2.2. **Definition of the distribution.** The X-scheme

$$\widetilde{J} = \underline{\mathrm{Iso}}_{\mathcal{O}_X}(\nu_*\mathcal{O}_Y, \nu_{0*}\mathcal{O}_{Y_0})$$

is both a left $\widetilde{G}$-torsor and a right $\widetilde{G}_1$-torsor. Similarly $J = \widetilde{J}/\mathbb{G}_m$ is both a left G-torsor and a right G_1-torsor. There are canonical identifications

$$A(F)\backslash J(F)/T(F) = \widetilde{A}(F)\backslash \widetilde{J}(F)/\widetilde{T}(F) = K_0^\times \backslash \mathrm{Iso}(K, K_0)/K^\times.$$

Thus, [3, §2] allows us to define the invariant map

$$A(F)\backslash J(F)/T(F) \xrightarrow{\mathrm{inv}} \{\xi \in K : \mathrm{Tr}_{K/F}(\xi) = 1\}. \tag{2.4}$$

In this case (where one quadratic algebra is split and the other is a field), inv is a bijection and has the following simple description when $\mathrm{char}\,\boldsymbol{k} \neq 2$. Choose F-algebra embeddings

$\alpha_1\colon K \hookrightarrow M_2(F)$ and $\alpha_2\colon K_0 \hookrightarrow M_2(F)$. Let $e = \alpha_2(1,0)$ and $f = \alpha_2(0,1)$ be the image of the two idempotents. Then $M_2(F) = e\alpha_1(K) + f\alpha_1(K)$. If

$$g \in K_0^\times \backslash \mathrm{Iso}(K, K_0)/K^\times \simeq A(F)\backslash G(F)/T(F)$$

is represented by $e\alpha + f\beta \in G(F)$, then $\mathrm{inv}(g) = \frac{2\alpha\bar{\beta}}{\mathrm{Tr}(\alpha\bar{\beta})}$.

Lemma 2.2. [3, Lem. 3.4] *There is a canonical homeomorphism*

$$U\backslash J(\mathbb{A})/U_1 \cong U\backslash G(\mathbb{A})/U. \tag{2.5}$$

Now fix $f \in \mathscr{H}$. Use the bijection of Lemma 2.2 to view f as a function

$$f : U\backslash J(\mathbb{A})/U_1 \to \mathbb{Q},$$

and define a function on $G(\mathbb{A}) \times G_1(\mathbb{A})$ by

$$\mathbb{K}_f(g, g_1) = \sum_{\gamma \in J(F)} f(g^{-1}\gamma g_1). \tag{2.6}$$

Recall the notation

$$[T] = T(F)\backslash T(\mathbb{A}), \quad [G_1] = G_1(F)\backslash G_1(\mathbb{A}),$$

from the introduction, and recall the normalization of Haar measures of §1.4. Define a distribution on $\mathscr{H}$ by

$$\mathbb{J}(f, s) = \int_{[A]\times[T]}^{\mathrm{reg}} \mathbb{K}_f(a, t)\, |a|^{2s} da\, dt. \tag{2.7}$$

Here, $|\cdot| : A(\mathbb{A}) \to \mathbb{R}^\times$ is the homomorphism $|\left(\begin{smallmatrix} a_1 & \\ & a_2 \end{smallmatrix}\right)| = |a_1/a_2|$.

The integral in (2.7) need not converge absolutely, so we regularize it. First define

$$A(\mathbb{A})_n = \left\{a \in A(\mathbb{A}) : |a| = q^{-n}\right\}$$

and $[A]_n = A(F)\backslash A(\mathbb{A})_n$, and set

$$\begin{aligned} \mathbb{J}_n(f, s) &= \int_{[A]_n\times[T]} \mathbb{K}_f(a, t)|a|^{2s}\, da\, dt \\ &= q^{-2ns} \int_{[A]_n\times[T]} \mathbb{K}_f(a, t)\, da\, dt. \end{aligned} \tag{2.8}$$

This integral is absolutely convergent, by compactness of $[A]_n$ and $[T]$.

Proposition 2.3. *The integral $\mathbb{J}_n(f, s)$ vanishes for $|n|$ sufficiently large.*

Proof. As in [3, Prop. 3.7]. □

Using Proposition 2.3, the regularized integral (2.7) is defined as

$$\mathbb{J}(f, s) = \sum_{n\in\mathbb{Z}} \mathbb{J}_n(f, s).$$

This is a Laurent polynomial in q^s. Define

$$\mathbb{J}_n(\gamma, f, s) = \int_{[A]_n\times[T]} \mathbb{K}_{f,\gamma}(a, t)|a|^{2s}\, da\, dt, \tag{2.9}$$

and

$$\mathbb{J}(\gamma, f, s) = \sum_{n \in \mathbb{Z}} \mathbb{J}_n(\gamma, f, s),$$

so that there are decompositions

$$\mathbb{J}(f, s) = \sum_{\gamma \in A(F)\backslash J(F)/T(F)} \mathbb{J}(\gamma, f, s) = \sum_{\substack{\xi \in K \\ \mathrm{Tr}_{K/F}(\xi)=1}} \mathbb{J}(\xi, f, s). \tag{2.10}$$

In the final expression, we have used (2.4) to define

$$\mathbb{J}(\xi, f, s) = \mathbb{J}(\gamma, f, s)$$

for the unique double coset $\gamma \in A(F)\backslash J(F)/T(F)$ satisfying $\mathrm{inv}(\gamma) = \xi$.

2.3. **Spectral decomposition.** Define for any $\phi \in \mathcal{A}_{\mathrm{cusp}}(G)^U$, the period integral

$$\mathscr{P}_A(\phi, s) = \int_{[A]} \phi(a)|a|^{2s}\, da.$$

This integral is absolutely convergent for all $s \in \mathbb{C}$. Using Lemma 2.2 to view $\phi \in \mathcal{A}_{\mathrm{cusp}}(G_1)^{U_1}$, define another period integral

$$\mathscr{P}_T(\phi) = \int_{[T]} \phi(t)\, dt.$$

As $[T]$ is compact, this integral is also absolutely convergent.

Recall the Eisenstein ideal $\mathcal{I}^{\mathrm{Eis}} \subset \mathscr{H}$ of (2.2).

Proposition 2.4. *Every $f \in \mathcal{I}^{\mathrm{Eis}}$ satisfies*

$$\mathbb{J}(f, s) = \sum_{\mathrm{unr.\,cusp.}\,\pi} \lambda_\pi(f) \frac{\mathscr{P}_A(\phi, s)\mathscr{P}_T(\overline{\phi})}{\langle \phi, \phi \rangle}, \tag{2.11}$$

where the sum is over all unramified cuspidal automorphic representations $\pi \subset \mathcal{A}_{\mathrm{cusp}}(G)$, and $\phi \in \pi^U$ is any nonzero vector. Moreover, $\mathbb{J}(f, s)$ only depends on the image of f under the quotient map $\mathscr{H} \to \mathscr{H}_{\mathrm{aut}}$.

Proof. View (2.6) as a function on $G(\mathbb{A}) \times G(\mathbb{A})$, and invoke the decomposition

$$\mathbb{K}_f(x, y) = \mathbb{K}_{f,\mathrm{cusp}}(x, y) + \mathbb{K}_{f,\mathrm{sp}}(x, y)$$

of [8, Theorem 4.3], to convert all three terms back into functions on $G(\mathbb{A}) \times G_1(\mathbb{A})$. The result is a decomposition

$$\begin{aligned}\mathbb{K}_f(g, g_1) = &\sum_{\mathrm{unr.\,cusp.}\,\pi} \lambda_\pi(f) \cdot \frac{\phi(g)\overline{\phi(g_1)}}{\langle \phi, \phi \rangle} \\ &+ \sum_{\mathrm{unr.\,quad.}\,\chi} \lambda_\chi(f) \cdot \chi(\det(G)) \cdot \chi(\det(g_1)).\end{aligned}$$

The first sum is over all unramified cuspidal representations π, and $\phi \in \pi^U$ is any nonzero vector. The second sum is over all unramified quadratic characters

$$\mathrm{Pic}(X) \cong F^\times \backslash \mathbb{A}^\times / \mathbb{O}^\times \xrightarrow{\chi} \{\pm 1\},$$

and

$$\lambda_\chi(f) = \int_{G(\mathbb{A})} f(g)\chi(\det(g))\, dg.$$

The distribution (2.8) now decomposes as

$$\mathbb{J}_n(f,s) = \sum_{\text{unr. cusp. } \pi} \mathbb{J}_n^\pi(f,s) + \sum_{\text{unr. quad. } \chi} \mathbb{J}_n^\chi(f,s),$$

where we have set

$$\mathbb{J}_n^\pi(f,s) = \frac{\lambda_\pi(f)}{\langle \phi, \phi \rangle} \left(\int_{[A]_n} \phi(a)|a|^{2s}\, da \right) \left(\int_{[T]} \overline{\phi(t)}\, dt \right)$$

and

$$\mathbb{J}_n^\chi(f,s) = \lambda_\chi(f) \left(\int_{[A]_n} \chi(\det(a))|a|^{2s}\, da \right) \left(\int_{[T]} \chi(\det(t))\, dt \right). \tag{2.12}$$

Next we show that $\mathbb{J}_n^\chi(f,s) = 0$ for all such χ. Note that when $\chi = 1$, *both* toric integrals in (2.12) are non-zero, so the proof from [3] does not carry over. The vanishing in all cases follows from:

Lemma 2.5. *If $f \in \mathcal{I}^{\mathrm{Eis}}$ and χ is unramified, then $\lambda_\chi(f) = 0$.*

Proof. Let $B \subset G$ be the Borel subgroup of upper triangular matrices. Following [8, §4], we consider the right translation representation ρ_χ of $G(\mathbb{A})$ on the space V_χ of functions

$$\phi\colon G(\mathbb{A}) \to \mathbb{C}$$

such that $\phi(bg) = \chi(b)\phi(g)$ for all $b \in B(\mathbb{A})$, $g \in G(\mathbb{A})$. The space V_χ is canonically identified (by restriction) with a space of functions on U. The latter space carries an inner product

$$(\phi, \phi') = \int_U \phi(u)\bar{\phi}'(y) du.$$

Now let $\phi = \mathbf{1}_U$. Since $f \in \mathcal{I}^{\mathrm{Eis}}$, we have [8, §4]

$$(\rho_\chi(f)\phi, \phi) = \mathrm{tr}\, \rho_\chi(f) = \chi(a_{\mathrm{Eis}}(f)) = 0.$$

On the other hand, we compute

$$\begin{aligned}(\rho_\chi(f)\phi, \phi) &= \int_U \int_{g = bu' \in G(\mathbb{A})} f(u^{-1}g)\chi(\det b) dg du \\ &= \int_{G(\mathbb{A})} f(g)\chi(\det g) dg \\ &= \lambda_\chi(f).\end{aligned}$$

We have used that χ is unramified, f is spherical, and U has volume 1. We conclude that $\lambda_\chi(f) = 0$. $\square$

We therefore have

$$\mathbb{J}_n(f,s) = \sum_{\mathrm{unr.\,cusp.}\,\pi} \mathbb{J}_n^\pi(f,s),$$

and (2.11) follows by summing both sides over n.

For the final claim, suppose f has trivial image under $\mathscr{H} \to \mathscr{H}_{\mathrm{aut}}$. This implies that f annihilates $\mathscr{A}_{\mathbb{C}}$, and lies in $\mathcal{I}^{\mathrm{Eis}}$. The first inclusion in (2.1) implies that $\lambda_\pi(f) = 0$ for all π, and so $\mathbb{J}(f,s) = 0$ by (2.11). $\square$

2.4. **Geometric expression.** Fix $d \geq 0$, and let i be an integer in the range $0 \leq i \leq 2d$. Recall from [3, §3] the commutative diagram of $\boldsymbol{k}$-schemes

$$\begin{tikzcd} N_{(i,2d-i)} \arrow[r, "\delta"'] \arrow[d, "\beta_i"'] & \Sigma_{i,2d-i}(Y) \arrow[d, "\otimes"] \\ A_d \arrow[r, "\nu^\sharp"'] \arrow[d, "\mathrm{Tr}"'] & \Sigma_{2d}(Y) \\ \Sigma_d(X) & \end{tikzcd} \tag{2.13}$$

in which the square is cartesian. We briefly recall the definitions of these schemes and maps in terms of their S-points, for any scheme S.

First, $\Sigma_d(X)(S)$ is the set of isomorphism classes of pairs (Δ, ζ) of

- a line bundle Δ on $X_S = X \times_{\boldsymbol{k}} S$ of degree d,
- a nonzero section $\zeta \in H^0(X_S, \Delta)$.

We have a canonical isomorphism

$$\Sigma_d(X) \cong \mathrm{Sym}^d(X) \cong S_d \backslash X^d, \tag{2.14}$$

and $\Sigma_d(X)$ is a smooth projective $\boldsymbol{k}$-scheme. We also set

$$\Sigma_{i,2d-i}(Y) = \Sigma_i(Y) \times_{\boldsymbol{k}} \Sigma_{2d-i}(Y),$$

parameterizing effective divisors of bidegree $(i, 2d-i)$ on $Y \coprod Y$.

Next, $A_d(S)$ is the set of isomorphism classes of pairs (Δ, ξ) consisting of

- a line bundle Δ on X_S of degree d,
- a section $\xi \in H^0(Y_S, \nu^*\Delta)$ with nonzero trace

$$\mathrm{Tr}_{Y/X}(\xi) = \xi + \xi^{\sigma_1} \in H^0(X_S, \Delta).$$

The arrows in (2.13) emanating from A_d are

$$\mathrm{Tr}(\Delta,\xi) = (\Delta, \mathrm{Tr}_{Y/X}(\xi)) \quad \text{and} \quad \nu^\sharp(\Delta,\xi) = (\nu^*\Delta, \xi).$$

A_d is a quasi-projective $\boldsymbol{k}$-scheme.

Finally, $\widetilde{N}_{(i,2d-i)}(S)$ is the groupoid of triples $(\mathcal{M}, \mathcal{L}, \phi)$ consisting of

- line bundles $\mathcal{M} = (\mathcal{M}', \mathcal{M}'') \in \mathrm{Pic}(Y_{0S})$ and $\mathcal{L} \in \mathrm{Pic}(Y_S)$ satisfying
$$2\deg(\mathcal{M}') - i = \deg(\mathcal{L}) = 2\deg(\mathcal{M}'') - (2d - i),$$
- a morphism $\phi : \nu_* \mathcal{L} \to \mathcal{M}' \oplus \mathcal{M}''$ of rank 2 vector bundles on X_S with nonzero determinant.

The Picard group $\mathrm{Pic}(X_S)$ acts on $\widetilde{N}_{i,2d-i}(S)$ by simultaneous twisting, and the quotient stack
$$N_{(i,2d-i)} = \widetilde{N}_{(i,2d-i)}/\mathrm{Pic}_X$$
is a scheme by [3, 3.12]. The map δ sends $(\mathcal{M}, \mathcal{L}, \phi)$ to $(\mathrm{div}(\boldsymbol{a}), \mathrm{div}(\boldsymbol{d}))$, where $\boldsymbol{a}$ and $\boldsymbol{d}$ are the diagonal matrix entries in the map $\nu^*\phi \colon \mathcal{L} \oplus \mathcal{L}^\sigma \to \nu^*\mathcal{M}' \oplus \nu^*\mathcal{M}''$.

Proposition 2.6. [3, 3.13] *Let g and g_1 be the genera of X and Y, respectively.*

(1) The morphisms β_i and $\otimes$ in (2.13) are finite.

(2) If $d \geq 2g_1 - 1$ then $N_{(i,2d-i)}$ is smooth over $\boldsymbol{k}$ of dimension $2d - g + 1$.

Now let D be an effective divisor on X of degree d. The constant function 1 defines a global section of $\mathcal{O}_X(D)$, and hence a point $(\mathcal{O}_X(D), 1) \in \Sigma_d(X)(\boldsymbol{k})$. Define A_D as the fiber product
$$\begin{array}{ccc} A_D & \longrightarrow & A_d \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\boldsymbol{k}) & \xrightarrow{(\mathcal{O}_X(D),1)} & \Sigma_d(X). \end{array}$$
Then there is a canonical bijection
$$A_D(\boldsymbol{k}) \cong \left\{ \xi \in K : \begin{array}{c} \mathrm{Tr}_{K/F}(\xi) = 1 \\ \mathrm{div}(\xi) + \nu^* D \geq 0 \end{array} \right\}. \tag{2.15}$$

The Hecke algebra $\mathscr{H}$ has a $\mathbb{Q}$-basis $\{f_D\}$ indexed by the effective divisors $D \in \mathrm{Div}(X)$, and defined as follows (see also [8, §3.1]). Let S_D be the image of the set
$$\{M \in \mathrm{Mat}_2(\mathbb{O}) \colon \mathrm{div}(\det M) = D\}$$
in $\mathrm{PGL}_2(\mathbb{A}) = G(\mathbb{A})$. Then $f_D \colon U \backslash G(\mathbb{A})/U \to \mathbb{Q}$ is the characteristic function of S_D.

We are now ready to give a geometric interpretation of the orbital integral $\mathbb{J}(\xi, f_D, s)$ appearing in (2.10). Using Lemma 2.2, we regard f_D as a compactly supported function
$$f_D : U \backslash J(\mathbb{A})/U_1 \to \mathbb{Q}. \tag{2.16}$$
Let ℓ be any prime different from the characteristic of $\boldsymbol{k}$. The following theorem is proved exactly as in [3, 3.17], this time with a trivial local system.

Theorem 2.7. *Fix $\xi \in K$ with $\mathrm{Tr}_{K/F}(\xi) = 1$, and view $A_D(\boldsymbol{k})$ as a subset of K via (2.15).*

(1) If $\xi \notin A_D(\boldsymbol{k})$ then $\mathbb{J}(\xi, f_D, s) = 0$.

(2) If $\xi \in A_D(\boldsymbol{k})$ then
$$\mathbb{J}(\xi, f_D, s) = \sum_{i=0}^{2d} q^{2(i-d)s} \cdot \mathrm{Tr}\big(\mathrm{Frob}_\xi;\ (\beta_{i*}\mathbb{Q}_\ell)_{\bar{\xi}}\big),$$
where $\bar{\xi}$ is a geometric point above $\xi : \mathrm{Spec}(\boldsymbol{k}) \to A_D \hookrightarrow A_d$.

3. Geometric distribution

Fix an integer $r \geq 0$, and an r-tuple $\mu = (\mu_1, \dots, \mu_r) \in \{\pm 1\}^r$ satisfying the parity condition $\sum_{i=1}^r \mu_i = 0$. In particular, r is even.

3.1. **Heegner-Drinfeld and Manin-Drinfeld cycles.** We recall some notation from [3, 8]. Recall that $G = \mathrm{PGL}_2$ over X.

Let Bun_G be the algebraic stack parametrizing G-torsors on X, and let Hk_G^μ be the Hecke stack parameterizing G-torsors on X with r modifications of type μ. It comes equipped with morphisms

$$p_0, \dots, p_r : \mathrm{Hk}_G^\mu \to \mathrm{Bun}_G$$

and $p_X : \mathrm{Hk}_G^\mu \to X^r$. For the definitions, see [8, §5.2].

The moduli stack Sht_G^μ of G-shtukas of type μ sits in the cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}_G^\mu \arrow[r] \arrow[d] & \mathrm{Hk}_G^\mu \arrow[d, "(p_0, p_r)"] \\ \mathrm{Bun}_G \arrow[r, "(\mathrm{id}, \mathrm{Fr})"] & \mathrm{Bun}_G \times \mathrm{Bun}_G. \end{tikzcd}$$

It is a Deligne-Mumford stack, locally of finite type over $\boldsymbol{k}$, and the morphism

$$\pi_G : \mathrm{Sht}_G^\mu \to X^r$$

induced by p_X is separated and smooth of relative dimension r.

The étale double covers $\nu_0 : Y_0 \to X$ and $\nu : Y \to X$ determine tori A and T, both rank 1 over X. Let Bun_T be the moduli stack of T-torsors on X. Denote by Hk_T^μ the Hecke stack parameterizing T-torsors with r modifications of type μ. It comes with morphisms

$$p_1, \dots, p_r : \mathrm{Hk}_T^\mu \to \mathrm{Bun}_T,$$

and $p_Y \colon \mathrm{Hk}_T^\mu \to Y^r$. See [8, §5.4] for the definitions.

The stack of T-shtukas of type μ is defined by the cartesian square

$$\begin{tikzcd} \mathrm{Sht}_T^\mu \arrow[r] \arrow[d] & \mathrm{Hk}_T^\mu \arrow[d, "(p_0, p_r)"] \\ \mathrm{Bun}_T \arrow[r, "(\mathrm{id}, \mathrm{Fr})"] & \mathrm{Bun}_T \times \mathrm{Bun}_T. \end{tikzcd} \tag{3.1}$$

It is Deligne-Mumford over $\boldsymbol{k}$, and the morphism

$$\pi_T : \mathrm{Sht}_T^\mu \to Y^r$$

induced by p_Y is finite étale. Thus, Sht_T^μ is smooth and proper over $\boldsymbol{k}$, and of dimension r.

For the diagonal torus A, we define A-shtukas of type μ, in a similar manner. Namely, define $\widetilde{\mathrm{Sht}}_A^\mu$ to be the moduli stack whose S-points are tuples $(\mathcal{M}, x_1, \dots, x_r, \iota)$, where $\mathcal{M} = (\mathcal{L}_1, \mathcal{L}_2)$ is a line bundle on Y_{0S}, $y_i \colon S \to Y_0$ are S-points of Y_0, and ι is an isomorphism

$$\iota \colon \mathcal{M}^{\mathrm{Fr}} \simeq \mathcal{M}\left(\sum_{i=1}^r \mu_i \Gamma_{y_i}\right).$$

Then Sht_A^μ is the quotient of $\widetilde{\mathrm{Sht}}_A^\mu$ modulo the twisting action of $\mathrm{Pic}_X(\boldsymbol{k})$.

Let $\pi_A : \mathrm{Sht}_A^\mu \to Y_0^r$ be the morphism which remembers the legs $y_i \in Y_0$ of the A-shtuka.

Lemma 3.1. *The morphism π_A is a $\mathrm{Pic}_X(\boldsymbol{k})$-torsor. In particular, Sht_A^μ is a smooth Deligne-Mumford stack over $\boldsymbol{k}$, locally of finite type.*

Proof. The proof is exactly as for T-shtukas [8, Lem. 5.13], using that

$$\mathrm{Pic}_X(\boldsymbol{k}) \simeq (\mathrm{Pic}_X(\boldsymbol{k}) \times \mathrm{Pic}_X(\boldsymbol{k}))/\Delta(\mathrm{Pic}_X(\boldsymbol{k})).$$

□

For each $d \geq 0$, the open substack $\mathrm{Sht}_A^{\mu,\leq d} \subset \mathrm{Sht}_A^\mu$ consisting of those $(\mathcal{M}, (y_i), \iota)$ with

$$|\deg(\mathcal{M})| := |\deg(\mathcal{L}_2) - \deg(\mathcal{L}_1)| \leq d,$$

is proper over $\boldsymbol{k}$. In fact, each of the substacks

$$\mathrm{Sht}_A^{\mu,d} = \langle(\mathcal{M}, (y_i), \iota)\colon |\deg(\mathcal{M})| = d\rangle,$$

are themselves proper and closed.

Push-forward of line bundles induces proper morphisms

$$\begin{tikzcd} \mathrm{Sht}_A^{\mu,\leq d} \arrow[dr, "\theta_A^{\mu,\leq d}"'] & & \mathrm{Sht}_T^\mu \arrow[dl, "\theta_T^\mu"] \\ & \mathrm{Sht}_G^\mu & \end{tikzcd}$$

since Sht_G^μ is separated. We therefore obtain two classes

$$[\mathrm{Sht}_T^\mu], [\mathrm{Sht}_A^{\mu,\leq d}] \in \mathrm{Ch}_{c,r}(\mathrm{Sht}_G^\mu),$$

by pushing forward the corresponding fundamental classes.

3.2. **Geometric distribution.** There is an intersection pairing

$$\langle\cdot,\cdot\rangle : \mathrm{Ch}_{c,r}(\mathrm{Sht}_G^\mu) \times \mathrm{Ch}_{c,r}(\mathrm{Sht}_G^\mu) \to \mathbb{Q},$$

as in [8, §A.1]. Recall the Hecke algebra $\mathscr{H}$ of §2.1 and its action $*$ on $\mathrm{Ch}_{c,r}(\mathrm{Sht}_G^\mu)$ [8, §5.3]. Recall also [8, §7] that Sht_G^μ can be written as the increasing union of open substacks of finite type:

$$\mathrm{Sht}_G^\mu = \bigcup_{\underline{d} \in \mathcal{D}} \mathrm{Sht}_G^{\leq \underline{d}}.$$

Here, $\mathcal{D}$ is the set of functions $\mathbb{Z}/r\mathbb{Z} \to \mathbb{Z}$, which is partially-ordered by pointwise comparison. The substack $\mathrm{Sht}_G^{\leq \underline{d}}$ parameterizes G-shtukas $\mathcal{E}$ such that the vector bundle $p_i(\mathcal{E})$ has index of instability less than or equal to $d(i)$, for all $i = 0, \ldots, r$.

For any $f \in \mathscr{H}$ define

$$\mathbb{I}_r(f) = \langle[\mathrm{Sht}_A^{\mu,\leq d}], f * [\mathrm{Sht}_T^\mu]\rangle \in \mathbb{Q}, \tag{3.2}$$

where d is any integer with the property that

$$f * [\mathrm{Sht}_T^\mu] \in \mathrm{Ch}_{c,r}(\mathrm{Sht}_G^\mu) = \varinjlim_{d \in \mathcal{D}} \mathrm{Ch}_r(\mathrm{Sht}_G^{\mu, \leq d})$$

is supported on $\mathrm{Sht}_G^{\mu, \leq d-r}$. In the last bit of notation, we view $d - r$ as a constant function in $\mathcal{D}$. Note that this intersection number is independent of the choice of such d, since

$$[\mathrm{Sht}_A^{\mu, \leq d+n}] = [\mathrm{Sht}_A^{\mu, \leq d}] + \sum_{i=1}^{n} [\mathrm{Sht}_A^{\mu, d+i}]$$

for any $n \geq 0$, and $\langle [\mathrm{Sht}_A^{\mu, d+i}], f * [\mathrm{Sht}_T^\mu] \rangle = 0$. In particular, it follows that the function $\mathbb{I}_r(f)$ is additive, and hence defines a distribution on $\mathcal{H}$.

3.3. **Intersections as traces.** Fix $d \geq 0$, and recall from §2 the morphism

$$\beta_i \colon N_{(i, 2d-i)} \to A_d,$$

defined for each non-negative $i \leq 2d$. Define

$$N_d = \coprod_{i=0}^{2d} N_{(i, 2d-i)}.$$

Then $N_d = \widetilde{N}_d / \mathrm{Pic}_X$, where $\widetilde{N}_d$ is the moduli stack of triples

$$(\mathcal{M} \in \mathrm{Pic}_Y, \mathcal{L} \in \mathrm{Pic}_{Y_0}, \phi \colon \nu_* \mathcal{L} \to \nu_{0*} \mathcal{M})$$

such that ϕ has determinant of degree d. Let $\beta \colon N_d \to A_d$ be the union of the β_i.

Let D be an effective divisor of degree d. To relate $\mathbb{I}_r(f_D)$ to the local system $\beta_* \mathbb{Q}_\ell$, we define the Yun-Zhang correspondence

$$\begin{tikzcd} & \mathrm{YZ}_d \arrow[dl, "\gamma_0"'] \arrow[dr, "\gamma_1"] & \\ N_d \arrow[dr, "\beta"'] & & N_d \arrow[dl, "\beta"] \\ & A_d & \end{tikzcd} \tag{3.3}$$

as follows. Let $\widetilde{\mathrm{YZ}}_d$ be the stack whose S-points is the groupoid of

- pairs of points $(\mathcal{M}_0, \mathcal{L}_0, \phi_0)$ and $(\mathcal{M}_1, \mathcal{L}_1, \phi_1)$ in $\widetilde{N}_d(S)$,
- one S-point (y_0, y_1) of $Y_{0S} \times_{X_S} Y_S$,
- and injective line bundle maps $s_0 : \mathcal{M}_0 \to \mathcal{M}_1$ and $s_1 : \mathcal{L}_0 \to \mathcal{L}_1$.

The cokernels of s_i are required to be invertible sheaves on the graphs of y_i. Moreover, we require that the diagram

$$\begin{tikzcd} \nu_*\mathcal{L}_0 \arrow[r, "s_1"] \arrow[d, "\phi_0"'] & \nu_*\mathcal{L}_1 \arrow[d, "\phi_1"] \\ \nu_{0*}\mathcal{M}_0 \arrow[r, "s_0"'] & \nu_{0*}\mathcal{M}_1 \end{tikzcd}$$

of $\mathcal{O}_{X_S}$-modules commutes. Then Pic_X acts on $\widetilde{\mathrm{YZ}}_d$ by simultaneously twisting, and we define $\mathrm{YZ}_d = \widetilde{\mathrm{YZ}}_d/\mathrm{Pic}_X$.

Let $W = Y \coprod Y$. Also let $\Sigma_{2d}(W)$ be the moduli stack of pairs $(\mathcal{K}, \boldsymbol{a})$, consisting of a line bundle $\mathcal{K}$ of degree $2d$ on W, together with a global section $\boldsymbol{a}$. Recalling the spaces $\Sigma_{i,j}(Y)$ introduced in §2, we have

$$\Sigma_{2d}(W) = \coprod_{i=0}^{2d} \Sigma_{i,2d-i}(Y).$$

Using the top horizontal arrow of (2.13), we may realize YZ_d as the pullback of a correspondence on $\Sigma_{2d}(W)$. For this, define the following automorphisns τ_i of W over X:

(1) τ_1 is σ on $Y \coprod \varnothing$ and the identity on $\varnothing \coprod Y$.
(2) τ_2 interchanges the copies.
(3) $\tau_3 = \tau_2 \circ \tau_1$, which sends $y \cup \varnothing \mapsto \varnothing \cup y^\sigma$ and $\varnothing \cup y \mapsto y \cup \varnothing$.

The first two are involutions, while τ_3 has order 4.

Then define the correspondence

$$\begin{tikzcd} & H_d(W) \arrow[dl] \arrow[dr] & \\ \Sigma_{2d}(W) \arrow[dr, "\otimes"'] & & \Sigma_{2d}(W) \arrow[dl, "\otimes"] \\ & \Sigma_{2d}(Y), & \end{tikzcd} \tag{3.4}$$

where, for any $\boldsymbol{k}$-scheme S, $H_d(W)(S)$ is the groupoid of:

- a pair of S-points $(\mathcal{K}_i, \boldsymbol{a}_i) \in \Sigma_{2d}(W)$, for $i = 0, 1$,
- an S-point $y \in W(S)$,
- an isomorphism
$$s : \mathcal{K}_0(y^{\tau_1} - y^{\tau_3}) \cong \mathcal{K}_1$$
of line bundles on W_S such that $s(\boldsymbol{a}_0) = \boldsymbol{a}_1$, where we view $\boldsymbol{a}_0$ as a rational section of $\mathcal{K}_0(y^{\tau_1} - y^{\tau_3})$.

This data is determined by $(\mathcal{K}_0, \boldsymbol{a}_0)$ and the point $y \in W(S)$, for from these we may recover the line bundle $\mathcal{K}_1 = \mathcal{K}_0(y^{\tau_1} - y^{\tau_3})$ and its rational section $\boldsymbol{a}_1 = \boldsymbol{a}_0$. The condition that $\boldsymbol{a}_1$ is a global section of $\mathcal{K}_1$, as opposed to merely a rational section, is equivalent to

$$\mathrm{div}(\boldsymbol{a}_0) + y^{\tau_1} - y^{\tau_3} \geq 0.$$

This is in turn equivalent to the condition that the effective Cartier divisor y^{τ_3} appears in the support of $\mathrm{div}(\boldsymbol{a}_0)$. In other words, we may realize

$$H_d(Y) \hookrightarrow \Sigma_{2d}(W) \times_k W$$

as the closed subscheme of triples $(\mathcal{K}_0, \boldsymbol{a}_0, y)$ for which y^{τ_3} appears in the support of $\mathrm{div}(\boldsymbol{a}_0)$, with z defined as above.

Remark 3.2. $H_d(W)$ does not preserve the substacks $\Sigma_{i,2d-i}(Y) \subset \Sigma_{2d}(W)$. In fact, it induces a correspondence from $\Sigma_{i,2d-i}(Y)$ to

$$\Sigma_{i+1,2d-i-1}(Y) \coprod \Sigma_{i-1,2d-i+1}(Y).$$

Accordingly, the correspondence YZ_d does not preserve $N_{(i,2d-i)} \subset N_d$.

Proposition 3.3. *The diagram (3.3) is canonically identified with*

$$\begin{tikzcd}[column sep=tiny] & A_d \times_{\Sigma_{2d}(Y)} H_d(W) \arrow[dl] \arrow[dr] & \\ A_d \times_{\Sigma_{2d}(Y)} \Sigma_{2d}(W) \arrow[dr] & & A_d \times_{\Sigma_{2d}(Y)} \Sigma_{2d}(W) \arrow[dl] \\ & A_d, & \end{tikzcd}$$

obtained from (3.4) by base change along the map $A_d \to \Sigma_{2d}(Y)$ in (2.13).

Proof. It is enough to show that $\mathrm{YZ}_d \cong A_d \times_{\Sigma_{2d}(Y)} H_d(W)$. We will use the construction in the proof of Proposition [3, 3.12]. Define a map

$$\mathrm{YZ}_d \to A_d \times_{\Sigma_{2d}(Y)} H_d(W)$$

as follows. Given

$$((\mathcal{M}_0, \mathcal{L}_0, \phi_0), (\mathcal{M}_1, \mathcal{L}_1, \phi_1), (y_0, y_1), s_0, s_1) \in \widetilde{\mathrm{YZ}}_d(S),$$

we obtain from the first two pieces of data, points $(\mathcal{K}_0, \boldsymbol{a}_0)$ and $(\mathcal{K}_1, \boldsymbol{a}_1)$ of $\Sigma_{2d}(W)(S)$. We have

$$\mathcal{K}_i \cong \underline{\mathrm{Hom}}((\mathcal{L}_i, \mathcal{L}_i^{\sigma}), \mathcal{M}_i|_{W_S}).$$

Let $y \in W(S)$ correspond to the point $(y_0, y_1) \in (Y_0 \times_X Y)(S)$. Then the isomorphisms $\mathcal{L}_0(y_1) \cong \mathcal{L}_1$ and $\mathcal{M}_0(y_0) \cong \mathcal{M}_1$ induce an isomorphism $\mathcal{K}_0(y^{\tau_1} - y^{\tau_3}) \cong \mathcal{K}_1$ sending $\boldsymbol{a}_0$ to $\boldsymbol{a}_1$. This gives a point in $H_d(W)(S)$. Since $(\mathcal{M}_0, \mathcal{L}_0, \phi_0)$ and $(\mathcal{M}_1, \mathcal{L}_1, \phi_1)$ lie over the same point in A_d, we obtain a map

$$\widetilde{\mathrm{YZ}}_d \to A_d \times_{\Sigma_{2d}(Y)} H_d(W),$$

which factors through YZ_d.

Next, we construct a map in the other direction. Suppose given an S-point

$$(\Delta, \xi, \mathcal{K}_0, \boldsymbol{a}_0, \mathcal{K}_1, \boldsymbol{a}_1, y, s) \in A_d \times_{\Sigma_{2d}(Y)} H_d(W).$$

By (2.13), we have points $(\mathcal{M}_i, \mathcal{L}_i, \phi_i) \in N_d(S)$ corresponding to $(\Delta, \xi, \mathcal{K}_i, \boldsymbol{a}_i)$, for $i = 0, 1$. Let's show that there are isomorphisms $\mathcal{M}_1 \cong \mathcal{M}_0(y_0)$ and $\mathcal{L}_1 \cong \mathcal{L}_0(y_1)$ inducing the given isomorphism

$$s : \mathcal{K}_0(y^{\tau_1} - y^{\tau_3}) \cong \mathcal{K}_1.$$

Suppose first that $y \in \varnothing \coprod Y$. Since we work modulo twisting, we may assume $\mathcal{M}_i = (\mathcal{O}_X, \mathcal{M}_i'')$. If we write $\mathcal{K}_i = (\mathcal{K}_i', \mathcal{K}_i'')$, then $\mathcal{L}_i = (\mathcal{K}_i')^{-1}$. By the proof of [3, 3.12], $\mathcal{M}_i''$ is a canonical descent of the line bundle $\mathcal{K}_i'' \otimes \mathcal{L}_i^\sigma$ on Y, down to X. Now, the isomorphism $\mathcal{K}_0(y^{\tau_1} - y^{\tau_3}) \simeq \mathcal{K}_1$ amounts to a pair of isomorphisms

$$\mathcal{K}_0'(-y_1) \simeq \mathcal{K}_1' \quad \text{and} \quad \mathcal{K}_0''(y_1) \simeq \mathcal{K}_1''. \tag{3.5}$$

We deduce from this an isomorphism $\mathcal{L}_0(y_1) \simeq \mathcal{L}_1$. To finish, we must produce an isomorphism $\mathcal{M}_0''(y_0) \simeq \mathcal{M}_1''$. Now, (3.5) gives us an isomorphism $\nu^*(\mathcal{M}_0''(y_0)) \simeq \nu^*\mathcal{M}_1''$, and the descent data defining $\mathcal{M}_0''$ and $\mathcal{M}_1''$ allows us to descend this to the desired isomorphism $\mathcal{M}_0''(y_0) \simeq \mathcal{M}_1''$. The case $y \in Y \coprod \varnothing$ is proved similarly. □

Corollary 3.4. *The stack* YZ_d *is a scheme, and* $\gamma_0, \gamma_1 \colon \mathrm{YZ}_d \to N_d$ *are finite and surjective. Hence, by Proposition* 2.6, *if* $d \geq 2g_1 - 1$ *then* $\dim \mathrm{YZ}_d = 2d - g + 1$.

The correspondence YZ_d induces an endomorphism

$$[\mathrm{YZ}_d] : \beta_*\mathbb{Q}_\ell \to \beta_*\mathbb{Q}_\ell$$

of sheaves on A_d, given by the composition

$$\beta_*\mathbb{Q}_\ell \to \beta_*\gamma_{0*}\gamma_0^*\mathbb{Q}_\ell \simeq \beta_*\gamma_{0*}\mathbb{Q}_\ell \simeq \beta_*\gamma_{1*}\mathbb{Q}_\ell \to \beta_*\mathbb{Q}_\ell.$$

The first and last maps are induced by adjunction, using that γ_0 and γ_1 are finite. Denote by $[\mathrm{YZ}_d]^r$ the r-fold composition of this endomorphism with itself.

Proposition 3.5. *Fix an effective divisor* $D \in \mathrm{Div}(X)$ *of degree* $d \geq 2g_1 - 1$*, and recall the closed subscheme* $A_D \subset A_d$ *and the inclusion* $A_D(\boldsymbol{k}) \subset K$ *of* (2.15)*. The distribution* $\mathbb{I}_r$ *(3.2) satisfies*

$$\mathbb{I}_r(f_D) = \sum_{\substack{\xi \in K \\ \mathrm{Tr}_{K/F}(\xi) = 1}} \mathbb{I}_r(\xi, f_D),$$

where

$$\mathbb{I}_r(\xi, f_D) = \begin{cases} \mathrm{Tr}\big([\mathrm{YZ}_d]^r_{\bar{\xi}} \circ \mathrm{Frob}_\xi;\, (\beta_*\mathbb{Q}_\ell)_{\bar{\xi}}\big) & \text{if } \xi \in A_D(\boldsymbol{k}) \\ 0 & \text{otherwise.} \end{cases}$$

Here $\bar{\xi}$ *is any geometric point above* $\xi : \mathrm{Spec}(\boldsymbol{k}) \to A_D$.

Proof. The proof is similar to the proof of [3, 4.7], so we omit it. One slight difference is the analogue of [8, Lemma 6.11(1)]. Specifically, we must show that the map $\mathrm{Hk}_T^\mu \to \mathrm{Hk}_G^\mu(y)$ discussed there remains a regular local immersion if we replace Hk_T^μ with Hk_A^μ. The tangent complex computations are similar except this time both tangent complexes have 1-dimensional H^0. In particular, it is no longer true in this context that the target is Deligne-Mumford in the neighborhood of a point in the image. Nevertheless, the induced map on H^0 (resp. H^1) is an isomorphism (resp. injection), which is enough to conclude that the map is indeed a regular local immersion of algebraic stacks. □

4. Comparison of traces

4.1. **Representation theory.** In this subsection, ℓ is any prime and all representations are over $\mathbb{Q}_\ell$.

Let $n \geq 1$ be an even integer. For each $0 \leq i \leq n$, view $S_i \times S_{n-i}$ as the subgroup of S_n preserving the subset $\{1, \ldots, i\}$. Let $\mathbf{1}$ denote the trivial representation $\mathbb{Q}_\ell$. Then the induced representation

$$V_i = \mathrm{Ind}_{S_i \times S_{n-i}}^{S_n} \mathbf{1}$$

is the permutation representation for the S_n-action on the ways of dividing the integers $\{1, \ldots, n\}$ into two bins of size i and $n - i$.

Let $V = \bigoplus_{i=0}^n V_i$. By the combinatorial description of V_i, we deduce an isomorphism of S_n-representations

$$V \simeq (\mathbb{Q}_\ell^2)^{\otimes n},$$

where the action of S_n on $(\mathbb{Q}_\ell^2)^{\otimes n}$ is by permuting coordinates.

Recall that the irreducible representations ρ_λ of S_n are indexed by partitions $\lambda \vdash n$. We write $\lambda = [\lambda_1, \ldots, \lambda_k]$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$ and $\sum_i \lambda_i = n$. We will also interpret the symbol $[n, 0]$ to mean $[n]$. Write

$$V = \bigoplus_{\lambda \vdash n} V_\lambda,$$

$$V_i = \bigoplus_{\lambda \vdash n} V_{i,\lambda}$$

where V_λ and $V_{i,\lambda}$ are the ρ_λ-isotypic components of V and V_i, respectively.

Proposition 4.1. *Let $\lambda \vdash n$. Then,*

(a) $V_{i,\lambda} \neq 0$ *if and only if* $\lambda = [k, n-k]$ *with* $n - k \leq i \leq k$, *and*
(b) *If* $V_{i,\lambda} \neq 0$, *then it has multiplicity* 1, *i.e.* $V_{i,\lambda} \simeq \rho_\lambda$.

Proof. This is a simple application of the Littlewood-Richardson rule. $\square$

Note that the diagonal action of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ on V commutes with the S_n action. There is also a lie algebra action of $\mathfrak{sl}_2$ on $V \simeq (\mathbb{Q}_\ell^2)^{\otimes 2n}$, given by

$$\begin{aligned} X \cdot v_1 \otimes & v_2 \otimes \cdots \otimes v_n \\ &= Xv_1 \otimes \cdots \otimes v_n + v_1 \otimes Xv_2 \otimes \cdots \otimes v_n + \ldots + v_1 \otimes \cdots \otimes Xv_n. \end{aligned}$$

This too commutes with the S_n action. By Schur-Weyl duality, each V_λ is isomorphic to $M_\lambda \otimes \rho_\lambda$, for some irreducible $\mathfrak{sl}_2$-representation M_λ (on which S_n acts trivially).

Corollary 4.2. *If* $\lambda = [k, n-k]$, *then* $V_\lambda \simeq \mathrm{Sym}^{2k-n}\mathbb{Q}_\ell^2 \otimes \rho_\lambda$.

Proof. By Proposition 4.1, we have $\dim V_\lambda = (2k - n + 1) \dim \rho_\lambda$. Since $\mathrm{Sym}^{2k-n}\mathbb{Q}_\ell^2$ is the unique irreducible representation of $\mathfrak{sl}_2$ of dimension $2k - n + 1$, we must have $M_\lambda \simeq \mathrm{Sym}^{2k-n}\mathbb{Q}_\ell^2$. $\square$

Let $\{e_+, e_-\}$ be a basis for $\mathbb{Q}_\ell^2$. For $\epsilon \in \{\pm\}^n$, let e_ϵ be the corresponding basis element of V. Also let ϵ_i be the element

$$(+,+,\ldots,+,-,+,\ldots,+,+) \in \{\pm\}^n,$$

with a minus sign in the i-th coordinate.

Let $H\colon V \to V$ be the $\mathbb{Q}_\ell$-linear map determined by

$$H(e_\epsilon) = \sum_{i=1}^n e_{\epsilon\epsilon_i}.$$

Lemma 4.3. *H commutes with the S_n action on V.*

In fact, the map H is simply the action of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2$ on V. Now suppose $\lambda = [k, n-k]$. Since H commutes with the S_n-action, it acts on V_λ as $H_\lambda \otimes 1$, for some $H_\lambda \in \mathrm{End}(\mathrm{Sym}^{2k}\mathbb{Q}_\ell^2)$.

Corollary 4.4. *Let $\lambda = [k, n-k]$. Then the eigenvalues of H_λ are*

$$\{-2k, -2k+2, \ldots, -2, 0, 2, \ldots, 2k-2, 2k\}$$

each appearing with multiplicity one.

Proof. The matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is usually denoted $e + f$ in the representation theory of $\mathfrak{sl}_2$. It is conjugate to the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, often denoted h. Thus, the characteristic polynomial of H_λ is the same as that of h acting on the space $\mathrm{Sym}^{2k}\mathbb{Q}_\ell^2$. The natural basis for $\mathrm{Sym}^{2k}\mathbb{Q}_\ell^2$ are eigenvectors for h with eigenvalues precisely the ones stated. □

4.2. **Local systems.** Now let ℓ be a prime different from char $\boldsymbol{k}$. Recall the split double cover $W = Y \coprod Y$ of Y. Let $U_{2d}(Y) \subset Y^{2d}$ be the open subscheme parametrizing $2d$-tuples of *distinct* points on Y, and let $U_{2d}(W) \subset W^{2d}$ be its preimage under the morphism $W^{2d} \to Y^{2d}$. Thus we have a cartesian diagram

$$\begin{tikzcd} U_{2d}(W) \arrow[r] \arrow[d] & W^{2d} \arrow[d] \\ U_{2d}(Y) \arrow[r] & Y^{2d}, \end{tikzcd}$$

in which the horizontal arrows are open immersions with dense image, and the vertical arrows are finite étale. Both W^n and $U_{2d}(W)$ are disconnected. The connected components are

$$W^{2d} = \coprod_{i=0}^{2d} (Y^i \times Y^{2d-i})$$

$$U_{2d}(W) = \coprod_{i=0}^{2d} U_{i,2d-i}(W).$$

Taking the appropriate quotients, and using the isomorphisms of (2.14), we obtain a cartesian diagram

$$\begin{tikzcd} S_i \times S_{2d-i}\backslash U_{i,2d-i}(W) \arrow[r] \arrow[d, "b_i"'] & \Sigma_{i,2d-i}(W) \arrow[d, "\mathrm{Nm}"] \\ S_{2d}\backslash U_{2d}(Y) \arrow[r, "u"'] & \Sigma_{2d}(Y), \end{tikzcd} \tag{4.1}$$

in which the horizontal arrows are open immersions, the vertical arrows are finite, and b_i is étale.

The Galois cover

$$U_{i,2d-i}(W) \to S_{2d}\backslash U_{2d}(Y),$$

has group S_{2d}, and the local system $b_{i*}\mathbb{Q}_\ell$ on $S_{2d}\backslash U_{2d}(Y)$ corresponds to the induced representation V_i from the previous section. We define

$$b_*\mathbb{Q}_\ell := \bigoplus_{i=0}^{2d} b_{i*}\mathbb{Q}_\ell.$$

This is a local system on $S_{2d}\backslash U_{2d}(Y)$ corresponding to the S_{2d}-representation $V \simeq (\mathbb{Q}_\ell^2)^{\otimes 2d}$. If $\lambda \vdash n$, we let $\widetilde{L}_\lambda$ be the local system on $S_{2d}\backslash U_{2d}(Y)$ corresponding to ρ_λ.

Define A_d° as the cartesian product

$$\begin{tikzcd} A_d^\circ \arrow[r, "v"] \arrow[d, "\pi"'] & A_d \arrow[d, "\nu^\sharp"] \\ S_{2d}\backslash U_{2d}(Y) \arrow[r, "u"'] & \Sigma_{2d}(Y), \end{tikzcd}$$

and set $L_\lambda = v_*\pi^*\widetilde{L}_\lambda$. Recall from §2 the maps $\beta_i \colon N_{(i,2d-i)} \to A_d$, for each $i = 0, \ldots, 2d$.

Proposition 4.5. *For each $i \in \{0, 1, \ldots, d\}$, there is an isomorphism*

$$\beta_{i*}\mathbb{Q}_\ell \simeq (\beta_{2d-i})_*\mathbb{Q}_\ell$$

and a decomposition

$$\beta_{i*}\mathbb{Q}_\ell = \bigoplus_{k=0}^{i} L_{[2d-k,k]}.$$

Proof. By Proposition 4.1, we have

$$b_{i*}\mathbb{Q}_\ell = \bigoplus_{k=0}^{i} \widetilde{L}_{[2d-k,k]}.$$

On the other hand, it follows from proper base change and the smoothness of $N_{(i,2d-i)}$ that

$$\beta_{i*}\mathbb{Q}_\ell \simeq v_*\pi^*b_{i*}\mathbb{Q}_\ell;$$

c.f. [3, Prop. 5.3]. The proposition now follows. $\square$

Proposition 4.6. *There is a decomposition*

$$\beta_*\mathbb{Q}_\ell = \bigoplus_{k=0}^{d} \left(\mathrm{Sym}^{2d-2k}\mathbb{Q}_\ell^2 \otimes L_{[2d-k,k]}\right),$$

and the endomorphism $[\mathrm{YZ}_d]$ *stabilizes each summand. The action of* $[\mathrm{YZ}_d]$ *on* $\mathrm{Sym}^{2d-2k}\mathbb{Q}_\ell^2 \otimes L_{[2d-k,k]}$ *is of the form* $H_k \otimes 1$*, for some* H_k *having characteristic polynomial*

$$\det(t - H_k) = \prod_{j=k-d}^{d-k} (t - 2j).$$

Proof. The proof of the first part is as in Proposition 4.5, this time using the decomposition

$$V = \bigoplus_{k=0}^{d} V_{[2d-k,k]}$$

and Corollary 4.2. The S_{2d}-map $H\colon V \to V$ corresponds to a map $H\colon b_*\mathbb{Q}_\ell \to b_*\mathbb{Q}_\ell$ of local systems. By Proposition 3.3, the map $[\mathrm{YZ}_d]|_{A_d^\circ}$ is identified with π^*H. Thus, the second part of the proposition follows from Corollary 4.4. $\square$

4.3. **The key identity.** Fix an auxiliary prime $\ell \neq \mathrm{char}(\boldsymbol{k})$, and define $\mathscr{H}_\ell = \mathscr{H} \otimes \mathbb{Q}_\ell$, the ℓ-adic analogue of the $\mathbb{Q}$-algebra $\mathscr{H}$ of §2.1.

The Hecke algebra $\mathscr{H}_\ell$ acts on the ℓ-adic cohomology group

$$V = H_c^{2r}(\mathrm{Sht}_G^\mu \otimes_{\boldsymbol{k}} \bar{\boldsymbol{k}}, \mathbb{Q}_\ell)(r),$$

as in [8, §7.1]. The cycle class map $\mathrm{cl}\colon \mathrm{Ch}_{c,r}(\mathrm{Sht}_G^r) \to V$ is $\mathscr{H}$-equivariant, and the cup product

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Q}_\ell \tag{4.2}$$

pulls back to the intersection pairing on the Chow group.

Recalling the map $\mathscr{H} \to \mathbb{Q}[\mathrm{Pic}_X(\boldsymbol{k})]^{\iota_{\mathrm{Pic}}}$ appearing in (2.2), define

$$\begin{aligned}
\widetilde{\mathscr{H}_\ell} &= \mathrm{Image}\big(\mathscr{H}_\ell \to \mathrm{End}_{\mathbb{Q}_\ell}(V) \times \mathrm{End}_{\mathbb{Q}_\ell}(\mathscr{A}_\ell) \times \mathbb{Q}_\ell[\mathrm{Pic}_X(\boldsymbol{k})]^{\iota_{\mathrm{Pic}}}\big)\\
\overline{\mathscr{H}_\ell} &= \mathrm{Image}\big(\mathscr{H}_\ell \to \mathrm{End}_{\mathbb{Q}_\ell}(V) \times \mathbb{Q}_\ell[\mathrm{Pic}_X(\boldsymbol{k})]^{\iota_{\mathrm{Pic}}}\big)\\
\mathscr{H}_{\mathrm{aut},\ell} &= \mathrm{Image}\big(\mathscr{H}_\ell \to \mathrm{End}_{\mathbb{Q}_\ell}(\mathscr{A}_\ell) \times \mathbb{Q}_\ell[\mathrm{Pic}_X(\boldsymbol{k})]^{\iota_{\mathrm{Pic}}}\big).
\end{aligned}$$

These are finite type $\mathbb{Q}_\ell$-algebras, related by surjections

$$\begin{tikzcd}
& \widetilde{\mathscr{H}_\ell} \arrow[dl] \arrow[dr] & \\
\overline{\mathscr{H}_\ell} \arrow[dr] & & \mathscr{H}_{\mathrm{aut},\ell} \arrow[dl] \\
& \mathbb{Q}_\ell[\mathrm{Pic}_X(\boldsymbol{k})]^{\iota_{\mathrm{Pic}}}. &
\end{tikzcd}$$

Recalling the $\mathbb{Q}$-algebra $\mathscr{H}_{\mathrm{aut}}$ of §2.1, there is a canonical isomorphism

$$\mathscr{H}_{\mathrm{aut}} \otimes \mathbb{Q}_\ell \cong \mathscr{H}_{\mathrm{aut},\ell}.$$

For any $f \in \mathscr{H}$, the function $\mathbb{J}(f,s)$ of (2.7) is a Laurent polynomial in q^s with rational coefficients. Setting

$$\mathbb{J}_r(f) = (\log q)^{-r} \frac{d^r}{ds^r} \mathbb{J}(f,s)\big|_{s=0},$$

we obtain a linear functional $\mathbb{J}_r : \mathscr{H} \to \mathbb{Q}$. The following result shows that this agrees with the linear functional $\mathbb{I}_r : \mathscr{H} \to \mathbb{Q}$ defined by (3.2).

Theorem 4.7. *If $f \in \mathscr{H}$, then $\mathbb{I}_r(f) = \mathbb{J}_r(f)$. Moreover, the $\mathbb{Q}_\ell$-linear extensions of $\mathbb{I}_r$ and $\mathbb{J}_r$ to $\mathscr{H}_\ell \to \mathbb{Q}_\ell$ factor through $\widetilde{\mathscr{H}_\ell}$.*

Proof. The compatibility of the cup product pairing (4.2) with the intersection pairing on the Chow group implies that the $\mathbb{Q}_\ell$-linear extension $\mathbb{I}_r : \mathscr{H}_\ell \to \mathbb{Q}_\ell$ factors through $\overline{\mathscr{H}_\ell}$. The final claim of Proposition 2.4 implies that the $\mathbb{Q}_\ell$-linear extension $\mathbb{J}_r : \mathscr{H}_\ell \to \mathbb{Q}_\ell$ factors through $\mathscr{H}_{\mathrm{aut},\ell}$. It follows that both $\mathbb{I}_r$ and $\mathbb{J}_r$ factor through the quotient $\widetilde{\mathscr{H}_\ell}$.

It remains to prove that $\mathbb{I}_r(f) = \mathbb{J}_r(f)$ for all $f \in \mathscr{H}$. Assume first that $f = f_D$ for some effective divisor $D \in \mathrm{Div}(X)$ of degree $d \geq 2g_1 - 1$. For each $\xi \in A_D(\boldsymbol{k})$, define

$$\mathbb{I}_r(\xi, f_D) = \mathrm{Tr}\big([\mathrm{YZ}_d]^r_{\bar{\xi}} \circ \mathrm{Frob}_\xi; (\beta_* \mathbb{Q}_\ell)_{\bar{\xi}}\big),$$

and

$$\mathbb{J}_r(\xi, f_D) = 2^r \sum_{i=0}^{2d} (i-d)^r \mathrm{Tr}\big(\mathrm{Frob}_\xi; (\beta_{(i,2d-i)*}\mathbb{Q}_\ell)_{\bar{\xi}}\big)$$

By Proposition 3.5 and Theorem 2.7,

$$\mathbb{I}_r(f_D) = \sum_{\xi \in A_D(\boldsymbol{k})} \mathbb{I}_r(\xi, f_D).$$

$$\mathbb{J}_r(f_D) = \sum_{\xi \in A_D(\boldsymbol{k})} \mathbb{J}_r(\xi, f_D),$$

so it suffices to show that $\mathbb{I}_r(\xi, f_D) = \mathbb{J}_r(\xi, f_D)$ for all ξ.

There is a decomposition

$$\beta_* \mathbb{Q}_\ell = \bigoplus_{i=0}^{2d} \beta_{(i,2d-i)*} \mathbb{Q}_\ell,$$

but the endomorphism $[\mathrm{YZ}_d]$ of $\beta_* \mathbb{Q}_\ell$ does not preserve this decomposition; see Remark 3.2. We instead consider the decomposition

$$\beta_* \mathbb{Q}_\ell = \bigoplus_{k=0}^{d} \left(\mathrm{Sym}^{2d-2k}(\mathbb{Q}_\ell^2) \otimes L_{[2d-k,k]} \right),$$

of Proposition 4.6, which has the property that $[\mathrm{YZ}_d]$ takes the form $H_k \otimes 1$ on each summand. We compute

$$\begin{aligned}
\mathbb{I}_r(\xi, f_D) &= \mathrm{Tr}\big([\mathrm{YZ}_d]^r_{\bar\xi} \circ \mathrm{Frob}_\xi; (\beta_* \mathbb{Q}_\ell)_{\bar\xi}\big) \\
&= \sum_{k=0}^{d} \mathrm{Tr}\left(H_k^r \otimes \mathrm{Frob}_\xi; \mathrm{Sym}^{2d-2k}(\mathbb{Q}_\ell^2) \otimes L_{[2d-k,k]_{\bar\xi}}\right) \\
&= \sum_{k=0}^{d} \mathrm{Tr}\left(H_k^r\right) \mathrm{Tr}\left(\mathrm{Frob}_\xi; L_{[2d-k,k]_{\bar\xi}}\right) \\
&= \sum_{k=0}^{d} \sum_{j=k-d}^{d-k} (2j)^r \mathrm{Tr}\left(\mathrm{Frob}_\xi; L_{[2d-k,k]_{\bar\xi}}\right) \\
&= 2^{r+1} \sum_{k=0}^{d} \sum_{j=0}^{d-k} j^r \mathrm{Tr}\left(\mathrm{Frob}_\xi; L_{[2d-k,k]_{\bar\xi}}\right)
\end{aligned}$$

where the second to last equality follows from the last statement in Proposition 4.6.

On the other hand, using Proposition 4.5 we compute

$$\begin{aligned}
\mathbb{J}_r(\xi, f_D) &= 2^r \sum_{i=0}^{2d} (d-i)^r \mathrm{Tr}\big(\mathrm{Frob}_\xi; (\beta_{(i,2d-i)*}\mathbb{Q}_\ell)_{\bar\xi}\big) \\
&= 2^{r+1} \sum_{i=0}^{d} (d-i)^r \mathrm{Tr}\big(\mathrm{Frob}_\xi; (\beta_{(i,2d-i)*}\mathbb{Q}_\ell)_{\bar\xi}\big) \\
&= 2^{r+1} \sum_{i=0}^{d} (d-i)^r \sum_{k=0}^{i} \mathrm{Tr}\big(\mathrm{Frob}_\xi; \big(L_{[2d-k,k]}\big)_{\bar\xi}\big) \\
&= 2^{r+1} \sum_{k=0}^{d} \sum_{j=0}^{d-k} j^r \mathrm{Tr}\big(\mathrm{Frob}_\xi; \big(L_{[2d-k,k]}\big)_{\bar\xi}\big).
\end{aligned}$$

We conclude that $\mathbb{I}_r(\xi, f_D) = \mathbb{J}_r(\xi, f_D)$, and hence $\mathbb{I}_r(f_D) = \mathbb{J}_r(f_D)$ for all effective D of degree $d \geq 2g_1 - 1$.

The proof of [8, Theorem 9.2] shows that the image of $\mathscr{H}_\ell \to \widetilde{\mathscr{H}_\ell}$ is generated as $\mathbb{Q}_\ell$-vector space by the images of $f_D \in \mathscr{H}$ as D ranges over all effective divisors on X of degree $d \geq 2g_1 - 1$. Therefore $\mathbb{I}_r = \mathbb{J}_r$. □

4.4. **Proof of main theorems.** The main theorems of the introduction follow from Theorem 4.7, by modifying the formal arguments of [3, §5]. The additional subtlety in our context is that the intersection pairing appearing in the definition of $\mathbb{I}_r(f)$ depends on the auxiliary integer d, which is itself a function of f. For the convenience of the reader, we spell out the argument below.

According to [8, (9.5)], there is a canonical $\mathbb{Q}_\ell$-algebra decomposition

$$\widetilde{\mathscr{H}_\ell} = \widetilde{\mathscr{H}}_{\ell,\mathrm{Eis}} \oplus \Big(\bigoplus_{\mathfrak{m}} \widetilde{\mathscr{H}}_{\ell,\mathfrak{m}} \Big), \tag{4.3}$$

where $\mathfrak{m}$ runs over the finitely many maximal ideals $\mathfrak{m} \subset \widetilde{\mathscr{H}_\ell}$ that do not contain the kernel of the projection

$$\widetilde{\mathscr{H}_\ell} \to \mathbb{Q}_\ell[\mathrm{Pic}_X(\boldsymbol{k})]^{\iota_{\mathrm{Pic}}}. \tag{4.4}$$

For each such $\mathfrak{m}$ the localization $\widetilde{\mathscr{H}}_{\ell,\mathfrak{m}}$ is a finite (hence Artinian) $\mathbb{Q}_\ell$-algebra. If we denote by $E_\mathfrak{m}$ its residue field, then Hensel's lemma implies that the quotient map $\widetilde{\mathscr{H}}_{\ell,\mathfrak{m}} \to E_\mathfrak{m}$ admits a unique section, which makes $\widetilde{\mathscr{H}}_{\ell,\mathfrak{m}}$ into an Artinian local $E_\mathfrak{m}$-algebra.

The decomposition (4.3) induces a decomposition of $\widetilde{\mathscr{H}_\ell}$-modules

$$V = V_{\mathrm{Eis}} \oplus \Big(\bigoplus_{\mathfrak{m}} V_\mathfrak{m} \Big), \tag{4.5}$$

in which each localization $V_\mathfrak{m}$ is a finite-dimensional $E_\mathfrak{m}$-vector space. It follows from [8, Corollary 7.15] that this decomposition is orthogonal with respect to the cup product pairing. Moreover, the self adjointness of the action of $\mathscr{H}_\ell$ with respect to the cup product pairing (4.2) implies that there is a unique symmetric $E_\mathfrak{m}$-bilinear pairing

$$\langle \cdot, \cdot \rangle_{E_\mathfrak{m}} : V_\mathfrak{m} \times V_\mathfrak{m} \to E_\mathfrak{m}$$

such that $\mathrm{Tr}_{E_\mathfrak{m}/\mathbb{Q}_\ell} \langle \cdot, \cdot \rangle_{E_\mathfrak{m}} = \langle \cdot, \cdot \rangle$.

Define $[\mathrm{Sht}^\mu_T]_\mathfrak{m} \in V_\mathfrak{m}$ to be the projection of the cycle class $\mathrm{cl}([\mathrm{Sht}^\mu_T]) \in V$. Next, define $[\mathrm{Sht}^\mu_A]_\mathfrak{m} \in V_\mathfrak{m}$ to be the projection of $\mathrm{cl}([\mathrm{Sht}^{\mu,\le d+r}_A]) \in V$, where $d \in \mathbb{Z} \subset \mathcal{D}$ is any integer such that

$$[\mathrm{Sht}^\mu_T]_\mathfrak{m} \in H^{2r}_c(\mathrm{Sht}^{\mu,\le d}_G \otimes_{\boldsymbol{k}} \bar{\boldsymbol{k}}, \mathbb{Q}_\ell)(r) \subset V.$$

We may form the intersection pairing

$$\langle [\mathrm{Sht}^\mu_A]_\mathfrak{m}, [\mathrm{Sht}^\mu_T]_\mathfrak{m} \rangle_{E_\mathfrak{m}} \in E_\mathfrak{m}.$$

Some maximal ideals $\mathfrak{m} \subset \widetilde{\mathscr{H}_\ell}$ appearing in (4.3) are attached to cuspidal automorphic forms. Fix an unramified cuspidal automorphic representation $\pi \subset \mathcal{A}_{\mathrm{cusp}}(G)$. As in §2.1, such a representation determines a homomorphism

$$\mathscr{H}_{\mathrm{aut}} \to \mathscr{H}_{\mathrm{cusp}} \xrightarrow{\lambda_\pi} \mathbb{C}$$

whose image is a number field E_π. The induced map

$$\widetilde{\mathscr{H}_\ell} \to \mathscr{H}_{\mathrm{aut},\ell} \xrightarrow{\lambda_\pi} E_\pi \otimes \mathbb{Q}_\ell \cong \prod_{\mathfrak{l} \mid \ell} E_{\pi,\mathfrak{l}},$$

determines, for every prime $\mathfrak{l} \mid \ell$ of E_π, a surjection $\lambda_{\pi,\mathfrak{l}} : \widetilde{\mathscr{H}_\ell} \to E_{\pi,\mathfrak{l}}$ whose kernel is one of those maximal ideals

$$\mathfrak{m} = \ker(\lambda_{\pi,\mathfrak{l}}) \tag{4.6}$$

appearing in the decomposition (4.5). This is a consequence of the isomorphism (2.3).

Recalling the period integrals $\mathscr{P}_A$ and $\mathscr{P}_T$ of §2.3, for every unramified cuspidal automorphic representation $\pi \subset \mathcal{A}_{\mathrm{cusp}}(G)$ define

$$C(\pi, s) = \frac{\mathscr{P}_A(\phi, s)\mathscr{P}_T(\overline{\phi}, \eta)}{\langle \phi, \phi \rangle_{\mathrm{Pet}}}.$$

Here, ϕ is any nonzero vector in π^U.

Proposition 4.8. *The complex number*

$$C_r(\pi) = (\log q)^{-r} \cdot \frac{d^r}{ds^r} C(\pi, s)\big|_{s=0}$$

satisfies $C_r(\pi)^\sigma = C_r(\pi^\sigma)$ *for all* $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$*. In particular, it lies in* E_π*.*

Proof. As in [3, Prop. 5.7]. □

Theorem 4.9. *Let* $\mathfrak{m} \subset \widetilde{\mathscr{H}_\ell}$ *be a maximal ideal that does not contain the kernel of (4.4).*

(1) If $\mathfrak{m}$ *is of the form (4.6) for an unramified cuspidal automorphic representation* π *and a place* $\mathfrak{l} \mid \ell$ *of* E_π*, the equality*

$$\langle [\mathrm{Sht}_A^\mu]_{\mathfrak{m}},\, [\mathrm{Sht}_T^\mu]_{\mathfrak{m}} \rangle_{E_{\mathfrak{m}}} = C_r(\pi)$$

holds in $E_{\mathfrak{m}} = E_{\pi,\mathfrak{l}}$*.*

(2) If $\mathfrak{m}$ *is not of the form (4.6) then*

$$\langle [\mathrm{Sht}_A^\mu]_{\mathfrak{m}},\, [\mathrm{Sht}_T^\mu]_{\mathfrak{m}} \rangle_{E_{\mathfrak{m}}} = 0.$$

Proof. Given Proposition 4.7, the proof follows that of [8, Theorem 1.6]. □

4.5. **The proof of Theorem 1.** As in the introduction, let $[\mathrm{Sht}_T^\mu]$ be the pushforward of the fundamental class under

$$\theta_T^\mu \colon \mathrm{Sht}_T^\mu \to \mathrm{Sht}_G^\mu,$$

and let $\widetilde{W}_T \subset \mathrm{Ch}_{c,r}(\mathrm{Sht}_G^\mu)$ be the $\mathscr{H}$-submodule it generates.

Let $d \in \mathbb{Z} \subset \mathcal{D}$ be any large enough integer so that the finite dimensional subspace

$$\Big(\bigoplus_{\mathfrak{m}} \widetilde{\mathscr{H}_{\ell,\mathfrak{m}}}\Big) \cdot \mathrm{cl}([\mathrm{Sht}_T^\mu]) \subset H^{2r}(\mathrm{Sht}_G^\mu \otimes_{\boldsymbol{k}} \bar{\boldsymbol{k}}, \mathbb{Q}_\ell(r))$$

is supported on $\mathrm{Sht}_G^{\mu, \leq d-r}$; c.f. (4.3). Then define $[\mathrm{Sht}_A^\mu]$ to be the pushforward of the fundamental class under

$$\theta_A^\mu \colon \mathrm{Sht}_A^{\mu, \leq d} \to \mathrm{Sht}_G^\mu.$$

Let $\widetilde{W}_A \subset \mathrm{Ch}_{c,r}(\mathrm{Sht}_G^\mu)$ be the $\mathscr{H}$-submodule generated by $[\mathrm{Sht}_A^\mu]$.

Define quotients

$$W_T = \widetilde{W}_T / \{c \in \widetilde{W}_T : \langle c, \widetilde{W}_A \rangle = 0\}$$
$$W_A = \widetilde{W}_A / \{c \in \widetilde{W}_A : \langle c, \widetilde{W}_T \rangle = 0\},$$

so that the intersection pairing descends to $\langle \cdot\,, \cdot \rangle : W_A \times W_T \to \mathbb{Q}$.

Proposition 4.10. *The actions of $\mathscr{H}$ on W_T and W_A factor through the quotient*

$$\mathscr{H} \to \mathscr{H}_{\mathrm{aut}} \cong \mathscr{H}_{\mathrm{cusp}} \times \mathbb{Q}[\mathrm{Pic}_X(\boldsymbol{k})]^{\iota_{\mathrm{Pic}}}$$

defined in §2.1.

Proof. By Proposition 2.4 the distribution $\mathbb{J}_r(f)$ only depends on the image of f under $\mathscr{H} \to \mathscr{H}_{\mathrm{aut}}$. By Proposition 4.7 the same is true of the distribution $\mathbb{I}_r(f)$ defined by (3.2), and the claim follows as in [8, Cor. 9.4]. □

It follows from the discussion of §2.1 that $\mathscr{H}_{\mathrm{cusp},\mathbb{R}} = \mathscr{H}_{\mathrm{cusp}} \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to a product of copies of $\mathbb{R}$, indexed by the unramified cuspidal automorphic representations π. For each such π, let $e_\pi \in \mathscr{H}_{\mathrm{cusp},\mathbb{R}}$ be the corresponding idempotent. Using Proposition 4.10, these idempotents induce a decomposition, for $* \in \{A, T\}$,

$$W_*(\mathbb{R}) = W_{*,\mathrm{cusp}} \oplus W_{*,\mathrm{Eis}} = \left(\bigoplus_\pi W_{*,\pi} \right) \oplus W_{*,\mathrm{Eis}}$$

with sum over unramified cuspidal π, and where $W_{*,\pi} \subset W_*(\mathbb{R})$ is the λ_π-eigenspace of $\mathscr{H}$.

The following is Theorem 1 of the introduction.

Theorem 4.11. *For $* \in \{A, T\}$, let $[\mathrm{Sht}^\mu_*]_\pi$ denote the projections of the images of $[\mathrm{Sht}^\mu_*]$ to the summand $W_{*,\pi}$. Then*

$$\langle [\mathrm{Sht}^\mu_A]_\pi, [\mathrm{Sht}^\mu_T]_\pi \rangle = C_r(\pi).$$

Proof. This follows from Theorem 4.9, as in [3, Thm. 5.10]. □

Theorems 2 and 3 are immediate corollaries of Theorem 1.

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