Transfer and Fourier Transform

Let K be a local field.

$\psi : K \to \mathbb{C}$ additive character, non-trivial

D: central division algebra over K. of dim n^2

$D \xrightarrow{tr} K$

$D^\times \xrightarrow{Nm} K^\times$

$$\left(\begin{array}{ccc} D(K) \hookrightarrow & D(\bar{K}) \cong \mathrm{Mat}_{n\times n}(\bar{K}) \\ & \downarrow Tr \\ & K \subseteq \bar{K} \end{array} \right)$$

Define for $d \in D$,

$$P_d(t) = Nm(t-d) \in K[t]$$

Lemma

$$\left\{ \begin{array}{c} D^\times\text{- conjugacy} \\ \text{classes on } D \end{array} \right\} \overset{bij}{\underset{\eta}{\longleftrightarrow}} \left\{ \begin{array}{c} \text{monic degree } n \text{ polys} \\ P(t) \in K[t] \text{ that are} \\ \text{powers of irreducibles} \end{array} \right\}$$

Pf: $d \in D \longmapsto P_d(t)$

Note: if $d \in D^\times$, then $K[d]$ is a field of degree $k | n$

$$q_d(t) = \text{minpoly}(d)$$

$$\deg(q_d) = k$$

$$P_d(t) = q_d(t)^{n/k}$$

η injective by Noether–Skolem

$$\begin{tikzcd} L \arrow[r, hook, "i"] & D \\ L \arrow[ur, hook, "j"'] & \end{tikzcd} \implies \exists\, \alpha \in D^\times \text{ such that}$$

$$j(\ell) = \alpha\, i(\ell)\, \alpha^{-1}.$$

$\underline{\eta \text{ surjective}}$: this is the claim that every field extension L/K of degree k, $k \mid n$, embeds in D.

Enough to show this for $k = n$.

$\iff$ every L/K of degree n <u>splits</u> D, i.e. $D \otimes_K L \cong M_{n\times n}(L)$.

$$Br(K) = \{\text{CSA over } K\}/\sim$$

$$= \{\text{Division algebras over } K\}/\sim$$

We have $\mathrm{Br}(K) \cong H^2(\mathrm{Gal}(K^{sep}/K), K^{sep,\times})$
$$\underset{\mathrm{inv}}{\cong} \mathbb{Q}/\mathbb{Z}$$

$$\begin{array}{ccc} \mathrm{Br}(K) & \xrightarrow{\mathrm{res}} & \mathrm{Br}(L) \\ D & \longmapsto & D \otimes_K L \\ \mathbb{Q}/\mathbb{Z} & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ \mathrm{inv}(D) & \longmapsto & \mathrm{inv}(D \otimes_K L) = [L:K]\,\mathrm{inv}(D) \end{array}$$

But D contains some maximal subfield L' of degree n.

So $\quad n \cdot \mathrm{inv}(D) = 0$

$$\Rightarrow 0 = [L:K]\ \mathrm{inv}(D) = \mathrm{inv}(D \otimes_K L)$$

$$\Rightarrow \quad D \otimes_K L \cong \mathrm{Mat}_{n\times n}(L)$$

□

Define Fourier transform $\mathcal{F}$ on $C_c^\infty(D)$

$f \in C_c^\infty(D)$

$$(\mathcal{F}f)(d) = \int \psi(\mathrm{Tr}(dx))\, f(x)\, dx$$

$x \in D$

dx chosen so that $\mathcal{F}$ is unitary.

Let $\tilde{C}(D) \subseteq C_c^\infty(D)$ of Ad-invariant functions.

By Lemma, if D, D' two division algebras of degree n over K, then $\tilde{C}(D) \underset{i}{\cong} \tilde{C}(D')$

Check: $\mathcal{F}_D$ preserves $\tilde{C}(D)$.

Write $\tilde{\mathcal{F}} = \mathcal{F}|_{\tilde{C}(D)}$

Thm: $i \circ \tilde{\mathcal{F}}_D = \tilde{\mathcal{F}}_{D'} \circ i$

Pf: Proof will be global.

Assume for simplicity that $K = \mathbb{F}_q((t))$

Let F be a global field.

Let V_F be its places.

For $v \in V_F$, F_v completion.

Fix F such that $\exists\, w_1, w_2 \in X_F$

such that $F_w \cong F_{w'} \cong K$.

Now choose division algebras

B, B' over F such that

$B_w = B \otimes_F F_w \cong D$ $\qquad$ $B'_w \cong D'$

$B_{w'} \cong D^{opp}$ $\qquad$ $B'_{w'} \cong D'^{opp}$

For $v \notin S := \{w, w'\}$

$$B_v \cong \mathrm{Mat}_{n\times n}(F_v)$$

$$B'_v \cong \mathrm{Mat}_{n\times n}(F_v)$$

$$\tilde{C}_S = \tilde{C}(D) \otimes \tilde{C}(D^{opp})$$

$$\tilde{C}'_S = \tilde{C}(D') \otimes \tilde{C}(D'^{opp})$$

$$i_S : \tilde{C}_S \xrightarrow{\sim} \tilde{C}'_S$$

$\tilde{\mathcal{F}}_S$ $\qquad$ $\tilde{\mathcal{F}}'_S$

Enough to show $\boxed{i_S \circ \tilde{\mathcal{F}}_S = \tilde{\mathcal{F}}'_S \circ i_S}$

$\mathbb{A} = \mathbb{A}_F$

$$\boxed{C = C_c^\infty(B(\mathbb{A}))}$$

$$C' = C_c^\infty(B'(\mathbb{A}))$$

$$\psi : \mathbb{A}/F \longrightarrow \mathbb{C}^\times$$

$$\mathcal{F} = \mathcal{F}_S \otimes \mathcal{F}^S$$

$$\mathcal{F}' = \mathcal{F}'_S \otimes \mathcal{F}^S$$

Define $\delta : C \longrightarrow \mathbb{C}$

$\delta' : C' \longrightarrow \mathbb{C}$

$$\delta(f) = \sum_{x \in B(F)} f(x)$$

$$\delta'(f) = \sum_{x \in B'(F)} f(x)$$

Poisson Summation:

$$\delta(\mathcal{F}f) = \delta(f)$$

$$\delta'(\mathcal{F}'f) = \delta'(f)$$

<u>Note</u>: $\mathcal{B} \cong F^{n^2}$ as F-spaces.
$\mathcal{B}' \cong F^{n^2}$

<u>Prop</u>: Let $f_S \in \tilde{C}_S$ be such that $\boxed{i_S(f_S) = f_S'}$.
$f_S' \in \tilde{C}_S'$

Let $\boxed{x \in B(F)}$ be such that $\boxed{P_x(t) = P_{x'}(t)}$
$x' \in B'(F)$

Then $\boxed{\tilde{\mathcal{F}}_S(f_S)(x_S) = \tilde{\mathcal{F}}_S'(f_S')(x_S')}$

Note: This implies the desired result $\boxed{i \circ \tilde{\mathcal{F}}_S = \tilde{\mathcal{F}}_S' \circ i}$ since $B(F)$ is dense in $B_S = B(F_w) \times B(F_{w'})$

(and since if $L \hookrightarrow B$ then $L \hookrightarrow B'$ as well).

<u>Proof of Prop</u>: Choose $h^S \in C^S$ so that:

a) $h^s(-x^s) = 1$

b) $\operatorname{supp}\left(\mathcal{F}_s(f_s) \otimes \underline{h^s}\right) \cap B(F) \subseteq \{x\}$.

c) $\operatorname{supp}\left(\mathcal{F}'_s(f'_s) \otimes \underline{h^s}\right) \cap B'(F) \subseteq \{x'\}$

Let $f = f_s \otimes \mathcal{F}^s(h^s)$

$f' = f'_s \otimes \mathcal{F}^s(h^s)$

Note that

$$\delta(f) = \sum_{\tilde{x} \in B(F)} f_s(\underline{\tilde{x}}) \otimes \underline{\mathcal{F}^s h^s(\tilde{x})}$$

$$||$$

$$\delta'(f') = \sum_{\tilde{y} \in B'(F)} f'_s(\underline{\tilde{y}}) \otimes \mathcal{F}^s \underline{h^s}(\tilde{y})$$

Thus, by Poisson summation, have:

$$\delta\left(\mathcal{F}_s f_s \otimes h^s\right) = \delta\left(\mathcal{F}'_s f'_s \otimes h^s\right)$$

$$\sum_{\tilde{x} \in B(F)} \qquad\qquad \sum_{\tilde{y} \in B(F)}$$

$$||$$

$$(\mathcal{F}_s f_s)(x_s) = \mathcal{F}'_s f'_s (x'_s)$$