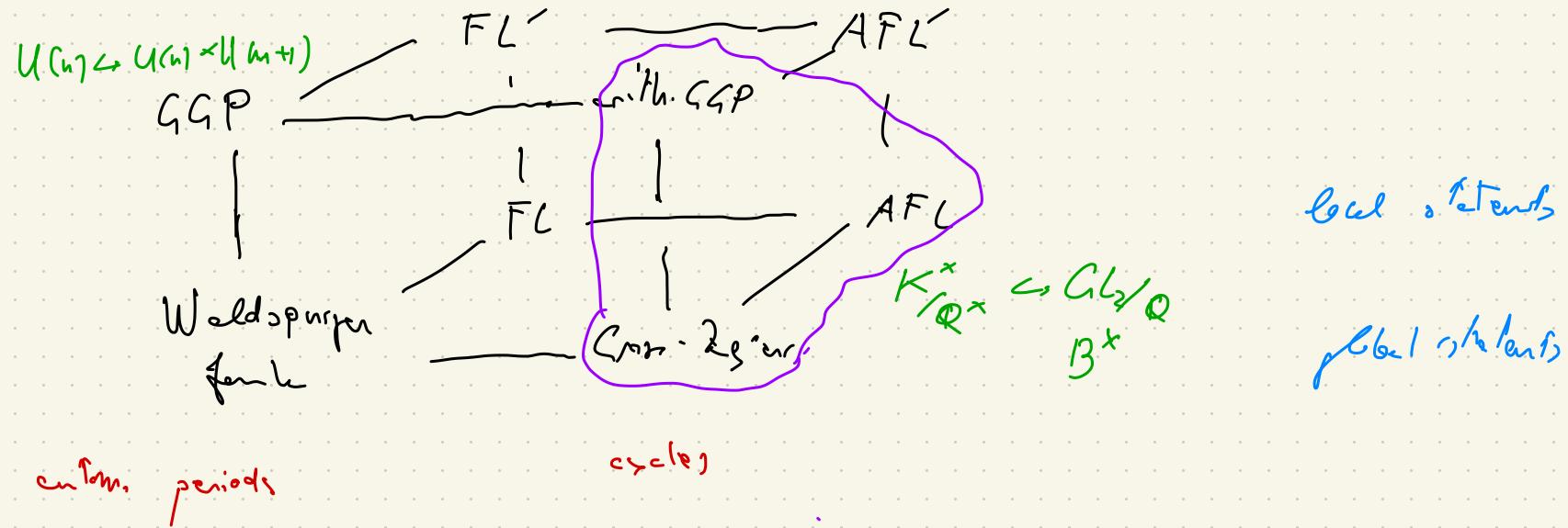


## AFL for CM points



$K/\mathbb{Q}$  imp. quadratic

$\Pi$  = repr. of  $\mathcal{G} = \text{PGL}_2(\mathbb{A})$  attached to a wt. 2 newform cusp, i.e.

$B(\Pi, K) = \left\{ \begin{array}{l} \text{quaternion algebras } B/A \text{ s.t. } B_{\infty} \cong H \\ \text{ & } \mathcal{O}_{K_A} \hookrightarrow B \text{ over } A \\ \bullet \Pi \text{ should admit a JL fix - sign } \pi_B \text{ of } B^{\times} \\ (\text{local conditions}) \end{array} \right\}$

$$B = B^+ \sqcup B^-$$

$$B^+ = \text{coherent } B \quad \text{i.e.} \quad B = B \otimes_{\mathbb{Q}} A \quad \exists! B/A$$

$B^- = \text{incoherent } B \quad \text{i.e.} \quad " \quad \text{does not hold}$

$$\varepsilon(B) = \underbrace{\varepsilon(B_J)}_{\substack{+1 \\ -1}} = +1 \iff B \subset B^+.$$

$\varepsilon(B_J)$  s/w,  
 $B_J$  non-split

Local theory:  $\mathcal{H}(\pi, \kappa, B) = \underset{T(A)}{\text{Hom}}(\pi_B, \mathbb{C}) \neq 0$  for  $\exists! B \in \mathcal{B}$ .

i)  $B \in \mathcal{B}^+$ ,  $\mathfrak{B} = B \otimes A$ .

$$\mathcal{L}(\pi_K, s) = \frac{L(s+s, \pi_K)}{L(1, \pi_{\text{ad}})}$$

Pick this  $\mathfrak{B}$ , then

Weldspurgen:  $P_T: \pi_B \rightarrow T$

$$\phi \mapsto \int \phi(t) dt$$

(T)

$$T = \mathbb{R}/K^\times$$

$$\zeta_B = PB^\times$$

$$|P_T(\phi)|^2 = \mathcal{L}(\pi_K, 0) \cdot (\text{Sel terms})$$

$\neq 0$

2) when about  $B \in \mathcal{P}$ ? ex.  $B_0 = M_2(\mathbb{A}^\infty) \otimes H$

$\rightsquigarrow$  tower of Shimura curves  $X_U/\mathbb{Q}$

$$X_U(\mathbb{C}) = \frac{\mathbb{H}^+}{\mathbb{P}B(\infty)^x} \times_{U^\vee \text{ (cusp)}} \mathbb{P}B^{\infty, x} \quad u \in \mathbb{P}B^{\infty, x}$$

↑ forget.

when  $v \neq v$ ,  $B(v)$  = the quaternion algebra  $/\mathbb{Q}$

s.t.  $B(v)_w \cong B_w$  if  $w \neq v$ .  
 $\not\cong B_w$  if  $w = v$ .

$B_0(\infty) = M_2(\mathbb{Q})$

$$\mathbb{K}^\times \cap \mathbb{P}_F^\times$$

$$K_A \hookrightarrow B \rightsquigarrow [Y_u] = \sum_{t \in [\tau]} [z_K, t]_u \in \text{Div}(X_{B,u})$$

$$CP_{c, d} \quad [Y_u]^\circ = \frac{[Y_u] - \deg(Y_u)[K_X]}{|(\tau)_u|} \in \text{Div}^\circ(X_{(B, u)}) \underset{\sim}{=} J_{(B, u)} \underset{\sim}{=} A^1(X_{(B, u)})$$

Moduli interpretation if  $B = B_0$

$$X_{u, K} = \left\{ \text{E elliptic curves, } \alpha: \hat{T} E \underset{\cong}{\sim} \hat{\mathbb{Z}}^2 / \langle u \rangle \right\} / \underset{\cong}{\sim} \otimes K$$

UI

$$Y_u = \left\{ E \text{ has CN by } K, \alpha: \hat{T} E \underset{\cong}{\sim} \hat{\mathcal{O}_K}^2 / \langle u \rangle \right\} \underset{\cong}{\sim}$$

$\hookrightarrow K_A \subset B_0$

$$u_7 \simeq u \cap K^\times$$

"Eichler-Shimura":  $J_{(B, u)} \sim \bigoplus A_{\pi_B} \otimes \underbrace{(\pi_B^u)^\circ}_{\text{Frob. v. } 1/\mathbb{Q}}$

$\hookrightarrow$   
Hecke ( $P|B^\times$ )

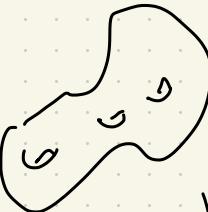
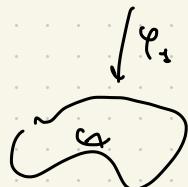
$V_{\ell} A_{\pi_B} \hookrightarrow \pi$  with legend

$$B = B_0$$

$$X_{B,u}$$

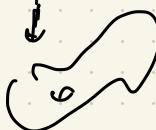
$$\downarrow$$

$$X_0(N)$$



$$\varphi_2 \in (\pi_B^u)^\vee$$

$$\downarrow$$



$$\downarrow$$

$$A_{\pi_B}$$

$$E_{f_1}$$

$$\{f_1, f_2\} \in S_2(\Gamma_0(N))$$

$$P_T : \pi_{B/R}^\infty \longrightarrow A_\pi(K)_R$$

$$P|B^{\Delta, \times}$$

$$\begin{array}{ccc} \psi & \longmapsto & \psi^*(Y)^\circ = \int_{(-1)}^{(1)} \psi([z_k, t]) \, dt \\ & \downarrow & \\ & & \end{array}$$

Herglotz pt.

Thm (Grosz-Zagier, S.Zhang, Yuan-Zhang-Zhang)

pick  $B/A \hookrightarrow \text{from local conditions, assume } \varepsilon(B) = -1$

$$\varepsilon(\pi_K, \chi)$$

$$\langle P_T(\varphi), P_T(\varphi') \rangle_{N_T} = \mathcal{L}'(\pi_K, \sigma) \circ \begin{pmatrix} \text{local term} \\ (\varphi, \varphi') \end{pmatrix}$$

If  $\text{Res}(\pi_K/K)$  are divisors, can  $\pi_K \nmid \varphi = \text{new Hecke}, \Rightarrow \approx 1$

$$\langle , \rangle_{N_T} : A(K) \otimes A^\vee(K) \rightarrow \mathbb{R}$$

$$A_{\pi_B}^\vee = A_{\pi_B^-}$$

$$A = \{y^2 = x^3 + ax + b\} \ni P = [x:y:1] \quad x = \frac{P}{q} \text{ reduced}$$

$$\langle P, P \rangle_{N_T} = \log(\max(|p|, |q|)) + O(1)$$

## RTF approach

$$\begin{aligned}
 \langle P_T(\varphi), P_T(\varphi') \rangle_A &= \langle [\gamma]^o, \varphi^* \varphi' ([\gamma]^o) \rangle_{J_{IB,u}} \\
 &= \langle [\gamma]^o, [\gamma]^o \cdot e_{\pi} T(f_{IB}) \rangle \\
 &= J_{\pi}(f_{IB})
 \end{aligned}$$

$f_{IB} \in \mathcal{H}_{IB}^*$

$$J(f_{IB}) = \langle (\gamma)^o, [\gamma]^o \cdot T(f_{IB}) \rangle = \sum_{\pi_{IB}} J_{\pi_{IB}}(f_{IB})$$

$$\begin{aligned}
 I(f, s) &= R \operatorname{Tr}_{P_{[A]}, s} \otimes P_{[A]}, \gamma, s (R(f)) = \sum I_{\pi} (f, s)
 \end{aligned}$$

$f \in \mathcal{H}_G$

$$\begin{aligned}
 \eta^* \eta_K \\
 A = [ \cdot, \cdot ] \subset \mathfrak{g} \\
 I_{\pi}(f, s) &= \sum_{a, b \in \mathfrak{g}} \int_{\pi} \int_{[A]} \phi(a)/|a|^s da \int_{[A]} R(f). \phi(b) \eta(b)/|b|^s db
 \end{aligned}$$

$$= \mathcal{L}(\pi_K, s) \cdot \prod_v \mathbb{I}_{\pi_v}(f_v, s)$$

Geometric expansion

$$\text{f.r.s. } I(f, s) = \sum_{\delta_f} \frac{\text{Orb}(\delta_f, s)}{A(G^{\text{rs}}/\Gamma)}$$

$$= \int f(a^{-1}\delta b) \gamma(b) \\ A(A)^2 \quad |a/b|^{\alpha} \text{ de} \\ db \\ = \prod_v \text{Orb}_v(\delta_f, f_v, s).$$

Recall Local Transfer:

$$\bullet \text{ local & global orbits}, \quad A(G^{\text{rs}}/\Gamma) = \bigsqcup_{K \hookrightarrow B} T(G_B^{\text{rs}}/\Gamma) \quad F = \text{Local} / \text{global field}$$

$$\delta_f \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{\text{der}(f)} \frac{b_c}{d_c} \in F, \{a_1\} \xrightarrow{\text{inv}_B} N(r) \setminus N(\beta_j) \xleftarrow{r = \alpha + \beta_j} B = K \oplus K_j$$

• Hecke Functions

$$f_v \leftrightarrow f_{B_v} \Leftrightarrow \text{Orb}_v(\delta, f, \sigma) = \begin{cases} \underline{\text{Orb}_v(r, f_B)} & \text{if } f \leftrightarrow f \text{ on } B \\ 0 & \text{else.} \\ \end{cases}$$

$$\int_{\mathbb{T}^2} f_B(t_1, r + t_2) dt_1 dt_2.$$

Suppose that  $B$  is incoherent,  $f \leftrightarrow f_B$

$$\delta \in \mathcal{G}_{A'/A} \hookrightarrow r \in \mathcal{G}_{B/\mathbb{T}} \quad B \text{ coherent.}$$

$$t \in \mathbb{V}_v, \quad B_v \neq B_v, \quad \text{Orb}_v(\delta, f, \sigma) = 0$$

• since  $\exists$  at least one such  $v$ ,  $\Rightarrow \text{Orb}(\delta, f, \sigma) = 0$

$$\Rightarrow \text{Orb}'(\delta, f, o) = \sum_v \text{Orb}^v(\delta, f, o) \cdot \text{Orb}_v'(\delta, f, o)$$

- Moreover, this is zero if  $B_v \neq B_v$  or more than one.
- so if nonzero,  $B = B(v)$

since  $K_v \hookrightarrow B_v$   
 $\hookrightarrow B_v$

This can't happen if  $v$  splits in  $K$

Conclusion:  $I'(\delta, o) = \sum_{\substack{v \\ \text{non-split} \\ \text{in } K}} \sum_{\delta \hookrightarrow r_{B(v)}} \underbrace{\text{Orb}^v(\delta, f, o) \cdot \text{Orb}_v'(\delta, f, o)}_{||}$

$\text{Orb}^v(r, f_{B(v)}, o).$

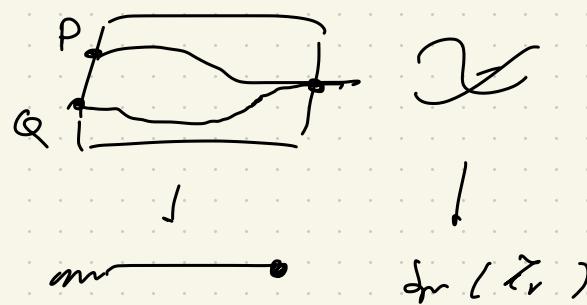
• Geometric expansion of  $J$

$$J(f_B) = \langle [Y]^0, [Y]^0 T(f_B) \rangle = \sum_i \langle [Y]^0, [Y]^0 T(f_{B,i}) \rangle,$$

where: if  $v$  is a fint. place of global Red. for  $X$  ( $U_v = \text{maximal}$

let  $X = \text{integral model} / \mathbb{Z}_v$   $x_{B,u}$   $B_v \text{ split}$ )

$$\langle \sum_i (P_i), \sum_j (Q_j) \rangle_p = \sum_{i,j} m_{X_{F_p}}(\bar{P}_i, \bar{Q}_j) \cdot b_p(n)$$



Assume •  $X = \text{modular curve } (\tilde{\mathbb{B}} = \mathbb{M}_2(\mathbb{A}^\infty))$

- forget  $[Y]^\circ$  v.s.  $[Y]$

• points in  $K$

• extend scalars to  $\tilde{\mathcal{O}}_{K,p} = \mathcal{O}$  completion of  
max'l unramified  
extn of  $\mathcal{O}_{K,p}$ .



$$\overline{\mathbb{F}_p}.$$

Fact : If  $E_2$  is ell. curve w/ CN b<sub>7</sub> K

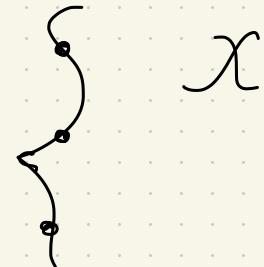
then  $\text{rep}_p(E_2)$  is ss.  $\hookrightarrow$  p nonsplit in K.

Rank : if p splits in K, then

$$\langle (z_k, t), [z_k, t \cdot r] \rangle_p \neq 0$$

only if they have same real.

$\Rightarrow r$  is not  $r_s$

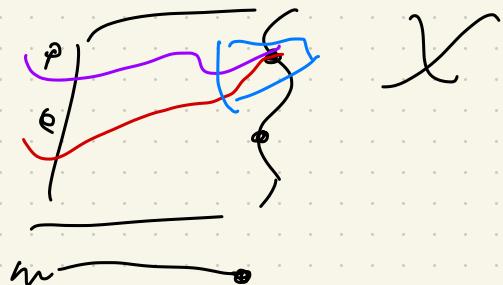


Conclusion

$$J = \sum_{v \text{ wpt in } K} J_v$$

$$\underline{P_{\text{cop}}} \quad \widehat{\chi^{ss}} = \frac{(N \times \mathbb{Z}) \times B^{\rho_{ss}, \times}}{pB(p)^{\times}} / U^p$$

as local schemes are  $\breve{\mathcal{O}}_p$   $\breve{\mathcal{O}}_{K_p}$



where:  $N(S) = \left\{ S = p\text{-div } \mathcal{I}^p / S, \text{ s.t. } S_{O_S} \otimes k \cong S_0 \otimes_{O_S} O_S \right\}$  over  $\tilde{\mathbb{Z}}$ ,  
 $B = B_0$

$$\cong \mathcal{I}_{pf}(\tilde{\mathbb{Z}}_p[[u]]).$$

$G_0$  = the  $p$ -div. group of a ss. eli. curve /  $\widehat{\mathbb{F}_p} = k$   
= the unique connected  $p$ -div gp of dim  $= 1$   
but  $\Rightarrow$

Proof: the map is  $(E, \alpha^\circ) \mapsto \left\{ \begin{array}{l} S = \widehat{E}, \text{ choose } \iota: S_0 \cong S_0 \otimes \\ (\text{unchoose } \iota) \end{array} \right.$   
 $B_0$   $\alpha \circ \iota: \text{level on } T^p S_0 / u$

$$\text{End}(G_0) \otimes \mathbb{Q} = B(p).$$

$$T^p S_0 \cong (\mathbb{Z}^p)^2$$

$$S L_2(\mathbb{A}^{p\infty}) \quad \square$$

$$[g] = [z_K, g]$$

$$m \left( f(t), f(t+r \cdot g) f_B(g) \right)$$

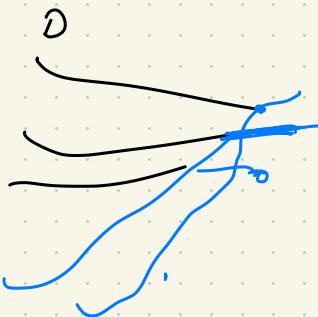
$$= \sum_{T \setminus B(p)^{\times} / T} \tilde{m} ( , )$$

$$= \text{Orb}^{\text{rc}}(r, f_B)$$

$$m_{n_Z}([t], [t, t+\delta] f_\delta(\delta))$$

$$\mathcal{X} = \underbrace{B(p)^{\times}}_{P(B(p)^{\times})} \times P(B^{p^{-1}})^{\times} \times U^n$$

$$|\chi^s| = \underbrace{Z}_{P(B(p)^{\times})} \times P(B^{p^{-1}})^{\times} / U^n$$



$$m_n([t], [r])$$

$$= m_n([1], [t \cdot r])$$

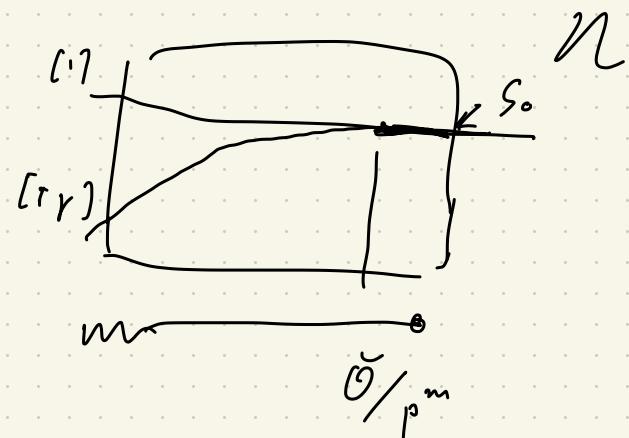
$$t = r_1/t_1$$

$$= m$$

$\uparrow$   $m$  is max'l.

$$f \in O_{B,p}^\times \text{ lift to } \text{Aut}(S/\tilde{\mathcal{O}}_{/p^m})$$

$$\text{Aut}(S_0)$$



$$\text{End}(G_p) \supset \dots \supset \text{End}(S/\tilde{\mathcal{O}}_{/p^m}) \supset \dots \supset \text{End}(G_{\sigma_r})$$

$$O_{B,p}^\times$$

• Gross' Theory of quasi  
canonic lifting.

$$\mathcal{O}_B = \mathcal{O}_K \cap \mathcal{O}_K^\times$$

$$\mathcal{O}_K + p^{m-1} \mathcal{O}_{B,\mathfrak{p}}$$

$$\gamma = \alpha + \beta j \quad \beta, \alpha \in \mathcal{O}_K, \quad \alpha \in \mathcal{O}_K^\times$$

• F. Saito's talk

$$tj \approx \bar{t} \quad t \in K$$

$$\text{val}(j) = 1.$$

$$\text{inv}(\alpha + \beta j) = \underbrace{2 \text{val}(\beta)}_{=1} - 1$$

$$m = \frac{1 + \text{inv}(-)}{2}$$

$$\underline{\text{AFL}} : \quad m_n([1], [\cdot, \cdot]) = \underset{\text{inv}}{\text{orb}}'(\overset{11}{\delta}, o).$$

$a_{inv(\delta), p}^{\prime} (\theta \circ E, o)$