CERESA CYCLES OF BIELLIPTIC PICARD CURVES

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Abstract. We show that the Ceresa cycle $\kappa(C_t)$ of the genus 3 curve $y^3 = x^4 + 2tx^2 + 1$ is torsion if and only if $Q_t = (\sqrt[3]{t^2-1}, t)$ is a torsion point on the elliptic curve $y^2 = x^3 + 1$. This shows that there are infinitely many smooth plane quartic curves over $\mathbb{C}$ (resp. $\mathbb{Q}$) with torsion (resp. infinite order) Ceresa cycle. Over $\overline{\mathbb{Q}}$, we show that the Beilinson-Bloch height of $\kappa(C_t)$ is proportional to the Néron-Tate height of Q_t. Thus, the height pairing on $\kappa(C_t)$ is nondegenerate and satisfies a Northcott property. To prove all this, we show that the Chow motive that controls $\kappa(C_t)$ is isomorphic to $\mathfrak{h}^1$ of an appropriate elliptic curve.

Contents

1. Introduction

Let C be a smooth, projective, geometrically integral curve of genus $g \geq 2$ over an algebraically closed field k. We embed C in its Jacobian J using a $(2g-2)$-th root of the canonical bundle. The (canonical) Ceresa cycle of C is the class

$$\kappa(C) = [C] - (-1)^*[C] \in \mathrm{CH}_1(J)$$

in the Chow group of 1-cycles on J with $\mathbb{Z}$-coefficients.[1] While $\kappa(C)$ is homologically trivial, Ceresa [12] famously showed that it need not be trivial in $\mathrm{CH}_1(J)$. Indeed, if $k = \mathbb{C}$ then the Ceresa cycle of a very general genus $g \geq 3$ curve has infinite order. On the other hand, if C is hyperelliptic then $\kappa(C) = 0$. Beyond this, very little is known about the set of genus g curves with vanishing or torsion Ceresa cycle, though see §1.4 for some recent results and examples.

In this paper, we study the Ceresa cycles of bielliptic Picard curves over fields of characteristic $\neq 2, 3$ [28]. As k is algebraically closed, such a curve has an affine model of the form

$$y^3 = x^4 + 2tx^2 + 1, \tag{1.1}$$

for some $t \in k \setminus \{\pm 1\}$, unique up to replacing t with $-t$. Our main result shows that the Ceresa cycles of these curves are governed by points on the single elliptic curve $\widehat{E}\colon y^2 = x^3 + 1$.

[1] This depends on our choice of $(2g-2)$-th root, but only up to a $(2g-2)$-torsion class; see Lemma 2.1.

Theorem 1.1. *Let C_t be the smooth projective curve with model (1.1), and let Q_t be the point $(\sqrt[3]{t^2-1}, t) \in \widehat{E}(k)$, for some choice of cube root. Then $\kappa(C_t)$ is torsion in $\mathrm{CH}_1(J_t)$ if and only if Q_t is torsion in $\widehat{E}(k)$.*

Theorem 1.1 (which is proved in §5.5) appears to be the first characterization of torsion Ceresa cycles in a positive-dimensional family of curves, aside from families where $\kappa(C)$ is identically torsion. We deduce from this result the following corollary (see §5.6).

Corollary 1.2. *There are infinitely many $\overline{\mathbb{Q}}$-isomorphism classes of genus 3 curves over $\mathbb{Q}$ with infinite order Ceresa cycle. In fact, $\kappa(C_t)$ has infinite order for all $t \in \mathbb{Q} \setminus \{0, \pm 1, \pm 3\}$.*

Corollary 1.2 is not surprising given Ceresa's result, but it seems to have been open until now and is still open in higher genus, as far as we know.

On the other hand, since $\widehat{E}$ has infinitely many torsion points, we also deduce from Theorem 1.1 that there are infinitely many plane quartic curves over $\mathbb{C}$ with torsion Ceresa cycle. In fact, our proof shows that there is an algebraic correspondence from J_t to $\widehat{E}$ that sends $\kappa(C_t)$ to a constant multiple of Q_t. This allows us to deduce the following stronger statement:

Theorem 1.3. *There exist infinitely many plane quartic curves over $\mathbb{C}$ with torsion Ceresa cycle. Moreover, their orders are unbounded.*

Note that it was only recently discovered that there exist *any* plane quartic curve with torsion Ceresa cycle: Qiu–Zhang [37, §4.1] have shown that $C_0 \colon y^3 = x^4 + 1$ has this property.

Interestingly, all known examples in Theorem 1.3 can be defined over $\overline{\mathbb{Q}}$; for the curves C_t, this follows immediately from Theorem 1.1. For $t \in \overline{\mathbb{Q}}$, we write $h(\kappa(C_t))$ for the Beilinson-Bloch height $\langle \kappa(C_t), \kappa(C_t) \rangle$ [6, 27] of $\kappa(C_t)$. We show that $h(\kappa(C_t))$ is proportional to the Neron–Tate height of Q_t (Theorem 5.19), which implies the following non-degeneracy and Northcott properties. Let $\deg(C_t)$ be the minimum degree of a number field over which C_t can be defined.

Theorem 1.4. *We have $h(\kappa(C_t))) = 0$ if and only if $\kappa(C_t)$ is torsion. Moreover, for any $X \in \mathbb{R}$ and $d \in \mathbb{N}$, the set*

$$\left\{ t \in \overline{\mathbb{Q}} \colon h(\kappa(C_t)) \leq X \quad \textit{and} \quad \deg(C_t) \leq d \right\}$$

is finite.

In particular, there are finitely many torsion Ceresa cycles $\kappa(C_t)$ defined over $\mathbb{Q}$. In fact, there are three such curves up to $\overline{\mathbb{Q}}$-isormorphism: C_0, $C_{\pm 3}$ and $C_{\pm\sqrt{-3}}$ (Proposition 5.18). Note that $C_{\sqrt{-3}}$ has $\mathbb{Q}$-model $y^3 = x^4 + 6x^2 - 3$.

Finally, we show that for all t, the image of $\kappa(C_t)$ in the Griffiths group of homologically trivial cycles modulo algebraic equivalence is torsion, in the following strong sense:

Theorem 1.5. *There exists an integer $N \geq 1$ such that for every bielliptic Picard curve C over a field of characteristic not 2 nor 3, the cycle $N \cdot \kappa(C) \in \mathrm{CH}_1(J)$ is algebraically trivial.*

Theorems 1.3-1.5 are quick corollaries of our proof of Theorem 1.1, which we discuss below.

1.1. **Relation with modified diagonal cycles.** All results above hold verbatim for the modified diagonal cycle $\Delta_{GS}(C_t) \in \mathrm{CH}_1(C_t^3)$ defined by Gross and Schoen [21]; see §2 for the definition. Indeed, $\kappa(C)$ is torsion if and only if $\Delta_{GS}(C)$ is torsion [44, Theorem 1.5.5], and our method of proof implies a variant of Theorem 1.4 for $\Delta_{GS}(C)$ as well.

S. Zhang has shown a similar Northcott property for the height of $\Delta_{GS}(C)$ in families of smooth curves over smooth projective bases [44, Theorem 1.3.5], i.e. compact subvarieties inside the moduli space $\mathcal{M}_g$. As the family C_t does not extend to a compact curve inside $\mathcal{M}_3$, Theorem 1.4 suggests that Zhang's Northcott property might very well extend to an open dense subset of $\mathcal{M}_g$.

1.2. **Summary of proof.** A curve C of the form (1.1) has an order 6 automorphism (or more canonically, a μ_6-action), and the class $\kappa(C)$ is fixed by the induced μ_6-action on J. On the other hand, over $\mathbb{C}$, the μ_6-invariant part of the Hodge structure $\mathrm{H}^3(J(\mathbb{C}),\mathbb{Q})(1)$ is two-dimensional and of type $(1,0)+(0,1)$. This suggests that there should exist an algebraic correspondence from J_t to an elliptic curve that sends $\kappa(C_t)$ to a point on this elliptic curve. Instead of attempting to construct this correspondence explicitly, we prove this by maneuvering within the flexible category of (pure) Chow motives.

Namely, we show that there is an isomorphism of Chow motives

$$\mathfrak{h}^3(J)^{\mu_6} \simeq \mathfrak{h}^1(E^\Delta)(-1), \tag{1.2}$$

where J is the Jacobian of any curve of the form $C\colon y^3 = x^4 + ax^2 + b$, E^Δ is a certain twist of $\widehat{E}$ depending on a and b, and the remainder of the notation will be explained later. It is crucial for our proof that we find a universal such isomorphism, in the category of relative Chow motives over the parameter space $S = \mathbb{A}^2_{\mathbb{Z}[1/6]} \setminus \{\Delta_{a,b} = 0\}$. The elliptic scheme $\mathcal{E}^\Delta \to S$ can then be thought of as the universal j-invariant 0 elliptic curve with a nonzero marked point $\mathcal{Q}$. The main ingredients in the construction of this universal isomorphism are the Chow–Kunneth decomposition for motives of abelian schemes (via Beauville's Fourier transform) [3, 13], Moonen's refinement of this decomposition in the presence of extra endomorphisms [32], and our study of the Galois action on the endomorphism algebra of the universal Prym surface $\mathcal{P} \to S$ [28]. The latter endomorphism algebra is a nonsplit quaternion algebra over $\mathbb{Q}$.

The Ceresa cycle of C naturally lives in $\mathrm{CH}^2(\mathfrak{h}^3(J)^{\mu_6})$, so the question becomes: what is the corresponding element of $\mathrm{CH}^2(\mathfrak{h}^1(E^\Delta(-1)) = E^\Delta(k) \otimes \mathbb{Q}$, under the isomorphism (1.2)? We show that it corresponds to a $\mathbb{Q}^\times$-multiple of the marked point $\mathcal{Q}$ by showing it universally over S, using two soft inputs. First, the elliptic scheme $\mathcal{E}^\Delta \to S$ (which after $\mathbb{G}_m$-scaling, is essentially an elliptic surface) has Mordell–Weil group of rank 1, generated by $\mathcal{Q}$. Second, we show by explicit computation that the universal Ceresa cycle has at least one infinite order specialization (see §2), so that $\kappa(\mathcal{C}/S)$ corresponds to a nonzero section of $\mathcal{E}^\Delta$, and hence a multiple of $\mathcal{Q}$. Specializing at geometric points, we recover Theorem 1.1. The construction allows us to quickly deduce the remaining results in the introduction as well.

The "only if" direction of Theorem 1.1 could be proved without the machinery of Chow motives, by implementing the above argument at the level of cohomology (via complex Abel-Jacobi maps and Hodge theory). However, this is not enough for the "if" direction since the injectivity of higher Abel-Jacobi maps for varieties over $\overline{\mathbb{Q}}$ is very much open. The motivic approach therefore seems crucial for an unconditional result and is in any case more direct.

1.3. **Geometric interpretation via bigonal duality.** $C_t\colon y^3 = x^4 + 2tx^2 + 1$ is a double cover of the elliptic curve $E_t\colon y^3 = x^2 + 2tx + 1$. Associated to any such double cover is the bigonal dual cover $\widehat{C}_t \to \widehat{E}_t$ constructed by Pantazis and Donagi [36, 14], whose Prym is dual to the Prym of $C_t \to E_t$; see [28, §2.7] for a construction in the situation considered here. In our case, the curve $\widehat{C}_t$ has model $y^3 = (x^2+t)^2 - 1$ so that $\widehat{E}_t$ is isomorphic to $\widehat{E}\colon y^2 = x^3 + 1$, for all t. The branch locus of $\widehat{C}_t \to \widehat{E}$ consists of ∞ and the three points $(\sqrt[3]{t^2-1}, t)$. Theorem 1.1 therefore says that the Ceresa cycle $\kappa(C_t)$ is torsion if and only if the bigonal dual $\widehat{C}_t$ is branched along torsion points of $\widehat{E}$. We find this formulation intriguing, but our proof does not actually use the bigonal dual construction in a direct way.

1.4. **Previous results.** Several examples of nonhyperelliptic curves with torsion Ceresa cycle were recently found, conditional on Beilinson-Bloch type conjectures [7, 20, 4, 5, 30]. Qiu and W. Zhang then gave some unconditional examples [37], including one in genus 3 and one-dimensional families in genus 4 and 5. Laterveer also found a two-dimensional family in genus 5 [29]. Subsequently, Qiu-Zhang found examples of plane quartic curves with trivial automorphism group and torsion Ceresa cycle arising as quotients of certain Shimura curves [38].

The proofs of [37] and [29] rely on the vanishing of the Chow motive $(\mathfrak{h}^1(C)^{\otimes 3})^{\mathrm{Aut}(C)}$, and hence of $\mathfrak{h}^3(J)^{\mathrm{Aut}(C)}$, for certain curves C, generalizing the observation that hyperelliptic curves have trivial Ceresa cycle. For our curves $C = C_t$ with $t \neq 0$, the motive $\mathfrak{h}^3(J)^{\mathrm{Aut}(C)} = \mathfrak{h}^3(J)^{\mu_6}$ does not vanish, but (1.2) shows that it is a direct summand of the motive of a curve, hence is still reasonably accessible. It would be interesting to study other families of curves where $\mathfrak{h}^3(J)^{\mathrm{Aut}(C)}$ is "small" in a suitable sense. More generally, one can try to study the non-zero isotypic components of $\kappa(C)$ with respect to the action of $\mathrm{End}(J)$ on $\mathfrak{h}^3(J)$. For example, even though some of the examples in [38] have trivial automorphism group, they all satisfy $\mathrm{End}(J) \neq \mathbb{Z}$.

Finally, Eskandari and Murty showed that infinitely many Fermat curves have infinite order Ceresa cycle [16], proving a weak version of Corollary 1.2 (where the genus is allowed to grow). If we fix g but allow for curves defined over $\overline{\mathbb{Q}}$ (and not necessarily over $\mathbb{Q}$), then the analogue of Corollary 1.2 holds for every genus $g \geq 3$ by [44, Theorem 1.3.5].

1.5. **Further questions.** For an integer $N \geq 1$, let $Z_g(N)$ be the subset of the coarse moduli space of curves M_g over $\mathbb{Q}$ consisting of those curves whose canonical Ceresa cycle (with respect to some choice of $(2g-2)$th root of the canonical bundle) is killed by N in the Chow group. What is the geometry of $Z_g(N)$? More specifically, what are its positive-dimensional components? Is the union of $Z_g(N)$ over all N dense in M_g?

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1.7. **Notation and conventions.**

- All fields considered in this paper will be of characteristic $\neq 2, 3$. Given a field k, we denote by $k^{\mathrm{sep}} \subset \bar{k}$ a choice of separable and algebraic closure, and the absolute Galois group by $\mathrm{Gal}_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$.
- A variety over a field k is a seperated scheme of finite type over k. A variety is nice if it is smooth, projective and geometrically integral.
- Let X be a smooth and quasi-projective scheme over a Dedekind domain (which might be a field). Then the Chow groups $\mathrm{CH}^p(X)$ of codimension p cycles on X (modulo rational equivalence and with $\mathbb{Z}$-coefficients) are defined and come equipped with flat pullbacks, proper pushforwards and an intersection product, see [19, §20.2]. If $Z \subset X$ is a codimension p reduced closed subscheme write $[Z]$ for its class in $\mathrm{CH}^p(X)$.
- If X is a nice variety over a field k we write $\mathrm{CH}^p(X)_{\mathrm{hom}\sim 0} \subset \mathrm{CH}^p(X)$ for the classes in the kernel of the ℓ-adic cycle class map for every prime ℓ invertible in k.
- If $T \to S$ is a morphism of schemes and X is an S-scheme we write X_T for the T-scheme $X \times_S T$ and $X(T)$ for the set of sections of $X_T \to T$. If $T = \mathrm{Spec}(R)$ is affine we sometimes write X_R and $X(R)$ instead of X_T and $X(T)$.
- If $A, B \to S$ are abelian schemes we denote by $\mathrm{Hom}(A/S, B/S)$ the set of S-homomorphisms between A and B and $\mathrm{Hom}^0(A/S) = \mathrm{Hom}(A/S) \otimes \mathbb{Q}$. We similarly write $\mathrm{End}(A/S) = \mathrm{Hom}(A/S, A/S)$ and $\mathrm{End}^0(A/S) = \mathrm{End}(A/S) \otimes \mathbb{Q}$. We write $A^\vee \to S$ for the dual abelian

scheme (see [18, Chapter 1, §1]) and write $\mathrm{Hom}^{\mathrm{sym}}(A, A^\vee)$ for the subset of all self-dual homomorphisms $A \to A^\vee$.

- If A/S is an abelian scheme we denote by (n) or $(n)_A$ the multiplication-by-n endomorphism $A \to A$.

2. Ceresa and modified diagonal cycles

We recall the definitions of Ceresa and modified diagonal cycles, and show, by means of an example, a way to verify that $\kappa(C)$ is infinite order, for genus 3 curves over $\overline{\mathbb{Q}}$. The latter is based on the methods of [11] and [15]. In this section, and for the rest of the paper, we work over a general (not necessarily algebraically closed) field k of characteristic $\neq 2, 3$.

2.1. **Ceresa and modified diagonal cycles.** Let C be a (nice) curve of genus $g \geq 2$ over k. If $e \in C(k)$, let $\iota_e \colon C \hookrightarrow J$ be the Abel-Jacobi map sending $x \mapsto x - e$, and define the Ceresa cycle based at e to be

$$\kappa_e(C) := [\iota_e(C)] - (-1)^*[\iota_e(C)] \in \mathrm{CH}^{g-1}(J).$$

Lemma 2.1. *The map $e \mapsto \kappa_e(C)$ extends to a homomorphism $\alpha_* \colon \mathrm{CH}^1(C) \to \mathrm{CH}^{g-1}(J)$.*

Proof. Let $f \colon C^2 \to C \times J$ be the morphism $(e, x) \mapsto (e, x - e)$ and let $\alpha' = f_*[C^2] \in \mathrm{CH}^2(C \times J)$. The correspondence $\alpha := \alpha' - (\mathrm{id} \times (-1))^*\alpha' \in \mathrm{CH}^2(C \times J)$ induces $\alpha_* \colon \mathrm{CH}^1(C) \to \mathrm{CH}^{g-1}(J)$ sending Z to $p_{2*}((p_1^*Z) \cdot \alpha)$, where p_1, p_2 denote the projections from $C \times J$. A computation shows that $\alpha_*([e]) = \kappa_e(C)$ for all $e \in C(k)$. $\square$

For a degree 1 divisor class $e \in \mathrm{CH}^1(C)$, we define $\kappa_e(C) := \alpha_*(e)$. When $(2g-2)e$ is canonical, we call $\kappa_e(C)$ a canonical Ceresa cycle and simply write $\kappa(C)$. There are finitely many such $e \in \mathrm{CH}^1(C)$, and they all give the same canonical Ceresa class up to $(2g-2)$-torsion, by Lemma 2.1.

The modified diagonal cycle of Gross and Schoen [21] is a close relative of the Ceresa cycle. For $e \in C(k)$, the modified diagonal cycle $\Delta_{GS,e} = \Delta_{GS,e}(C)$ based at e is

$$\Delta_{GS,e} = \Delta_{123} - \Delta_{12} - \Delta_{13} - \Delta_{23} + \Delta_1 + \Delta_2 + \Delta_3 \in \mathrm{CH}^2(C^3),$$

where $\Delta_{123} = \{(c, c, c) \colon c \in C\}$, $\Delta_{12} = \{(c, c, e) \colon c \in C\}$, etc. As before, one can define a cycle $\Delta_{GS,e}$ for any divisor e of degree 1 on C, and if $(2g-2)e$ is a canonical divisor, then we call $\Delta_{GS,e}$ a canonical modified diagonal cycle and write $\Delta_{GS} = \Delta_{GS,e}$. Again this is well-defined up to torsion in the Chow group; see [21, eq. 3.5].

Some useful general facts about these cycles are:

Proposition 2.2. *The cycles $\kappa_e(C)$ and $\Delta_{GS,e}(C)$ are homologically trivial.*

Proof. For $\kappa_e(C)$, this is immediate from the fact that $(-1)^*$ acts trivially on even dimensional cohomology. For $\Delta_{GS,e}(C)$, this is [21, Proposition 3.1]. $\square$

Theorem 2.3. [44, Theorem 1.5.5] *$\kappa(C)$ has infinite order if and only if $\Delta_{GS}(C)$ has infinite order.*

The following proposition shows that the canonical modified diagonal cycle $\Delta_{GS}(C)$ is "minimal up to torsion" in a certain sense.

Proposition 2.4. *If $\Delta_{GS,e}(C)$ is torsion, then $(2g-2)e - K_C$ is torsion in $J(k)$, where K_C is the canonical divisor class.*

Proof. This follows from [37, Proposition 2.3(1)], whose proof is valid over any field k. $\square$

2.2. **A general divisibility criterion.** Assume now that k is a number field. Since $\Delta_{GS,e}$ is homologically trivial, we may consider for every prime ℓ its image under the ℓ-adic Abel-Jacobi map

$$\mathrm{AJ}_\ell\colon \mathrm{CH}^2(C^3)_{\mathrm{hom}\sim 0} \longrightarrow \mathrm{H}^1(\mathrm{Gal}_k, \mathrm{H}^3(C^3_{k^{\mathrm{sep}}}, \mathbb{Z}_\ell(2)));$$

see [22, §9] for more details concerning this map. The following proposition is based on the methods of [11] and [15].

Proposition 2.5. *Let $e \in C(k)$ and let v be a finite place of k of good reduction for C. Let $\ell > 3$ be a prime coprime to the norm of v. Let (C_v, e_v) be the smooth reduction of (C, e) over the finite residue field $\mathbb{F}_v$. Define*

$$D := \sum_{c\in C_v(\mathbb{F}_v)} (c - e_v) \in J(\mathbb{F}_v). \tag{2.1}$$

If the order of D is a multiple of ℓ, then $\mathrm{AJ}_\ell(\Delta_{GS,e}) \neq 0$.

Proof. Consider the correspondence

$$\Gamma = \{(a, \mathrm{Fr}(a), b, b)\colon a, b \in C_v\} \subset C_v^3 \times C_v,$$

where Fr is the Frobenius morphism. This gives a homomorphism

$$\Gamma_*\colon \mathrm{CH}^2(C_v^3) \to \mathrm{CH}^1(C_v) \simeq \mathrm{Pic}(C_v),$$

defined by $\Gamma_*\beta = p_{2*}([\Gamma]\cdot p_1^*\beta)$, where p_i are the projections from $C_v^3\times C_v$. Let $\Delta_{GS,v}$ be the modified diagonal cycle on C_v^3 with respect to e_v. A short computation yields $\Gamma_*(\Delta_{GS,v}) = \sum_{c\in C_v(\mathbb{F}_v)}(c - e_v) = D \in J(\mathbb{F}_v)$.

If $\mathrm{AJ}_\ell(\Delta_{GS,e}) = 0$, then a specialization argument using [11, Proposition 3.1] shows that $\mathrm{AJ}_\ell(D) \in \mathrm{H}^1(\mathrm{Gal}_{k_v}, \mathrm{H}^1(C_{k_v^{\mathrm{sep}}}, \mathbb{Z}_\ell(1)))$ is zero too. By [11, Lemma 1.12] this implies that D is zero in $J(k_v)\otimes\mathbb{Z}_\ell$, so D has order coprime to ℓ. We conclude that if D has order divisible by ℓ then $\mathrm{AJ}_\ell(\Delta_{GS,e}) \neq 0$. ☐

2.3. **An infinite order example.** We now show by example how Proposition 2.5 can be used to verify that $\kappa(C)$ has infinite order. For the rest of the section, let $k = \mathbb{Q}$, let C be the curve with plane model $y^3 = x^4 + x^2 + 1$, and let $\ell = 7$. Consider the modified diagonal cycle Δ_{GS} based at the unique point ∞ at infinity; note that $(2g-2)\infty = 4\infty$ is canonical.

Lemma 2.6. $\mathrm{AJ}_7(\Delta_{GS}) \neq 0$.

Proof. We take $v = 41$ in Proposition 2.5 and show that $D \in J(\mathbb{F}_{41})$ of (2.1) has order divisible by 7 by explicit computation. Let $\pi\colon C \to E\colon y^3 = x^2 + x + 1, (x, y) \mapsto (x^2, y)$ be the double cover and let $\tau \in \mathrm{Aut}(C)$ be the corresponding covering involution. Pairing points c and c^τ together, we see that D is a sum of points on E plus some leftover ramification points. Doubling this sum, we obtain a sum of points on E. More precisely, we have

$$2D = (0, 1) + 2 \sum_{q\in E(\mathbb{F}_{41})_{\mathrm{lift}}} q$$

where $E(\mathbb{F}_{41})_{\mathrm{lift}}$ is the set of non-branch points in the image of $\pi\colon C(\mathbb{F}_{41}) \to E(\mathbb{F}_{41})$. We compute in Sage that $2D = (37, 15)$ has order 7, so the order of D is divisible by 7. ☐

Let $\mathrm{Sym}^3(C) := C^3/S_3$ be the symmetric cube of C, and let $V := \mathrm{H}^3(\mathrm{Sym}^3(C_{\overline{\mathbb{Q}}}), \mathbb{Z}_7(2))$.

Lemma 2.7. $V \simeq \mathrm{H}^3(J_{\overline{\mathbb{Q}}}, \mathbb{Z}_7(2)) \oplus \mathrm{H}^1(C_{\overline{\mathbb{Q}}}, \mathbb{Z}_7(1))$ *as* $\mathrm{Gal}_{\mathbb{Q}}$*-representations.*

Proof. Since C is a nonhyperelliptic genus 3 curve, the Abel–Jacobi map $\Sigma\colon \mathrm{Sym}^3(C) \to J$ sending a degree 3 divisor D to $D - 3\infty$ is the blow-up of J along the image of those D with the property that $\dim \mathrm{H}^0(C, \mathcal{O}(D)) = 2$ [33, Theorem 2.3]. By Riemann-Roch, these are exactly the degree 3 divisors linearly equivalent to $4\infty - x$ for some unique $x \in C$. We conclude that Σ is the blow-up of J along the closed subvariety $\{\infty - x \mid x \in C\} \simeq C$. The lemma now follows from the description of the cohomology of a blow-up [1, VII, Theoreme 8.1.1]. $\square$

Lemma 2.8. *The class Δ_{GS} has infinite order.*

Proof. Suppose, for the sake of contradiction, that Δ_{GS} has finite order. Then its Abel-Jacobi image $\mathrm{AJ}_7(\Delta_{GS}) \in \mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, \mathrm{H}^3(C^3_{\overline{\mathbb{Q}}}, \mathbb{Z}_7))$ has finite order. By Lemma 2.6 it is nonzero. Let $h\colon C^3 \to \mathrm{Sym}^3(C)$ be the quotient map and let $\tilde{\Delta}_{GS} = h_*\Delta_{GS}$. Use the same notation as before for the Abel-Jacobi map $\mathrm{AJ}_\ell\colon \mathrm{CH}^2(\mathrm{Sym}^3(C))_{\mathrm{hom}\sim 0} \to \mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, V)$. Since $h^*\tilde{\Delta}_{GS} = 6\Delta_{GS}$, we conclude that $\mathrm{AJ}_\ell(\tilde{\Delta}_{GS})$ is also nonzero and torsion. On the other hand, we use Lemma 2.7 and Magma to check that the action of the geometric Frobenius Fr_{11} at 11 on V satisfies

$$\det(\mathrm{Fr}_{11} - 1) = 29049104246323668435011663307177984 \not\equiv 0 \pmod{7}.$$

Thus $\mathrm{H}^1(\mathrm{Gal}_{\mathbb{Q}}, V)$ is torsion-free by [43, 6.1], which is a contradiction. $\square$

Combining Lemma 2.8 and Theorem 2.3, we have proven:

Corollary 2.9. *The Ceresa cycle $\kappa(C)$ of the curve $C\colon y^3 = x^4 + x^2 + 1$ has infinite order.*

3. Bielliptic Picard curves

We recall some results on bielliptic Picard curves from [28]; we refer to that paper for more details. In particular, we collect the results we will need on the endomorphism algebra of the Prym variety and its monodromy in §3.3.

3.1. **Definitions.** A bielliptic Picard curve C over a field k (of characteristic $\neq 2, 3$) is a nice plane quartic curve over k with affine model of the form

$$y^3 = x^4 + ax^2 + b, \tag{3.1}$$

for some $a, b \in k$. We often write $C = C_{a,b}$. Smoothness of C implies that

$$\Delta_{a,b} := 16b(a^2 - 4b) \neq 0,$$

and any plane quartic of the form (3.1) with $\Delta_{a,b} \neq 0$ is smooth. These curves admit a μ_6-action given by $\zeta \cdot (x, y) = (\zeta^3 x, \zeta^4 y)$ for every $\zeta \in \mu_6$. The unique fixed point of this action is the unique point ∞ at infinity. Since ∞ intersects the line at infinity with multiplicity 4, the divisor 4∞ is canonical.

Define the j-invariant $j(C_{a,b}) = (4b - a^2)/4b \in k^\times$.

Lemma 3.1. [28, §2.2-3] *Let $C_{a,b}$ and $C_{a',b'}$ be bielliptic Picard curves over k.*

(1) $C_{a,b} \simeq C_{a',b'}$ if and only if there exists $\lambda \in k^\times$ such that $(a', b') = (\lambda^6 a, \lambda^{12} b)$.
(2) $C_{a,b}$ is isomorphic to $C_{a',b'}$ over $\overline{k}$ if and only if $j(C_{a,b}) = j(C_{a',b'})$.

Lemma 3.2. *Let $C_{a,b}$ be a bielliptic Picard curve over k^{sep}. Then C is defined over k (in other words, there exists a curve X over k such that $X_{k^{\mathrm{sep}}} \simeq C_{a,b}$) if and only if $j(C_{a,b}) \in k$.*

Proof. Suppose that $j = j(C_{a,b}) \in k$. If $j = 1$, then $a = 0$ and $C_{a,b}$ is k^{sep}-isomorphic to $C_{0,1}$ and defined over k. If $j \neq 1$, then a computation using Lemma 3.1(1) shows that $C_{a,b}$ is k^{sep}-isomorphic to $C_{2,(1-j)^{-1}}$ and hence also defined over k. Conversely, suppose that $C_{a,b}$ is defined over k. Then $C_{a,b} \simeq \sigma^*(C_{a,b}) = C_{\sigma(a),\sigma(b)}$ for all $\sigma \in \mathrm{Gal}_k$. By Lemma 3.1(2), $j(C_{a,b}) = \sigma(j(C_{a,b}))$ for all $\sigma \in \mathrm{Gal}_k$, in other words $j(C_{a,b}) \in k$. ☐

3.2. **The Prym variety.** The curve C admits an involution $\tau(x,y) = (-x,y)$, and the quotient by this involution is a double cover $\pi\colon C \to E$ to an elliptic curve E with origin the image of ∞. This elliptic curve has equation $y^3 = x^2 + ax + b$ and short Weierstrass model $E\colon y^2 = x^3 + 16(a^2 - 4b)$. This double cover decomposes the Jacobian variety J of C. Let

$$P := \ker(1 + \tau^*\colon J \to J) \tag{3.2}$$

be the Prym variety associated to the double cover $C \to E$.

Lemma 3.3. [28, §2.5] *P is an abelian surface and J is isogenous to $P \times E$.*

3.3. **Galois action on the Neron–Severi group.** The main ingredient we need from [28] for our proof of Theorem 1.1 is an explicit description of the Gal_k-action on the Néron-Severi group $\mathrm{NS}(P_{k^{\mathrm{sep}}})$, as well as the fact that $P_{k^{\mathrm{sep}}}$ has quaternionic multiplication.

We will need a universal version of this story over the parameter space of all bielliptic Picard curves, so let

$$S := \mathbb{A}^2_{\mathbb{Z}[1/6]} \setminus \{\Delta_{a,b} = 0\} = \mathrm{Spec}(\mathbb{Z}[1/6, a, b, \Delta_{a,b}^{-1}]).$$

Let $\mathcal{C} \to S$ be the universal bielliptic Picard curve with equation (3.1). The morphism $\mathcal{C} \to S$ is smooth, proper and of relative dimension 1 and every bielliptic Picard curve occurs as a fiber of this morphism. Let $\mathcal{J} = \mathbf{Pic}^0_{\mathcal{C}/S}$ be its relative Jacobian [10, §9.4, Proposition 4], an abelian scheme over S. The definition (3.2) of the Prym variety works in families and we obtain an abelian scheme $\mathcal{P} \to S$ of relative dimension 2 whose fibers are the Prym varieties described in §3.2.

We need to pass to a cover of S to "see" all endomorphisms of $\mathcal{P}$. Let $\tilde{S} \to S$ be the finite etale cover given by adjoining a primitive third root of unity ω and a sixth root of $\Delta_{a,b}$. More formally, consider the ring extension

$$R = \mathbb{Z}[a, b, \Delta_{a,b}^{-1}] \to \tilde{R} = \mathbb{Z}[\omega][1/6, a, b, \Delta_{a,b}^{-1}, \varepsilon]/(\varepsilon^6 - \Delta_{a,b})$$

and define $\tilde{S} \to S$ to be the induced morphism of schemes. This is finite, etale, and Galois, with Galois group $G := \mathrm{Aut}(\tilde{S}/S)$ a dihedral group of order 12. The group G naturally acts on $\mathrm{End}^0(\mathcal{P}_{\tilde{S}}/\tilde{S})$. Let $\mathrm{triv}\colon G \to \mathrm{GL}_1(\mathbb{Q})$ be the trivial representation and let $\mathrm{std}\colon G \to \mathrm{GL}_2(\mathbb{Q})$ be a model for the reflection representation, i.e. the unique irreducible 2-dimensional representation of G defined over $\mathbb{Q}$. See §1.7 for our notations concerning abelian schemes.

Theorem 3.4. *The endomorphism algebra $B := \mathrm{End}^0(\mathcal{P}_{\tilde{S}}/\tilde{S})$ is a quaternion algebra of discriminant 6 over $\mathbb{Q}$. Moreover, the G-representation $\mathrm{Hom}^{\mathrm{sym}}(\mathcal{P}_{\tilde{S}}, \mathcal{P}^\vee_{\tilde{S}}) \otimes \mathbb{Q}$ is three-dimensional and isomorphic to $\mathrm{triv} \oplus \mathrm{std}$.*

Proof. The schemes S and $\tilde{S}$ are integral and normal; let K and $\tilde{K}$ denote their function fields and let $P = \mathcal{P}_K$ denote the generic fiber of $\mathcal{P}$. By [17, §2, Lemma 1], the natural map $\mathrm{End}(\mathcal{P}_{\tilde{S}}/\tilde{S}) \to \mathrm{End}(P_{\tilde{K}})$ is an isomorphism. By [41, Tag 0BQM], the natural map $G = \mathrm{Aut}(\tilde{S}/S) \to \mathrm{Gal}(\tilde{K}/K)$ is also an isomorphism. It therefore suffices to prove all claims over the generic point $\mathrm{Spec}(K)$. By [28, Lemma 6.16], P is geometrically simple. If $\bar{K}$ denotes an algebraic closure of K then [28, Lemma 6.2] implies that $\mathrm{End}^0(P_{\bar{K}})$ is a discriminant 6 quaternion algebra, and [28, Theorem 6.5] shows that $\mathrm{End}^0(P_{\bar{K}}) = \mathrm{End}^0(P_{\tilde{K}})$. The claim concerning $\mathrm{Hom}^{\mathrm{sym}}(\mathcal{P}_{\tilde{S}}, \mathcal{P}^\vee_{\tilde{S}}) \otimes \mathbb{Q} = \mathrm{Hom}(P_{\tilde{K}}, P^\vee_{\tilde{K}}) \otimes \mathbb{Q}$ follows from [28, Corollary 6.8]. ☐

The μ_3-action on $\mathcal{J}$ restricts to a μ_3-action on $\mathcal{P}$, which by functoriality induces a μ_3-action on $\mathrm{Hom}^{\mathrm{sym}}(\mathcal{P}_{\tilde{S}}, \mathcal{P}_{\tilde{S}}^{\vee})$ via $\omega \cdot \lambda = \omega^{\vee} \circ \lambda \circ \omega^{-1}$ for all $\lambda \in \mathrm{Hom}^{\mathrm{sym}}(\mathcal{P}_{\tilde{S}}, \mathcal{P}_{\tilde{S}}^{\vee}) \otimes \mathbb{Q}$ and $\omega \in \mu_3$. This μ_3-action commutes with the G-action, so preserves the isotypic components triv and std.

Lemma 3.5. *In the decomposition* $\mathrm{Hom}^{\mathrm{sym}}(\mathcal{P}_{\tilde{S}}, \mathcal{P}_{\tilde{S}}^{\vee}) \otimes \mathbb{Q} = \mathrm{triv} \oplus \mathrm{std}$ *of Theorem 3.4,* μ_3 *acts trivially on* triv *and* μ_3 *acts on* $\mathrm{std} \otimes \overline{\mathbb{Q}}$ *as a direct sum of the two distinct nontrivial characters.*

Proof. The proof of [28, Corollary 6.8] shows that if we write $B = \mathrm{span}\{1, i, j, ij\}$ with $i^2 = -3$, $j^2 = 2$, $ij = -ji$ and $\omega = (-1+i)/2$, then $\mathrm{Hom}^{\mathrm{sym}}(\mathcal{P}_{\tilde{S}}, \mathcal{P}_{\tilde{S}}^{\vee}) \otimes \mathbb{Q} \simeq \mathrm{span}\{1, j, ij\}$, triv is spanned by 1, std is spanned by $\{i, ij\}$ and $\omega \in \mu_3$ acts via conjugation by $\omega \in B$. The description of the μ_3-action on triv and std now follow from an explicit calculation. ☐

4. Background on Chow motives

Our proof of Theorem 1.1 uses the language of relative Chow motives, so we recall the necessary definitions and prove some basic statements on motives for which we could not find a reference. The most important sections are §4.2 and §4.7 and the others can be referred back to when needed in §5.

We call a scheme S adequate if it is smooth and quasiprojective over a Dedekind domain. For the remainder of this section fix such a scheme S.

4.1. Correspondences.

We denote by $\mathsf{SmProj}(S)$ the category of smooth projective S-schemes. For X in $\mathsf{SmProj}(S)$ the Chow groups $\mathrm{CH}^p(X)$ are well defined, see §1.7. Write $\mathrm{CH}^p(X;\mathbb{Q}) := \mathrm{CH}^p(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and if $X \to S$ has equidimensional fibers write $d(X/S)$ for the relative dimension of $X \to S$. If X and Y are in $\mathsf{SmProj}(S)$ we define the $\mathbb{Q}$-vector space of correspondences by $\mathrm{Corr}(X,Y) := \mathrm{CH}^*(X \times_S Y;\mathbb{Q})$. We say a correspondence has degree r if it lies in the subspace

$$\mathrm{Corr}^r(X,Y) := \bigoplus_i \mathrm{CH}^{r+d(X_i/S)}(X_i \times_S Y;\mathbb{Q})$$

where $X = \sqcup_i X_i$ is a decomposition such that $X_i \to S$ is equidimensional. If $p \in \mathrm{Corr}^r(X,Y)$ and $q \in \mathrm{Corr}^s(Y,Z)$ are correspondences, denote by $q \circ p = \pi_{XZ,*}(\pi_{XY}^* p \cdot \pi_{YZ}^* q) \in \mathrm{Corr}^{r+s}(X,Z)$ their composition. If $f\colon X \to Y$ is an S-morphism write $\Gamma_f \subset X \times_S Y$ for its graph and ${}^t\Gamma_f \subset Y \times_S X$ for its transpose. Write $\Delta_{X/S}$ for the graph of the identity $X \to X$.

4.2. Relative Chow motives.

We denote by $\mathsf{Mot}(S)$ the category of relative Chow motives with respect to graded correspondences over S, see [13, §1] for more details. The results of [13] are stated under the standing assumption that S is smooth and quasi-projective over a field, but their results continue to hold in our more general set-up since intersection theory is developed in this generality, see [25, §1, Remark 1.1].

We recall that an object of $\mathsf{Mot}(S)$ is a triple $(X/S, p, m)$, where X/S is a smooth projective S-scheme, $p \in \mathrm{Corr}^0(X,X)$ is an idempotent correspondence and m is an integer. Morphisms are given by

$$\mathrm{Hom}((X/S,p,m),(Y/S,q,n)) := q \circ \mathrm{Corr}^{m-n}(X,Y) \circ p.$$

Given a motive $M = (X/S,p,m)$ and $n \in \mathbb{Z}$ we write $M(n) := (X/S,p,m+n)$. There is a contravariant functor $\mathsf{SmProj}(S) \to \mathsf{Mot}(S)$ sending X/S to $\mathfrak{h}(X/S) := (X/S, [\Delta_{X/S}], 0)$ and sending morphisms $f\colon X \to Y$ to $[{}^t\Gamma_f]$. Write $\mathbf{1} = \mathbf{1}_S := \mathfrak{h}(S/S)$ for the unit motive and $\mathbb{L} := \mathbf{1}(-1)$ for the Lefschetz motive. There is a notion of direct sums and tensor products in $\mathsf{Mot}(S)$.

4.3. **Base changing motives.** Suppose that T is another adequate scheme and $f\colon T \to S$ a morphism. There are two situations in which we have a good theory of pullbacks[2] [19, §20.1]:

(1) If f is flat, there are pullback maps $f^*\colon \mathrm{CH}^p(X) \to \mathrm{CH}^p(X_T)$ for every X/S in $\mathsf{SmProj}(S)$ and $p \in \mathbb{Z}_{\geq 0}$.
(2) If f is an lci morphism, there are Gysin homomorphisms $f^*\colon \mathrm{CH}^p(X) \to \mathrm{CH}^p(X_T)$ for every X/S in $\mathsf{SmProj}(S)$ compatible with flat pullbacks.

We will say that f admits pullbacks if it is a finite composition $f_1 \circ \cdots \circ f_n$ of morphisms of the above form, and for such a morphism we define $f^* = f_n^* \circ \cdots \circ f_1^*\colon \mathrm{CH}^p(X) \to \mathrm{CH}^p(X_T)$ for every X/S in $\mathsf{SmProj}(S)$. (This does not depend on the choice of f_i's.) For example, if T is the spectrum of a field then f admits pullbacks. When f admits pullbacks we define a functor

$$f^*\colon \mathsf{Mot}(S) \to \mathsf{Mot}(T),$$

which on objects sends $(X/S, p, m)$ to $(X_T/T, f^*(p), m)$ and sends a correspondence q to $f^*(q)$.

4.4. **Descending direct summands of motives.** Let $\pi\colon S' \to S$ be a finite etale morphism of adequate schemes. Suppose additionally that it is Galois and write G for the group of Deck transformations, so we have an isomorphism of S-schemes $g\colon S' \to S'$ for every $g \in G$. Using the pullback functor from §4.3, we obtain functors $g^*\colon \mathsf{Mot}(S') \to \mathsf{Mot}(S')$. Let M, N be objects of $\mathsf{Mot}(S)$ and write $M' = \pi^*M$, $N' = \pi^*N$ for the corresponding objects of $\mathsf{Mot}(S')$. Then we have canonical identifications $M' = g^*M'$ and $N' = g^*N'$ for all $g \in G$. It follows by functoriality that $\mathrm{Hom}(M', N')$ inherits a G-action.

Lemma 4.1. *In the above notation, the natural map* $\mathrm{Hom}(M, N) \to \mathrm{Hom}(M', N')^G$ *is an isomorphism.*

Proof. This is well known when $S = \mathrm{Spec}(k)$ [39, Lemma 1.17]; we sketch the generalization of the classical proof to the relative setting. Write $M = (X/S, p, m)$ and $N = (Y/S, q, n)$. Then an element of $\mathrm{Hom}(M', N')^G$ is the same as a G-invariant element α of $q \circ \mathrm{CH}^{d(X/S)+m-n}(X \times_S Y; \mathbb{Q}) \circ p$. Choose a cycle Z on $X_{S'} \times_{S'} Y_{S'}$ representing α. Then $\sum_{g \in G} g^*Z$ descends to a unique cycle W of $X \times_S Y$. Moreover the rational equivalence class $[W]$ is independent of the choice of Z representing α. It follows that $\alpha \mapsto \frac{1}{|G|}[W]$ is a well-defined inverse to the base change map above. ☐

This implies that Galois-invariant summands also satisfy descent:

Corollary 4.2. *Let M be an object of* $\mathsf{Mot}(S)$ *and $U' \oplus V'$ a direct sum decomposition of $M' = \pi^*M$ such that we have an equality of direct summands $g^*U' = U'$ and $g^*V' = V'$ in $M' = g^*M$ for all $g \in G$. Then there exists a unique direct sum decomposition $M = U \oplus V$ such that $\pi^*U = U'$ and $\pi^*V = V'$.*

Proof. Since direct sum decompositions correspond bijectively to a pair of orthogonal idempotents in the endomorphism ring, this immediately follows from Lemma 4.1. ☐

4.5. **Group actions on motives.** Let M be a motive in $\mathsf{Mot}(S)$ and G a finite group acting on M. Then every idempotent in the group algebra $\mathbb{Q}[G]$ gives rise to an idempotent in M. For example, if ρ is an irreducible (but not necessarily absolutely irreducible) representation defined over $\mathbb{Q}$ and e_ρ is the corresponding central idempotent, we get a direct summand M_ρ of M, called the ρ-isotypic component. Applying this to $\rho =$ the trivial representation, we see that the motive of G-fixed points $M^G = M_\rho$ is well-defined and a direct summand of M.

[2]Pullbacks can be defined in greater generality, see for example [19, Example 20.1.2], but we restrict ourselves to the concrete cases we need in this paper.

We may also consider actions by nonconstant finite group schemes, using Lemma 4.1. For example, if n is invertible in S, a μ_n-action on a motive M in $\mathsf{Mot}(S)$ is an $\mathrm{Aut}(\tilde{S}/S)$-equivariant map $\mu_n(\tilde{S}) \to \mathrm{Aut}(f^*M)$, where $f\colon \tilde{S} = S \times_{\mathbb{Z}[1/n]} \mathbb{Z}[\zeta_n][1/n] \to S$. The $\mathrm{Aut}(\tilde{S}/S)$-equivariance implies that the μ_n-fixed points of f^*M descend to a well-defined direct summand M^{μ_n} of M.

4.6. **Relative Artin motives.** Assume that S is connected. An Artin motive in $\mathsf{Mot}(S)$ is a direct summand of a motive of the form $\mathfrak{h}(X/S)$, where $X \to S$ in $\mathsf{SmProj}(S)$ has relative dimension zero or equivalently, is finite etale. When S is the spectrum of a field, Artin motives are the same as Galois representations on finite-dimensional $\mathbb{Q}$-vector spaces; we give a similar (presumably well known) description in the relative setting. Let $f\colon \tilde{S} \to S$ be a connected finite etale cover with Galois group $G = \mathrm{Aut}(\tilde{S}/S)$. We say an Artin motive M in $\mathsf{Mot}(S)$ is trivialized by f if $f^*M \simeq \mathbf{1}_{\tilde{S}}^{\oplus n}$ for some $n \in \mathbb{Z}_{\geq 0}$. In that case the n-dimensional $\mathbb{Q}$-vector space $\mathrm{Hom}(\mathbf{1}_{\tilde{S}}, f^*M)$ has natural G-action, by the same reasoning as §4.4.

Lemma 4.3. *Every Artin motive in $\mathsf{Mot}(S)$ is trivialized by some connected finite etale Galois cover $f\colon \tilde{S} \to S$. Fixing such an f, the assignment $M \mapsto V_M := \mathrm{Hom}(\mathbf{1}_{\tilde{S}}, f^*M)$ induces an equivalence between the full subcategory of Artin motives in $\mathsf{Mot}(S)$ trivialized by f and the category of $\mathrm{Aut}(\tilde{S}/S)$-representations on finite-dimensional $\mathbb{Q}$-vector spaces.*

Proof. (Sketch) If M is a direct summand of $\mathfrak{h}(X/S)$, then M is trivialized by the compositum of the Galois closures of the connected components of X, justifying the first sentence. Fully faithfulness of $M \mapsto V_M$ follows from Lemma 4.1 and the fact that $\mathrm{Hom}(\tilde{M}, \tilde{N}) \to \mathrm{Hom}(V_M, V_N)$ is an isomorphism when $\tilde{S}$ is connected. Moreover, again since $\tilde{S}$ is connected $V_{\mathfrak{h}(\tilde{S}/S)}$ is isomorphic to the regular representation of G. Therefore essential surjectivity follows from the fact that every G-representation is a direct summand of copies of the regular representation. □

4.7. **Motives of abelian schemes.** Let $A \to S$ be an abelian scheme of relative dimension g. Deninger–Murre [13, §3] have shown that the motive $\mathfrak{h}(A/S)$ has a canonical decomposition into Chow–Kunneth components.

Theorem 4.4 (Deninger–Murre)**.** *There exists a unique direct sum decomposition*

$$\mathfrak{h}(A/S) = \bigoplus_{i=0}^{2g} \mathfrak{h}^i(A/S) \tag{4.1}$$

in $\mathsf{Mot}(S)$ with the property that $[{}^t\Gamma_{(n)}]$ acts as n^{2i} on $\mathfrak{h}^i(A/S)$ for every $0 \leq i \leq 2g$ and every $n \in \mathbb{Z}$.

We call the decomposition (4.1) the canonical Chow–Kunneth decomposition of $\mathfrak{h}(A/S)$. In what follows $\mathfrak{h}^i(A/S)$ will always denote the i-th component of this decomposition. We will use the next proposition frequently when doing explicit calculations with $\mathfrak{h}^1$ of abelian schemes.

Proposition 4.5. [24, Proposition 2.2.1] *Let $A, B \to S$ be abelian schemes of the same relative dimension. Then the pullback map*

$$\mathrm{Hom}^0(A/S, B/S) \to \mathrm{Hom}(\mathfrak{h}^1(B/S), \mathfrak{h}^1(A/S)) \tag{4.2}$$

is an isomorphism of $\mathbb{Q}$-vector spaces .

We also include the following result on abelian schemes, which follows from [17, §2, Lemma 1] and the fact that adequate schemes are normal.

Proposition 4.6. *Suppose that S is integral with generic point $\mathrm{Spec}(K)$. Then the map $A/S \mapsto A_K/K$ identifies the category of abelian schemes over S with a full subcategory of the category of abelian varieties over K.*

5. Proof of main theorem

This section is the technical heart of the paper. In §5.1 we state and prove the isomorphism of motives (1.2) promised in the introduction. Next in §5.2 we translate this isomorphism into a concrete statement about Chow groups, and in §5.3 we show that one of these Chow groups contains the Ceresa cycle. In §5.4 we identify the image of the Ceresa cycle under our constructed isomorphism. In the last two subsections we combine everything and prove the theorems and corollaries stated in the introduction.

5.1. **An isomorphism of motives.** Let $C = C_{a,b}$ be a bielliptic Picard curve (3.1) over a field k with Jacobian J. The μ_6-action on C induces, via pullback, a μ_6-action on J and $\mathfrak{h}(J)$. The uniqueness of the Chow–Kunneth decomposition (4.1) shows that this μ_6-action restricts to a μ_6-action on $\mathfrak{h}^i(J)$ for each $0 \leq i \leq 6$. Let $E^\Delta \colon y^2 = x^3 + 4b(a^2-4b)^2$ be the sextic twist of $E : y^2 = x^3 + 16(a^2 - 4b)$ by $\Delta = \Delta_{a,b} = 16b(a^2-4b)$. Recall from §4.5 that $\mathfrak{h}^3(J)^{\mu_6}$ is a direct summand of $\mathfrak{h}^3(J)$.

Theorem 5.1. *There is an isomorphism in* $\mathsf{Mot}(k)$*:* $\mathfrak{h}^3(J)^{\mu_6} \simeq \mathfrak{h}^1(E^\Delta)(-1)$.

In fact, it will be crucial for our proof of Theorem 1.1 to construct this isomorphism universally. We will use the notation of §3.3, so $S = \mathbb{A}^2_{\mathbb{Z}[1/6]} \setminus \{\Delta_{a,b} = 0\}$ is the base scheme, $\mathcal{C} \to S$ the universal bielliptic Picard curve with Jacobian $\mathcal{J} \to S$ and Prym variety $\mathcal{P} \to S$. Let $\mathcal{E} \to S$ be the relative elliptic curve with equation $y^2 = x^3 + 16(a^2-4b)$ and let $\mathcal{E}^\Delta \to S$ be the relative elliptic curve with equation $y^2 = x^3 + 4b(a^2-4b)^2$. There is again a μ_6-action on $\mathcal{C}$, $\mathcal{J}$ and $\mathfrak{h}^3(\mathcal{J}/S)$. The goal of this subsection is to show:

Theorem 5.2. *There is an isomorphism in* $\mathsf{Mot}(S)$*:* $\mathfrak{h}^3(\mathcal{J}/S)^{\mu_6} \simeq \mathfrak{h}^1(\mathcal{E}^\Delta/S)(-1)$.

Specializing this isomorphism at k-points $\mathrm{Spec}(k) \to S$ recovers Theorem 5.1. The proof of Theorem 5.2 will be given at the end of this subsection after combining a few different ingredients.

The starting point is the decomposition

$$\mathfrak{h}^1(\mathcal{J}/S) = \mathfrak{h}^1(\mathcal{P}/S) \oplus \mathfrak{h}^1(\mathcal{E}/S)$$

induced by the isogeny $\mathcal{J} \sim \mathcal{P} \times_S \mathcal{E}$ of Lemma 3.3. Via the isomorphism $\mathfrak{h}^3(\mathcal{J}/S) \simeq \bigwedge^3 \mathfrak{h}^1(\mathcal{J}/S)$ [26] and similarly for $\mathcal{P}$ and $\mathcal{E}$, we obtain a μ_6-equivariant decomposition

$$\mathfrak{h}^3(\mathcal{J}/S) = \mathfrak{h}^3(\mathcal{P}/S) \oplus (\mathfrak{h}^2(\mathcal{P}/S) \otimes \mathfrak{h}^1(\mathcal{E}/S)) \oplus (\mathfrak{h}^1(\mathcal{P}/S) \otimes \mathfrak{h}^2(\mathcal{E}/S)). \tag{5.1}$$

Lemma 5.3. $\mathfrak{h}^3(\mathcal{J}/S)^{\mu_6} = (\mathfrak{h}^2(\mathcal{P}/S) \otimes \mathfrak{h}^1(\mathcal{E}/S))^{\mu_3}$.

Proof. By definition of $\mathcal{P}$ and $\mathcal{E}$ in §3.2, if $\mu_2(S) = \{1, \tau\}$ then τ acts as (-1) on $\mathcal{P}$ and as the identity on $\mathcal{E}$. Since $[{}^t\Gamma_{(n)}]$ acts as n^i on $\mathfrak{h}^i$ of an abelian scheme, we see that τ acts on $\mathfrak{h}^i(\mathcal{P}/S) \otimes \mathfrak{h}^{3-i}(\mathcal{E}/S)$ as -1 for $i = 1, 3$ and as the identity for $i = 2$. It follows that $(\mathfrak{h}^i(\mathcal{P}/S) \otimes \mathfrak{h}^{3-i}(\mathcal{E}/S))^{\mu_6} = (\mathfrak{h}^i(\mathcal{P}/S) \otimes \mathfrak{h}^{3-i}(\mathcal{E}/S))^{\mu_2} = 0$ if $i = 1,3$ and $(\mathfrak{h}^2(\mathcal{P}/S) \otimes \mathfrak{h}^1(\mathcal{E}/S))^{\mu_6} = (\mathfrak{h}^2(\mathcal{P}/S) \otimes \mathfrak{h}^1(\mathcal{E}/S))^{\mu_3}$, so we conclude by (5.1). $\square$

It remains to determine $(\mathfrak{h}^2(\mathcal{P}/S) \otimes \mathfrak{h}^1(\mathcal{E}/S))^{\mu_3}$. To this end, we decompose $\mathfrak{h}^2(\mathcal{P}/S)$ further using results of Moonen [32] and the quaternionic action on $\mathcal{P}$.

Recall the finite etale cover $\tilde{S} \to S$ from §3.3 given by adjoining a sixth root of unity and a sixth root of the discriminant Δ. This cover has Galois group $G = \mathrm{Aut}(\tilde{S}/S) \simeq D_6$. Also recall from Theorem 3.4 the G-representation $\mathrm{Hom}^{\mathrm{sym}}(\mathcal{P}_{\tilde{S}}, \mathcal{P}^\vee_{\tilde{S}}) \otimes \mathbb{Q}$ of self-dual homomorphisms; let $\mathrm{NS}_{\mathcal{P}}$ be the corresponding Artin motive in $\mathsf{Mot}(S)$ under the equivalence of Lemma 4.3. The μ_3-action on $\mathcal{P}$ induces via pullback a μ_3-action on $\mathrm{NS}_{\mathcal{P}}$ which is described in Lemma 3.5.

Proposition 5.4. *There exists a μ_3-stable decomposition*

$$\mathfrak{h}^2(\mathcal{P}/S) = \mathfrak{h}^2_{\mathrm{alg}}(\mathcal{P}/S) \oplus \mathfrak{h}^2_{\mathrm{tr}}(\mathcal{P}/S) \tag{5.2}$$

in $\mathsf{Mot}(S)$ with the following two properties:

(1) there is a μ_3-equivariant isomorphism $\mathfrak{h}^2_{\mathrm{alg}}(\mathcal{P}/S) \simeq \mathrm{NS}_{\mathcal{P}}(-1)$;

(2) The μ_3-action on $\mathfrak{h}^2_{\mathrm{tr}}(\mathcal{P})$ is trivial.

Proof. By Corollary 4.2, it suffices to find a G-stable decomposition of $\mathfrak{h}^2(\mathcal{P}_{\tilde{S}}/\tilde{S})$ with similar properties to (5.2). Theorem 3.4 shows that $B = \mathrm{End}^0(\mathcal{P}_{\tilde{S}}/\tilde{S})$ is a nonsplit quaternion algebra. By functoriality we obtain a homomorphism $B^{\mathrm{op}} \to \mathrm{End}(\mathfrak{h}^1(\mathcal{P}_{\tilde{S}}/\tilde{S}))$, where B^{op} is the opposite algebra of B. We identify B^{op} with B using the canonical involution on B. Moonen [32, Theorem 7.2] has shown that this B-action on $\mathfrak{h}^1(\mathcal{P}_{\tilde{S}}/\tilde{S})$ can be used to refine the Chow–Kunneth decomposition of $\mathfrak{h}(\mathcal{P}_{\tilde{S}}/\tilde{S})$. Our specific situation is explained in detail in [32, §8.2]: let

$$\mathfrak{h}^2(\mathcal{P}_{\tilde{S}}/\tilde{S}) = R^{(2,0)} \oplus R^{(1,1)} \tag{5.3}$$

be the decomposition described in [32, Bottom of p. 104]. There exist algebraic representations ρ_1, ρ_2 of $B^\times$ such that $\mathrm{Hom}(M, R^{(2,0)})$ is ρ_1-isotypic and $\mathrm{Hom}(M, R^{(1,1)})$ is ρ_2-isotypic for every motive M in $\mathsf{Mot}(S)$. The representation ρ_1 is a twist of the conjugation action of $B^\times$ on trace zero elements in B, and the representation ρ_2 is the norm map $\mathrm{Nm}\colon B^\times \to \mathbb{Q}^\times$. The μ_3-action on $\mathcal{P}_{\tilde{S}}$ is induced from restricting the $B^\times$-action to a quadratic field $\mathbb{Q}(\omega)^\times \subset B^\times$ and restricting to its subgroup $\mu_3 = \langle \omega \rangle \subset \mathbb{Q}(\omega)^\times$. Since Nm is trivial on μ_3, it follows that μ_3 acts trivially on $R^{(1,1)}$. By construction of $R^{(2,0)}$ [32, §6.4], there is a canonical isomorphism $\alpha\colon \mathbb{L} \otimes (\mathrm{Hom}^{\mathrm{sym}}(\mathcal{P}_{\tilde{S}}, \mathcal{P}^\vee_{\tilde{S}}) \otimes \mathbb{Q}) \xrightarrow{\sim} R^{(2,0)}$. Since G preserves the isomorphism classes of ρ_1 and ρ_2, the decomposition (5.3) is G-stable. Therefore by Corollary 4.2, $R^{(2,0)} \oplus R^{(1,1)}$ descends to a decomposition $\mathfrak{h}^2_{\mathrm{alg}}(\mathcal{P}/S) \oplus \mathfrak{h}^2_{\mathrm{tr}}(\mathcal{P}/S)$. Since the isomorphism α is G-equivariant, it descends to an isomorphism $\mathrm{NS}_{\mathcal{P}}(-1) \simeq \mathfrak{h}^2_{\mathrm{alg}}(\mathcal{P}/S)$, as desired. $\square$

Remark 5.5. Let $\bar{K}$ be the algebraic closure of the function field of S. Then the base change of the decomposition (5.2) along $\mathrm{Spec}(\bar{K}) \to S$ is the decomposition of $\mathfrak{h}^2(\mathcal{P}_{\bar{K}})$ into its algebraic and transcendental part [23], explaining our choice of notation.

Corollary 5.6. $(\mathfrak{h}^2(\mathcal{P}/S) \otimes \mathfrak{h}^1(\mathcal{E}/S))^{\mu_3} \simeq (\mathrm{NS}_{\mathcal{P}} \otimes \mathfrak{h}^1(\mathcal{E}/S))^{\mu_3}(-1)$

Proof. By Proposition 5.4, μ_3 acts trivially on $\mathfrak{h}^2_{\mathrm{tr}}(\mathcal{P}/S)$. Since the μ_3-action on $\mathcal{E}$ is faithful, $\mathfrak{h}^1(\mathcal{E}/S)^{\mu_3} = 0$. We conclude that $(\mathfrak{h}^2_{\mathrm{tr}}(\mathcal{P}/S) \otimes \mathfrak{h}^1(\mathcal{E}/S)))^{\mu_3} = 0$ and so $(\mathfrak{h}^2(\mathcal{P}/S) \otimes \mathfrak{h}^1(\mathcal{E}/S))^{\mu_3} = (\mathfrak{h}^2_{\mathrm{alg}}(\mathcal{P}/S) \otimes \mathfrak{h}^1(\mathcal{E}/S))^{\mu_3} \simeq (\mathrm{NS}_{\mathcal{P}} \otimes \mathfrak{h}^1(\mathcal{E}/S))^{\mu_3}(-1)$, again by Proposition 5.4. $\square$

To analyze the piece $\mathrm{NS}_{\mathcal{P}} \otimes \mathfrak{h}^1(\mathcal{E}/S)$, we define and study an auxiliary abelian scheme $\mathcal{E} \otimes \mathrm{std}$. Recall from §3.3 the finite etale cover $\tilde{S} \to S$ with Galois group $G \simeq D_6$ and its two-dimensional representation $\mathrm{std}\colon G \to \mathrm{GL}_2(\mathbb{Q})$, which in fact lands in $\mathrm{GL}_2(\mathbb{Z})$. Let $\mathcal{E} \otimes \mathrm{std}$ be the (up to isomorphism) unique abelian scheme over S that is isomorphic to $\mathcal{E}^2$ over $\tilde{S}$ with descent data given by $\mathrm{std}\colon G \to \mathrm{GL}_2(\mathbb{Z}) \subset \mathrm{End}(\mathcal{E}^2/S)$. (This is the analogue of the construction of [31] in the relative setting.)

Lemma 5.7. *There is an isogeny $\mathcal{E} \otimes \mathrm{std} \sim \mathcal{E}^{\Delta} \times \mathcal{E}^{\Delta^{-1}}$.*

Proof. Let $K = \mathbb{Q}(a, b)$ be the function field of S and denote the base change of $\mathcal{E}$ to K by E. Proposition 4.6 shows that it suffices to prove the lemma over K. Let $\tilde{K} = K(\omega, \sqrt[6]{\Delta})$ denote the function field of $\tilde{S}$. For $d \in K^\times$, write $E^d : y^2 = x^3 + 16(a^2 - 4b)d$ for the sextic twist of E by d. Then $E \otimes \mathrm{std}$ is an isogeny factor of

$$\mathrm{Res}_K^{\tilde{K}} E \sim \mathrm{Res}_K^{K(\sqrt[6]{\Delta})}(E^2) \sim \left(\mathrm{Res}_K^{K(\sqrt[6]{\Delta})} E\right)^2 \sim \left(\prod_{i=0}^{5} E^{\Delta^i}\right)^2,$$

where the first isogeny is because E is isogenous to its own (-3)-quadratic twist, this being true for any j-invariant 0 elliptic curve. The decomposition on the right hand side corresponds (by base change) to the six characters of $H = \mathrm{Gal}(\tilde{K}/K(\omega))$. Since $\mathrm{std}|_H$ is the sum of the two characters of order 6, we must have

$$E \otimes \mathrm{std} \sim E^{\Delta} \times E^{\Delta^5} \simeq E^{\Delta} \times E^{\Delta^{-1}},$$

as claimed. □

Lemma 5.8. $(\mathrm{NS}_{\mathcal{P}} \otimes \mathfrak{h}^1(\mathcal{E}/S))^{\mu_3} \simeq \mathfrak{h}^1(\mathcal{E}^{\Delta}/S)$.

Proof. Use Theorem 3.4 to decompose the G-representation $\mathrm{NS}_{\mathcal{P}}$ as $\mathrm{triv} \oplus \mathrm{std}$. By Lemma 3.5, μ_3 acts trivially on triv and as a direct sum of two nontrivial characters on $\mathrm{std} \otimes \overline{\mathbb{Q}}$. The μ_3-actions on $\mathcal{E}$ and std combine to a μ_3-action on $\mathcal{E} \otimes \mathrm{std}$. Write $\mathcal{A} := (\mathcal{E} \otimes \mathrm{std})^{\mu_3, \circ}$. We then have

$$(\mathrm{NS}_{\mathcal{P}} \otimes \mathfrak{h}^1(\mathcal{E}/S))^{\mu_3} \simeq (\mathrm{std} \otimes \mathfrak{h}^1(\mathcal{E}/S))^{\mu_3} \simeq (\mathfrak{h}^1((\mathcal{E} \otimes \mathrm{std})/S))^{\mu_3} \simeq \mathfrak{h}^1(\mathcal{A}/S), \tag{5.4}$$

where the first isomorphism follows from the fact that $\mathfrak{h}^1(\mathcal{E}/S)^{\mu_3} = 0$, and the last two follow from Lemma 4.1 and Proposition 4.5. To prove the lemma, it remains to show that the abelian scheme $\mathcal{A}$ is isogenous to $\mathcal{E}^{\Delta}$.

Let $\mathcal{A}'$ be the quotient abelian scheme $(\mathcal{E} \otimes \mathrm{std})/\mathcal{A}$. A tangent space calculation (using the μ_3-action on std and $\mathcal{E}$) shows that $\mathcal{A} \to S$ has relative dimension 1, hence so does the quotient $\mathcal{A}' \to S$. By complete reducibility and Lemma 5.7, $\mathcal{A}$ is isogenous to either $\mathcal{E}^{\Delta} \colon y^2 = x^3 + 4b(a^2 - 4b)^2$ or $\mathcal{E}^{\Delta^{-1}} \colon y^2 = x^3 + b^{-1}$. We will exclude the latter possibility using a rather indirect argument.

Fix a unit $d \in \mathcal{O}(S)^\times$, consider the isomorphism $\lambda_d \colon S \to S$ mapping (a, b) to $(da, d^2 b)$ and let $\mathcal{J}^d = \lambda_d^* \mathcal{J}$. Then $\mathcal{J}$ and $\mathcal{J}^d$ are sextic twists in the sense that they become isomorphic over the μ_6-cover $f_d \colon S_d \colon \to S$ given by adjoining a sixth root δ of d. There exists an isomorphism $f_d^* \mathcal{J} \xrightarrow{\sim} f_d^* \mathcal{J}^d$ of abelian schemes over S_d induced by the isomorphism $(x, y) \mapsto (\delta^3 x, \delta^4 y)$ between $\mathcal{C}$ and $\lambda_d^* \mathcal{C}$. This in turn induces an isomorphism $\mathfrak{h}^3(f_d^* \mathcal{J}^d / S_d) \xrightarrow{\sim} \mathfrak{h}^3(f_d^* \mathcal{J}/S_d)$. Lemma 4.1 shows that the induced morphism on μ_6-fixed points descends to an isomorphism $\mathfrak{h}^3(\mathcal{J}/S)^{\mu_6} \simeq \mathfrak{h}^3(\mathcal{J}^d/S)^{\mu_6}$ over S.

Now assume, for the sake of contradiction, that $\mathcal{A} \sim \mathcal{E}^{\Delta^{-1}}$. Then $\mathfrak{h}^3(\mathcal{J}/S)^{\mu_6} \simeq \mathfrak{h}^1(\mathcal{E}^{\Delta^{-1}}/S)(-1)$ by (5.4). Pulling back this isomorphism along λ_d shows that $\mathfrak{h}^3(\mathcal{J}^d/S)^{\mu_6} \simeq \mathfrak{h}^1(\mathcal{E}^{d^{-2}\Delta^{-1}}/S)(-1)$ and hence $\mathfrak{h}^1(\mathcal{E}^{\Delta^{-1}}/S) \simeq \mathfrak{h}^1(\mathcal{E}^{d \cdot \Delta^{-1}}/S)$. By Proposition 4.5, $\mathcal{E}^{\Delta^{-1}}$ and $\mathcal{E}^{d^{-2} \cdot \Delta^{-1}}$ are isogenous for all such d. This is a contradiction, since setting $s = (a, b) = (1, 1)$ and $d = 3^{-1}$ produces two curves $\mathcal{E}^{\Delta}_{(a,b)} \colon y^2 = x^3 + 1$ and $\mathcal{E}^{9\Delta}_{(a,b)} \colon y^2 = x^3 + 9$ that are not isogenous over $\mathbb{Q}$ (by counting $\mathbb{F}_7$-points, for example). □

Proof of Theorem 5.2. We have

$$\mathfrak{h}^3(\mathcal{J}/S)^{\mu_6} = (\mathfrak{h}^2(\mathcal{P}/S) \otimes \mathfrak{h}^1(\mathcal{E}/S))^{\mu_3} \simeq (\mathrm{NS}_{\mathcal{P}} \otimes \mathfrak{h}^1(\mathcal{E}/S))^{\mu_3}(-1) \simeq \mathfrak{h}^1(\mathcal{E}^{\Delta}/S)(-1).$$

using Lemma 5.3, Corollary 5.6 and Lemma 5.8 respectively. □

5.2. **Chow groups.** Our next goal is to translate the isomorphism of Theorem 5.2 into a concrete statement about Chow groups. The (covariant) Chow groups of a motive M in $\mathsf{Mot}(S)$ are by definition the $\mathbb{Q}$-vector spaces $\mathrm{CH}^p(M) := \mathrm{Hom}(\mathbb{L}^{\otimes p}, M)$ for $p \in \mathbb{Z}$, where we recall from §4.2 that $\mathbb{L}$ denotes the Lefchetz motive. For example, $\mathrm{CH}^p(\mathfrak{h}(X/S)) = \mathrm{CH}^p(X)$ for all X/S in $\mathsf{SmProj}(S)$. We will compute CH^2 of $\mathfrak{h}^3(\mathcal{J}/S)^{\mu_6}$ and $\mathfrak{h}^1(\mathcal{E}^\Delta/S)(-1)$ below. To describe the former we will use the Beauville decomposition of the Chow groups of an abelian variety [3], extended by Deninger–Murre to the relative setting [13, Theorem 2.19].

Proposition 5.9 (Beauville decomposition)**.** *Let A/S be an abelian scheme and $p \in \mathbb{Z}_{\geq 0}$. For $i \in \mathbb{Z}$ let*

$$\mathrm{CH}^p_{(i)}(A;\mathbb{Q}) := \{\alpha \in \mathrm{CH}^p(A;\mathbb{Q}) \mid (n)^*\alpha = n^{2p-i}\alpha \text{ for all } n \in \mathbb{Z}\}.$$

Then $\mathrm{CH}^p_{(i)}(A;\mathbb{Q})$ is nonzero for only finitely many i and we have a direct sum decomposition

$$\mathrm{CH}^p(A;\mathbb{Q}) = \bigoplus_{i\in\mathbb{Z}} \mathrm{CH}^p_{(i)}(A;\mathbb{Q}). \tag{5.5}$$

Proposition 5.10. *Let T be an adequate scheme and $T \to S$ a morphism. Then there are isomorphisms:*

1. $\mathrm{CH}^2(\mathfrak{h}^1(\mathcal{E}^\Delta_T/T)(-1)) = \mathcal{E}^\Delta(T) \otimes \mathbb{Q}$.
2. $\mathrm{CH}^2(\mathfrak{h}^3(\mathcal{J}_T/T)^{\mu_6}) = \mathrm{CH}^2_{(1)}(\mathcal{J}_T;\mathbb{Q})^{\mu_6}$.

Moreover if $T' \to S$ is another such morphism and if $f\colon T' \to T$ is an S-morphism that admits pullbacks (see §4.3) then the above isomorphisms are compatible with pullback along f.

Proof. (1) We have $\mathrm{CH}^2(\mathfrak{h}^1(\mathcal{E}^\Delta_T/T)(-1)) = \mathrm{CH}^1(\mathfrak{h}^1(\mathcal{E}^\Delta_T/T)$. We will prove the stronger statement that if $X \to T$ is an abelian scheme of relative dimension 1 then $\mathrm{CH}^1(\mathfrak{h}^1(X/T)) = X(T) \otimes \mathbb{Q}$. This is the relative version of a classical curve computation [34, Theorem 2.7.2]; we briefly sketch the details. Write $\Gamma_e \subset X \times_T X$ for the graph of the zero section $e\colon T \to X$. Then $\mathfrak{h}^1(X/T) = (X/T, [\Delta_{X/T}] - [\Gamma_e] - [{}^t\Gamma_e], 0)$. Using the equivalence between Cartier divisors and line bundles and unwinding the definition of $\mathrm{CH}^1(\mathfrak{h}^1(X/T))$, we see that it is exactly the $\mathbb{Q}$-tensor product of the set of isomorphism classes of line bundles on X, fiberwise of degree zero and rigidified along the zero section. In other words, $\mathrm{CH}^1(\mathfrak{h}^1(X/T)) = \mathbf{Pic}^0_{X/T}(T) \otimes \mathbb{Q}$, where $\mathbf{Pic}^0_{X/T}$ denotes the identity component of the Picard functor [10, §8.1, Proposition 4]. Since X is a relative elliptic curve, the map $P \mapsto [P] - [e]$ induces an isomorphism $X \simeq \mathbf{Pic}^0_{X/T}$.

(2) By definition of the Chow–Kunneth decomposition of Theorem 4.4 and the Beauville decomposition (5.5), $\mathrm{CH}^p(\mathfrak{h}^i(A/S)) = \mathrm{CH}^p_{(2p-i)}(A)$ for every abelian scheme A/S and $i, p \in \mathbb{Z}$. Therefore $\mathrm{CH}^2(\mathfrak{h}^3(\mathcal{J}_T/T)) = \mathrm{CH}^2_{(1)}(\mathcal{J}_T;\mathbb{Q})$. Since $\mathrm{CH}^p(M^G) = \mathrm{CH}^p(M)^G$ for every finite group G acting on a motive M, the result follows from taking μ_6-fixed points.

The statement about compatibility with pullbacks follows from the construction of the isomorphisms. □

Corollary 5.11. *Let T be an adequate scheme and $T \to S$ a morphism that admits pullbacks (see §4.3). Then there exists an isomorphism of $\mathbb{Q}$-vector spaces*

$$\Phi_T\colon \mathrm{CH}^2_{(1)}(\mathcal{J}_T;\mathbb{Q})^{\mu_6} \xrightarrow{\sim} \mathcal{E}^\Delta(T) \otimes \mathbb{Q}$$

with the following property: If $T' \to S$ is another such morphism and $f\colon T' \to T$ a morphism of S-schemes that admits pullbacks then the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{CH}^2_{(1)}(\mathcal{J}_T;\mathbb{Q})^{\mu_6} \arrow[r, "\Phi_T"] \arrow[d, "f^*"] & \mathcal{E}^\Delta(T)\otimes\mathbb{Q} \arrow[d, "f^*"] \\
\mathrm{CH}^2_{(1)}(\mathcal{J}_{T'};\mathbb{Q})^{\mu_6} \arrow[r, "\Phi_{T'}"] & \mathcal{E}^\Delta(T')\otimes\mathbb{Q}
\end{tikzcd}$$

Proof. Choose an isomorphism $\phi\colon \mathfrak{h}^3(\mathcal{J}/S)^{\mu_6} \xrightarrow{\sim} \mathfrak{h}^1(\mathcal{E}^\Delta/S)$ using Theorem 5.2. This defines an isomorphism ϕ_T in $\mathsf{Mot}(T)$ after base change, see §4.3. Taking CH^2 of this isomorphism and using Proposition 5.10, we get the desired isomorphism Φ_T. ∎

5.3. **The image of the Ceresa cycle.** The defining equation (3.1) of $\mathcal{C}$ shows there exists a unique section $\infty\colon S\to\mathcal{C}$ at infinity. Use ∞ to view $\mathcal{C}$ as a closed subscheme of $\mathcal{J}$ using the Abel–Jacobi map based at ∞. We define the universal Ceresa cycle to be the cycle

$$\kappa(\mathcal{C}) = [\mathcal{C}] - (-1)^*[\mathcal{C}] \in \mathrm{CH}^2(\mathcal{J}).$$

For every field-valued point $s\colon \mathrm{Spec}(k)\to S$, the pullback $s^*(\kappa(\mathcal{C}))\in\mathrm{CH}^2(\mathcal{J}_s)$ equals the (canonical) Ceresa cycle $\kappa(\mathcal{C}_s)$ of the curve $\mathcal{C}_s$ over k with respect to the basepoint at infinity defined in §2.1. The next lemma shows that CH^2 of $\mathfrak{h}^3(\mathcal{J}/S)^{\mu_6}$ naturally contains the Ceresa cycle.

Lemma 5.12. *For every field k and $s\in S(k)$, we have $\kappa(\mathcal{C}_s)\in\mathrm{CH}^2_{(1)}(\mathcal{J}_s;\mathbb{Q})^{\mu_6} = \mathrm{CH}^2(\mathfrak{h}^3(\mathcal{J}_s)^{\mu_6})$.*

Proof. Write $C=\mathcal{C}_s$ and $J=\mathcal{J}_s$. Since J is defined over a field, Beauville has shown [3, Theoreme and Proposition 3(a)] that $\mathrm{CH}^2_{(1)}(J;\mathbb{Q})$ is zero unless $i\in\{0,1,2\}$. So using (5.5) we may write $[C]=[C]_0+[C]_1+[C]_2$ where $(n)^*$ acts on $[C]_i$ as n^{4-i}. Therefore $\kappa(C)=[C]-(-1)^*[C] = ([C]_0+[C]_1+[C]_2)-([C]_0-[C]_1+[C]_2) = 2[C]_1\in\mathrm{CH}^2_{(1)}(J;\mathbb{Q})$. Since $\kappa(C)$ is also μ_6-invariant, we conclude the lemma. ∎

Let $\kappa(\mathcal{C})_{(1)}$ be the component in $\mathrm{CH}^2_{(1)}(\mathcal{J})$ of $\kappa(\mathcal{C})$ in the Beauville decomposition (5.5) of $\mathrm{CH}^2(\mathcal{J})$. (It might very well be that $\kappa(\mathcal{C})=\kappa(\mathcal{C})_{(1)}$, but this is not needed in our analysis.) Since $\kappa(\mathcal{C})$ is μ_6-invariant, $\kappa(\mathcal{C})_{(1)}\in\mathrm{CH}^2_{(1)}(\mathcal{J})^{\mu_6} = \mathrm{CH}^2(\mathfrak{h}^3(\mathcal{J}/S)^{\mu_6})$. Choose for the remainder of the paper a collection of isomorphisms Φ_T satisfying the conclusions of Corollary 5.11. Define

$$\sigma_0 := \Phi_S(\kappa(\mathcal{C})_{(1)})\in\mathcal{E}^\Delta(S)\otimes\mathbb{Q}.$$

Let $\mathrm{den}(\sigma_0)$ be the smallest positive integer such that $\mathrm{den}(\sigma_0)\cdot\sigma_0\in\mathcal{E}^\Delta(S)$. Let $\sigma=\mathrm{den}(\sigma_0)\cdot\sigma_0$, a section of the abelian scheme $\mathcal{E}^\Delta\to S$.

Corollary 5.13. *For every field k and $s\in S(k)$, $\sigma(s)\in\mathcal{E}^\Delta_s(k)$ has infinite order if and only if $\kappa(\mathcal{C}_s)\in\mathrm{CH}^2(\mathcal{J}_s)$ is of infinite order.*

Proof. Follows from Corollary 5.11 and Lemma 5.12. ∎

5.4. **Identifying σ.** Our next goal is to explicitly identify the section σ of $\mathcal{E}^\Delta\to S$, at least up to integer multiples. The equation $y^2=x^3+4b(a^2-4b)^2$ allows us to define $\mathbb{G}_m$-actions on $\mathcal{E}^\Delta$ and $S\subset\mathbb{A}^2$ by the formulas $\lambda\cdot(x,y)=(\lambda^2x,\lambda^3y)$ and $\lambda\cdot(a,b)=(\lambda a,\lambda^2 b)$. With these actions, the morphism $\mathcal{E}^\Delta\to S$ is $\mathbb{G}_m$-equivariant. Write $\mathrm{MW}^{\mathbb{G}_m}(\mathcal{E}^\Delta/S)$ for the set of $\mathbb{G}_m$-equivariant sections of $\mathcal{E}^\Delta\to S$. It is an abelian group, the $\mathbb{G}_m$-equivariant Mordell–Weil group of $\mathcal{E}^\Delta/S$.

Lemma 5.14. $\mathrm{MW}^{\mathbb{G}_m}(\mathcal{E}^\Delta/S)$ *is free of rank 1, generated by the section $\mathcal{Q}:=((a^2-4b),a(a^2-4b))$.*

Proof. Let T be the closed subscheme of S given by setting $a=1$. Restricting sections to T induces an injection $M\hookrightarrow\mathcal{E}^\Delta(T)$, so it suffices to prove that $\mathcal{Q}|_T$ generates the Mordell–Weil group of the elliptic surface $\mathcal{E}^\Delta|_T\to T$.

Let $\pi\colon \mathcal{X} \to \mathbb{P}^1_{\mathbb{Q}}$ be the minimal regular model of $\mathcal{E}^{\Delta}_T \to T = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1/4, \infty\}$. Using Tate's algorithm, we see that the three singular fibers above $b = 0, 1/4, \infty$ have Kodaira type II, IV and I_0^* respectively. By [40, Lemma 7.8], there are no nonzero torsion sections of π. By the Shioda–Tate formula [40, Theorem 6.3, Proposition 6.6 and §8.8], the Mordell–Weil group of $\mathcal{X}_{\bar{\mathbb{Q}}} \to \mathbb{P}^1_{\bar{\mathbb{Q}}}$ is free of rank 2. The μ_3-action on $\mathcal{E}^{\Delta}$ shows that it is also a $\mathbb{Z}[\omega]$-module, necessarily of rank 1. Taking Galois invariants, it follows that the Mordell–Weil group of $\mathcal{X} \to \mathbb{P}^1_{\mathbb{Q}}$ is free of rank 1 over $\mathbb{Z}$. To show that it is generated by (the closure of) $\mathcal{Q}|_T$, it suffices to find a single $b \in T(\mathbb{Q})$ for which $\mathcal{Q}_b$ is primitive in $\mathcal{E}^{\Delta}_b(\mathbb{Q})$, i.e. not divisible by n for every $n \geq 2$. One may check that this holds for $b = 1$. □

Theorem 5.15. *There exists an integer $N \geq 1$ such that $\sigma = N \cdot \mathcal{Q}$.*

Proof. The section $\sigma \in \mathcal{E}^{\Delta}(S)$ is $\mathbb{G}_m$-equivariant. Moreover by our concrete computation with $y^3 = x^4 + x^2 + 1$ in Lemma 2.8 and Corollary 5.13, it has a nonzero specialization, so it is not identically zero. We conclude using Lemma 5.14. □

5.5. **Proof of the main theorem.** Putting everything together, we can prove the following theorem which implies Theorem 1.1 of the introduction.

Theorem 5.16. *Let k be a field of characteristic $\neq 2, 3$ and let C/k be a smooth projective curve with equation $y^3 = x^4 + ax^2 + b$ and Jacobian J. Then the Ceresa cycle $\kappa(C) \in \mathrm{CH}^2(J)$ with respect to the point at infinity is torsion if and only if $\mathcal{Q}_{(a,b)} = ((a^2 - 4b), a(a^2 - 4b))$ is a torsion point of the elliptic curve $E^{\Delta}\colon y^2 = x^3 + 4b(a^2 - 4b)^2$.*

Proof of Theorem 5.16. The curve C is isomorphic to the pullback $\mathcal{C}_s$ of $\mathcal{C} \to S$ along the k-point $s = (a, b)\colon \mathrm{Spec}(k) \to S$. By Corollary 5.13, $\kappa(C)$ is torsion if and only if $\sigma(s) \in \mathcal{E}^{\Delta}_s(k)$ is torsion. By Theorem 5.15, $\sigma(s)$ is a multiple of $\mathcal{Q}_s = \mathcal{Q}_{(a,b)}$, so $\sigma(s)$ is torsion if and only if $\mathcal{Q}_s = (a^2 - 4b, a(a^2 - 4b))$ is torsion. □

Proof of Theorem 1.1. We take $(a, b) = (2t, 1)$. A simple computation (using that $k = \bar{k}$) shows that there is an isomorphism from $E^{\Delta}_{a,b}$ to $\widehat{E}\colon y^2 = x^3 + 1$, sending $\mathcal{Q}_{(a,b)} = (4t^2 - 4, 2t(4t^2 - 4))$ to $Q_t = (\sqrt[3]{t^2 - 1}, t)$, for some choice of cube root. Thus $\kappa(C_t)$ is torsion if and only if Q_t is torsion. □

5.6. **Corollaries of Theorem 1.1.** First we prove the classification of torsion Ceresa cycles over $\mathbb{Q}$ claimed in the introduction. We use the following classification of rational torsion points on the elliptic curves $\widehat{E}_d\colon y^2 = x^3 + d$ of j-invariant 0.

Lemma 5.17. *Let $d \in \mathbb{Q}^{\times}$. Then*

(1) $\widehat{E}_d(\mathbb{Q})_{\mathrm{tors}}$ is a subgroup of $\mathbb{Z}/6\mathbb{Z}$;
(2) $\widehat{E}_d(\mathbb{Q})$ contains a point of order 2 if and only if d is a cube;
(3) $\widehat{E}_d(\mathbb{Q})$ contains a point of order 3 if and only if d is a square or $d = -2^4 3^3 m^6$.

Proof. Since $\widehat{E}_d$ is a CM elliptic curve, it has potentially good reduction at every prime p. The prime-to-p part of $\widehat{E}_d(\mathbb{Q})_{\mathrm{tors}}$ therefore injects into $B(\mathbb{F}_p)$ for some j-invariant 0 elliptic curve $B/\mathbb{F}_p$. Thus (1) follows from the list of $\mathbb{F}_p$-points of j-invariant 0 elliptic curves over $\mathbb{F}_3$ and $\mathbb{F}_5$. Parts (2) and (3) can be read off from the 2- and 3- division polynomials of X_d, which are $x^3 + d$ and $x(x^3 + 4d)$, respectively. □

In the following proposition we use the notation of the introduction and denote the bielliptic Picard curve $C_{2t,1}$ by C_t.

Proposition 5.18. *Suppose C is a nice curve over $\mathbb{Q}$ that is $\overline{\mathbb{Q}}$-isomorphic to some bielliptic Picard curve $C_{a,b}$ and has torsion Ceresa cycle. Then C is $\overline{\mathbb{Q}}$-isomorphic to either C_0, C_3, or $C_{\sqrt{-3}}$.*

Proof. By Lemma 3.2, C is $\overline{\mathbb{Q}}$-isomorphic to some $C_{a,b}$ with rational j-invariant, and the proof of that lemma shows that a, b can be chosen to be in $\mathbb{Q}$. Since $C_{a,b}$ has torsion Ceresa cycle, the point $\mathcal{Q}_{a,b} = (a^2 - 4b, a(a^2 - 4b))$ is a $\mathbb{Q}$-rational torsion point on the j-invariant 0 elliptic curve $E^{\Delta}_{a,b} : y^2 = x^3 + 4b(a^2 - 4b)^2$. It follows from Lemma 5.17 that $\mathcal{Q}_{a,b}$ has order $2, 3$, or 6. If $\mathcal{Q}_{a,b}$ has order 2, then the y-coordinate $a(a^2 - 4b)^2$ of $\mathcal{Q}_{a,b}$ is zero, so since $\Delta_{a,b} \neq 0$ we have $a = 0$. Hence C is isomorphic to $C_{0,b}$, which is $\overline{\mathbb{Q}}$-isomorphic to $C_{0,1} = C_0$. Similarly, if $\mathcal{Q}_{a,b}$ has order 3, we compute that $a^2 + 12b = 0$ hence C is $\overline{\mathbb{Q}}$-isomorphic to $C_{\sqrt{-3}}$. If $\mathcal{Q}_{a,b}$ has order 6, then Lemma 5.17 shows that there exists an isomorphism between $E^{\Delta}_{a,b}$ and $y^2 = x^3 + 1$ that sends $\mathcal{Q}_{a,b}$ to $(2, 3)$. A calculation shows that $a^2 - 36b = 0$, in other words C is $\overline{\mathbb{Q}}$-isomorphic to C_3. □

Proof of Corollary 1.2. Follows from Proposition 5.18. □

Proof of Theorem 1.3. Recall from §5.3 that $\Phi_S(\kappa(\mathcal{C})_{(1)}) = \sigma/d$ in $\mathcal{E}^{\Delta}(S) \otimes \mathbb{Q}$, where d is the denominator of σ_0. Moreover $\sigma = N \cdot \mathcal{Q}$ in $\mathcal{E}^{\Delta}(S)$ by Theorem 5.15. So $d \cdot \Phi_S(\kappa(\mathcal{C})_{(1)}) - N \cdot \mathcal{Q}$ is torsion in $\mathcal{E}^{\Delta}(S)$, so

$$M \cdot (d \cdot \Phi_S(\kappa(\mathcal{C})_{(1)}) - N \cdot \mathcal{Q}) = 0 \text{ in } \mathcal{E}^{\Delta}(S) \tag{5.6}$$

for some $M \geq 1$. The orders of the torsion specializations $\mathcal{Q}_s \in \mathcal{E}^{\Delta}_s(\mathbb{C})$, where s varies in $S(\mathbb{C})$, are unbounded. The relation (5.6) implies that the same is true for the orders of the torsion specializations of $\kappa(\mathcal{C}_s)_{(1)}$, which equals $\kappa(\mathcal{C}_s)$ by Lemma 5.12. □

Theorem 1.4 will follow from the following corollary of Theorem 5.15. If A is an abelian variety over $\overline{\mathbb{Q}}$ of dimension g, then the Beilinson-Bloch height pairing [6]

$$\langle\, , \rangle_{\mathrm{BB}} \colon \mathrm{CH}^i(A)_0 \times \mathrm{CH}^{g+1-i}(A)_0 \to \mathbb{R}$$

has been constructed unconditionally by Künnemann [27, Corollary 1.7].

Theorem 5.19. *There is a rational number N such that for all $a, b \in \overline{\mathbb{Q}}$ with $\Delta_{a,b} \neq 0$, we have*

$$\langle \kappa(C_{a,b}), \kappa(C_{a,b}) \rangle_{\mathrm{BB}} = N^2 \langle \mathcal{Q}_{a,b}, \mathcal{Q}_{a,b} \rangle_{\mathrm{NT}},$$

where $\langle\, , \rangle_{\mathrm{NT}}$ is the Néron-Tate height pairing on $E^{\Delta}(\overline{\mathbb{Q}})$.

Proof. This follows from Theorem 5.15 and Lemma 5.12, using the fact that the Beilinson-Bloch height is compatible with correspondences [6, 4.0.3] and agrees with the Néron-Tate height pairing for divisors on curves [6, 4.0.8]. □

Proof of Theorem 1.4. The theorem follows from Theorem 5.19 and the nondegeneracy and Northcott properties of the Néron-Tate height of an elliptic curve. Note that the field of definition of Q_t is $\mathbb{Q}(\sqrt[3]{t^2 - 1}, t)$, whereas the field of definition of C_t is $\mathbb{Q}(1 - t^2)$ by Lemma 3.2. □

6. Algebraic triviality and the Griffiths group

Let C be a nice curve over a field k and J be its Jacobian. The Griffiths group of 1-cycles on J is the group $\mathrm{Gr}_1(J) = \mathrm{Gr}^{g-1}(J) = \mathrm{CH}_1(J)_0/A_1(J)$, where $A_1(J)$ is the subgroup of algebraically trivial 1-cycles. The very general Ceresa cycle (of a genus $g \geq 3$ curve over $\mathbb{C}$) is known to be of infinite order in $\mathrm{Gr}_1(J)$ [35]. In particular, the Ceresa cycle of a very general curve is not algebraically trivial, nor is any multiple of it. In this section we show that this fails for bielliptic Picard curves. We also discuss the Beilinson-Bloch conjectures in this context, and show that they imply that $\mathrm{Gr}^2(J)$ is finite for certain bielliptic Picard curves.

6.1. **Proof of Theorem 1.5.** We keep the notation of §5.1. Theorem 1.5 follows from the following proposition. It essentially follows from the fact that homological and algebraic equivalence on a curve coincide.

Proposition 6.1. *There exists an integer N, a smooth projective relative curve $\mathcal{X} \to S$ with two sections σ_1, σ_2 and a cycle $\mathcal{Z}$ on $\mathrm{CH}^1(\mathcal{J} \times_S X)$, with the property that $\mathcal{X}_{\sigma_1} = N \cdot \kappa(\mathcal{C})_{(1)}$ and $\mathcal{X}_{\sigma_2} = 0$.*

This proposition implies Theorem 1.5, since for every k-point $s\colon \mathrm{Spec}(k) \to S$, corresponding to the bielliptic Picard curve $\mathcal{C}_s = C$ with $X = \mathcal{X}_s$, $p_i = \sigma_i(s)$ and $Z = \mathcal{Z}_s$, the cycle $Z \in \mathrm{CH}^1(J \times_k X)$ has the property that $Z_{p_1} = N \cdot \kappa(C)_{(1)}$ and $Z_{p_2} = 0$. Since $\kappa(C)_{(1)} = \kappa(C)$ by Lemma 5.12, $N \cdot \kappa(C)$ is algebraically equivalent to zero.

Proof. We will use the isomorphisms Φ from Corollary 5.11. Recall from §5.3 that there exists an N_1 such that $\Phi_S(N_1 \cdot \kappa(\mathcal{C})_{(1)}) = \sigma \in \mathcal{E}^\Delta(S)$. We can deform the section σ of $\mathcal{E}^\Delta$ to the identity section $\infty\colon S \to \mathcal{E}^\Delta$. More precisely, there exists a smooth projective curve $\mathcal{X} \to S$, a section $\mathcal{V} \in \mathcal{E}^\Delta(\mathcal{X})$ and sections σ_1, σ_2 of $\mathcal{X} \to S$, such that $\mathcal{V}_{\sigma_1} = \sigma$ and $\mathcal{V}_{\sigma_2} = \infty$. In fact, $\mathcal{X} = \mathcal{E}^\Delta$ with $\sigma_1 = \sigma$, $\sigma_2 = \infty$ and with $\mathcal{V} = \Delta_{\mathcal{X}/S} \subset \mathcal{X} \times_S \mathcal{X} = \mathcal{E}^\Delta \times_S \mathcal{X}$ will do. There exists an N_2 such that $\Phi_{\mathcal{X}}^{-1}(N_2 \cdot \mathcal{V}) \in \mathrm{CH}^2_{(1)}(\mathcal{J} \times_S \mathcal{X}; \mathbb{Q})$ is the image of an integral cycle $\mathcal{W} \in \mathrm{CH}^2(\mathcal{J} \times_S \mathcal{X})$. The commutativity of the diagram of Corollary 5.11 applied to $\mathcal{X} \to S$ (which is flat, so admits pullbacks) shows that $\mathcal{W}_{\sigma_1} = N_1 N_2 \cdot \kappa(\mathcal{C})$ and $\mathcal{W}_{\sigma_2} = 0$ in $\mathrm{CH}^2(\mathcal{J}; \mathbb{Q})$. In other words, the elements $\mathcal{W}_{\sigma_1} - N_1 N_2 \kappa(\mathcal{C})$ and $\mathcal{W}_{\sigma_2}$ are torsion in $\mathrm{CH}^2(\mathcal{J})$. There exists an integer N_3 such that these elements are killed by N_3, so the elements $N = N_1 N_2 N_3$ and $\mathcal{Z} = N_3 \cdot \mathcal{W}$ satisfy the conclusion of the proposition. $\square$

6.2. **Beilinson-Bloch conjectures.** We briefly describe some consequences of the Beilinson-Bloch conjecture [6, 8] in our setting. Let X be a nice variety over a number field k. Fix a prime ℓ and denote the ℓ-adic cohomology $\mathrm{H}^i(X_{\bar{k}}, \mathbb{Q}_\ell)$ simply by $\mathrm{H}^i(X)$. The conjecture predicts that for each $i \in \mathbb{Z}$, the group of homologically trivial cycles $\mathrm{CH}^i(X)_{\mathrm{hom}\sim 0}$ is finitely generated, the L-function $L(\mathfrak{h}^{2i-1}(X), s)$ has analytic continuation and functional equation, and

$$\mathrm{rk}\,(\mathrm{CH}^i(X)_{\mathrm{hom}\sim 0}) = \operatorname*{ord}_{s=i} L(\mathfrak{h}^{2i-1}(X), s).$$

Similarly, if $\epsilon \in \mathrm{CH}^{\dim X}(X \times X)$ is an idempotent cutting out a direct summand of the Chow motive $\mathfrak{h}(X)$, then

$$\mathrm{rk}\,(\epsilon \cdot \mathrm{CH}^i(X)_{\mathrm{hom}\sim 0}) = \operatorname*{ord}_{s=i} L(\epsilon \cdot \mathfrak{h}^{2i-1}(X), s).$$

Let $C = C_{a,b}$ be a bielliptic Picard curve over k with Jacobian J. By Theorem 5.1 and Proposition 5.10, the Beilinson-Bloch conjecture for the motive $\mathfrak{h}^3(J)^{\mu_6}$ reads:

Conjecture 6.2. $\mathrm{rk}\,\mathrm{CH}^2_{(1)}(J)^{\mu_6} = \mathrm{ord}_{s=1} L(\mathfrak{h}^1(E^\Delta), s)$.

The (weak) Birch and Swinnerton-Dyer conjecture states that $\mathrm{ord}_{s=1} L(\mathfrak{h}^1(E^\Delta), s) = \mathrm{rk}\, E^\Delta(k)$. Therefore the Beilinson-Bloch conjecture for $\mathfrak{h}^3(J)^{\mu_6}$ is equivalent to the weak Birch and Swinnerton-Dyer conjecture for E^Δ (by Corollary 5.11).

6.3. **The Griffiths group.** As detailed in [9, 1.3], the Beilinson-Bloch conjecture should also be compatible with the coniveau filtrations on $\mathrm{CH}^i(X)$ and $\mathrm{H}^{2i-1}(X)(i)$. In our setting, this allows us to predict the rank of the Griffiths group $\mathrm{Gr}^2(J)$, at least in some cases. To formulate it, we use the decomposition $\mathfrak{h}^2(P) = \mathfrak{h}^2_{\mathrm{alg}}(P) + \mathfrak{h}^2_{\mathrm{tr}}(P)$ from [23] which has the property that the ℓ-adic realization of $\mathfrak{h}^2_{\mathrm{tr}}(P)$ is

$$V := (\mathrm{NS}(P)(-1) \otimes \mathbb{Q}_\ell)^\perp \subset \mathrm{H}^2(P),$$

the transcendental part of $\mathrm{H}^2(P)$.

Remark 6.3. If P does not have CM then V is three dimensional. If P is moreover of GL_2-type, then V is (up to twist) the symmetric square of the associated 2-dimensional Gal_k-representation. If P has CM by an imaginary quadratic field K, then V is 2-dimensional and the L-function $L(V,s)$ is a product of Hecke L-functions attached to the field K.

Using the isogeny decomposition $J \sim P \times E$ and the Kunneth formula, one checks that the first graded piece $\mathrm{gr}^0 \, \mathrm{H}^3(J)(2)$ in the coniveau filtration is a nonzero quotient of $V \otimes \mathrm{H}^1(E)(2)$. Define

$$\mathfrak{h}^3_{\mathrm{Gr}}(J) := \mathfrak{h}^2_{\mathrm{tr}}(P) \otimes \mathfrak{h}^1(E).$$

The following prediction then follows from [9, 1.3].

Conjecture 6.4. *Let C be a bielliptic Picard curve over a number field k. Then*

$$\mathrm{rk}\, \mathrm{Gr}^2(J) \leq \operatorname*{ord}_{s=2} L(\mathfrak{h}^3_{\mathrm{Gr}}(J), s),$$

with equality if $V \otimes \mathrm{H}^1(E)$ is an irreducible Gal_k-representation.

Assuming this conjecture, we exhibit below some new examples of nonhyperelliptic Jacobians over a number fields with finite Griffiths group. First we describe some known examples.

Example 6.5. Following up on [7], Gross found two nonhyperelliptic Jacobians with finite Griffiths group (assuming the Beilinson-Bloch conjecture): the genus 7 Fricke-Macbeath curve and its genus 3 quotient [20], both defined over $\mathbb{Q}$.

The other previously known example is a certain "special" bielliptic Picard curve. To put it in context, note that if P is defined over $\mathbb{Q}$ and has CM (over $\overline{\mathbb{Q}}$) by an imaginary quadratic field $\mathbb{Q}(\sqrt{d}) \neq \mathbb{Q}(\omega)$, then $L(\mathfrak{h}^3_{\mathrm{Gr}}(J), s)$ is a Hecke L-function for the biquadratic CM field $\mathbb{Q}(\omega, \sqrt{d})$.

Example 6.6. The curve $C_{0,1} \colon y^3 = x^4+1$ has CM by $\mathbb{Q}(i)$ [28, Example 6.15]. In [30, Theorem 1.3], Lilienfeldt and the second author show that the corresponding Hecke L-function has nonvanishing central value, and hence $\mathrm{Gr}^2(J_{0,1})$ is conjecturally finite.

Using Theorem 3.4 and mod p point counts of E and J, one can compute the good Euler factors for the L-function $L(\mathfrak{h}^3_{\mathrm{Gr}}(J), s)$, even for non-CM bielliptic Picard curves $C_{a,b}$. These L-functions are not known to have analytic continuation in general but we can still compute their analytic rank in Magma assuming that they do.

Example 6.7. Using [2, 42], we have verified (conditional on the analytic continuation of the L-function) that for both $C_{6,-3}$ and $C_{6,1}$, the central value $L(\mathfrak{h}^3_{\mathrm{Gr}}(J), 2)$ is nonzero. Hence $\mathrm{Gr}^2(J_{6,-3})$ and $\mathrm{Gr}^2(J_{6,1})$ are conjecturally finite groups.

Theorem 1.5 suggests that the behavior in Examples 6.6 and 6.7 is not so special, and we expect that many bielliptic Picard Jacobians over $\mathbb{Q}$ have finite Griffiths group.

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