Exercise set 1

November 20, 2016

To be handed in by December 7th, directly to me or via email to perin@math.huji.ac.il.

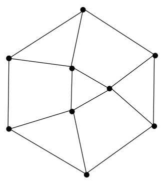
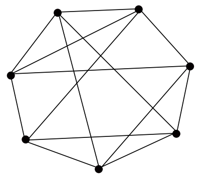
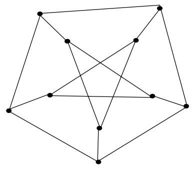
Solutions in english or typeset much appreciated, but of course hebrew and handwritten are fine (please write as clearly as possible).

You are required to hand in solutions for **6 out of the following 8** exercises.

Exercise 1: Draw the Cayley graph $X(G, S)$ for:

1. $G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and $S = \{(\bar{1}, \bar{0}), (\bar{0}, \bar{1})\}$
2. $G = S_3$ (permutations on 3 elements) and $S = \{(123), (23)\}$.

Exercise 2: Show the following graphs cannot be Cayley graphs:

Exercise 3: Show that any finitely generated group can be realized as the **full** symmetry group of an undirected unlabelled graph.

Exercise 4: Let $F(a, b)$ be the free group on $\{a, b\}$.

1. Show that the set $\{a, b^2\}$ is free, yet cannot be extended to a basis.
2. Show that $U = \{ab, ba^{-1}, b^3\}$ forms a generating set for $F(a, b)$, but that no proper subset of U forms a basis.
3. Show that the subgroup H generated by $S = \{h_n = b^n a b^{-n}, n \in \mathbb{N}\}$ is free on S.

Exercise 5: 1. Show that the group $\langle a, b \mid a^{-1}bab^{-2}, b^{-1}aba^{-2}\rangle$ is trivial.

2. Is the group $\langle xy \mid xyx = yxy\rangle$ trivial?

Exercise 6: Consider the subset of $SL_3(\mathbb{Z})$ consisting of matrices of the form $\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$ for $a, b, c \in \mathbb{Z}$

1. Check this is indeed a subgroup of $SL_3(\mathbb{Z})$, and that it is generated by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Show that it admits the presentation $\langle A, B, C \mid AC = CA; BC = CB; ABA^{-1}B^{-1} = C\rangle$.
3. Draw a neighborhood of the identity of the Cayley graph of this group with respect to $\{A, B, C\}$

Exercise 7: Show that the infinite dihedral group D_∞ is isomorphic to the free product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

Recall: D_∞ is the group of isometries of the "infinite-sided" polygon, which can be thought of as the real line with vertices at each integer point - thus elements of D_∞ are isometries of the real line which send integer points to integer points.

Hint: play ping-pong.

Exercise 8: Let $G = \langle A \mid R \rangle$, and let w be a word in A. Prove that if there exists a Van Kampen diagram whose boundary is labelled by w, then w represents the identity in G.