Optimal generators for matrix groups (DRAFT)

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Abstract

For any $d \geq 2$ and prime power q, we construct k elements in $G_\ell = PSL_d(q^\ell)$, such that the associated sequence of Cayley graphs exhibits optimal bounded cutoff. Namely, for any $\varepsilon > 0$ there exists C, independent of ℓ, such that the random walk on G_ℓ is ε-close to the uniform distribution, in total-variation distance, at time $\log_k |G_\ell| + C$.

This is achieved by two separate results of a different nature: (a) proving that Ramanujan digraphs with certain properties (almost-normality and logarithmic collision-radius) exhibit optimal bounded cutoff, and (b) constructing elements in G_ℓ whose associated Cayley digraphs are Ramanujan digraphs with these properties. For (a), we develop a new method to bound the norms of walk operators in directed graphs, which should be of independent interest. For (b), we realize G_ℓ as a congruence quotient of a specially chosen arithmetic semigroup in a division algebra over $\mathbb{F}_q(t)$. The crucial point for us is that in the action of the algebra on its associated Bruhat-Tits building, the generators of the semigroup act by geodesic flow on an appropriate set of edges.

1 Introduction

The study of group expansion concerns the following problem: given a generating set S for a finite group G, how quickly do words in S cover all, or most of G? Sets of generators which "expand" rapidly are especially interesting, as well as useful for many applications in mathematics and computer science [Lub12]. There are many variants to the question of expansion – for example, asking what is the expansion rate of a group with respect to its worst, random, average, and best-case generators. Expansion in finite simple groups in particular has attracted the attention of many mathematicians, and we refer the reader to the recent survey [BL22] and the monograph [Tao15] for the history and open problems in the field.

One measure of group expansion is given by mixing time, namely, the speed at which a random walk on the Cayley graph of the group convergence to the uniform distribution, in some chosen metric. For L^2 metric, the mixing rate is dictated by the spectrum of the adjacency operator of the graph, and a k-regular graph whose spectrum is bounded by $2\sqrt{k-1}$ (save for the trivial eigenvalue k) was named *Ramanujan graph* in [LPS88]. By the Alon-Boppana theorem, the bound $2\sqrt{k-1}$ is asymptotically optimal, and in [LPS88] Lubotzky, Phillips and Sarnak (LPS) show that for any prime $p \equiv 1 \pmod 4$ and infinitely many q, the groups $PSL_2(q)$ have an explicit set of $k = p+1$ generators which yield a Ramanujan graph. This gives optimal expansion in L^2 metric, but for the more informative L^1, also known as *total-variation distance*, the story is more complicated. Only in [LP16] it was shown that Ramanujan graphs indeed have optimal L^1 expansion, and furthermore, that they exhibit Diaconis' cutoff phenomenon: the L^1-distance to uniform distribution drops abruptly from almost maximal to almost zero, over a short period of time. The recent work [NS23] shows that for some Ramanujan graphs, such as the LPS graphs, this period of time is even independent of the size of the graph (which is called *bounded cutoff*).

Subsequent works [Chi92, Mor94, DSV03] have generalized the LPS construction to give k Ramanujan generators for other values of k, but always for the groups $PSL_2(q)$ (and $PGL_2(q)$). A high-dimensional analogue of the LPS construction was established in [LSV05b, Sar07], yielding generators for $PSL_d(q)$ such that the resulting Cayley graph is the one-dimensional skeleton of a *Ramanujan complex*. However, the expansion in these complexes deteriorates as d grows – they still exhibit cutoff, but not at the optimal time [CP22].

There is no reason to suspect that $PSL_d(q)$ has Ramanujan generators, let alone that there exists an explicit description of such generators, valid for infinitely many q. Nevertheless, in this paper we give an optimal expansion result for families of the form $PSL_d(q^\ell)$, with d and q fixed and ℓ varying. For this, we make use of an undirected analogue of Ramanujan graphs:

Definition 1.1 ([PS18, LLP20]). A k-regular digraph (directed graph) $\mathcal{D}$ is a *Ramanujan digraph* if every eigenvalue $\lambda \in \mathbb{C}$ of its adjacency operator satisfies either $|\lambda| = k$, or $|\lambda| \leq \sqrt{k}$.

Note that the spectral bound $\sqrt{k}$ is better than that of Ramanujan graphs: this reflects the fact that a free semigroup expands faster than a free group, as there are no cancellations, or "backtracks". In accordance, in some sense it is harder to find Ramanujan digraph generators for a group. In [PS18, §5] we study this problem, and find explicit generators that give $PSL_2(q)$ a Ramanujan-digraph structure. Unlike the LPS construction, we manage to do this only for a *finite* number of possible degrees k (specifically, $k \in \{2, 3, 4, 5, 7, 11, 23, 59\}$ – each case corresponds to a Platonic symmetry group of size $k + 1$, lying in the unit group of some quaternion order). In this paper we present a different construction, using the notion of geodesic flow on buildings introduced in [LLP20]. This is a branching process which can be thought of as a p-adic analogue of the geodesic flow on the unit bundle of a Riemannian manifold. Remarkably, this gives us both infinitely many values of k, and in addition applies to PGL_d for any $d \geq 2$.

Let us fix $d \geq 2$ and a prime power q, and let $\ell \in \mathbb{N}$ vary, excluding the special case $q^\ell = 2$. For each such ℓ we construct in Section 2 an explicit set S_ℓ of size q^{d-1} in $PGL_d(q^\ell)$. Denoting $G_\ell = \langle S_\ell \rangle$, each G_ℓ is an intermediate subgroup

$$PSL_d(q^\ell) \trianglelefteq G_\ell \trianglelefteq PGL_d(q^\ell),$$

and we denote by $\mathcal{D}_\ell = D_{d,q,\ell}$ the Cayley digraph $Cay\,(G_\ell, S_\ell)$, which has vertices G_ℓ and edges $\{g \to sg \,|\, g \in G_\ell, s \in S_\ell\}$.

Theorem 1.2. *The digraph $\mathcal{D}_{d,q,\ell}$ is a connected $k = q^{d-1}$-regular Ramanujan digraph, whose periodicity $\mathfrak{p}_\ell$ divides $\left[G_\ell : PSL_d(q^\ell)\right]$. In fact, its spectrum satisfies*

$$\operatorname{Spec}\left(\mathcal{D}_{d,q,\ell}\right) \subseteq \langle e^{2\pi i/\mathfrak{p}_\ell} \rangle\, k \cup \left\{ z \,\middle|\, |z| = \sqrt{k} \quad or \quad |z| = k^{\frac{d-2}{2d-2}} \right\}.$$

It is easily deduced from this that $S_\ell^r = \{s_1 \cdot \ldots \cdot s_d \,|\, s_1, \ldots, s_d \in S_\ell\}$ is a generating set of size $d(d-1)$ for $PSL_d(q^\ell)$, whose Cayley graph $Cay(PSL_d(q^\ell), S_\ell^d)$ is a connected aperiodic Ramanujan digraph. Some examples of $\mathcal{D}_{d,q,\ell}$ and their adjacency spectrum are shown in Figure 1.1.

Allowing for directed graphs has a price: the spectral analysis becomes trickier, as the adjacency operator A is not self-adjoint, and is not even normal in the cases which we are interested in. In these settings, the Ramanujan property alone tells us little: as A is not normal, its operator norm is controlled by its singular values, rather than its eigenvalues, and it turns out that Ramanujan digraphs can have abysmal singular values [Par20, §3.5]. We gain some control of the situation by the notion of almost-normality: A matrix is *r-normal* if it is unitarily equivalent to a block-diagonal matrix with blocks of size at most $r \times r$. A digraph is called r-normal if its adjacency matrix is r-normal, and a family of matrices (or digraphs) is said to be *almost-normal* if its members are r-normal for some fixed $r < \infty$. A main ingredient in [LP16, LLP20] is the spectral analysis on almost-normal digraphs, which leads to cutoff with logarithmic window (see definition below). The digraph $\mathcal{D}_{d,q,\ell}$ which we construct in theorem 1.2 are d-normal, so the results from [LLP20] shows that our generators indeed have optimal cutoff with a logarithmic window. However, in this paper we develop a new approach (Section 3), which ultimately leads to bounded cutoff.

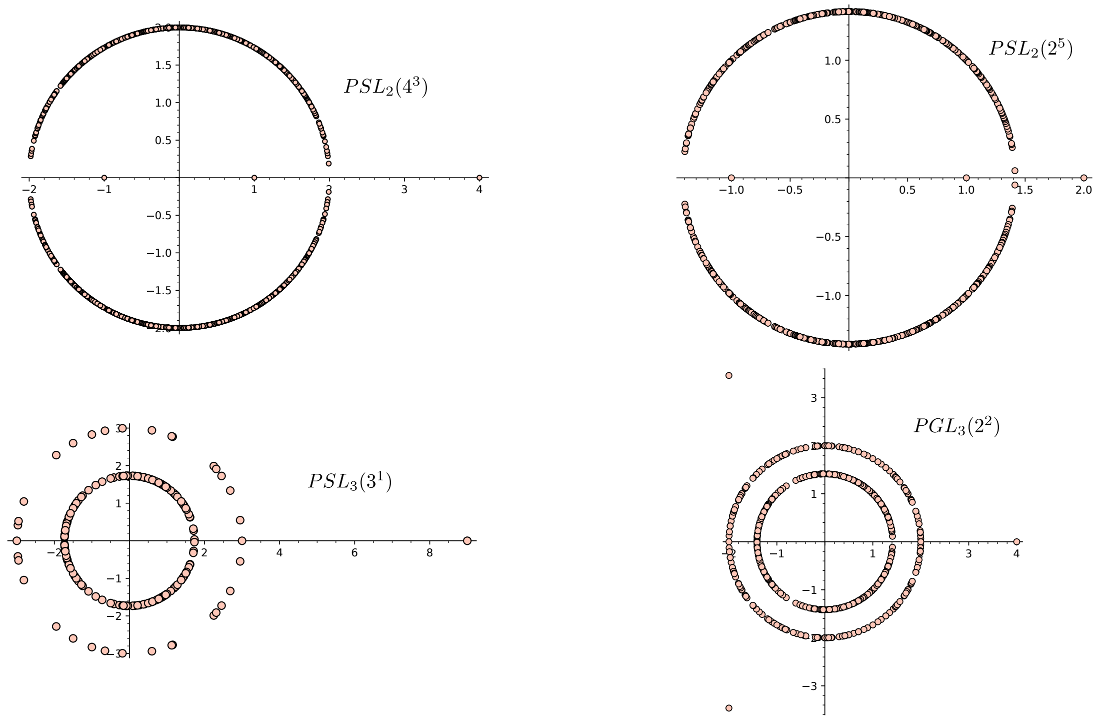

Figure 1.1: The spectrum of some q^{d-1}-regular Cayley digraphs $\mathcal{D}_{d,q,\ell}$, and the underlying group $G_\ell \leq PGL_d(q^\ell)$.

A sequence of digraphs $\{\mathcal{D}_n\}$ is said to exhibit *cutoff* at time $t = t(n)$ with window $w = w(n, \varepsilon)$ if for any $\varepsilon > 0$, the total-variation mixing time $t_\star(\mathcal{D}_n)$ satisfies

$$t(n) - w(n,\varepsilon) < t_{1-\varepsilon}(\mathcal{D}_n) < t_\varepsilon(\mathcal{D}_n) < t(n) + w(n,\varepsilon)$$

for n large enough. If $\mathcal{D}_n$ are k-regular, we say that the cutoff is *optimal* if $t(n) = \log_k |\mathcal{D}_n|$, and that the window is *logarithmic* if $w(n) = O(\log |\mathcal{D}_n|)$, and *bounded* if $w(n) = O(1)$. In [LLP20, §3] it is shown that almost-normal Ramanujan digraphs satisfy optimal cutoff, with logarithmic cutoff window. Combining this with Theorem 1.2 would already give us optimal logarithmic cutoff for $PGL_d(q^\ell)$, but our goal is to obtain bounded cutoff. We introduce two definitions:

Definition 1.3. The *collision radius* of a digraph $\mathcal{D}$ is

$$\operatorname{crad}(\mathcal{D}) = \min \left\{ \ell \,\middle|\, \begin{array}{l} \exists \text{ vertices } v, w \in \mathcal{D} \text{ with two different} \\ \text{directed paths of length } \ell \text{ from } v \text{ to } w \end{array} \right\},$$

and $\mathcal{D}$ is called *collision-free* if $\operatorname{crad}(\mathcal{D}) = \infty$, i.e. for any $v, w \in \mathcal{D}$ there is at most one directed path from v to w.

A branching operator T on a simplicial complex $\mathcal{B}$ is called *geometric* if it commutes with $\operatorname{Aut}(\mathcal{B})$, in which case for any quotient X of $\mathcal{B}$ it defines a branching operator on X, denoted by $T\big|_X$. We observe that if T is collision-free (where we identify T with the corresponding digraph $\mathcal{D}_T = \{v \to w \,|\, v, w \in \mathcal{B}, w \in T(v)\}$), then the collision-radius of the $T\big|_X$ is at least the injectivity radius of X as a quotient of $\mathcal{B}$. We shall make use of the geodesic flow $\mathcal{F}$, which is a collision-free branching operator on certain edges in the Bruhat-Tits building $\mathcal{B}$ (associated with the group $PGL_d(\mathbb{F}_q((y)))$). Our digraphs $\mathcal{D}_\ell$ will be obtained from the action of $\mathcal{F}$ on certain quotients of $\mathcal{B}$. Finally, arithmetic considerations show that these quotients have logarithmic injectivity radius [LM07]. In section 3 we prove the following:

Theorem 1.4. *If $\mathcal{D}$ is an aperiodic, r-normal, k-regular, transitive Ramanujan digraph on n vertices with* $\operatorname{crad}(\mathcal{D}) > \frac{r-1}{m-1} \log_k n$ *(where $m \geq 2$), then*

$$\log_k n - \log_k \tfrac{1}{\varepsilon} < t_{1-\varepsilon}(\mathcal{D}) < t_\varepsilon(\mathcal{D}) < \log_k n + \log_k \frac{r^2 m^{2r}}{4\varepsilon^2}.$$

It seems that transitivity is actually not necessary, but it holds for Cayley digraphs, and makes the proof much simpler. To apply this theorem to our $\mathcal{D}_\ell$ we need to establish r-normality. This comes from the representation theory of $PGL_d(\mathbb{F}_q((y)))$, which shows that for any geometric operator T on $\mathcal{B}$ the digraph $\mathcal{D}_{T\curvearrowright X}$ is r-normal for some r which depends only on $\mathcal{B}$ [LLP20, Thm. 3]. Specifically, for the geodesic flow operator on $\mathcal{B}$ we have $r = d$ (see Section 2). From the arguments in [LM07] we obtain $\operatorname{crad}(\mathcal{D}_{d,q,\ell}) \geq \frac{\log_k n-1}{d+1}$, which implies we can take $m = d^2+1$. Combining everything together, we arrive at bounded cutoff for Cayley digraphs:

Theorem 1.5. *If $\mathcal{D}_\ell = Cay(G_\ell, S_\ell)$ is aperiodic (in particular, when $\mathcal{D}_\ell = PSL_d\left(q^\ell\right)$), then*

$$\log_k n - \log_k \tfrac{1}{\varepsilon} < t_{1-\varepsilon}(\mathcal{D}_\ell) < t_\varepsilon(\mathcal{D}_\ell) < \log_k n + \log_k \frac{d^2\left(d^2+1\right)^{2d}}{4\varepsilon^2}.$$

When $\mathcal{D}_\ell$ is $\mathfrak{p}$-periodic, $\mathcal{D}_\ell = \bigsqcup_{i=1}^{\mathfrak{p}} V_i$, the walk is of course not mixing, but the same result holds if one starts from a randomly chosen block V_i, or if one restrict his attention to convergence to $\mathbf{u}_{V_i}$ at times which are multiples of $\mathfrak{p}$. Returning to the original question of group covering, we have two easy consequences of total-variation cutoff:

Corollary 1.6. *If $\mathcal{D}_\ell = Cay(G_\ell, S_\ell)$ is aperiodic, then almost every element in G_ℓ is obtained by a word of length $\log_k |G_\ell| + C$ in S_ℓ, and every element is obtained by a word of length $2\log_k |G_\ell| + C$.*

Let us explain a little where Theorem 1.4 comes from. It is inspired by the main result in [NS23], which shows bounded cutoff for non-backtracking walk on Ramanujan graphs of logarithmic girth. The main idea there is to "bootstrap" expansion from early time to later one: at time $t = \frac{\operatorname{girth}(X)}{2}$, the non-backtracking walk is optimally expanding, as no edge is encountered twice. This shows that an appropriately chosen operator norm of B^t is small[(†)]. To show optimal cutoff one needs to study $B^{t'}$, where $t' \approx \log_k n \approx mt$, so the challenge is to relate the norm of B^t to that of B^{tm}. Both norms can be computed by applying a combinations of Chebyshev polynomials to the eigenvalues of the graph [NS23, Lem. 4.3], and the relation between the norms of B^t and of B^{tm} is obtained in [NS23] from a careful analysis involving trigonometry and recursion relations of Chebyshev polynomials (see also [ABLS07] for earlier usage of this idea).

In our digraphs $\mathcal{D}$, we have optimal expansion at time t being the injectivity radius of our complexes, since the geodesic flow operator $\mathcal{F}$ is collision-free on the building. Again, this means that $\mathcal{F}^t|_{\mathcal{D}}$ has "small" norm, and we need to bound that of $\mathcal{F}^{tm}|_{\mathcal{D}}$. In Section 4, we relate the eigenvalues of $\mathcal{F}^t$ to those of the "colored adjacency operators" on the complex, by means of multivariate polynomials satisfying a recursive relation (4.1). The problem is that the (multivariate) trigonometry which arises is the analysis of the norms quickly becomes intractable. We therefore develop in Section 3 a completely new approach, which relates the norms of several different powers of a matrix, and which we expect to be of use in other situations not involving graphs or buildings:

Proposition (3.3). *If $A \in M_r(\mathbb{C})$ has eigenvalues bounded by 1, then $\|A^m\| \leq m^r \sum_{t=0}^{r-1} \|A^t\|$ for $m \geq 1$ and any matrix norm $\|\cdot\|$.*

This proposition allows us gives a bound for A^m in terms of several the first $r-1$ powers $A, \ldots, A^{r-1}$. Once we move from graphs to higher dimensions (i.e. $r \geq 3$), we need more than A itself, and our bootstrapping uses several points in time for which we already know that the expansion is optimal. Even though eventually we make no use of our high-dimensional Chebyshev polynomials in the analysis of cutoff, we have left them in this paper as we think they merit a study for their own sake.

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(†) In [NS23] the norm is a weighted average of the squares of the eigenvalues. For transitive graphs the weights are all equal and one obtains the Frobenius norm.

2 Geodesic flow generators

For a nonarchimedean local field F, the Bruhat-Tits building $\mathcal{B}_{d,F}$ associated with the group $G = PGL_d(F)$ is a contractible, $(d-1)$-dimensional, simplicial flag complex. Its vertices may be identified with G/K for the maximal compact subgroup $K = PGL_d(\mathcal{O}_F)$, and if π is a uniformizer in F, two vertices $gK, g'K$ in $\mathcal{B}_{d,F}$ are neighbors if $g' \in KgsK$ for some non-scalar $s = \operatorname{diag}(1,\ldots,1,\pi,\ldots,\pi)$. The directed edges in $\mathcal{B}_{d,F}$ are colored by $\operatorname{col}(gK \to g'K) = \operatorname{ord}_\pi(\det(g^{-1}g'))$, which is a well defined element in $(\mathbb{Z}/d\mathbb{Z})\backslash\{0\}$, and the action of G preserves edge colors. Ramanujan complexes were defined in [Li04, LSV05a] as quotients of $\mathcal{B}_{d,F}$ for which the spectrum of the "colored adjacency operators"

$$(A_i f)(gK) = \sum_{\operatorname{col}(gK\to g'K)=i} f(g'K)$$

is contained in the L^2-spectrum of the corresponding operator on the building, save for the Perron-Frobenius eigenvalues.

In this section we construct a π-arithmetic group Δ in a division algebra D of degree d over $\mathbb{F}_q(y)$, with a special set of generators S. For the y-adic completion $F = \mathbb{F}_q((y))$, the completion $D_y = D \otimes_{\mathbb{F}_q(y)} F$ splits, so that $D_y^\times / Z(D_y^\times) \cong PGL_d(F)$, and the group Δ acts simply-transitively on the set of (directed) edges of color one in $\mathcal{B}_{d,F}$. Furthermore, the set S takes a certain edge e_0 to all the edges which are connected to it by geodesic flow, as defined in [LLP20, §5], which in particular implies by the non-collision property that the semigroup generated by S is free.

Let E/F be a cyclic Galois extension of degree d with $\operatorname{Gal}(E/F) = \langle\sigma\rangle$, and $a \in F^\times$. The *cyclic algebra* (E,σ,a) is defined by

$$(E,\sigma,a) = \frac{E\{x\}}{\langle x^d = a, \varepsilon x = x\sigma(\varepsilon) \mid \varepsilon \in E\rangle}, \tag{2.1}$$

where $E\{x\}$ are polynomials in a variable x which does not commute with the scalars E. This is always an F-CSA (central simple F-algebra) of dimension d^2; conversely, every F-CSA of dimension d^2 which contains a cyclic field extension of degree d of F is of this form (for example, Hamilton's quaternions are $(\mathbb{C}, z \mapsto \overline{z}, -1)$). If F is a global field and $a \in \mathcal{O}_F$, then an integral model for (E,σ,a) is given by the $\mathcal{O}_F$-algebra

$$(\mathcal{O}_E,\sigma,a) = \frac{\mathcal{O}_E\{x\}}{\langle x^d = a, \varepsilon x = x\sigma(\varepsilon) \mid \varepsilon \in \mathcal{O}_E\rangle},$$

namely $(E,\sigma,a) = (\mathcal{O}_E,\sigma,a) \otimes_{\mathcal{O}_F} F$. We denote by $\widetilde{\mathbf{G}}$ the group scheme of $(\mathcal{O}_E,\sigma,a)$-units, namely

$$\widetilde{\mathbf{G}}(R) = ((\mathcal{O}_E,\sigma,a) \otimes_{\mathcal{O}_F} R)^\times$$

for any $\mathcal{O}_F$-algebra R, and define $\mathbf{G} \overset{def}{=} \widetilde{\mathbf{G}}/\mathbf{U}$, where $\mathbf{U}(R) = R^\times$. For the cases that will concern us we have

$$\mathbf{G}(R) = {\left((\mathcal{O}_E,\sigma,a)\otimes_{\mathcal{O}_F} R\right)^\times}/{R^\times},$$

but we warn the reader that for general R the scheme-theoretic quotient can be larger. Next, for $d \geq 2$ and a prime-power q, we take $F = \mathbb{F}_q(y)$ and $E = \mathbb{F}_{q^d}(y)$, so that $\operatorname{Gal}(E/F) = \langle\phi\rangle$ where ϕ fixes y and acts on $\mathbb{F}_{q^d}$ by $\alpha \mapsto \alpha^q$. We observe the cyclic algebra

$$D = (\mathbb{F}_{q^d}(y), \phi, 1+y),$$

and the associated $\mathbb{F}_q[y]$-group scheme $\mathbf{G}(R) = {\left((\mathbb{F}_{q^d}[y],\phi,1+y)\otimes R\right)^\times}/{R^\times}$. Let $\mathcal{V}$ denote the valuations of F; these are ν_π for every irreducible polynomial $\pi \in \mathbb{F}_q[y]$, and the valuation at infinity, $\nu_{1/y}(f/g) = \deg g - \deg f$ (for $f,g \in \mathbb{F}_q[y]$). We denote by $D_\pi = D \otimes_F F_\pi$ the completion of D at ν_π (including $\pi = 1/y$), and similarly $\mathbf{G}_\pi = \mathbf{G}(F_\pi) = {D_\pi^\times}/{F_\pi^\times}$. It is a standard exercise to check that D_π splits for any π, except for $D_{1/y}$ and D_{1+y} which are division algebras. In particular we have $D_y \cong M_d(\mathbb{F}_q((y)))$ and $\mathbf{G}_y \cong PGL_d(\mathbb{F}_q((y)))$, whose Bruhat-Tits building was described above.

The S-arithmetic group $\Gamma = \mathbf{G}\left(\mathbb{F}_q\left[y, \frac{1}{y}, \frac{1}{1+y}\right]\right)$ embeds discretely in $\mathbf{G}_y \times \mathbf{G}_{1/y} \times \mathbf{G}_{1/(1+y)}$, and since $D_{1/y}$ and $D_{1/(1+y)}$ are division algebras $\mathbf{G}_{1/y} \times \mathbf{G}_{1/(1+y)}$ is compact, so that Γ is already discrete in $\mathbf{G}_y$. Our goal is to find subgroups of Γ which act nicely on the building $\mathcal{B}_{d,F}$ of $\mathbf{G}_y \cong PGL_d\left(\mathbb{F}_q\left((y)\right)\right)$. An example for such a subgroup is the lattice Γ_{cs} constructed by Cartwright and Steger in [CS98]:

$$\Gamma_{cs} = \left\langle \left\{ \alpha^{-1}\left(1 - \tfrac{1}{x}\right)\alpha \,\middle|\, \alpha \in \mathbb{F}_{q^d}^{\times}/\mathbb{F}_q^{\times} \right\} \right\rangle,$$

where x is the indeterminate from (2.1), and $\mathbb{F}_{q^d}^{\times}/\mathbb{F}_q^{\times}$ indicates any set of coset representatives – since $\mathbb{F}_q \subseteq Z(D)$, the elements of $\mathbb{F}_q^{\times}$ are trivial in $\mathbf{G}_y$. This lattice acts simply-transitively on the vertices of the building [CS98], and was used in [LSV05b] to give explicit constructions of Ramanujan complexes. If we denote $G = \mathbf{G}_y \cong PGL_d\left(\mathbb{F}_q\left((y)\right)\right)$ and $K = \mathbf{G}(\mathbb{F}_q\left[[y]\right]) \cong PGL_d\left(\mathbb{F}_q\left[[y]\right]\right)$, the vertices of $\mathcal{B}$ correspond to the cosets G/K, and Γ_{cs} acting simply-transitively on the vertices means that it is a transversal for G/K.

For our purposes, we take a bigger lattice:

$$\Delta \overset{def}{=} \left\langle \left\{ \left(1 - \tfrac{1}{x}\right), \alpha \,\middle|\, \alpha \in \mathbb{F}_{q^d}^{\times}/\mathbb{F}_q^{\times} \right\} \right\rangle.$$

The lattice Δ acts simply-transitively on the set of directed edges of color 1 in $\mathcal{B}_{n,F}$. This is equivalent to stating that after identifying G with $PGL_d\left(\mathbb{F}_q\left((y)\right)\right)$, Δ is a transversal for the cosets G/P, where P is the parahoric group

$$P = \left\{ g \in K \,\middle|\, g \equiv \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \pmod{y} \right\},$$

which is the stabilizer of the edge $e_0 = K \to \mathrm{diag}\left(y, 1 \ldots, 1\right) K$ in $\mathcal{B}_{n,F}$. We are interested in a special set of q^{d-1} elements in Δ:

$$S = \left\{ s \in \Delta \,\middle|\, \bullet \overset{e_0}{\to} \bullet \overset{se_0}{\to} \bullet \text{ is a geodesic path of color } 1 \right\}.$$

The elements S generate a free semigroup (but not a free group). Denoting by $\mathcal{B}_1^1$ the edges of color one in $\mathcal{B}_{n,F}$, the digraph $\mathcal{D} = Cay\left(\Delta, S\right)$ is isomorphic to the digraph whose vertices are $\mathcal{B}_1^1$, and whose edges correspond to the geodesic flow. Furthermore, as Δ acts simply transitively on $\mathcal{B}_1^1$, we obtain that in the isomorphism

$$L^2\left(V_{\mathcal{D}}\right) = L^2\left(\mathcal{B}_1^1\right) \cong L^2\left(\Delta\right) \cong L^2\left(G/P\right) \cong L^2\left(G\right)^P,$$

the element $\mathbb{1}_{PSP}$ in the Parahori-Hecke algebra $\mathscr{H}_P^G$ acting on the r.h.s. coincides with the adjacency operator in $\mathcal{D}$ acting on the l.h.s.. For a finite index subgroup $\Lambda \trianglelefteq \Delta$, we observe the quotient digraph $\mathcal{D}_\Lambda = \Lambda \backslash \mathcal{D}$, which coincides with $Cay\left(\Lambda \backslash \Delta, S\right)$. Note that if some $s \in S$ is in Λ, or two $s, s' \in \Lambda$ are equivalent modulo Λ, then $\mathcal{D}_\Lambda$ will have loops or multiple edges, which we allow. We obtain natural identifications:

$$L^2\left(V_{\mathcal{D}_\Lambda}\right) = L^2\left(\Lambda \backslash \mathcal{B}_1^1\right) \cong L^2\left(\Lambda \backslash \Delta\right) \cong L^2\left(\Lambda \backslash G/P\right) \cong L^2\left(\Lambda \backslash G\right)^P,$$

where compactness of P is used to preserve L^2-condition in the last step, and again the action of $\mathbb{1}_{PSP} \in \mathscr{H}_P^G$ on $L^2\left(\Lambda \backslash G\right)^P$ coincides with the adjacency operator $A_{\mathcal{D}_\Lambda}$ acting on $L^2\left(V_{\mathcal{D}_\Lambda}\right)$. As $\Lambda \backslash G$ is compact, we can decompose $L^2\left(\Lambda \backslash G\right) = \widehat{\bigoplus}_i V_i$ as a Hilbert sum of orthogonal irreducible representations of G. This induces a decomposition of $\mathscr{H}_P^G$-modules $L^2\left(V_{\mathcal{D}_\Lambda}\right) \cong \bigoplus_i V_i^P$ ($V_i^P = 0$ for almost all i in the sum as $|V_{\mathcal{D}_\Lambda}| < \infty$), and in particular, we obtain an orthogonal block-diagonal decomposition for $\mathbb{1}_{PSP} = A_{\mathcal{D}_\Lambda}$. In addition, the P-fixed part of a P-spherical representation of G is of dimension at most d [LLP20, Prop. 5.3], so that $\mathcal{D}_\Lambda$ is d-normal. Our goal is to show that for specific Λ it is a Ramanujan digraph, and that choosing Λ correctly gives G_ℓ from the introduction. These follow from:

(1) Except for the special case $q^\ell = 2$, there exists an irreducible $\pi \in \mathbb{F}_q[y]$ of degree ℓ which is coprime to y and to $y+1$. Taking quotient by π give $\mathbb{F}_q\left[y, \frac{1}{y}, \frac{1}{1+y}\right] \to \mathbb{F}_{q^\ell}$, and thus $\Gamma \to \mathbf{G}_{\mathbb{F}_{q^\ell}}$. We denote $\Lambda_\ell = \ker\left(\Delta \to \mathbf{G}_{\mathbb{F}_{q^\ell}}\right)$ and $G_\ell = \operatorname{im}\left(\Delta \to \mathbf{G}_{\mathbb{F}_{q^\ell}}\right)$. By strong approximation, $G_\ell \cong \Lambda_\ell \backslash \Delta$ contains $\mathbf{G}^1_{\mathbb{F}_{q^\ell}}$ (the elements of reduced norm 1), and by Wedderburn's little theorem $D_{\mathbb{F}_{q^\ell}}$ splits, so that $\mathbf{G}^1_{\mathbb{F}_{q^\ell}} \cong PSL_d\left(q^\ell\right)$ and $\mathbf{G}_{\mathbb{F}_{q^\ell}} \cong PGL_d\left(q^\ell\right)$. Finally, the generating set S_ℓ for G_ℓ is the image of S under $\Delta \to \mathbf{G}_{\mathbb{F}_{q^\ell}} \cong PGL_d\left(q^\ell\right)$.

(2) By the Ramanujan conjecture in positive characteristic [Laf02], the Jacquet-Langlands correspondence [BR17], and the classification of cuspidal automorphic representations of $D\left(\mathbb{A}_F\right)$ [MW89], every P-spherical representation which appears in $L^2\left(\Lambda \backslash \Delta\right)$ is either one-dimensional or tempered.

From the fact that geodesic flow is collision-free, it follows that its eigenvalues are of size $|k|$ on one-dimensional representations [LLP20, Prop. 4.3], and of size at most $\sqrt{k}$ on tempered representations [LLP20, Prop. 4.1 and 2.3]. This is enough for our purposes, but we can also study the P-spherical representations directly, and show that their size must equal either $\sqrt{k} = q^{(d-1)/2}$ or $q^{(d-2)/2}$, as stated in Theorem 1.2.

3 Bounded Cutoff

In this section we prove Theorem 1.4, which shows that almost-normal transitive Ramanujan digraphs with logarithmic collision radius have bounded cutoff. The proof follows from bounding the Frobenius norm of powers of almost-normal Ramanujan digraphs:

Theorem 3.1. *Let $\mathcal{D}$ be an r-normal k-regular p-periodic Ramanujan digraph on n vertices, with adjacency matrix A. If $\ell < \frac{\operatorname{crad}(\mathcal{D})}{r-1}$, then $\left\|A^{\ell m}\right\|_F^2 < pk^{2\ell m} + r^2 m^{2r} n k^{\ell m}$ for any m.*

The next two propositions prepare the ground for the proof, though each of them seems interesting in its own rights.

Proposition 3.2. *If A is a $r \times r$ matrix over an integral domain, then for any $j \geq 1$*

$$A^{r-1+j} = \sum_{t=0}^{r-1} (-1)^t s_{(j,1^{\times t})}\left(z_1, \ldots, z_r\right) A^{r-1-t} \tag{3.1}$$

where s_λ is the Schur polynomial of the partition λ, and $z_1, \ldots, z_r$ are the eigenvalues of A with algebraic multiplicities.

Proof. We proceed by induction on j. Since $s_{(1^{\times t})}$ is the t-th elementary symmetric polynomial e_t, for $j = 1$ eq. (3.1) becomes

$$A^r = \sum_{t=0}^{r-1} (-1)^t e_{t+1}\left(z_1, \ldots, z_r\right) A^{r-1-t},$$

which is just the Cayley-Hamilton theorem. The product $s_\lambda e_j$ is given by Pieri's 1893 formula [Pie93]: $s_\lambda e_j = \sum_\mu s_\mu$ where the sum is over all μ whose Young table is obtained from that of λ by adding j boxes, each in a different row. In particular, we find that

$$s_{(j)} e_{t+1} = s_{(j+1,1^{\times t})} + s_{(j,1^{\times(t+1)})}. \tag{3.2}$$

Writing e_t, s_λ as a shorthand for $e_t(z_1,\dots,z_r), s_\lambda(z_1,\dots,z_r)$, we assume (3.1) for j and prove it for $j+1$:

$$\begin{aligned}\sum_{t=0}^{r-1}(-1)^t s_{(j+1,1^{\times t})}A^{r-1-t} &= \sum_{t=0}^{r-1}(-1)^t\left(s_{(j)}e_{t+1}-s_{(j,1^{\times(t+1)})}\right)A^{r-1-t}\\ &= s_{(j)}\sum_{t=0}^{r-1}(-1)^t e_{t+1}A^{r-1-t}+A\sum_{t=1}^{r}(-1)^t s_{(j,1^{\times t})}A^{r-1-t}\\ &\overset{(3.1)}{=} s_{(j)}A^r + A\left(A^{r-1+j}-s_{(j)}A^{r-1}+(-1)^r s_{(j,1^{\times r})}A^{-1}\right)\\ &= A^{r+j}+(-1)^r s_{(j,1^{\times r})},\end{aligned}$$

and we are done since $s_\lambda(x_1,\dots,x_r)$ vanishes when the length of λ is greater than r. $\square$

Proposition 3.3. *If $A\in M_r(\mathbb{C})$ has eigenvalues bounded by 1, then $\|A^m\|\le m^r\sum_{t=0}^{r-1}\|A^t\|$ for $m\ge1$ and any matrix norm $\|\cdot\|$.*

Proof. For $m<r$ this is immediate, so we assume $m\ge r$ and apply Proposition 3.2 with $j=m-r+1$:

$$\|A^m\|=\left\|A^{r-1+j}\right\|\le\sum\nolimits_{t=0}^{r-1}\left|s_{(j,1^{\times t})}(z_1,\dots,z_r)\right|\left\|A^{r-1-t}\right\|.$$

The coefficients of Schur polynomials are the Kostka numbers, and since they are non-negative, $\left|s_{(j,1^{\times t})}(z_1,\dots,z_r)\right|\le s_{(j,1^{\times t})}(1^{\times r})$ for any $z_1,\dots,z_r$ in the complex unit disc. As $s_\lambda(1^{\times r})$ is the number of semistandard Young tableaux of shape λ with entries in $\{1..r\}$, it is well studied (see e.g. [Sta99, (7.106)]), but for our simple partition λ we can compute it directly using $s_{(t)}=e_t$ and (3.2), obtaining

$$s_{(j,1^{\times t})}(1^{\times r})=\frac{1}{j+t}\binom{j+r-1}{j-1,t,r-1-t,1}\le(j+r-1)^r=m^r$$

(here $\binom{n}{m_1,\dots,m_k}$ is the k-multinomial coefficient), so that $\|A^m\|\le m^r\sum_{t=0}^{r-1}\|A^t\|$. $\square$

We can now prove Theorems 1.4 and 3.1:

Proof of Theorem 3.1. As $\mathcal{D}$ is r-normal, A is unitarily equivalent to a block-diagonal matrix $\mathrm{diag}(B_0,\dots,B_N)$ with each B_j of size at most $r\times r$. We assume $B_0,\dots,B_{p-1}$ are the "trivial" blocks, namely, those corresponding to constant or periodic eigenfunctions; they are of size 1×1 and satisfy $\left\|B_j^\ell\right\|_F=k^\ell$ for any $\ell\ge0$. For $j\ge p$, $B=B_j$ is a block of size $s\times s$ for some $s\le r$, and as $\mathcal{D}$ is Ramanujan, the eigenvalues of $\frac{B^\ell}{k^{\ell/2}}$ lie in the unit disc, being nontrivial eigenvalues of $\frac{A}{\sqrt{k}}$ raised to the ℓ-th power. Applying Proposition 3.3 to $\frac{B^\ell}{k^{\ell/2}}$ we get

$$\left\|\frac{B^{\ell m}}{k^{\ell m/2}}\right\|_F^2\le m^{2s}\left(\sum_{t=0}^{s-1}\left\|\frac{B^{\ell t}}{k^{\ell t/2}}\right\|_F\right)^2\le sm^{2s}\sum_{t=0}^{s-1}\left\|\frac{B^{\ell t}}{k^{\ell t/2}}\right\|_F^2\le rm^{2r}\sum_{t=0}^{r-1}\left\|\frac{B^{\ell t}}{k^{\ell t/2}}\right\|_F^2$$

for any $m\ge1$. Thus,

$$\begin{aligned}\left\|A^{\ell m}\right\|_F^2&=pk^{2\ell m}+\sum_{j=p}^{N}\left\|B_j^{\ell m}\right\|_F^2\le pk^{2\ell m}+rm^{2r}\sum_{j=p}^{N}\sum_{t=0}^{r-1}k^{\ell(m-t)}\left\|B_j^{\ell t}\right\|_F^2\\ &=pk^{2\ell m}+rm^{2r}\sum_{t=0}^{r-1}k^{\ell(m-t)}\left(\left\|A^{\ell t}\right\|_F^2-pk^{2\ell t}\right)=\bigstar.\end{aligned}$$

For $0\le t\le r-1$ we have $\ell t<\mathrm{crad}(\mathcal{D})$, which means that the operator $A^{\ell t}$ takes each vertex to $k^{\ell t}$ different ones. Computing$A^{\ell t}$ w.r.t. the standard basis of $L^2(V_{\mathcal{D}})$ we obtain that $\left\|A^{\ell t}\right\|_F^2=nk^{\ell t}$, hence

$$\bigstar=pk^{2\ell m}+rm^{2r}\sum\nolimits_{t=0}^{r-1}k^{\ell(m-t)}\left(nk^{\ell t}-pk^{2\ell t}\right)<pk^{2\ell m}+r^2m^{2r}nk^{\ell m}.$$ $\square$

This leads us to bounded cutoff:

Proof of Theorem 1.4. At time $t \leq \frac{m}{m-1}\log_k n$ we have $\frac{t}{m} < \frac{\mathrm{crad}(\mathcal{D})}{r-1}$, hence

$$\left\| A^t|_{L_0^2} \right\|_F^2 = \left\| A^{\frac{t}{m}m}|_{L_0^2} \right\|_F^2 \leq r^2 m^{2r} n k^t$$

by the previous Theorem with $\ell = \frac{t}{m}$. Denoting by $\mathbf{J}$ the all-one matrix, by $\mathbf{P}_{v_0}^t = \left(\frac{A}{k}\right)^t \mathbf{1}_{v_0}$ the distribution of the walk at time t (starting from a vertex v_0), and by $\mathbf{u}$ the uniform distribution $\frac{1}{n}\mathbf{1}$, we have

$$\begin{aligned} \left\| \mathbf{P}_{v_0}^t - \mathbf{u} \right\|_2^2 &= \frac{1}{k^{2t}} \left\| A^t \mathbf{1}_{v_0} - \frac{k^t}{n}\mathbf{1} \right\|_2^2 = \frac{1}{nk^{2t}} \sum_{v \in V_{\mathcal{D}}} \left\| A^t \mathbf{1}_v - \frac{k^t}{n}\mathbf{1} \right\|_2^2 \\ &= \frac{1}{nk^{2t}} \left\| A^t - \frac{k^t}{n}\mathbf{J} \right\|_F^2 = \frac{1}{nk^{2t}} \left\| A^t|_{\mathbf{1}^\perp} \right\|_F^2 \leq \frac{r^2 m^{2r}}{k^t} \end{aligned}$$

(the passage to $A^t|_{\mathbf{1}^\perp}$ is since the nonzero singular values of $A^t - \frac{k^t}{n}\mathbf{J}$ and of $A^t|_{\mathbf{1}^\perp}$ are the same). Thus,

$$\left\| \mathbf{P}_{e_0}^t - \mathbf{u} \right\|_{TV} \leq \frac{\sqrt{n}}{2} \left\| \mathbf{P}_{v_0}^t - \mathbf{u} \right\|_2 \leq \frac{rm^r}{2}\sqrt{\frac{n}{k^t}},$$

so at

$$t = \log_k n + \log_k \frac{r^2 m^{2r}}{4\varepsilon^2}$$

(rounded up to a multiple of m, and for n large enough to have $t \leq \frac{m}{m-1}\log_k n$), we get $\left\| \mathbf{P}_{v_0}^t - \mathbf{u} \right\|_{TV} \leq \varepsilon$. The lower bound for $t_{1-\varepsilon}$ holds for all k-regular graphs: denoting $S = \mathrm{supp}\left(A^t \mathbf{1}_{v_0}\right)$ once has $|S| \leq k^t$, so that for $t \leq \log_k n - \log_k(\frac{1}{\varepsilon})$

$$\left\| \mathbf{P}_{v_0}^t - \mathbf{u} \right\|_{TV} \geq \left| \mathbf{P}_{v_0}^t(V \backslash S) - \mathbf{u}(V \backslash S) \right| = \mathbf{u}(V \backslash S) \geq \frac{n - k^t}{n} \geq 1 - \varepsilon. \qquad \square$$

4 Multivariate Chebyshev and flow on buildings

In [NS23], bounded cutoff for the non-backtracking walk on the vertices of a Ramanujan graph is proved. The authors study the operator $(K_t f)(v) = \sum_w f(w)$, where the sum is over all vertices w connected to v by a non-backtracking path of length t. The operator K_t is self-adjoint, so the spectral analysis depends only on its eigenvalues (and eigenfunctions, in the non-transitive case). Furthermore, K_t can be expressed as a polynomial in the adjacency operator A, using Chebyshev polynomials of the first and second kind [NS23, Lem. 4.3]. A fundamental ingredient in the analysis are the recursion relations satisfied by these polynomials, and in this section we explore the possibility to generalize this picture to higher dimensions.

We begin with the two-dimensional case. Let F be a non-archimedean field with residue field $\mathbb{F}_q$, and let $\mathcal{B} = \mathcal{B}_{3,F}$ be the Bruhat-Tits building associated with $G = PGL_3(F)$. Let A_1, A_2 be the "colored adjacency operators" on the vertices on $\mathcal{B}$ (see [LSV05a, Li04]). These operators generate the Hecke algebra $\mathscr{H}_K^G$, which is the algebra of all finitely-supported operators on the vertices of $\mathcal{B}$ who commute with the action of G. This includes in particular the operator $(F_\ell f)(v) = \sum_w f(w)$ which sums over all w connected to v by a geodesic flow of length ℓ (as defined in [LLP20, 5]), so we should be able to express F_ℓ as a polynomial in A_1, A_2. We define the polynomials:

$$\begin{aligned} T_0(x,y) &= x \\ T_1(x,y) &= x^2 - (q+1)\,y \\ T_2(x,y) &= x^3 - (2q+1)\,xy + q\left(q^2+q+1\right) \\ \forall j \geq 3 : T_j(x,y) &= xT_{j-1}(x,y) - qyT_{j-2}(x,y) + q^3 T_{j-3}(x,y)\,. \end{aligned}$$

One can verify that the operator $T_j(A_1, A_2)$ sums over all vertices connected to v by geodesic flows of length ℓ *and color 1*. This is not a self-adjoint operator, since the opposite direction of a path of color 1 is a path of color 2 (however, $T_j(A_1, A_2)$ is a normal operator, as $\mathscr{H}$ is a commutative *-algebra). The operator F_ℓ, which is self-adjoint, can be expressed as

$$F_\ell = T_j(A_1, A_2) + T_j(A_1, A_2)^* = T_j(A_1, A_2) + T_j(A_2, A_1).$$

The connection between geodesic flow at time t and previous times is therefore

$$T_j + T_j^*\Big|_{x=A_1, y=A_2} = A_1\left(T_{j-1} - qT_{j-2}^*\right) + A_2\left(T_{j-1}^* - qT_{j-2}\right) + q^3\left(T_{j-3} + T_{j-3}^*\right).$$

We do not see how to connect $T_j + T_j^*$ directly with $T_{j-1} + T_{j-1}^*$ and $T_{j-2} + T_{j-2}^*$, in the same manner that Chebyshev polynomial connect NBRW at time t with earlier times. As the operator $T_j(A_1, A_2)$ (which only goes along edges of color 1) is already normal, perhaps this is the right analogue for the NBRW in higher dimension. Moving to higher dimensions, one can construct similar recursion relations using the Hecke operators $A_1, \dots, A_{d-1}$ for PGL_d. Sadly, even though the geodesic walk on edges in higher dimension still depends only on the triangle structure, the recursion formula become more involved: for geodesic edge flow on PGL_d (see [LLP20, §5.1]), we obtain (for $j \geq d$)

$$T_j\left(x_1, \dots, x_{d-1}\right) = \left[\sum_{m=1}^{d-1} (-1)^{m+1} q^{\binom{m}{2}} x_m T_{j-m}\right] + (-1)^{d+1} q^{\binom{d}{2}} T_{j-d}. \tag{4.1}$$

The methods developed in this paper serve us to avoid having to analyze these "higher Chebyshev" relations, but it seems to us that they merit further study in their own right.

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