Golden Gates in $PU(n)$
(DRAFT)

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Abstract

Our goal in this paper is to construct optimal topological generators for compact unitary Lie groups, extending the works of [Sar15b, PS18] on golden and super-golden gates to higher dimensions. To do so we state and prove a variant of the Sarnak–Xue Density Hypotheses [Sar90, SX91] in the weight aspect for definite projective unitary groups.

Contents

1 Introduction

In [Sar15b, PS18], motivated by applications to quantum computation, the notions of golden gate sets and super-golden gate sets for $PU(2)$ were introduced. These are topological generating sets which possess optimal covering properties as well as an efficient algorithm for navigation and approximation (see the definition below). In [PS18] and [EP18], golden and super-golden gate sets were constructed for $PU(2)$ and $PU(3)$, respectively. The main goal of this paper is to construct golden and super-golden gate sets on $PU\left(2^b\right)$ for $b = 2, 3$.

We begin by defining the notions of golden and super-golden gate sets for a general compact Lie group. Let L be a compact Lie group equipped with a probability Haar measure $\mu = \mu_L$ and a bi-invariant metric $d = d_L$. For $\varepsilon > 0$, $x \in L$ and $X \subseteq L$, let $B(x, \varepsilon)$ be the ball around x of *volume* ε, and let $B(X, \varepsilon) := \bigcup_{x \in X} B(x, \varepsilon)$. For $S \subseteq L$ and $\ell \in \mathbb{N}$, let $S^\ell \subset L$ (resp. $S^{(\ell)} \subset L$) be the set of words with shortest representation of length at most (resp. precisely) ℓ in S, and let $\langle S \rangle$ (resp. $\langle S \rangle_{sg}$) be the group (resp. semigroup) generated by S.

Definition 1.1. A finite subset $S \subset L$ (resp. also generated by elements of finite order) is said to be a *Golden Gate Set* (resp. *Super Golden Gate Set*) if it satisfies the following conditions:

(1) Covering: The covering rate of $S^{(\ell)}$ in L is optimal up to a polylogarithmic factor; namely, there exists a fixed $c \geq 1$, such that

$$\mu\left(L \setminus B\left(S^{(\ell)}, \varepsilon_\ell\right)\right) \overset{\ell \to \infty}{\longrightarrow} 0, \qquad \varepsilon_\ell = \frac{\left(\log \left|S^{(\ell)}\right|\right)^c}{\left|S^{(\ell)}\right|}.$$

(2) Growth: The size of $S^{(\ell)}$ grows exponentially in ℓ.

(3) Navigation: There is an efficient algorithm such that, given $g \in \langle S \rangle_{sg} \subset L$, the algorithm writes g as a word of shortest possible length in S.

(4) Approximation: There exists $N \geq 1$ and a (heuristic, randomized) efficient algorithm such that given $g \in L$, $\varepsilon > 0$, and ℓ satisfying $B(g, \varepsilon) \cap S^{(\ell)} \neq \emptyset$, the algorithm outputs an element from $B(g, \varepsilon) \cap S^{(\ell \cdot N)}$.

For the motivation behind this definition and its connection to quantum computation, we refer the interested reader to [Sar15b, PS18]—see in particular §4.2.2 for details on the analogy to [PS18]. Following our main interest in quantum computation, in this paper, we shall only concern ourselves with the case where the compact Lie group is the group of unitary or projective unitary $2^b \times 2^b$ matrices:

$$U(2^b) := \{g \in GL_{2^b}(\mathbb{C}) \mid g^* g = I\} \quad , \quad PU(2^b) := U(2^b)/\{c \cdot I \mid c \in U(1)\}.$$

Following [Sar15b, PS18, EP18], our constructions of golden and super-golden gate sets for $PU(n)$ come from certain arithmetic groups of unitary matrices that we call "golden adelic groups":

Theorem 1.2. *Let $n = 2^b = 4, 8$ and K' be a golden adelic group of a rank-n, definite arithmetic unitary group that is golden (resp. super-golden) at some prime $\mathfrak{p}$ as in Definition 4.1. Then there is a corresponding set $S_{\mathfrak{p}}$ of golden (resp. super-golden) gates of $PU(n)$.*

By finding an explicit example of a golden adelic group for $n = 4$:

Theorem 1.3. *Define the following Hermitian positive definite 4×4 matrix,*

$$H = 2 \cdot \begin{pmatrix} I_2 & A \\ -A & I_2 \end{pmatrix}, \quad A = \frac{\sqrt{-3}}{3} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and let $B \in GL_4(\mathbb{C})$ be such that $H = B^*B$. For any prime $p \neq 2, 3$, denote

$$S_p := \left\{ g \in M_4\left(\mathbb{Z}\left[\frac{1-\sqrt{-3}}{2}\right]\right) \mid \begin{array}{c} g^*Hg = p' \cdot H, \ g \equiv I_4 \mod 2 \\ g \text{ is not a scalar matrix} \end{array} \right\}$$

$$\text{where} \qquad p' = \begin{cases} p & p \equiv 1 \mod 3 \\ p^2 & p \equiv 2 \mod 3 \end{cases}.$$

Then the set, $S'_p = \left\{ BgB^{-1} \mid g \in S_p \right\}$, is a golden gate set of $PU(4)$.

The paper [MSG12] provides examples of golden adelic groups for $n = 2^b = 8$ and shows that no such exist for larger n. We leave to later work both the determination of the $\mathfrak{p}$ at which these groups are golden or super-golden and the computations of the resulting gate sets.

The key property of golden adelic groups we use is that they determine lattices acting simply transitively on $G_{\mathfrak{p}}$-orbits in a Bruhat-Tits building. For example:

Theorem 1.4. *In the notations of Theorem 1.3, let Λ_p be the group generated by S_p in $PU(4)$. Then Λ_p is a p-arithmetic subgroup and for $p \neq 2$ and Λ_p acts simply transitively on the G_p-orbit of a hyperspecial vertex of the corresponding Bruhat-Tits building.*

Theorem 1.4, combined with the Ross-Selinger algorithm [RS15] (see Subsection 4.3), yields gate sets for $PU(4)$ that satisfy the properties of growth, navigation and approximation (Theorem 4.13). The optimal covering property would follow if we could prove the naïve Ramanujan conjecture for the underlying algebraic group of Λ_p (this was the method of proof used in [PS18, EP18]). However, one can construct counterexamples for the naïve Ramanujan conjecture for $n \geq 4$ (see Theorem 1.4 in [LSV05]).

To overcome this obstacle, we proceed according to the strategy suggested in [PS18] of replacing the naïve Ramanujan conjecture with a variant of the Sarnak-Xue Density Hypothesis ([Sar90, SX91]) in the weight aspect. We consider certain *automorphic families* $\mathcal{F}$: weighted subsets of the discrete automorphic spectrum $\mathcal{AR}_{\text{disc}}(G)$ defined by assigning numbers $\mathcal{F}(\pi)$ to each π. We also consider for any representation π_v of G_v, the matrix coefficient decay

$$\sigma(\pi_v) := \inf\{p \ : \ p \geq 2, \ \pi_v \text{ has matrix coefficients in } L^p(G_v) \text{ mod center}\}.$$

Then:

Theorem 1.5. *Let G be a definite, F-algebraic unitary group for some number field F and let v be a prime such that $G(F_v)$ is not compact. For the automorphic families $\mathcal{F}_\delta$ for small $\delta > 0$ defined in §7 roughly representing the decomposition over the automorphic spectrum of the indicator function of a ball of volume δ in G_∞, denote*

$$|\mathcal{F}_\delta| = \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G)} \mathcal{F}_\delta(\pi), \qquad |\mathcal{F}_\delta(\sigma, v)| = \sum_{\substack{\pi \in \mathcal{AR}_{\mathrm{disc}}(G) \\ \sigma(\pi_v) \geq \sigma}} \mathcal{F}_\delta(\pi).$$

Then, for any $\epsilon > 0$, there exist $c_\epsilon > 0$ such that for any $\sigma \geq 2$ and small enough δ:

$$|\mathcal{F}_\delta(\sigma, v)| \leq c_\epsilon |\mathcal{F}_\delta|^{\frac{2}{\sigma} - \epsilon} |\mathcal{F}_\delta(\infty, v)|^{1 - \frac{2}{\sigma} - \epsilon}$$

(where we note that both numbers under the exponents are ≤ 1).

Theorem 1.5 is proven through the heavy use of recent advances in the Langlands program, especially Arthur's work on the endoscopic classification of automorphic representations of classical groups.

This paper is organized as follows: In Sections 2 and 3, we collect some basic facts about arithmetic unitary groups and Bruhat-Tits buildings while pointing out specific details particularly important to this application. In Section 4, we define the notions of golden and super-golden adelic groups and show how they give rise to gate sets that satisfy the last three properties of Definition 1.1. In Section 5, we review material about automorphic representations of general reductive groups, the Generalized Ramanujan Conjectures, endoscopic classifications, automorphic families, and the Sarnak-Xue density hypothesis. We also make a key definition of the shape of an automorphic representation π. In Section 6, we relate the shape of π to the local matrix coefficient decay $\sigma(\pi_v)$ at finite places v. We combine this with a bound on counts of π with a given shape to prove Theorem 1.5 in Section 7. Finally, in Section 8, we use Therorem 1.5 to prove that gate sets coming from golden adelic groups satisfy the optimal covering property, thus proving Theorem 1.3.

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2 Arithmetic unitary groups

In this section, we collect some facts about arithmetic unitary groups and their class number.

Let F be a totally real number field and let $\mathcal{O}$ be its ring of integers. Let V be the set of places of F and let V_f and ∞ be the subsets of finite and infinite places, respectively. For any $\mathfrak{p} \in V$, let $F_\mathfrak{p}$ be the $\mathfrak{p}$-completion of F and, if $\mathfrak{p} \in V_f$, let $\mathcal{O}_\mathfrak{p}$ be the ring of integers of $F_\mathfrak{p}$ and $q_\mathfrak{p}$ be the residue degree of $F_\mathfrak{p}$. For any $S \subset V$, let $\mathcal{O}[1/S] = F \bigcap_{\mathfrak{p} \in V_f \setminus S} \mathcal{O}_\mathfrak{p}$ be the ring of S-integers of F. When $S = \{\mathfrak{p}\} \bigcup V_\infty$, we abbreviate $\mathcal{O}[1/\mathfrak{p}] = \mathcal{O}[1/S]$, called the ring of $\mathfrak{p}$-integers of F. Note that for $S = V_\infty$, we have $\mathcal{O}[1/S] = \mathcal{O}$, the ring of integers of F.

Let E be a totally imaginary quadratic extension of F and let $\mathcal{O}_E$ be its ring of integers. Let $3 \leq n \in \mathbb{N}$ and let $H \in GL_n(E)$ be a non-degenerate, Hermitian, totally positive-definite matrix. Assume for simplicity that $H \in M_n(\mathcal{O}_E)$ and that $\gcd(H_{ij}) \in \mathcal{O}_E^\times$. Denote by $U_n^{E,H}$ the unitary group scheme over $\mathcal{O}$ with respect to E and H defined for any $\mathcal{O}$-algebra A by

$$U_n^{E,H}(A) = \{g \in GL_n(A \otimes_{\mathcal{O}} \mathcal{O}_E) \,:\, g^* H g = H\}.$$

Define the special unitary and projective unitary group schemes $SU_n^{E,H}$ and $PU_n^{E,H}$ over $\mathcal{O}$ with respect to E and H by:

$$SU_n^{E,H} = \left\{g \in U_n^{E,H} \,:\, \det(g) = 1\right\}, \qquad PU_n^{E,H} = U_n^{E,H}/U_1^E,$$

where $U_1^E(A) = \left\{x \in (A \otimes_{\mathcal{O}} \mathcal{O}_E)^\times \,:\, \bar{x}x = 1\right\}$ is identified with the scalar matrices of $U_n^{E,H}$. Finally, define

$$GU_n = \{g \in GL_n(A \otimes_{\mathcal{O}} \mathcal{O}_E) \,:\, g^* H g = \alpha H, \alpha \in (A \otimes_{\mathcal{O}} \mathcal{O}_E)^\times\}.$$

We have equalities on points $PU_n(\mathbb{R}) = U_n(\mathbb{R})/U_1(\mathbb{R})$ and $PU_n(S) = GU_n(S)/(S \otimes_{\mathcal{O}} \mathcal{O}_E)^\times$ for local and global fields S. However, this does not hold for general $\mathcal{O}$-algebras.

Remark 2.1. In our final construction, we will exclusively use $U_n^{E,H}$ since it is the only case where we have access to the endoscopic classification and can prove optimal covering. However, the endoscopic classification is expected to be true for the other groups as well and these may be better suited to the construction of super-golden gates at non-split primes. We therefore present a more comprehensive overview for future-proofing.

2.1 Basic Structure

Let G be either $U_n^{E,H}$, $SU_n^{E,H}$, $PU_n^{E,H}$, or $GU_n^{E,H}$. For any $v \in V$, denote $G_v := G(F_v)$, and for any $\mathfrak{p} \in V_f$, denote $K_{\mathfrak{p}} := G(\mathcal{O}_{\mathfrak{p}})$ and $\Gamma_{\mathfrak{p}} := G(\mathcal{O}[1/\mathfrak{p}])$. Call $\Gamma_{\mathfrak{p}}$ the principal $\mathfrak{p}$-arithmetic subgroup of G; any finite index subgroup of it is called a $\mathfrak{p}$-arithmetic subgroup. Denote $G_\infty := \prod_{v \in V_\infty} G_v$ and $K := G(\mathcal{O})$. Call K the principal arithmetic subgroup of G; any finite index subgroup of it is called an arithmetic subgroup.

Lemma 2.2. *The following hold for $G \neq GU_n^{E,H}$:*

(1) Any arithmetic subgroup of G is finite.

(2) Any $\mathfrak{p}$-arithmetic subgroup of G is a cocompact lattice of $G_{\mathfrak{p}}$.

(3) Any $\mathfrak{p}$-arithmetic subgroup of G is a dense subgroup of G_∞.

Proof. First note that, since H is totally positive-definite, we get that for any $v \in V_\infty$, $G_v \cong U(n)$, $SU(n)$ or $PU(n)$, the unitary, special unitary, or projective unitary compact Lie groups; hence, G_∞ is a compact Lie group. By Borel–Harish-Chandra theory [BHC62], we get that any finite arithmetic subgroup of G is a cocompact lattice of G_∞ and any $\mathfrak{p}$-arithmetic subgroup of G is a cocompact lattice of $G_\infty \times G_{\mathfrak{p}}$.

Claim 1 follows from the fact that a discrete subgroup of a compact group is finite. Claim 2 follows from the fact that the projection of a cocompact lattice of $G_\infty \times G_{\mathfrak{p}}$ onto the second component remains a cocompact lattice of $G_{\mathfrak{p}}$ (since G_∞ is compact). Claim 3 follows from the fact that, since $\mathrm{rank}_{F_{\mathfrak{p}}}(G_{\mathfrak{p}}) \geq \frac{n}{2} - 1 > 0$ (since $n \geq 3$), $G_{\mathfrak{p}}$ is non-compact and therefore projecting a cocompact lattice of $G_\infty \times G_{\mathfrak{p}}$ to the first component is dense. □

Let $\mathbb{A} := \prod'_{v \in V} F_v$ be the ring of adeles of F, let $\widehat{\mathcal{O}} := \prod'_{\ell \in V_f} \mathcal{O}_\ell$, let $F_\infty = \prod_{v \in V_\infty} F_v$, and consider F embedded diagonally in $\mathbb{A}$. Then $\mathbb{A}$ is a locally compact ring, F (embedded diagonally) is a discrete subring, $\widehat{\mathcal{O}}$ and $F_\infty \widehat{\mathcal{O}}$ (embedded coordinate-wise) are compact and open subrings, respectively, and $\mathbb{A} = F \cdot F_\infty \widehat{\mathcal{O}}$.

Lemma 2.3. *Let E be of class number one. Then*

$$U_1^E(\mathbb{A}) = U_1^E(F) \cdot \prod_{v \in V_\infty} U_1^E(F_v) \prod_{\ell \in V_f} U_1^E(\mathcal{O}_\ell).$$

Proof. Let $x = (x_v) \in U_1^E(\mathbb{A})$, i.e. $\overline{x_v} x_v = 1$ for any v and $x_\ell \in U_1^E(\mathcal{O}_\ell)$ for almost all finite places ℓ. Note that if ℓ is non-split in E, then $U_1^E(F_\ell) = U_1^E(\mathcal{O}_\ell)$. If $\ell = \ell_1 \ell_2$ is split in E, then $F_\ell \otimes_F E = E_{\ell_1} \times E_{\ell_2}$ so we can write $x_\ell = (x_{\ell_1}, x_{\ell_2})$ for $x_{\ell_i} \in E_{\ell_i}$.

Let S be the finite set of split primes ℓ such that $|x_{\ell_1}|_{\ell_1} \neq 1$. By the class-number-one hypothesis, there exists $\alpha \in E$ such that $|\alpha|_{\ell_1} = |x_{\ell_1}|_{\ell_1}$ and $|\alpha_{\ell_2}| = 1$ for all $\ell \in S$ and $|\alpha|_{\ell_i} = 1$ for all $\ell \notin S$. Then $\alpha/\bar{\alpha} \in U_1^E(F)$ and satisfies $|\alpha_{\ell_i}| = |x_{\ell_i}|_{\ell_i}$ for $i = 1, 2$ and all split ℓ.

Putting it all together, $x \in (\alpha/\bar{\alpha}) \prod_{v \in V_\infty} U_1^E(F_v) \prod_{\ell \in V_f} U_1^E(\mathcal{O}_\ell)$. □

Lemma 2.4. *Let G be either $U_n^{E,H}$, $SU_n^{E,H}$, $PU_n^{E,H}$, or $GU_n^{E,H}$. If $G \neq SU_n^{E,H}$ assume that E is of class number one. Then for any prime $\mathfrak{p}$,*

$$G(\mathbb{A}) = G(F) \cdot G_\infty G_\mathfrak{p} \prod_{\mathfrak{p} \neq \ell \in V_f} K_\ell.$$

Proof. If $G = SU_n^{E,H}$, then by the strong approximation property we get that $G(F) G_\mathfrak{p}$ is dense in $G(\mathbb{A})$, so since $K = G_\infty \prod_{\ell \in V_f} K_\ell$ is open, we get the claim. If $G = U_n^{E,H}$, then by the claim for $SU_n^{E,H}$, we get that $G(F) \cdot G_\infty G_\mathfrak{p} \prod_{\mathfrak{p} \neq \ell \in V_f} K_\ell$ contains $SU_n^{E,H}(\mathbb{A})$. Then, since $U_n = SU_n^{E,H} U_1^E$, the claim follows from Lemma 2.3.

If $G = GU_n^{E,H}$, then we similarly use $GU_n^{E,H} = SU_n^{E,H} \operatorname{Res}_{\mathcal{O}_F}^{\mathcal{O}_E} \mathbb{G}_m$ and that E has class number one. Finally, the result for $G = PU_n^{E,H}$ follows from that for $GU_n^{E/H}$: by Shapiro's lemma, we have surjections $GU_n^{E/H}(S) \twoheadrightarrow PU_n^{E/H}(S)$ for $S = \mathbb{A}, F_v, F$ under which $PU_n^{E,H}(\mathcal{O}_{F_v})$ contains the image of $GU_n^{E,H}(\mathcal{O}_{F_v})$. □

Lemma 2.5. *Let E be of class number one. Then for any prime $\mathfrak{p}$, there is a bijective map from $\Gamma_\mathfrak{p} \backslash G_\mathfrak{p} / K_\mathfrak{p}$ to $G(F) \backslash G(\mathbb{A}) / K$. In particular, $c_\mathfrak{p}(G) = c(G)$, for any prime $\mathfrak{p}$.*

Proof. The map from $G_\mathfrak{p}$ to $G(\mathbb{A})$, defined by

$$g \mapsto (g_v)_{v \in V}, \; g_v = \begin{cases} g & v = \mathfrak{p} \\ 1 & v \neq \mathfrak{p} \end{cases}$$

induces a well defined map from $\Gamma_\mathfrak{p} \backslash G_\mathfrak{p} / K_\mathfrak{p}$ to $G(F) \backslash G(\mathbb{A}) / K$. This map is obviously injective, so the claim boils down to proving it is surjective, which follows from the strong approximation type result of Lemma 2.4. □

Corollary 2.6. *Let E be of class number one. The following are equivalent:*

- *There exists $\mathfrak{p}$ such that $G_\mathfrak{p} = \Gamma_\mathfrak{p} K_\mathfrak{p}$.*
- *For any $\mathfrak{p}$, $G_\mathfrak{p} = \Gamma_\mathfrak{p} K_\mathfrak{p}$.*

2.2 Class Number of G

Let G be either $U_N^{E,H}$, $SU_N^{E,H}$, or $PU_n^{E,H}$. By [BHC62], $G(F)$ is a cocompact lattice of $G(\mathbb{A})$ and note that $K := G_\infty G(\widehat{\mathcal{O}}) \leq G(\mathbb{A})$ is a compact open subgroup. Hence the double quotient space $G(F)\backslash G(\mathbb{A})/G_\infty G(\widehat{\mathcal{O}})$, is finite.

Definition 2.7. Let G be either $U_n^{E,H}$, $SU_n^{E,H}$, or $PU_n^{E,H}$. Define the class and $\mathfrak{p}$-class numbers of G to be

$$c(G) := |G(F)\backslash G(\mathbb{A})/K|, \qquad c_{\mathfrak{p}}(G) := |\Gamma_{\mathfrak{p}}\backslash G_{\mathfrak{p}}/K_{\mathfrak{p}}|.$$

For any CM field E/F and any Hermitian form $H \in M_n(E)$, denote by $\mathrm{disc}(E)$ and $\mathrm{disc}(H)$ the discriminants of E and H and by $\mathrm{Ram}(E)$ and $\mathrm{Ram}(H)$ the sets of primes which divide these discriminants, respectively. Denote by $\zeta_F(s)$ and $L(s, \chi_{E/F})$ the (analytic continuations of the) Dedekind zeta function of F and the Dirichlet L-function of $\chi_{E/F}$, the quadratic Dirichlet character associated to E/F by class field theory.

We now specialize to the case of $G = U_n^{E/H}$.

Definition 2.8. Let $G = U_n^{E,H}$. Define the set of ramified primes of G to be

$$\mathrm{Ram}(G) := \mathrm{Ram}(E) \bigcup \mathrm{Ram}(H),$$

define the λ-constant of G to be

$$\lambda(G) := \prod_{\ell \in \mathrm{Ram}(G)} \lambda_\ell, \quad \lambda_\ell = \begin{cases} 1/2 & 2 \nmid n,\ \ell \in \mathrm{Ram}(E) \\ (q_\ell^{\lfloor n/2 \rfloor} + 1)/(q_\ell + 1) & 2 \nmid n,\ \ell \notin \mathrm{Ram}(E) \\ (q_\ell^{\lfloor n/2 \rfloor} - 1)/(q_\ell + 1) & 2 \mid n,\ \ell \notin \mathrm{Ram}(E) \\ (q_\ell^{\lfloor n/2 \rfloor} - 1)/2(q_\ell + 1) & 2 \mid n,\ \ell \in \mathrm{Ram}(E) \end{cases},$$

define the L-special value of G to be

$$L(G) := 2^{-n+1} \cdot \prod_{r=1}^{n} L\left(1 - r, \chi_{E/F}^r\right) = 2^{-n+1} \cdot \prod_{r=1}^{\lfloor n/2 \rfloor} \zeta_F(1 - 2r) \cdot \prod_{r=1}^{\lfloor n+1/2 \rfloor} L(2 - 2r, \chi_{E/F}),$$

and finally, define the mass constant of G to be

$$R(G) := L(G) \cdot \lambda(G).$$

The next lemma give us a computable criterion for determining whether a group has class number one:

Lemma 2.9. *Let $G = U_n^{E,H}$. Then*

$$c(G) = 1 \qquad \Leftrightarrow \qquad |G(\mathcal{O})|^{-1} = R(G).$$

Proof. Let μ_{tam} be the Tamagawa measure of $G(\mathbb{A})$ and denote the mass of G by

$$\mathrm{Mass}(G) := \frac{\mu_{\mathrm{tam}}(G(F)\backslash G(\mathbb{A}))}{\mu_{\mathrm{tam}}(K)}.$$

On the one hand, Propositions 4.4 and 4.5 in [GHY01], yields

$$\mathrm{Mass}(G) = R(G).$$

On the other hand, by Siegel mass formula, the mass of G is equal the sum over the representatives of the genus of G weighted by the reciprocal of the size of the associated arithmetic finite group:

$$\mathrm{Mass}\,(G) = \sum_{g \in G(F)\backslash G(\mathbb{A})/K} \left|G\,(F) \cap gKg^{-1}\right|^{-1} = \left|G\,(\mathcal{O})\right|^{-1} + \sum_{1 \neq g \in G(F)\backslash G(\mathbb{A})/K} \cdots .$$

Since each member in the sum is positive rational number, comparing both estimates shows that $|G\,(\mathcal{O})\,|^{-1} = R\,(G)$ if and only if $c\,(G) = 1$. □

Remark 2.10. The are only finitely many values of n, $\mathrm{disc}\,(E)$, and $\mathrm{disc}\,(H)$ such that the definite unitary group $G = U_N^{E,H}$ has class number one (see [BP89]). In fact, Mohammadi and Salehi-Golsefidy in [MSG12] showed that $n = 8$ is the threshold for definite unitary groups of class number one; namely, for any $n > 8$, there are no class number one definite unitary groups of the form $G = U_N^{E,H}$ and for any $4 < n \leq 8$, there exists class number one definite unitary groups of the form $G = U_N^{E,H}$.

3 Bruhat-Tits buildings

This subsection contains a quick introduction to the theory of Bruhat–Tits buildings, focusing on unitary and general linear groups. The standard reference summarizing the theory is [Tit79]. A modernized textbook treatment of the full details can be found in [KP23].

3.1 Description

Let $\mathbb{F}$ be a non-Archimedean local field, $\mathcal{O}_{\mathbb{F}} \subset \mathbb{F}$ its ring of integers, $\varpi_{\mathbb{F}} \in \mathcal{O}_{\mathbb{F}}$ a choice uniformizer and $q_{\mathbb{F}} = |\mathcal{O}_{\mathbb{F}}/\varpi_{\mathbb{F}}\mathcal{O}_{\mathbb{F}}|$ its residue degree (e.g. $\mathbb{F} = \mathbb{Q}_p$, $\mathcal{O}_{\mathbb{F}} = \mathbb{Z}_p$, $\varpi_{\mathbb{F}} = p$ and $q_{\mathbb{F}} = p$). Let H be the group of $\mathbb{F}$-rational points of an $\mathbb{F}$-almost-simple, connected reductive group $\mathcal{H}$. Then Bruhat–Tits theory constructs a pure, simplicial, infinite, locally-finite, contractible complex $\mathcal{B} := \mathcal{B}\,(H)$ called the Bruhat-Tits building of H such that H acts simplicially on $\mathcal{B}\,(H)$ and transitively on its maximal faces (called chambers). The dimension of $\mathcal{B}\,(H)$ is equal to the semisimple rank $r = \mathrm{rank}_{\mathbb{F}}\left(H^{\mathrm{der}}\right)$.

The building is equipped with a natural type function from its 0-simplices to $[r] := \{0, 1, \ldots, r\}$, which is a bijection on the vertices of each chamber. We say the type of a k-simplex is the set consisting of the k types of its vertices. If $H = \mathbf{H}(\mathbb{F})$ for $\mathbf{H}$ semisimple and simply connected, then the H-orbit of a simplex of type τ is the set of all simplices of type τ. In general, it might be bigger.

Call H unramified if $H = \mathbf{H}\,(\mathbb{F})$, where $\mathbf{H}$ is quasi-split over $\mathbb{F}$ and splits over an unramified extension of $\mathbb{F}$. In this case, there is known to be a reductive model of $\mathbf{H}$ over $\mathcal{O}_{\mathbb{F}}$; then $K_H = \mathbf{H}\,(\mathcal{O}_{\mathbb{F}})$ is called a hyperspecial maximal compact subgroup of H. Such a K_H is the stabilizer of a particular kind of vertex v_0 in $\mathcal{B}\,(H)$ called hyperspecial. We fix a choice of v_0 and without loss of generality assume to be of type 0.

We now give two important examples of the Bruhat-Tits buildings of unramified groups:

Example 3.1. Let $H = PGL_n\,(\mathbb{F})$ be the projective general linear group of $\mathbb{F}$-points. The Bruhat-Tits building $\mathcal{B} = \mathcal{B}\,(PGL_n\,(\mathbb{F}))$ is the $(n-1)$-dimensional simplicial complex, whose vertices are homotetic classes $[L] = \{\alpha L \,|\, \alpha \in \mathbb{F}^{\times}\}$ of $\mathcal{O}_{\mathbb{F}}$-lattices L in $\mathbb{F}^n$ and a collection of vertices $\sigma = \{v_0, \ldots, v_k\}$, forms a face in $\mathcal{B}$ if there exists representatives $L_i \in v_i$, $0 \leq i \leq k$ such that $L_0 \subsetneq \ldots \subsetneq L_k \subsetneq \varpi_{\mathbb{F}}^{-1} L_0$.

The group $PGL_n\,(\mathbb{F})$ acts on the homotetic classes of lattices by matrix multiplication and this action extends to the entire complex. The subgroup $PGL_n\,(\mathcal{O}_{\mathbb{F}})$ is the stabilizer of the hyperspecial vertex $v_0 = [\mathcal{O}_{\mathbb{F}}^n]$. The group $PGL_n(\mathbb{F})$ acts transitively on the vertices of the building, hence all vertices are hyperspecial. The degree of (i.e. the number of chambers containing) any $(n-2)$-dimensional face is $q_{\mathbb{F}}+1$.

For example, $\mathcal{B}(PGL_2(\mathbb{F}))$ is the infinite $(q_{\mathbb{F}}+1)$-regular tree, and $\mathcal{B}(PGL_3(\mathbb{F}))$ is the infinite 2-dimensional simplicial complex, all of whose edges are of degree $q_{\mathbb{F}}+1$ and all of whose vertices are contained in $2\left(q_{\mathbb{F}}^2+q_{\mathbb{F}}+1\right)$ edges.

Example 3.2. Let $\mathbb{E}$ be the unique unramified quadratic extension of $\mathbb{F}$ and let $c \mapsto \bar{c}$ be the non-trivial element in $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$. Let $H = PU_n(\mathbb{F})$ be the unramified projective unitary group, where

$$PU_n(\mathbb{F}) = \{g \in GL_n(\mathbb{E}) \,:\, g^* J g = J\} / \left\{cI_n \,:\, \bar{c}c = 1,\, c \in \mathbb{E}^\times\right\},$$

where $g^* = \bar{g}^t$ and $J = (\delta_{i,n+1-j})_{i,j}$ is the anti-diagonal Hermitian form. Let $\sharp$ be the involution on $PGL_n(\mathbb{E})$ defined by $g^\sharp = J(g^*)^{-1} J$. Note that $PU_n(\mathbb{F})$ is the subgroup of $\sharp$-fixed elements of $PGL_n(\mathbb{E})$:

$$PU_n(\mathbb{F}) = PGL_n(\mathbb{E})^\sharp := \left\{g \in PGL_n(\mathbb{E}) \,:\, g^\sharp = g\right\}.$$

Similarly, define the order 2 automorphism $\sharp$ on $\mathcal{B}(PGL_n(\mathbb{E}))$ as follows: on the $\mathcal{O}_{\mathbb{E}}$-lattices L of $\mathbb{E}^n$, define

$$L^\# := \{v \in \mathbb{E}^n \mid vJ\bar{u} \in \mathcal{O}_{\mathbb{E}},\ \forall u \in L\},$$

on the homothetic class of $\mathcal{O}_{\mathbb{E}}$-lattices (i.e. the vertices) $v = [L]$, define $v^\sharp = \left[L^\sharp\right]$, and on the faces $\sigma = \{v_0, \ldots, v_k\}$, define $\sigma^\sharp = \left\{v_0^\sharp, \ldots, v_k^\sharp\right\}$.

Then the Bruhat-Tits building of $PU_n(\mathbb{F})$ is the subcomplex of $\sharp$-fixed faces of the Bruhat-Tits building of $PGL_n(\mathbb{E})$: i.e,

$$\mathcal{B}(PU_n(\mathbb{F})) = \mathcal{B}(PGL_n(\mathbb{E}))^\sharp := \left\{\sigma \in \mathcal{B}(PGL_n(\mathbb{E})) \,:\, \sigma^\sharp = \sigma\right\}.$$

The group $PU_n(\mathbb{F}) = PGL_n(\mathbb{E})^\sharp$ acts naturally on $\mathcal{B}(PU_n(\mathbb{F})) = \mathcal{B}(PGL_n(\mathbb{E}))^\sharp$. The vertices of the buildings of $\mathcal{B}(PU_n(\mathbb{F}))$ are either $\sharp$-fixed vertices, which are the hyperspecial vertices, or edges whose endpoints are swapped by $\sharp$, which are non-hyperspecial. Note that $v_0 = [\mathcal{O}_{\mathbb{F}}^n] \in \mathcal{B}(PGL_n(\mathbb{E}))^\sharp$ is a hyperspecial vertex and its stabilizer in $PU_n(\mathbb{F})$ is $PU_n(\mathcal{O}_{\mathbb{F}})$, the subgroup of elements with coefficients in $\mathcal{O}_{\mathbb{F}}$. The dimension of $\mathcal{B}(PU_n(\mathbb{F}))$, which is equal to the $\mathbb{F}$-rank $PU_n(\mathbb{F})$, is $\left\lfloor \frac{n}{2} \right\rfloor$.

For example, $\mathcal{B}(PU_3(\mathbb{F}))$ is the infinite $\left(q_{\mathbb{F}}^3+1, q_{\mathbb{F}}+1\right)$-biregular tree, where the hyperspecial vertices are those of degrees are $q_{\mathbb{F}}^3+1$.

Example 3.3. For $G = GL_n$, the building $\mathcal{B}$ is that same as that of PGL_n. Since the center intersect a maximally split torus is in this case connected, the action of GL_n on $\mathcal{B}$ factors through PGL_n.

Example 3.4. For $G = U_n$, the building $\mathcal{B}$ is the same as that of PU_n. However, when n is even, the natural map $U_n(\mathbb{F}) = GL_n(\mathbb{E})^\sharp$ to $PU_n(\mathbb{F})$ is not a surjection so the action might be smaller.

These are the examples that will come up for $G = U_n^{E,H}$.

Remark 3.5. For $H = PGL_n(\mathbb{F})$, $H = PU_n(\mathbb{F})$, or $H = U_{2n+1}(\mathbb{F})$, the group acts transitively on the hyperspecial vertices of $\mathcal{B}(H)$. However this is not true in general.

Lemma 3.6. *Let $G = U_n^{E,H}$ and let $\mathfrak{p} \in V_f \setminus \mathrm{Ram}\,(G)$. Then*

$$G_{\mathfrak{p}} = G\left(F_{\mathfrak{p}}\right) \cong \begin{cases} GL_n\left(F_{\mathfrak{p}}\right) & \mathfrak{p} \text{ split in } E \\ U_n\left(F_{\mathfrak{p}}\right) & \mathfrak{p} \text{ inert in } E \end{cases}.$$

Forthermore, $K_{\mathfrak{p}} = G\left(\mathcal{O}_{\mathfrak{p}}\right)$ is a hyperspecial maximal parahoric subgroup of $G_{\mathfrak{p}}$.

Proof. If $\mathfrak{p}$ splits in E , then $E_{\mathfrak{p}} \cong F_{\mathfrak{p}} \times F_{\mathfrak{p}}$ with $\overline{(a,b)} = (b,a)$. Therefore $GL_n\left(E_{\mathfrak{p}}\right) = GL_n\left(F_{\mathfrak{p}}\right) \times GL_n\left(F_{\mathfrak{p}}\right)$, so we get

$$G_{\mathfrak{p}} = \left\{g = (g_1, g_2) \in GL_n\left(E_{\mathfrak{p}}\right) \mid g_1 = H^{-1}(g_2^*)^{-1}H\right\} \overset{g \mapsto g_1}{\cong} GL_n\left(F_{\mathfrak{p}}\right).$$

Since $\mathfrak{p} \notin \mathrm{Ram}\,(H)$, then (up to similarity) $H \in GL_n\left(\mathcal{O}_{\mathbb{E}}\right)$, and we therefore get that $K_{\mathfrak{p}} \cong GL_n\left(\mathcal{O}_{\mathfrak{p}}\right)$.

If $\mathfrak{p}$ inert in E , then $E_{\mathfrak{p}}/F_{\mathfrak{p}}$ is an unramified extension and we get that $G_{\mathfrak{p}} \cong U_n\left(F_{\mathfrak{p}}\right)$ by the classification of Hermitian forms over p-adic fields [Jac62]. Since $\mathfrak{p} \notin \mathrm{Ram}\,(H)$, then (up to similarity) $H \in GL_n\left(\mathcal{O}_{E_{\mathfrak{p}}}\right)$, and we get that $K_{\mathfrak{p}} \cong U_n\left(\mathcal{O}_{\mathfrak{p}}\right)$. □

3.2 Cartan Invariants

Assume H is unramified and K_H is the stabilizer of the hyperspecial vertex v_0 of $\mathcal{B}\,(H)$. Let $A_H \cong \left(\mathbb{F}^{\times}\right)^r$ be a maximally split torus of H and let $X_+(A_H)$ be a positive Weyl chamber in the cocharacter lattice $X_*(A_H)$. Then the following Cartan decomposition holds:

$$H = K_H \cdot X_+(A_H) \cdot K_H := \bigsqcup_{a \in X_+(A_H)} K_H a(\varpi_{\mathbb{F}}) K_H. \tag{3.1}$$

For any $h \in H$, define $a_h \in X_+(A_H)$ to be the unique element such that $h \in K_H a\,(h)\,K_H$.

To resolve a technicality when H has non-anisotropic center, let

$$\bar{A}_H := A_H/Z_H^{\mathrm{spl}} := A_H/(A_H \cap Z_H)^0,$$

where Z_H^{spl} is the maximal split torus in Z_H. Define $\bar{a}_h$ to be the image of a_h in this quotient. There is a dominance ordering on $X_+(\bar{A}_H)$ given by

$$a \preceq b \iff b - a \in X_+(\bar{A}_H).$$

Let Σ be the set of minimal non-zero elements of $(X_+(\bar{A}_H), \preceq)$. Then $X_+(\bar{A}_H)$ is exactly the non-negative integer linear combinations of elements of Σ. In general, $|\Sigma|$ might be larger than the semisimple rank $r_{\mathrm{ss}}(H)$ (e.g. $H = \mathrm{SL}_n$, $n \geq 3$). However, when they are equal, $X_+(\bar{A}_H) = \mathbb{Z}_{\geq 0}^{\Sigma}$ as semigroups. In this case, index a dual basis to Σ by elements α_i' that we will call modified simple roots. These are the same as the (non-multipliable) simple roots α_i when $Z_H \cap A_H$ is connected. Otherwise, they are scalings of the α_i.

We can now make the key definition that will let us construct gate sets.

Definition 3.7. Define the Cartan norm on H to be:

$$\|\cdot\|_0 = \|\cdot\|_H \colon H \to \mathbb{N}_0 \quad , \quad \|h\|_{H,0} = \|a_h\|_{H,0} = \sum_{1 \leq i \leq s} n_i |\alpha_i(a_h)|.$$

where n_i is the coefficient of α_i in the highest root of H.

In the case where $\bar{X}_+(A_H) = \mathbb{Z}_{\geq 0}^{\Sigma}$, define the modified Cartan norm to be:

$$\|\cdot\| = \|\cdot\|_H' \colon H \to \mathbb{N}_0 \quad , \quad \|h\|_H = \|a_h\|_H = \sum_{1 \leq i \leq s} |\alpha_i'(a_h)|.$$

Given two points $v_1, v_0 \in \mathscr{B}$, we can also define norms of $\|v_0 - v_1\|$, where $v_0 - v_1$ is always interpreted as an element of $X_+(\bar{A}_H)$ in a common apartment. Note also that

$$\Sigma = \{\alpha \in X_+(\bar{A}_H) \ : \ \|\alpha\|'_H = 1\}. \tag{3.2}$$

The Cartan norm $\|a\|$ has a clean interpretation in terms of $\mathcal{B}$. Let dist be the graph metric on the vertices in the 1-skeleton of $\mathcal{B}$.

Lemma 3.8. *For any $h \in H$, its Cartan norm $n = \|h\|_H$ satisfies the following:*

$$\operatorname{dist}(h.v_0, v_0) = \|h\|_H.$$

Proof. Note that for any $h \in H$ and any $k_1, k_2 \in K_H$, we get

$$\operatorname{dist}(k_1 h k_2.v_0, v_0) = \operatorname{dist}\left(h.\left(k_2.v_0\right), k_1^{-1}.v_0\right) = \operatorname{dist}(h.v_0, v_0).$$

Hence by the Cartan decomposition, it suffice to assume that $h = a \in A_H$. The split torus A_H acts by translation on the apartment of $\mathcal{B}$ corresponding to it, which we assume to v_0. As a fact about buildings, one chamber of this apartment is the convex hull of 0 and the λ_i/n_i where λ_i are the (non-divisible) fundamental coweights corresponding to the α_i.

Therefore, $\operatorname{dist}(h.v_0, v_0)$ is the sum of the coordinates of $h.v_0 - v_0$ in the basis of λ_i/n_i. This is exactly $\sum_{1 \le i \le s} n_i |\alpha_i(h)|$. □

Similarly, the modified Cartan norm is a weighted graph distance whenever it can be defined.

Example 3.9. Let $H = PGL_n(\mathbb{F})$ or $GL_N(\mathbb{F})$ and $K_H = PGL_n(\mathcal{O}_\mathbb{F})$ or $GL_N(\mathcal{O}_\mathbb{F})$, respectively. Then $\bar{A}_H$ is a quotient of the group of diagonal matrices $\operatorname{diag}(x_1, \ldots, x_n)$ where $x_i \in \mathbb{F}^\times$.

The relative root system is the absolute root system which is type-A_{n-1}, so

$$\bar{X}_+(A_H) = \{(m_1, \ldots, m_n) \in \mathbb{Z}^n/\langle(1, \ldots, 1)\rangle \mid m_1 \ge \ldots \ge m_n\}.$$

The Cartan norm is the same as the modified Cartan norm and is defined on $X_+(A_H)$ by

$$\|(m_1, \ldots, m_n)\|_H = \sum_{1 \le i \le n-1} (m_i - m_{i+1}) = m_1 - m_n.$$

The set Σ is the set of standard fundamental weights.

Example 3.10. Let H be quasisplit $U_n(\mathbb{F})$ with respect to the unramified quadratic extension $\mathbb{E}$ and $K_H = U_n(\mathcal{O}_\mathbb{F})$. Then we can choose a maximal torus consisting of elements $\operatorname{diag}(x_1, \ldots, x_n)$ with $x \in \mathbb{E}^\times$ and $x_i \overline{x_{n-i+1}} = 1$. Inside this, $A_H = \bar{A}_H$ is those elements with $x_i \in \mathbb{F}^\times$. Then,

$$X_*(A_H) = \{(m_1, \ldots, m_n) \in \mathbb{Z}^n \mid m_i = -m_{n-i+1}\}.$$

The relative root system is the restriction of the absolute root system type-A_{n-1} to A_H which is type-$C_{\lfloor n/2 \rfloor}$ if n is even and (non-reduced!) type-$BC_{\lfloor n/2 \rfloor}$ when n is odd. Either way,

$$X_+(A_H) = \{(m_1, \ldots, m_n) \in X_*(A_H) \mid m_i \in \mathbb{Z},\ m_1 \ge \ldots \ge m_{\lfloor n/2 \rfloor} \ge 0\}$$

and Cartan norm is defined on $X_+(A_H)$ by

$$\|(m_1, \ldots, m_n)\|_0 = H = \left(\sum_{i=1}^{\lfloor n/2 \rfloor - 1} 2(m_i - m_{i+1})\right) + 2m_{\lfloor n/2 \rfloor} = 2m_1.$$

The set Σ is the all elements of the form $(1,\dots,1,0,\dots,0,-1,\dots,-1)\in X_*(A)$ which is of the correct size to define modified Cartan norm:

$$\|(m_1,\dots,m_n)\|'_H = \left(\sum_{i=1}^{\lfloor n/2\rfloor-1}(m_i-m_{i+1})\right)+m_{\lfloor n/2\rfloor}=m_1.$$

Then Σ is exactly the elements of modified Cartan norm 1.

In this specific case, we get a clean replacement for lemma 3.8: the modified Cartan norm is exactly half the graph distance.

3.3 Discrete Actions and General Gate Sets

Our construction of gate sets will have to do with an analysis of discrete subgroups acting on $\mathcal{B}$. Fix $\Lambda\subseteq H$ such that $\Lambda K_H=H$. Recall the set Σ of minimal non-zero elements of $(\bar{X}_+(A_H),\preceq)$. We will make two technical assumptions for this section (that will always be satisfied in our applications):

- $|\Sigma|=\operatorname{rank}_{\mathrm{ss}} H$ so we can define a modified Cartan norm $\|\cdot\|$.
- For when H has non-anisotropic center, $\Lambda\cap K_H\cap Z_H^{\mathrm{spl}}=1$.

As more technicalities when Z_H isn't anisotropic, let $\bar\Lambda=\Lambda/(\Lambda\cap Z_H^{\mathrm{spl}})$ and note that $\gamma\mapsto\bar a_\gamma$ is well-defined on $\bar\Lambda$ and the action of Λ on $\mathscr{B}$ factors through $\bar\Lambda$.

Also, for each $\alpha\in\Sigma$, choose once and for all a lift $\widetilde\alpha\in X_*(A_H)$ and let $\widetilde\Sigma$ be the set of lifts. Since under our assumptions, $X_+(A_H)=\mathbb{Z}^{\Sigma}_{\geq 0}$, we can extend this to well-defined map

$$X_+(\bar A_H)\to X_+(A_H):\alpha\mapsto\widetilde\alpha.$$

Definition 3.11. Let discrete subgroup $\Lambda\le H$ such that $H=\Lambda\cdot K_H$. Then the corresponding Gate set is:

$$S_\Lambda:=\{\gamma\in\bar\Lambda\ :\ \bar a_\gamma\in\Sigma\}.$$

The corresponding lifted Gate set is

$$\widetilde S_\Lambda=\{\gamma\in\Lambda\ :\ a_\gamma\in\widetilde\Sigma\}.$$

Finally, let

$$C_\Lambda=\widetilde C_\Lambda:=\Lambda\cap K_H$$

(this is the same as its image in $\bar\Lambda$ since $\Lambda\cap K_H\cap Z_H^{\mathrm{spl}}=1$).

Since $\Lambda\cap K_H\cap Z_H^{\mathrm{spl}}=1$, the map $\Lambda\to\bar\Lambda$ restricted to a fiber of $\gamma\mapsto a_\gamma$ is a bijection. In particular, we can choose a bijection

$$\Sigma\to\widetilde\Sigma: s_i\mapsto\widetilde s_i.$$

such that $\widetilde{\bar a}_{s_i}=a_{\widetilde s_i}$. Then:

Proposition 3.12. *In the notation of definition 3.11,*

(1) All $\gamma\in\bar\Lambda$ are of the form $cs_1\cdots s_k$ for $c\in C_\Lambda$ and $s_i\in S_\Lambda$.

(2) Let $S_\Lambda^{[\ell]}=C_\Lambda S_\Lambda^{(\ell)}$ be $\gamma\in\bar\Lambda$ for which the minimum possible k in an expansion as above is exactly ℓ. Then,

$$S_\Lambda^{(\ell)}=\{\gamma\in\bar\Lambda:\|\bar a_\gamma\|'=\ell\},$$

where $\|\bar a_\gamma\|$ is the modified Cartan norm.

(3) Let $\widetilde{S}_\Lambda^{[\ell]} = \{c\widetilde{s}_1 \cdots \widetilde{s}_\ell \ : \ cs_1 \cdots s_\ell \in S_\Lambda^{(\ell)}\}$*. Then*

$$\widetilde{S}_\Lambda^{[\ell]} = \{\gamma \in \bar{\Lambda} : a_\gamma = \widetilde{\alpha} \text{ for some } \|\alpha\|' = \ell\}.$$

Proof. Let $\gamma \in \Gamma_z$ and pick a common apartment $\mathcal{A}$ of v_0 and γv_0. Choose an isomorphism $\mathcal{A} \simeq X_*(\bar{A}_H)$ by setting $v_0 = 0$ and orienting it in some way such that γv_0 corresponds to the point $v_1 = \bar{a}_\gamma \in X_*(\bar{A}_H)$.

If $v_1 \neq 0$, then there is $0 \preceq v_1' \in X_*(\bar{A}_H)$ such that $v_1 - v_1' \in \Sigma$. Since $H = \Lambda \cdot K_H$, without loss of generality, there is $\gamma' \in \bar{\Lambda}$ such that $v_1 = \gamma'.v_0$. Then, $\bar{a}_{(\gamma')^{-1}\gamma} = \bar{a}_\gamma - \bar{a}_{\gamma'} \in \Sigma$. In total, we have produced $s \in S_\Lambda$ such that $\gamma' s = \gamma$, $\gamma' \in \Lambda$, and $\bar{a}_{\gamma'} \prec \bar{a}_\gamma$.

Inductively reducing $\bar{a}_{\gamma'}$ further, we can find $s_1, \ldots, s_k \in S_\Lambda$ such that $\gamma = s_1 \cdots s_k$, $\gamma(s_1 \ldots s_k)^{-1} \in \bar{\Lambda}$, and $\bar{a}_{\gamma(s_1 \ldots s_k)^{-1}} = 0$. Since $a_x = 0$ if and only if $x \in K_H(Z_H \cap A_H)^0$, this implies that $\gamma(s_1 \ldots s_k)^{-1} \in C_\Lambda$ which proves (1)

For (2), the induction for (1) and equation (3.2) also gives that the claimed set is contained in $S_\Lambda^{(\ell)}$. Containment the other way follows from the triangle inequality $a_{xy} \preceq a_x + a_y$.

For (3), the inequality-squeezing for (2) forces that if $\gamma = cs_1 \cdots s_\ell \in S_\Lambda^{[\ell]}$, then $\bar{a}_\gamma = \bar{a}_{s_1} + \cdots + \bar{a}_{s_\ell}$. Therefore, $a_\gamma = a_{\widetilde{s}_1} + \cdots + a_{\widetilde{s}_\ell} = \widetilde{\bar{a}}_\gamma$. □

Remark 3.13. Proposition 3.12(1) holds even if we only allow the s_i to be chosen from amongst a single distinguished element in each coset $C_\Lambda \backslash \Lambda$.

4 Golden adelic subgroups

In this section we introduce the notions of golden and super-golden adelic groups, use them to construct gate sets of $U(n)$, and prove that these gate sets satisfy the first three properties of Definition 1.1; namely, growth, navigation and approximation (Subsection 4.3). We end this section by giving examples of golden and super-golden groups for $n = 4$ (Subsection 4.4) and noting that all previous constructions comes from such golden and super-golden groups ([Sar15b, PS18] for $n = 2$ and [EP18] for $n = 3$).

4.1 Definition and First Properties

Let $G = U_n^{E,H}$, $SU_n^{E,H}$ or $PU_n^{E,H}$.

Let $K' = \prod_\ell K_\ell' \leq G(\widehat{\mathcal{O}})$ be a finite index subgroup. Denote by $\mathrm{Ram}(K') \supseteq \mathrm{Ram}(G)$ the finite set of primes ℓ such that $K_\ell' \neq K_\ell$. For any $\ell \in \mathrm{Ram}(K')$ let $n_\ell(K') \in \mathbb{N}$ be the minimal n such that K_ℓ' contains the level ϖ_ℓ^n principal congruence subgroup

$$K_\ell(\varpi_\ell^n) = \{g \in K_\ell \ : \ g \equiv I \mod \varpi_\ell^n\}$$

and let $q(K') = \prod_{\ell \in \mathrm{Ram}(K')} \varpi_\ell^{n_\ell(K')} \in \mathcal{O}$, be the conductor of K'.

Definition 4.1. The adelic subgroup, $K' \leq G(\widehat{\mathcal{O}})$, is said to be golden if

$$G(\mathbb{A}) = G(F) \cdot G_\infty K' \qquad \text{and} \qquad G(F) \cap K' = \{1\}.$$

We say it is golden at $\mathfrak{p}$ if $\mathfrak{p} \notin \mathrm{Ram}(K')$. We say it is super-golden at $\mathfrak{p}$ if $\mathfrak{p} \in \mathrm{Ram}(K')$ such that $K_\mathfrak{p}'$ is a proper parahoric subgroup of $K_\mathfrak{p}$ (i.e. stabilizer of a facet τ of dimension > 0 in the building).

Beware that the first condition is much harder to satisfy—in particular, it requires G and therefore E to have class number one. Remark 2.10 therefore makes golden adelic subgroups quite rare.

Given $K' \leq G(\widehat{\mathcal{O}})$ and prime $\mathfrak{p}$, define

$$\Lambda_{\mathfrak{p}} := \Lambda_{\mathfrak{p}}^{K'} := G(F) \cap (K')^{\mathfrak{p}}.$$

As the key properties we will use:

Lemma 4.2. *If $K' \leq G(\widehat{\mathcal{O}})$ is a golden adelic group, then:*

(1) We have an isomorphism of G_∞-sets

$$G_\infty \to G(F) \backslash G(\mathbb{A}) / K', \qquad g \mapsto G(F)(g, 1, 1, \ldots) K'$$

inducing the following isomorphism of G_∞-representations

$$L^2(G_\infty) \cong L^2(G(F) \backslash G(\mathbb{A}))^{K'}.$$

(2) For primes $\mathfrak{p}$, we have an isomorphism of $G_\infty \times G_{\mathfrak{p}}$-sets:

$$\Lambda_{\mathfrak{p}} \backslash G_\infty \times G_{\mathfrak{p}} / K'_{\mathfrak{p}} \to G(F) \backslash G(\mathbb{A}) / K', \qquad (g_\infty, g_{\mathfrak{p}}) \mapsto G(F)(g_\infty, g_{\mathfrak{p}}, 1, \ldots,) K'$$

inducing the following isomorphism of $G_\infty \times G_{\mathfrak{p}}$-representations

$$L^2(\Lambda_{\mathfrak{p}} \backslash G_\infty \times G_{\mathfrak{p}})^{K'_{\mathfrak{p}}} \cong L^2(G(F) \backslash G(\mathbb{A}))^{K'}.$$

Proof. For both claims, surjectivity follows since $G(\mathbb{A}) = G(F) G_\infty K'$ and injectivity since $G(F) \cap K' = 1$. □

Lemma 4.3. *Let $K' \leq G(\widehat{\mathcal{O}})$ be a golden adelic subgroup and let $\mathfrak{p}$ be a prime. If K' is golden at $\mathfrak{p}$ (resp. super-golden stabilizing τ), then:*

(1) $\Lambda_{\mathfrak{p}}$ acts simply transitively on $G_{\mathfrak{p}} / K'_{\mathfrak{p}}$,

(2) $\bar{\Lambda}_{\mathfrak{p}}$ acts simply transitively on $G_{\mathfrak{p}} v_0$ (resp. $G_{\mathfrak{p}} \tau$).

(3) $\Lambda_{\mathfrak{p}} \cap Z_{G_{\mathfrak{p}}}^{\mathrm{spl}}$ acts simply transitively on the fibers of $\Lambda_{\mathfrak{p}} \to \bar{\Lambda}_{\mathfrak{p}}$.

Proof. Since $K'_{\mathfrak{p}}$ is a stabilizer of v_0 or τ, respectively, (1) follows from the set equalities

$$G_{\mathfrak{p}} = \Lambda_{\mathfrak{p}} \cdot K'_{\mathfrak{p}} \qquad \text{and} \qquad \Lambda_{\mathfrak{p}} \cap K'_{\mathfrak{p}} = \{1\},$$

which in turn follow from the assumption on the group K' (similarly to lemma 2.5).

(2) follows since τ is has stabilizer exactly $K'_{\mathfrak{p}} Z_{G_{\mathfrak{p}}}^{\mathrm{spl}}$ and (3) follows from (1) and (2). □

4.2 Golden Gate Sets

We now discuss how to construct gate sets from golden adelic subgroups. The first assumption from §3.3 holds since we only consider $G_v = GL_n$ or U_n.

4.2.1 Golden Case

Definition 4.4. Let K' be a golden arithmetic subgroup of $U_n^{E,H}$ that is golden at $\mathfrak{p}$. Then define the gate set and lifted gate set

$$S_{\mathfrak{p}} := S_{\mathfrak{p}}^{K'} := S_{\Lambda_{\mathfrak{p}}^{K'}}, \qquad \widetilde{S}_{\mathfrak{p}} := \widetilde{S}_{\mathfrak{p}}^{K'} := \widetilde{S}_{\Lambda_{\mathfrak{p}}^{K'}}$$

as in Definition 3.11. Note that since K is golden, the lifts $\widetilde{s}_i$ are uniquely defined for any choice of $\widetilde{\Sigma}$.

Proposition 3.12 immediately gives that

- $S_{\mathfrak{p}}$ generates $\bar{\Lambda}_{\mathfrak{p}}$,
- $S_{\mathfrak{p}}^{(\ell)} = \{\gamma \in \bar{\Lambda}_{\mathfrak{p}} \ : \ \|a_\gamma\|'_{G_{\mathfrak{p}}} = \ell\}$.

Example 4.5. If $\mathfrak{p}_{\mathrm{spl}}$ is split, then by example 3.9, $\|\cdot\|'_{G_{\mathfrak{p}_{\mathrm{spl}}}}$ is the graph distance on the 1-skeleton of $\mathcal{B}$ so

$$S_{\mathfrak{p}_{\mathrm{spl}}} = \{\gamma \in \bar{\Lambda}_{\mathfrak{p}_{\mathrm{spl}}} \ : \ \mathrm{dist}(\gamma.v_0, v_0) = 1\}.$$

If $\mathfrak{p}_{\mathrm{ns}}$ in non-split, then by example 3.10, $\|\cdot\|'_{G_{\mathfrak{p}_{\mathrm{ns}}}}$ is half the graph distance on the 1-skeleton of $\mathcal{B}$. In addition, $Z^{\mathrm{spl}}_{G_{\mathfrak{p}_{\mathrm{ns}}}} = 1$ so we can ignore center technicalities. Therefore,

$$S_{\mathfrak{p}_{\mathrm{ns}}} = \widetilde{S}_{\mathfrak{p}_{\mathrm{ns}}} = \{\gamma \in \Lambda_{\mathfrak{p}_{\mathrm{ns}}} \ : \ \mathrm{dist}(\gamma.v_0, v_0) = 2\}.$$

4.2.2 Super-golden Case

Moving on to the super-golden case, first note that any parahoric $K'_{\mathfrak{p}}$, $K'_{\mathfrak{p}} \cap Z^{\mathrm{spl}}_{G_{\mathfrak{p}}} = K_{\mathfrak{p}} \cap Z^{\mathrm{spl}}_{G_{\mathfrak{p}}}$, so if K' is super-golden at $\mathfrak{p}$, then $\Lambda_{\mathfrak{p}}$ automatically satisfies the second assumption of Section 3.3.

If $v_1 \in \mathcal{B}$ is in the $G_{\mathfrak{p}}$-orbit of v_0 and $\mathcal{A} \cong X_*(\bar{A}_H)$ is an apartment containing v_0 and v_1, then there is a finite order automorphism r_{v_1} of $\mathcal{A}$ realized by an element of H acting on $\mathcal{A}$ such that $r_{v_1}(v_0) = v_1$ (this is a rotation of a chamber when $\mathfrak{p}$ is split and a reflection when $\mathfrak{p}$ is inert). We also restrict our choice of τ:

Assumption 4.6. *If $\mathfrak{p}$ is non-split, then τ is a full chamber.*

Definition 4.7. Let K' be a golden arithmetic subgroup of $U_n^{E,H}$ that is super-golden at $\mathfrak{p}$ stabilizing facet τ. Then, following definition 3.11, set

$$C_{\mathfrak{p}} := C_{\mathfrak{p}}^{K'} := C_{\Lambda_{\mathfrak{p}}^{K'}}.$$

For any set of representatives v_i of the orbits $C_{\mathfrak{p}} \backslash \{v \in Gv_0 \ : \ \|v - v_0\|'_{G_{\mathfrak{p}}} = 1\}$, define

$$T_{\mathfrak{p}} := T_{\mathfrak{p}}^{K'} := \{\gamma \in \bar{\Lambda} \ : \ \gamma\tau = r_{v_i}(\tau) \text{ for some } i\}.$$

Finally, let

$$S_{\mathfrak{p}} := S_{\mathfrak{p}}^{K'} := S_{\Lambda_{\mathfrak{p}}^{K'}}, \qquad \widetilde{S}_{\mathfrak{p}} := \widetilde{S}_{\mathfrak{p}}^{K'} := \widetilde{S}_{\Lambda_{\mathfrak{p}}^{K'}}.$$

and, given $\widetilde{\Sigma}$, make the unique (by lemma 4.3) choice of lifts $\widetilde{s}_i$ such that $\widetilde{s}_i\tau = s_i\tau$.

As some quick properties:

Lemma 4.8. *The following hold*

(1) $C_{\mathfrak{p}}$ is a finite group.

(2) For each v_i, there is exactly one $t_i \in T_{\mathfrak{p}}$ with $t_i\tau = r_{v_i}(\tau)$ by lemma 4.3. By assumption 4.6 on τ, there is some minimal $o_i > 1$ such that that $t_i^{o_i}\tau = \tau$:

- *If $\mathfrak{p}$ is split, t_i stabilizes a chamber containing τ setwise, so $o_i|n$.*
- *If $\mathfrak{p}$ is non-split, t_i fixes a vertex $x \in \tau$, so o_i divides the order of automorphism group of the spherical building at x.*

(3) Each $t_i \in T_{\mathfrak{p}}$ has order o_i by lemma 4.3.

(4) $S_{\mathfrak{p}} = \{c_1c_2tc_2^{-1} \ : \ c_1, c_2 \in C_{\mathfrak{p}}, t \in T_{\mathfrak{p}}\}$. In particular, $S_{\mathfrak{p}} \cup C_{\mathfrak{p}}$ is generated by finite-order elements.

(5) By lemma 4.3, there is unique set $\widetilde{T}_{\mathfrak{p}}$ of lifts $\widetilde{t}_i \in \Lambda_{\mathfrak{p}}$ for each $t_i \in T_{\mathfrak{p}}$ such that $\widetilde{S}_{\mathfrak{p}} = \{c_1c_2\widetilde{t}c_2^{-1} \ : \ c_1, c_2 \in C_{\mathfrak{p}}, t \in T_{\mathfrak{p}}\}$.

We reduce to a set containing a choice of coset representatives for $C_{\mathfrak{p}}\backslash S_{\mathfrak{p}}$:

$$ {}^0S_{\mathfrak{p}} := \{ctc^{-1} \ : \ c \in C_{\mathfrak{p}}, t \in T_{\mathfrak{p}}\}, \qquad {}^0\widetilde{S}_{\mathfrak{p}} := \{c\widetilde{t}c^{-1} \ : \ c \in C_{\mathfrak{p}}, t \in T_{\mathfrak{p}}\}. $$

Proposition 3.12 and remark 3.13 then give that

- ${}^0S_{\mathfrak{p}} \cup C_{\mathfrak{p}}$ generates $\bar{\Lambda}_{\mathfrak{p}}$,
- $C_{\mathfrak{p}}{}^0S_{\mathfrak{p}}^{(\ell)} = \{\gamma \in \bar{\Lambda}_{\mathfrak{p}} \ : \ \|a_\gamma\|'_{G_{\mathfrak{p}}} = \ell\}$.

4.2.3 Sizes

To understand the sizes of gate sets $S_{\mathfrak{p}}$ and $S_{\mathfrak{p}}^{(\ell)}$ for golden K', we recall the following well-known fact about buildings:

Proposition 4.9. *The degree $\deg(v_0)$ of a hyperspecial vertex in the 1-skeleton of the building $\mathcal{B}$ for group $H/\mathbb{F}$ is the number of maximal proper parabolics in $H(k_{\mathbb{F}})$.*

We also recall

Proposition 4.10 ([Cas95, Prop. 1.5.2])**.** *Let $\lambda \in X_+(\bar{A})$. Let B_λ be the number of points $v \in \mathcal{B}$ such that $v - v_0 = \lambda$ for the invariant $v - v_0 \in X_+(\bar{A}_H)$. Then*

$$ C_1 q_{\mathbb{F}}^{\langle \lambda, 2\rho_G\rangle} \leq B_\lambda \leq C_2 q_{\mathbb{F}}^{\langle \lambda, 2\rho_G\rangle} $$

for some constants C_1, C_2 depending on H and where ρ_G is the half-sum of positive roots.

Corollary 4.11. *Let $K' < G^\infty$ be golden at $\mathfrak{p}$ (resp. super-golden) and $S_{\mathfrak{p}}$ the corresponding gate set (resp ${}^0S_{\mathfrak{p}}$). Let $M = \max_{\alpha\in\Sigma}\langle \alpha, 2\rho_{G_{\mathfrak{p}}}\rangle$. Then $|S_{\mathfrak{p}}^{(\ell)}| \asymp q_{\mathfrak{p}}^{M\ell}$ (resp. $|C_{\mathfrak{p}}{}^0S_{\mathfrak{p}}^{(\ell)}|$).*

Proof. By lemma 4.3 and the versions of Proposition 3.12 as used above,

$$ |S_{\mathfrak{p}}^{(\ell)}| = |\{v \in gv_0 \subseteq \mathcal{B} \ : \ \|v - v_0\|' = \ell\}| $$

(resp. $|C_{\mathfrak{p}}{}^0S_{\mathfrak{p}}^{(\ell)}|$ is $|C_{\mathfrak{p}}|$ times this). By Proposition 4.10, this is

$$ \asymp \sum_{\substack{\alpha \in X_+(\bar{A}_{G_{\mathfrak{p}}}) \\ \|\alpha\|' = \ell}} q^{\langle \alpha, 2\rho_G\rangle} \asymp q^{\langle \ell\alpha_0, 2\rho_G\rangle}, $$

where α_0 realizes the maximum value of $\langle \alpha, 2\rho_G\rangle$. $\square$

Example 4.12. If $\mathfrak{p}$ is split. then $G_\mathfrak{p} = GL_n/F_\mathfrak{p}$. If K' is golden at $\mathfrak{p}$, then $|S_\mathfrak{p}| = \deg(v_0)$. This is the sum of the sizes of the Grassmanians $G(n,k)$ for $1 \le k \le n-1$ over the residue field of $F_\mathfrak{p}$ giving

$$|S_\mathfrak{p}| = \sum_{i=1}^{n-1} \binom{n}{k}_{q_\mathfrak{p}} := \sum_{i=1}^{n-1} \frac{(1-q_\mathfrak{p}^n)\cdots(1-q_\mathfrak{p}^{n-k+1})}{(1-q_\mathfrak{p})\cdots(1-q_\mathfrak{p}^k)} = (1+\mathbf{1}_{n \text{ odd}})q_\mathfrak{p}^{\lfloor n^2/4\rfloor} + O(q_\mathfrak{p}^{\lfloor n^2/4\rfloor-1}).$$

In comparison, $2\rho_G = (n-1, n-3, \dots, -n+1)$, which has maximized pairing with the middle fundamental weight in Σ. This gives

$$|S_\mathfrak{p}^{(\ell)}| \asymp q_\mathfrak{p}^{\lfloor n^2/4\rfloor \ell}.$$

4.3 Growth, Navigation, and Approximation

Next we summarize how the gate sets that correspond to golden adelic groups satisfy the last three properties in the Definition 1.1 of golden gate sets; namely, growth, navigation and approximation.

Theorem 4.13. *Let $G = U_n^{E,H}$, let $K' \le G(\widehat{\mathcal{O}})$ be a golden adelic subgroup, let $\mathfrak{p}$ be a prime at which K' is golden (resp. super-golden satisfying assumption 4.6), and let $S_\mathfrak{p}$ (resp. ${}^0S_\mathfrak{p} \cup C_\mathfrak{p}$ which is generated by elements of finite order by 4.8(4)) be the gate set corresponding to K' as in definition 4.4 (resp. 4.7). Then $S_\mathfrak{p}$ satisfies the following properties:*

(1) Growth: $|S_\mathfrak{p}^{(\ell)}|$ grows exponentially in ℓ.

(2) Navigation: The group generated by $S_\mathfrak{p}$ is $\bar{\Lambda}_p := \bar{\Lambda}_\mathfrak{p}^{K'}$ and it has the following efficient solution for its word problem: Given $1 \ne g \in \bar{\Lambda}_\mathfrak{p}$, find an element $s \in S_\mathfrak{p}$ (resp. ${}^0S_\mathfrak{p}$) such that $a_{sg} \prec a_g$, and proceed by induction on a_g. The algorithm will terminate in $O(|S_\mathfrak{p}| \cdot \|a_g\|'))$ time (resp $O(|C_\mathfrak{p}| \cdot |T_\mathfrak{p}| \cdot \|a_g\|')$ time) which (for a fixed $\mathfrak{p}$) is polynomial in the input g.

(3) Approximation: If $\mathcal{O}_F$ is Euclidean, there exists a heuristic polynomial algorithm such that given $g \in PU(n)$, $\varepsilon > 0$, and $\ell \in \mathbb{N}$, if $B(g,\varepsilon) \cap S^{(\ell)} \ne \emptyset$, then the algorithm outputs an element in $B(g,\varepsilon) \cap S^{(\ell \cdot N)}$, where $N = \dim PU(n)$.

Proof. In order:
Growth:

This follows from Corollary 4.11.

Navigation:

Both termination and run time follow by the proof of 3.12(1). By uniqueness of the Cartan decomposition 3.1, we can compute a_g at each step by the integer normal form algorithm on $G_v = GL_n(F_v)$ when v is split or on the bigger group $GL_n(E_w) \supset G_v$ when w lies over v and v is unramified non-split.

Approximation:

This follows as a consequence (Corollary 4.15) of the algorithm of Ross and Selinger [RS15, PS18] for approximating elements in $U(2)$ by matrices with integer coefficients. We sketch this in the subsequent lemmas.

To apply these lemmas, by examples 3.9 and 3.10, we can find i so that $S_\mathfrak{p}^{(k)} \subseteq \varpi_\mathfrak{p}^{-i} U_n^{E/H}(\mathcal{O}_F)$ if and only if $k \le \ell$. Furthermore, by Gram-Schmidt, we can find a diagonal H' such that $U_n^{E,H}/F \cong U_n^{E,H'}/F$ so we can find $r \in \mathbb{N}$ with only ramified factors such that $U_n^{E/H}(\mathcal{O}_F) \subseteq r^{-1}M_n(\mathcal{O}_E) \cap U^{H'}(n)$. These two together provide the inputs m, H' in the lemmas.

Since $\Gamma_{\mathfrak{p}}^{(\ell)}$ is then a finite fraction of $\varpi_{\mathfrak{p}}^{-i} r^{-1} M_n(\mathcal{O}_E) \cap U^{H'}(n)$, sampling enough outputs will heuristically generate one in $\Gamma_{\mathfrak{p}}^{(\ell)}$ with high probability. □

Theorem 4.14 ([RS15], [PS18, Thm 2.6, §2.3]). *Let K be a totally real number field such that its ring of integers $\mathcal{O}_K$ is Euclidean and let $d \in \mathbb{N}$. Let H' be a diagonal definite Hermitian form for $K[\sqrt{-d}]/K$. Then there is a randomized, heuristic efficient algorithm, such that given $\varepsilon > 0$, $k, m \in \mathbb{N}$, and a diagonal unitary matrix*

$$g = \begin{pmatrix} x_1 + ix_2 & \\ & x_1 - ix_2 \end{pmatrix} \in U^{H'}(2),$$

it finds up to k different

$$h \in M_2(\mathcal{O}_K[\sqrt{-d}]) \quad : \quad \tilde{h} = m^{\frac{-1}{2}} \cdot h \in U^{H'}(2) \quad , \quad 1 - \frac{Trace(g^* \cdot \tilde{h})}{2} < \varepsilon.$$

if that many exist and all such h otherwise.

Proof. The original statement of the Theorem 2.6 in [PS18] is with $d = 1$ and $H = I$, but their arguments works just as well for any fixed d by modifying the lattices studied and any fixed diagonal H' by the discussion after (3.13) therein. □

The above algorithm can be generalized to arbitrary elements of higher dimensional unitary groups:

Corollary 4.15. *K be a totally real number field such that its ring of integers $\mathcal{O}_K$ is Euclidean and let $d \in \mathbb{N}$. Let H' be a diagonal definite Hermitian form for $K[\sqrt{-d}]/K$. Then, there is a heuristic efficient algorithm such that given $\varepsilon > 0$, $k, m \in \mathbb{N}$, and a unitary matrix, $g \in U(n)$, it finds k different tuples of matrices*

$$h_1, \dots, h_N \in M_n(\mathbb{Z}[\sqrt{-d}]) \quad : \quad \tilde{h}_i = m^{\frac{-1}{2}} \cdot h_i \in U(n) \quad , \quad 1 - \frac{|\,Trace(g^* \cdot (\tilde{h}_1 \cdots \tilde{h}_N))|}{n} < \varepsilon$$

if that many exist and all such tuples otherwise.

Proof. The claim follows from the fact that for each, $g \in U(n)$, there exists N unitary matrices, $g_1, \dots, g_N$ which are 1-parameter elements (namely, $g_i \in H_i \leq U(n)$, where $H_i \cong U(1)$), such that $g = g_1 \cdot \ldots \cdot g_N$ (see also [NC11] section 4.5.1, for a similar though longer decomposition and the discussion after (3.13) in [PS18]). Using the fact that

$$d : U(n) \times U(n) \to \mathbb{R}_{\geq 0} \quad , \quad d(g,h) = 1 - \frac{|\mathrm{Trace}(g^* \cdot h)|}{n},$$

is a bi-invariant metric on $U(n)$, we get that it is enough to approximate each g_i individually, which is done by Theorem 4.14. □

4.4 Explicit Construction of Golden Adelic Groups

Let us present several examples of golden and super-golden adelic groups.

Example 4.16. In dimension $n = 2$, several constructions were given in [PS18, Sar15b]. Let us present just one example of a golden adelic group which is super-golden at $p = 3$:

$$G = U_2^{\mathbb{Q}[\sqrt{-1}],I}/\mathbb{Q}, \quad K' = \left\{ g \in G\left(\widehat{\mathbb{Z}}\right) \mid g \equiv I \mod 2 \right\}.$$

Example 4.17. In dimension $n = 3$, several constructions were given in [EP18, BEF+18]. Here are two such examples of golden adelic groups:

$$G = U_3^{\mathbb{Q}[\sqrt{-1}],I}/\mathbb{Q}, \quad K' = \left\{ g \in G\left(\widehat{\mathbb{Z}}\right) \mid \forall i,\ g_{i,i} \equiv 1 \mod 2+2i \right\},$$

$$G = U_3^{\mathbb{Q}[\sqrt{-3}],I}/\mathbb{Q}, \quad K' = \left\{ g \in G\left(\widehat{\mathbb{Z}}\right) \mid \forall i,\ g_{i,i} \equiv 1 \mod 2+2i \right\}.$$

Furthermore, as a consequence of the work of Mumford [Mum79] (see [BEF+18] for the details), one get that for the 3×3 Hermitian positive definite matrix

$$H = \begin{pmatrix} 3 & \lambda & \lambda \\ \bar{\lambda} & 3 & \lambda \\ \bar{\lambda} & \bar{\lambda} & 3 \end{pmatrix} \quad \text{where} \quad \lambda = \frac{1+\sqrt{-7}}{2},$$

the following is a golden adelic group which is super-golden at $p = 2$:

$$G = U_3^{\mathbb{Q}[\sqrt{-7}],H}/\mathbb{Q}, \quad K' = \left\{ g \in G\left(\widehat{\mathbb{Z}}\right) \mid \forall i > j,\ g_{i,j} \equiv 0 \mod 2 \right\}.$$

Below we present our constructions of a golden groups. Proposition 4.18 produces the example in Theorem 1.4.

Proposition 4.18. *For the 4×4 Hermitian positive definite matrix,*

$$H = 2 \cdot \begin{pmatrix} I & A \\ -A & I \end{pmatrix} \quad \text{where} \quad A = \frac{\sqrt{-3}}{3} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

the following is a golden adelic group

$$G = U_4^{\mathbb{Q}[\sqrt{-3}],H}/\mathbb{Q}, \quad K' = \left\{ g \in G\left(\widehat{\mathbb{Z}}\right) \mid g \equiv I \mod 2 \right\}.$$

Proof. A direct calculation with Bernoulli numbers shows that

$$R(G) = 155560^{-1} = |G(\mathbb{Z})|^{-1},$$

so by lemma 2.9, G has class number one.

Since K' is the kernel of the reduction map $r_2 : G(\widehat{\mathbb{Z}}) \to G(\mathbb{Z}/2)$, it therefore suffices to show that $r_2|_{G(\mathbb{Z})}$ is an isomorphism. The prime 2 is unramified in G, hence $G(\mathbb{Z}/2\mathbb{Z}) \cong U_4(\mathbb{F}_2)$ and $|U_4(\mathbb{F}_2)| = 155560 = |G(\mathbb{Z})|$. A computer check finally shows that $G(\mathbb{Z}) \cap \ker r_2 = \{1\}$. □

We end this section by discussing the well studied Clifford+T gates and their generalizations from the point of view of arithmetic unitary groups and golden adelic groups (see [Sar15b] and the references there).

The classical Clifford+T gates are a finite set of unitary 2×2 matrices with coefficients in the ring $\mathbb{Z}\left[\sqrt{-1}, \sqrt{2}^{\pm 1}\right]$. By [KMM13], the set of elements in $U(2)$ that are synthesisable (i.e. generated by matrix multiplication and tensoring) by the Clifford+T gates is precisely the full 2-arithmetic group Λ of unitary 2×2 matrices with coefficients in the ring $\mathbb{Z}[\sqrt{-1}, \sqrt{2}^{\pm 1}]$; i.e. $\Lambda = G(\mathbb{Z}[\sqrt{2}^{\pm 1}])$ where

$$G = U_2^{\mathbb{Q}[\sqrt{2},\sqrt{-1}],I}/\mathbb{Q}[\sqrt{2}].$$

The group G has class number one and moreover gives rise to super golden gate sets (see Section 4.1.3 in [PS18]).

The multiqubit Clifford+T gates are a finite set of unitary $2^n \times 2^n$ matrices with coefficients in the ring $\mathbb{Z}[\sqrt{-1}, \sqrt{2}^{\pm 1}]$. By [GS13], the group of element in $U(2^n)$ that are synthesisable by the multiqubit Clifford+T gates is $\Lambda = G(\mathbb{Z}[\sqrt{2}^{\pm 1}])$, where

$$G = U_{2^n}^{\mathbb{Q}[\sqrt{2},\sqrt{-1}],I}/\mathbb{Q}[\sqrt{2}].$$

However, for $n \geq 2$, the group G is not of class number one—in particular Λ does not act transitively on the special vertices of the corresponding Bruhat-Tits building of $G(\mathbb{Q}[\sqrt{2}]_{\sqrt{2}})$.

The Clifford+cyclotomic gates are a finite set of unitary 2×2 matrices with coefficients in the ring $R_m = \mathbb{Z}[\zeta_m]\left[\frac{1}{2}\right]$, where $\zeta_m = e^{\frac{2\pi i}{m}}$, $m \in \mathbb{N}$. These matrices sit inside the full 2-arithmetic subgroup of

$$G = U_2^{\mathbb{Q}[\zeta_m],I}/\mathbb{Q}[\zeta_m + \zeta_m^{-1}].$$

By [FGKM15, IJK+19], the group of elements in $U(2)$ that are synthesizable by the Clifford+cyclotomic gates is a 2-arithmetic subgroup of G if and only if $m = 4, 8, 12, 16$ or 24. Note that being a 2-arithmetic subgroup only implies that the class number is finite, not necessarily that the class number is one (it is one for $m = 4, 8$).

5 Automorphic Representations Background

In this section we recall some basic facts and notations concerning automorphic representations.

5.1 Automorphic representations

Throughout this section, F is a number field with ring of integers $\mathcal{O} = \mathcal{O}_F$ and adele ring $\mathbb{A} = \mathbb{A}_F$. Let v denote a place of F, let F_v be the v-completion of F, and, when v is finite, let $\mathcal{O}_v$ be the ring of integers of F_v with uniformizer ϖ_v and order of residue field $q_v := |\mathcal{O}_v/\mathfrak{p}_v\mathcal{O}_v|$. We use standard upper and lower-index notation: if S is a finite set of places and X a variety over F, then $X_S := X(F_S)$ including only the places in S and $X^S := X(\mathbb{A}^S)$ including all places except those in S.

Let G be a connected reductive group over F. For simplicity, assume that the maximal split torus in the center of the real group G_∞ is trivial so that $G(F)\backslash G(\mathbb{A})$ has finite volume. Fix a k-embedding $G \hookrightarrow GL_n(k)$. For any place v, denote $G_v := G(k_v)$. When v is finite, denote $K_v = G(k_v) \cap GL_n(\mathcal{O}_v)$ and, for any $m \in \mathbb{N}$, denote $K_n(\mathfrak{p}_v^m) := \ker(K_v \to GL_n(\mathcal{O}_v/\mathfrak{p}_v^m\mathcal{O}_v))$.

Consider the right regular $G(\mathbb{A})$-representation on $L^2(G(F)\backslash G(\mathbb{A}))$. An F-automorphic representation of G is an irreducible $G(\mathbb{A})$-representation π which is weakly contained in $L^2(G(F)\backslash G(\mathbb{A}))$ and whose central character is unitary. Denote by $\mathcal{AR}(G)$ the set of F-automorphic representations of G. Consider the decomposition of $\mathcal{AR}(G)$ into its cuspidal $\mathcal{AR}_{cusp}(G)$, residual $\mathcal{AR}_{res}(G)$, discrete $\mathcal{AR}_{\text{disc}}(G)$, and continuous $\mathcal{AR}_{cont}(G)$ parts:

$$\mathcal{AR}(G) = \mathcal{AR}_{cusp}(G) \oplus \mathcal{AR}_{res}(G) \oplus \mathcal{AR}_{cont}(G),$$
$$\mathcal{AR}_{\text{disc}}(G) = \mathcal{AR}_{cusp}(G) \oplus \mathcal{AR}_{res}(G).$$

Any $\pi \in \mathcal{AR}(G)$ decomposes as a restricted tensor product $\pi = \otimes'_v \pi_v$, where π_v, called the *local-factor* of π at v, is an irreducible admissible G_v-representation (cf. [Fla79]). Let $\sigma(\pi_v)$ be the infimum over $\sigma \geq 2$, such that each K_v-finite matrix coefficient of π_v is in $L^{\sigma+\epsilon}(G_v)$ for any $\epsilon > 0$. Say that π is tempered at v if $\sigma(\pi_v) = 2$. The Generalized Ramanujan Conjecture for $G = GL_n$ states the following (see [Sar05]):

Conjecture 5.1. *(GRC) Let F be a number field, $N \in \mathbb{N}$, and let $\pi \in \mathcal{AR}_{cusp}(GL_n)$. Then the local component π_v is tempered at every place v of F.*

The conjecture is open for any $N \geq 2$ and any number field F. However, there are special cases of cuspidal automorphic representations for which it is a theorem:

Definition 5.2. Let $\pi \in \mathcal{AR}(n)$. Say that π is cohomological if, for an Archimdean place v of F, the v-factor of π has an infinitesimal character of a finite dimensional representation. When F is totally real (resp. CM), say that π is F self dual (resp. F-conjugate self-dual) if it is isomorphic to its (resp. F-conjugate of its) contragredient representation $\tilde{\pi}(g) := \pi((g^t)^{-1})$.

Theorem 5.3. [Shi11, Clo13] *Let F be either a totally real or a CM field, $n \in \mathbb{N}$, and let $\pi \in \mathcal{AR}_{cusp}(GL_n)$ be both cohomological and F-conjugate self-dual (see Definition 5.2). Then π_v is tempered at every place v of F.*

Remark 5.4. For $F = \mathbb{Q}$ and $n = 2$, this Theorem was first proved by Eichler [Eic54], for weight $k = 2$, and by Deligne [Del74], for general weights.

According to the Langland functoriality conjecture, for any G with dual $\widehat{G} \leq GL_n$, the set $\mathcal{AR}(G)$ should be encoded in $\mathcal{AR}(GL_n/F)$. We therefore begin by describing the classification of automorphic representations of $G = GL_n$ over a number field F. Fix a global field F and, for any $N \in \mathbb{N}$, denote $\mathcal{AR}(n) := \mathcal{AR}(GL_n/F)$ and $\mathcal{AR}_\star(n) := \mathcal{AR}_\star(GL_n, k)$ for $\star = cusp$, res or $cont$.

Definition 5.5. Define an (unrefined) shape of n to be a sequence of pairs of positive integers, $\square = ((T_1, d_1), \ldots, (T_k, d_k))$, such that $\sum_i T_i \cdot d_i = n$. Let $M_\square := \prod_{i=1}^k GL_{T_i}^{d_i} \leq P_\square \leq GL_n$ be the corresponding Levi (block diagonal) and parabolic (block upper triangular) subgroups of shape $\square$.

Theorem 5.6. [Lan06, MW89] *For any shape of n, $\square = ((T_i, d_i))_{i=1}^k$, there is a map, $I_\square : \prod_{i=1}^k \mathcal{AR}_{cusp}(T_i) \to \mathcal{AR}(n)$, called the automorphic parabolic induction of shape $\square$ and satisfying:*

(1) For any $\pi \in \mathcal{AR}(n)$, there exist a unique shape $\square = ((T_i, d_i))_{i=1}^k$ and a unique (up to order) sequence of cuspidal representations $(\pi_i)_{i=1}^k \in \prod_{i=1}^k \mathcal{AR}_{cusp}(T_i)$ such that $\pi = I_\square((\pi_i)_{i=1}^k)$, in which case π is said to be of shape $\square$. Moreover, π lies in the discrete (resp. cuspidal) part of $\mathcal{AR}(n)$ if and only if $k = 1$ (resp. $k = 1$ and $d_1 = 1$).

(2) Let $\pi = I_\square((\pi_i)_{i=1}^k)$, where $\square = ((T_i, d_i))_{i=1}^k$ and $(\pi_i)_{i=1}^k \in \prod_{i=1}^k \mathcal{AR}_{cusp}(T_i)$. Then, for any place v of k, the local component π_v is a subqoutient of the (unitary) parabolic induction

$$Ind_{P_\square(\mathbb{Q}_v)}^{GL_n(\mathbb{Q}_v)} \left(\bigotimes_{i=1}^k \left(|\cdot|_v^{\frac{d_i-1}{2}} \pi_{i,v} \otimes |\cdot|_v^{\frac{d_i-3}{2}} \pi_{i,v} \otimes \ldots \otimes |\cdot|_v^{\frac{1-d_i}{2}} \pi_{i,v} \right) \right).$$

Definition 5.7. Let $\square = ((T_i, d_i))_{i=1}^k$. Then we shorthand $I_\square(\tau_1, \ldots, \tau_k)$ by the formal expression

$$I_\square(\tau_1, \ldots, \tau_k) =: \bigoplus_{i=1}^k \tau_i[d_i].$$

5.2 Endoscopic Classification and Shapes

The unitary endoscopic classification of [Mok15] extended to non-quasisplit unitary groups in [KMSW14] lets us decompose $\mathcal{AR}_{\text{disc}}(G)$ for our unitary groups $U_n^{E/F,H}$ into pieces corresponding to shapes $\square$ for GL_n. Fix CM quadratic extension E/F.

Definition 5.8. An automorphic representation $\bigoplus_i \tau_i[d_i]$ of $\operatorname{Res}_F^E GL_n$ (this is the same as one of GL_n/E) is said to be *elliptic* if each τ_i is conjugate self-dual and the individual $\tau_i[d_i]$ are all distinct.

Let $\widetilde{\Psi}_{\text{ell}}(n)$ be the set of elliptic automorphic representations ψ of $\operatorname{Res}_F^E GL_n$. These are usually referred to as elliptic (global) Arthur parameters.

Attached to $\operatorname{Res}_F^E GL_n$ are a set of elliptic twisted endoscopic groups $G^* \in \widetilde{\mathcal{E}}_{\text{ell}}^E(n)$ described in [Mok15, §2.1]. These each come with L-embeddings

$$^L G^* \hookrightarrow {}^L\text{Res}_F^E GL_n.$$

We will only care about a specific "simple" element: $U_n^+ \in \widetilde{\mathcal{E}}_{\text{ell}}^E(n)$. The main result of [Mok15] is a decomposition

$$\widetilde{\Psi}_{\text{ell}}(n) = \bigsqcup_{G^* \in \widetilde{\mathcal{E}}_{\text{ell}}^E(n)} \Psi_{\text{ell}}(G^*)$$

and a description of discrete automorphic representations of each $G^* \in \widetilde{\mathcal{E}}_{\text{ell}}^E(n)$ in terms of $\Psi_{\text{ell}}(G^*)$.

The paper [KMSW14] generalizes Mok's classification to "extended pure inner" forms G of each G^*. These are enumerated in [KMSW14, §0.3.3]. In particular, the extended pure inner forms of U_n^+ include all the definite unitary groups $U_n^{E,H}$ we consider here. We recall that if G is an inner form of G^*, then $^L G = {}^L G^*$.

We recall all parts of [KMSW14]'s classification that are needed to explain our results and point readers to [DGG23, §2] for a full summary geared towards trace formula applications.

Theorem 5.9 ([KMSW14] partial summary of main result)**.** *Let $G^* \in \widetilde{\mathcal{E}}_{\text{ell}}^E(n)$ and G an extended pure inner form of G^*. Then*

(1) To each $\psi \in \Psi_{\text{ell}}(G^*)$ there is subset Π_ψ^G called the (global) Arthur packet such that

$$\mathcal{AR}_{\text{disc}}(G) = \bigsqcup_{\psi \in \Psi_{\text{ell}}(G^*)} \Pi_\psi^G.$$

This Π_ψ^G is empty unless ψ satisfies a condition of being relevant as in [KMSW14, §0.4,1.2].

(2) Let $\psi = \bigoplus_{i=1}^k \tau_i[d_i]$ and fix a place V. Through the local Langlands correspondence, each τ_i is associated to a (local) L-parameter

$$\tau_i : WD_{F_v} \to {}^L(\operatorname{Res}_F^E GL_{T_i})_v$$

from the Weil-Deligne group of F_v. Define the local A-parameter

$$\psi_v = \bigoplus_i \tau_i \boxtimes [d_i] : WD_{F_v} \times \mathrm{SL}_2 \to {}^L(\operatorname{Res}_F^E GL_n)_v$$

where $[d_i]$ is the d_i-dimensional representation of SL_2. Then ψ_v factors through $^L G$.

(3) There is a finite set of representations $\Pi^G_{\psi_v}$, depending only on ψ_v, called the local A-packet such that for all $\pi \in \Pi^G_\psi$, we have $\pi_v \in \Pi^G_{\psi_v}$.

(4) If ψ_v is generic (i.e. all $d_i = 1$), then the assignment $\psi_v \to \Pi_{\psi_v}$ satisfies all the desired properties of a local Langlands correspondence. In particular:

(a) If $v|\infty$, then the infinitesimal character of $\pi_v \in \Pi_{\psi_v}$ is the same as that of ψ_v through the embedding $\widehat{G} \hookrightarrow GL_n\mathbb{C} \times GL_n\mathbb{C}$.

(b) $\pi_v \in \Pi_{\psi_v}$ is tempered if and only if ψ_v is (for $v \nmid \infty$, this means that the τ_i are bounded/correspond to unitary supercuspidals).

We can now make our key definition, following [DGG24, §5]:

Definition 5.10. Let $\square$ be a shape for GL_n/E. If $\pi \in \mathcal{AR}_{\mathrm{disc}}(G)$ with parameter $\psi \in \Psi_{\mathrm{ell}}(n)$ such that $\psi \in \square$, then we say π has shape $\square$ or $\pi \in \square$.

Remark 5.11. The definition of shape in [DGG24, §5] is actually a list of triples $\square = ((T_i, d_i, \eta_i))_{i=1}^k$ for some signs η_i. In general, the η_i are needed to determine the G such that $\psi \in \Psi_{\mathrm{ell}}(G^*)$ for all $\psi \in \square$. However, when G^* is a simple twisted endoscopic group (e.g. U_n^+), there is always a unique choice of η_i that determines $\psi \in \Psi_{\mathrm{ell}}(G^*)$.

In our case, we require a priori that $\psi \in \Psi_{\mathrm{ell}}(U_n^+)$ and can therefore ignore the data of the η_i. Nevertheless, the induction in the black-boxed proof of Theorem 7.1 requires keeping track of them.

As in [DGG24], we also associate to shape $\square = ((T_i, d_i, \eta_i))_{i=1}^k$ the group

$$G_F(\square) := \prod_{i=1}^k U^+_{T_i}. \tag{5.1}$$

This is not in general an element of $\widetilde{\mathcal{E}}_{\mathrm{ell}}(n)$ and can be thought of as the smallest group through which $\psi \in \square$ functorially factor through. It will appear in bounds on sizes of families intersect $\square$.

Finally,

Definition 5.12. Let

$$L^2_\square := \bigoplus_{\pi \in \square} m_\pi \pi,$$

where m_π is the multiplicity of π in $L^2(G(F)\backslash G(\mathbb{A}))$, and

$$\mathbb{P}_\square : L^2(G(F)\backslash G(\mathbb{A})) \to L^2_\square$$

be the orthogonal projection operator.

If K is an open compact subgroup of $G(\mathbb{A})$, we will also use $\mathbb{P}_\square$ to denote the restriction of this projection operator to the subspace $L^2(G(F)\backslash G(\mathbb{A}))^K$.

5.3 Infinitesimal Characters

5.3.1 Definitions and Relation to Shapes

Formulas later on will involve infinitesimal characters. Consider again G that is an extended pure inner form of $G^* \in \widetilde{\mathcal{E}}_{\mathrm{ell}}(n)$.

Any finite dimensional representation π_∞ of G_∞ has an associated infinitesimal character λ that is a semisimple conjugacy class in $\widehat{\mathfrak{g}}_\infty$. Since $G^* \in \widetilde{\mathcal{E}}_{\mathrm{ell}}(n)$, there is a map

$\widehat{G} \hookrightarrow GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ restricted from ${}^L G \hookrightarrow {}^L \mathrm{Res}_F^E GL_n$. It is in particular determined by its first coordinate so the infinitesimal character can also be represented by a semisimple conjugacy class in $M_N(F_\infty \otimes_{\mathbb{R}} \mathbb{C})$. This can then be represented as an unordered sequence of eigenvalues,

$$\lambda = (\lambda_1, \dots, \lambda_N),$$

with each $\lambda_i = (\lambda_{i,v})_{v|\infty} \in F_\infty \otimes_{\mathbb{R}} \mathbb{C}$.

It is also sometimes useful to package the tuple $\lambda_v = (\lambda_{1,v}, ..., \lambda_{n,v})$ as the generating function $\sum_j X^{\lambda_{j,v}}$, which, by abuse of notation, we will also denote by λ_v. In this way, if $\bigoplus_i \tau_{i,v}[d_i]$ is a local Arthur parameter such that each $\tau_{i,v}$ has infinitesimal character $\lambda_v^{(i)} = \sum_{j=1}^{n_i} X^{\lambda_{j,v}^{(i)}}$, then we have infinitesimal character assignment

$$\left(\bigoplus_i \tau_{i,v}[d_i]\right)_\infty \mapsto \square((\lambda_v^{(i)})_i) := \sum_i \lambda_v^{(i)} \sum_{l=1}^{d_i} X^{\frac{d_i+1}{2} - l}. \tag{5.2}$$

It can be seen from this that the character of $\tau[d]$ determines that of τ.

Definition 5.13. If λ matches the infinitesimal character of a finite-dimensional representation, we say that it is *regular integral*.

Regular integral is equivalent to two conditions:

- (Regular) For each v, the $\lambda_{i,v}$ are distinct.
- (Integral) If N is even, the $\lambda_{i,v} \in \mathbb{Z}$ and if N is odd, the $\lambda_{i,v} \in \mathbb{Z} + 1/2$.

We without loss of generality order regular integral λ:

$$\lambda_{1,v} > \cdots > \lambda_{N,v}.$$

We also make some convenient defintions:

Definition 5.14. Let $\lambda_\square$ be the set of possible regular, integral infinitesimal characters of ψ_∞ with $\psi \in \square$ (i.e. the image of (5.2)). We say $\lambda \in \square$ as shorthand for $\lambda \in \lambda_\square$.

Let $\square^{-1}(\lambda)$ be the set of possible assignments of infinitesimal characters $(\lambda^{(i)})_{i=1}^k$ to each $(\tau_{i,\infty})_{i=1}^k$ so that $\bigoplus_{i=1}^k \tau_{i,\infty}[d_i]$ has infinitesimal character λ.

5.3.2 Norms of Infinitesimal Characters

Choose distinguished infinite place v_0 and consider infinitesimal character λ of G_{v_0} for G an extended pure inner form of $G^* \in \widetilde{\mathcal{E}}_{\mathrm{ell}}^E(n)$.

Let $\Phi_+(G)$ be the standard set of positive coroots of G_{v_0}. We will need to compare/recall three different norms of λ_{v_0}:

- The dimension of the finite dimensional representation corresponding to λ:

$$\dim \lambda = C_{G,1} \prod_{\alpha \in \Phi_+(G)} \langle \alpha, \lambda \rangle,$$

- λ paired with itself with by the Killing form:

$$\|\lambda\| = C_{G,2} \left(\sum_{\alpha \in \Phi_*(G)} \langle \alpha, \lambda \rangle^2 \right)^{1/2},$$

- A minimum
$$m(\lambda) := \min_{\alpha \in \Phi_+(G)} \langle \alpha, \lambda \rangle = \min_{v|\infty} \min_{1 \le i \le N-1} (\lambda_{i,v} - \lambda_{i+1,v}), \tag{5.3}$$

where the constants $C_{G,i}$ only depend on G.

Given unrefined shape $\square$, we can also define

$$\dim_\square \lambda := \max_{(\lambda_i)_i \in \square^{-1}(\lambda)} \prod_i \dim \lambda_i.$$

Note that this can be 0 if $\square^{-1}(\lambda)$ is empty. All such definitions can be made analogously for G_∞ as a whole.

Given a group G, there are three key dimensions to keep track of $N_G^{\mathrm{der}} = \dim G^{\mathrm{der}}$, $r_G^{\mathrm{der}} = \operatorname{rank} G^{\mathrm{der}}$ and $z_G = \dim Z_G$. From these we can compute the number of positive roots:

$$P_G := \frac{1}{2}(N_G^{\mathrm{der}} - r_G^{\mathrm{der}}).$$

As some bounds (recalling the definition (5.1) of $G_F(\square)$):

Lemma 5.15.
$$\dim_\square \lambda \le (\dim \lambda) m(\lambda)^{P_{G_F(\square)} - P_G}.$$

Proof. The factors $\langle \alpha, \lambda \rangle$ in the Weyl dimension formula for $\dim_\square \lambda$ are always a subset of those in $\dim \lambda$. However, $\dim_\square \lambda$ has $P_G - P_{G_F(\square)}$ fewer factors. $\square$

Lemma 5.16.
$$\dim_\square \lambda \le C \|\lambda\|^{P_{G_F(\square)}}$$

for some constant C depending only on G and $\square$.

Proof. $\dim_\square \lambda$ is a product of some subset of size $P_{G_F(\square)}$ of the $\langle \alpha, \lambda \rangle$ for $\alpha \in \Phi_+(G)$. Therefore, by the RMS-AM-GM math-contest inequality, $(\dim_\square \lambda)^{1/P_{G_F(\square)}}$ is bounded above by the root-mean-square of this subset. This is further bounded above by a constant times $\|\lambda\|$ where the constant depends only on $P_{G_F(\square)}$ and P_G. $\square$

For $\square = \Sigma_\eta$, the bound 5.16 has the optimal exponent on $\|\lambda\|$ when there is C such that $C \min_{\alpha \in \Phi_+(G)} \langle \alpha, \lambda \rangle \ge \max_{\alpha \in \Phi_+(G)} \langle \alpha, \lambda \rangle$. This is an asymptotically positive proportion of all λ in a $\| \cdot \|$-ball as the ball's radius goes to infinity.

However, the $\lambda \in \square$ do not satisfy this property if $\square$ has non-trivial SL_2 as some of the $\langle \alpha, \lambda \rangle$ are then bounded. Therefore, we will also need a slight variant of the bound:

Lemma 5.17. *Choose constant m and subset $S \subseteq \Phi_+(G)$. Then for all λ such that $\langle \alpha, \lambda \rangle \le m$ for all $\alpha \in S$,*
$$\dim \lambda \le C m^{|S|} \|\lambda\|^{P_G - |S|}$$

for some constant C depending only on G and $|S|$.

Proof. This is a slight variant of the argument of lemma 5.16 where we apply RMS-AM-GM to the set of $\langle \alpha, \lambda \rangle$ for $\alpha \in \Phi_+(G) - S$. $\square$

Applying this to $\lambda_\square$, define for $\square = ((T_i, d_i))_i$ a correction

$$e(\square) := \sum_i \frac{1}{2} T_i d_i (d_i - 1). \tag{5.4}$$

Then we get our tightening:

Corollary 5.18. *Choose unrefined shape $\square$. Then for all $\lambda \in \square$ and all $\delta > 0$,*

$$\dim \lambda \leq C\|\lambda\|^{P_G - e(\square)}$$

for some constant C depending only on G and $\square$.

Proof. By inspecting formula (5.2), we see that for each (T_i, d_i) pair making up $\square$, any $\lambda \in \square$ has $1/2T_i d_i(d_i - 1)$ different $\alpha \in \Phi_+$ with $\langle \alpha, \lambda \rangle \leq d_i - 1$. The result then follows from lemma 5.17 after noting that the m and $|S|$ just depend on $\square$. $\square$

5.4 Automorphic Families and the Density Hypothesis

In this subsection, we introduce the notion of an automorphic family (following [SST16]) and the statements of the Ramanujan conjecture and density hypothesis for such automorphic families (following [SX91, Sar90]).

Let $n \in \mathbb{N}$, and let $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$. Denote $f \lesssim g$ if for any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that $f(x) \leq c_\epsilon \cdot \max\{g(x)^{1+\epsilon}, g(x)^{1-\epsilon}\}$ for any $x \geq 0$. Denote $f \sim g$ if both $f \lesssim g$ and $g \lesssim f$.

Definition 5.19. Let G/F be a reductive group over a number field. A (discrete) *automorphic family* $\mathcal{F}$ for G is a weighted subset of $\mathcal{AR}_{\text{disc}}(G)$: i.e. a function

$$\mathcal{F} : \mathcal{AR}_{\text{disc}}(G) \to \mathbb{R}_{\geq 0}.$$

Example 5.20. Let $K' < G^\infty$ be compact open. Then the family of automorphic forms at level K' is

$$\mathcal{F}_{K'}(\pi) = m_\pi \dim \left((\pi^\infty)^{K'} \right).$$

This models the vector space of automorphic forms on G of level K'

Definition 5.21. We say automorphic family $\mathcal{F}$ satisfies the *Ramanujan conjecture* if all $\pi \in \mathcal{AR}_{\text{disc}}(G)$ with $\mathcal{F}(\pi) \neq 0$ are tempered.

If $\mathcal{F}$ is the family of all cuspidal automorphic representations, this is called the "naïve Ramanujan conjecture" and it was shown to be false even just on Sp_4 in [HPS79]. The expected correction is that Ramanujan holds for the families of *generic* automorphic representations—this is the generalized Ramanujan conjecture. In the case of GL_n, cuspidal implies generic so this difference is irrelevant.

Definition 5.22. Let G/F be a reductive group over a number field. An asymptotic family $\mathcal{F}$ for G is an indexed sequence of automorphic families $(\mathcal{F}_\lambda)_{\lambda \in \Lambda}$ together with a "conductor" function $m : \Lambda \to \mathbb{R}$ such that:

- Each $\mathcal{F}_\lambda$ has finite total weight.
- The size of the support of $\mathcal{F}_\lambda$ goes to infinity as $m(\lambda) \to \infty$.

Example 5.23. Let K' a compact open subgroup of G^∞. Then the *weight-aspect family of level-K' automorphic forms* on G is

$$\mathcal{F}_{K',\lambda}(\pi) = m_\pi \mathbf{1}_{\text{infchar}(\pi_\infty)=\lambda} \dim \left((\pi^\infty)^{K'} \right)$$

indexed over the set of regular, integral infinitesimal characters λ of G_∞. Its conductor function is the m from (5.3).

This models the space of automorphic forms on G of level K' as the infinitesimal character at infinity gets larger and larger.

Example 5.24. Let $I \subseteq \infty$ contain all infinite places at which G is non-compact. Let $K = K^\infty G_{\infty \setminus I}$ for open compact $K^\infty \subseteq G^\infty$. Then the connected component of the identity in $G(F)\backslash G(\mathbb{A})/K$ is $\Gamma\backslash G_I$ for some discrete, cofinite volume Γ.

Pick an invariant metric on G_I and let $B(\delta)$ be the ball of volume δ around the identity in $\Gamma\backslash G_I$. Then define the δ-ball family in G_I by:

$$\mathcal{F}_\delta(\pi) := m_\pi \frac{\| \mathbb{P}_{\pi_I}(\mathbf{1}_{B(\delta)})\|_2^2}{\|\mathbf{1}_{B(\delta)}\|_2^2} \mathbf{1}_{\pi_{\infty\setminus I} \text{ triv.}} \dim\left((\pi^\infty)^{K^\infty}\right) = m_\pi \frac{1}{\|\mathbf{1}_{B(\delta)}\|_2^2} \operatorname{tr}_\pi((f_\epsilon \star f_\epsilon^*)\bar{\mathbf{1}}_{G_{\infty\setminus I}} \bar{\mathbf{1}}_K),$$

with $m(\delta) = 1/\delta$.

This models the decompositions of the indicators of smaller and smaller balls in the automorphic spectrum.

Definition 5.25. We say an asymptotic family $\mathcal{F}_\lambda$ eventually satisfies the Ramanujan conjecture if there L such that $\mathcal{F}_\lambda$ satisfies the Ramanujan conjecture whenever $m(\lambda) \geq L$.

Fix place v so that $(G_{\text{sc}})_v$ has no anisotropic factors. We say an asymptotic family $\mathcal{F}_\lambda$ satisfies the density hypothesis at v if for all $\sigma \geq 2$ and $\epsilon > 0$,

$$\sum_{\substack{\pi \in \mathcal{AR}_{\text{disc}}(G) \\ \sigma(\pi,v)\geq\sigma}} \mathcal{F}_\lambda(\pi) \lesssim \left(\sum_{\pi\in\mathcal{AC}(G)} \mathcal{F}_\lambda(\pi)\right)^{1-\frac{2}{\sigma}} \left(\sum_{\pi\in\mathcal{AR}_{\text{disc}}(G)} \mathcal{F}_\lambda(\pi)\right)^{\frac{2}{\sigma}},$$

where $\mathcal{AC}(G)$ is the set of automorphic characters (1-dimensional automorphic representations of G) and $\lesssim$ is interpreted asymptotically in $m(\lambda)$.

Note that $\sigma(\pi_v) = \infty$ is equivalent to π_v being a character which, under our conditions on G_v, is further equivalent to π being a character (see e.g. [KST16, Lem 6.2]). Therefore, this can be thought of as an interpolation between the case $\sigma = 2$ and $\sigma = \infty$.

The automorphic density hypothesis was raised as a conjecture in [SX91, Sar90] as a possible substitute for the failure of the naïve Ramanujan conjecture. In recent years this conjecture was proven in several special instances [Blo19, Mar14, MS18, GK19, Sar15a]. Here, we will specifically be applying methods from [DGG23, DGG24].

Example 5.26. Let G be one of the $U_n^{E,H}$. Then weight-aspect families on G can be seen to eventually satisfy the Ramanujan conjecture through the endoscopic classification [KMSW14]: if $m(\lambda) \gg 1$, then formula (5.2) shows that all π with infinitesimal character λ at infinity are necessarily of shape $\square = ((T_i, d_i))_{i=1}^k$ with all $d_i = 1$ (i.e. they have generic parameters). Then, Theorem 5.3 can be used to show that they are all tempered.

Example 5.27. The density hypothesis for a slight variant of the δ-ball family will be the key input towards proving the optimal covering property for our set of gates.

6 Matrix Coefficient Decay

We next need to understand how the shape of an A-parameter of an automorphic representation controls the decay of the matrix coefficients of its local components at finite places. This will require the very serious black-box inputs of Theorem 5.3 and results from explicit constructions of A-packets.

Fix some finite place v of quadratic extension E/F and unitary group $G = U_n^{E/H}$. Assume first that G_v is unramified and v is non-split. A much simpler version of this argument works for v split (see remark 6.15)—we only present the full details of the more complicated non-split case for brevity.

6.1 Exponents

Fix a minimal parabolic P_0 of G_v and a corresponding set of standard Levis M and parabolics P_M.

Let π be an irreducible representation of G_v. Then by the Langlands classification, there is a standard Levi M of G_v, tempered irreducible representation σ of M, and unramified character λ of M such that π is a subrepresentation of the normalized parabolic induction $\mathcal{I}_{P_M}^G(\sigma \otimes \lambda)$.

Since G_v is unitary splitting over E_w, M is of the form

$$M = \mathrm{Res}_{F_v}^{E_w} GL_{n_1} \times \cdots \times \mathrm{Res}_{F_v}^{E_w} GL_{n_k} \times G'_v,$$

where G'_v is a smaller unitary group splitting over E_w. Therefore, by the Bernstein-Zelevinsky classification, we can actually choose (M, σ) so that $\sigma \otimes \lambda$ is of the form

$$\sigma \otimes \lambda = \mathrm{St}(\rho_1, a_1)|\det|^{-x_1} \boxtimes \cdots \boxtimes \mathrm{St}(\rho_k, a_k)|\det|^{-x_k} \boxtimes \pi_{\mathrm{temp}}, \tag{6.1}$$

where $\mathrm{St}(\rho_i, a_i)$ are Steinberg representations built out of supercuspidals ρ_i of $GL_{T_i}E_w$ with $T_i a_i = n_i$, π_{temp} is a tempered representation[(†)] for G'_v, and $x_1 \geq \cdots \geq x_k > 0$. Such $\sigma \otimes \lambda$ is unique up to permuting factors i with equal x_i's.

Furthermore, by the endoscopic classification, π_{temp} has a tempered L-parameter:

$$\varphi_{\pi_{\mathrm{temp}}} = \bigoplus_j \tau_j \boxtimes [b_j] \tag{6.2}$$

for τ_j unitary supercuspidals of some $GL_{R_j}E_w$.

Definition 6.1. For π an irreducible representation of G_v as above, define:

(1) The Langlands data for π is the data $((\rho_i, a_i, x_i)_i, \pi_{\mathrm{temp}})$ from (6.1),

(2) The extended supercuspidal support for π is the multiset produced by taking a union of

$$(\rho_i|\det|^{l-x_i}, \bar{\rho}_i^\vee|\det|^{l+x_i} : l \in \{(a_i-1)/2, (a_i-3)/2, \dots, (1-a_i)/2\})$$

over the i from (6.1) together with

$$(\tau_j|\det|^l : l \in \{(b_j-1)/2, (b_j-3)/2, \dots, (1-b_j)/2\})$$

over the j from (6.2).

Our bounds on matrix coefficient decay will require slightly different information:

Definition 6.2. For π an irreducible representation of G_v as above, define:

(1) the coarse exponents $\overline{L}_\pi$ to be the list of x_i from (6.1) in non-increasing order,

(2) the exponents L_π to be the list of each x_i from (6.1) repeated n_i times in non-increasing order.

Lists of exponents can be compared:

Definition 6.3. If $L = (l_i)_{i=1}^k$ is a non-increasing list of numbers:

(1) define

$$\sigma_i(L) := \sum_{j=1}^{i} x_j,$$

where for indexing purposes, $x_j = 0$ when j is out-of-bounds.

(2) We say that $L_1 \succeq L_2$ if for all i, $\sigma_i(L_1) \geq \sigma_i(L_2)$.

(†) we say $\pi_{\mathrm{temp}} = 0$ to cover the case when the last factor of M doesn't appear

6.2 Exponents and Matrix Coefficient Decay

The exponents of a representation control its matrix coefficient decay. Pick representation π of G_v and recall the λ, σ, M as above, where

$$\lambda : M = \mathrm{Res}_{F_v}^{E_w} GL_{n_1} \times \cdots \times \mathrm{Res}_{F_v}^{E_w} GL_{n_k} \times G'_v \to \mathbb{C} : \\ g_1 \times \cdots \times g_k \times g \mapsto |\det g_1|^{-x_1} \cdots |\det g_k|^{-x_k}.$$

For any Levi's $M_1 \supseteq M_2$, let $\delta_{M_2}^{M_1}$ be modulus character of $P_{M_2} \cap M_1$—i.e taking a product over sets of positive roots Φ^+:

$$\delta_{M_2}^{M_1} := \prod_{\alpha \in \Phi^+(M_1) \setminus \Phi^+(M_2)} |\alpha|,$$

which extends to a character of M_2. Note that for $z \in Z_{M_2}$:

$$\delta_{M_2}^{M_1}(z) = \delta^{M_1}(z) = \prod_{\alpha \in \Phi^+(M_1)} |\alpha(z)|,$$

and for $z \in Z_{M_1}$, $\delta^{M_1}(z) = 1$.

Let T be a maximal torus of the minimal standard Levi M_0 and $T^{\mathcal{O}}$ be the subset on which all algebraic characters take values with norm 1. For any M, define $Z_M^{\mathcal{O}}$ similarly in the center of M and let Z_M^- be the set of all $z \in Z_M$ such that $|\alpha(z)| \leq 1$ for all $\alpha \in \Phi^+(G) \setminus \Phi^+(M)$.

Lemma 6.4. *In the notation above, let N be another standard Levi of G. Let χ be the central character of an irreducible subquotient of the normalized Jacquet module $\mathcal{R}_N^G \mathcal{I}_M^G(\sigma \otimes \lambda)$. Then there is Weyl element w such that $\chi = \chi_1(\lambda \circ w)|_{Z_N}$, where $|\chi_1(z)| \leq 1$ for all $z \in Z_N^- \setminus Z_G T^{\mathcal{O}}$.*

Proof. This follows from Bernstein's geometric lemma [BZ77, pg 448]:

Any χ is, in the notation therein, a central character of a subquotient of some

$$F_w(\sigma \otimes \lambda) = \mathcal{I}_{N'}^N \circ w \circ \mathcal{R}_{M'}^M(\sigma \otimes \lambda),$$

with w some Weyl element and $w(M') = N'$. Computing, $\mathcal{R}_{M'}^M I_M^G(\sigma \otimes \lambda)$ has central characters of the form $\chi'_1 \lambda|_{Z'_M}$ where χ'_1 is a central character of $\mathcal{R}_{M'}^M \sigma$. Further applying w produces those of the form $\chi_1(w \circ \lambda)|_{Z_{N'}}$ where χ_1 is a character in $\mathcal{R}_{N'}^{wM}(w\sigma)$. The induction finally gives those of the form $\chi_1(w \circ \lambda)(\delta^N)^{1/2}|_{Z_N} = \chi_1(\lambda \circ w)|_{Z_N}$.

Finally, [Cas95, Cor 4.4.6] gives the desired property of χ_1. Note we cancel out the $(\delta_{N'}^{wM})^{-1/2}$ in the reference since we are using normalized Jacquet modules. □

Proposition 6.5. *In the notation above, the matrix coefficients of $\mathcal{I}_M^{G_v}(\sigma \otimes \lambda)$ are $L^{p+\epsilon}$ mod center for all $\epsilon > 0$ if for all negative dominant $\nu \in X_*(Z_{M_0}) \setminus X_*(Z_G)$,*

$$|\lambda(\nu(\varpi))\delta^G(\nu(\varpi))^{1/2-1/p}| \leq 1.$$

Proof. The parameter p in the application of Corollary 4.4.5 in the proof of Theorem 4.4.6 in [Cas95] can be any value instead of just 2 producing a test for L^p matrix coefficients.

It therefore suffices to check that for any Levi N, the central characters χ of $\mathcal{R}_N^G \mathcal{I}_M^G(\sigma \otimes \lambda)$ satisfy (with normalized Jacquet modules) that

$$|\chi(a)\delta^G(a)^{1/2-1/p}| \leq 1$$

for all $a \in Z_N \setminus Z_G T(\mathcal{O}_v)$, where T is a maximal torus of M_0. However, by lemma 6.4, there is a Weyl element w such that

$$|\chi(a)\delta^G(a)^{1/2-1/p}| \leq |\lambda(wa)\delta^G(a)^{1/2-1/p}| \leq |\lambda(a)\delta^G(a)^{1/2-1/p}|$$

by the ordering of the x_i. Then, since λ is unramified, we only need to check the condition for the listed coset representatives $\nu(\varpi)$ of $Z_N^-/Z_N^{\mathcal{O}}$, ignoring those intersecting Z_G.

The result follows noting that $Z_N \subseteq Z_{M_0}$. □

We rephrase this slightly:

Corollary 6.6. *In the notation above,*

$$\frac{2}{\sigma(\pi)} \geq 1 - \max_{1 \leq i \leq (N-1)/2} \frac{2\sigma_i(L_\pi)}{i(n-i)}.$$

Proof. Since $\pi \subseteq \mathcal{I}_M^{G_v}(\sigma \otimes \lambda)$, it suffices to check which p satisfy condition of Proposition 6.5. We also without loss of generality consider all negative dominant $\nu \in X_*(A) \setminus X_*(G)$ for a maximally split torus A.

Parameterize A as diagonal matrices $(t_1, \dots, t_n)$ for $t_i \in E_w$ with $t_i^{-1} = \bar{t}_{n-i}$ so that $X_+(A) \setminus X_*(Z_G)$ is generated as a semigroup by the fundamental weights for $1 \leq i \leq (n-1)/2$:

$$\xi_i : F_v^\times \to T : t \mapsto (t, \dots, t, 1 \dots, 1, t^{-1}, \dots, t^{-1})$$

where the breakpoints are at indices i and $N - i$.

By the inequality,

$$b, d > 0 \text{ and } \frac{a}{b} \leq \frac{c}{d} \implies \frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d},$$

it suffices to only check $\nu = -\xi_i$ in 6.5. Then

$$\log_{q_v} |\lambda(-\xi_i(\varpi))| = 2\sigma_i(L_\pi), \qquad \log_{q_v} |\delta^G(-\xi_i(\varpi))| = -2i(n-i),$$

so π has matrix coefficients in L^p if

$$\sigma_i(L_\pi) \leq \left(\frac{1}{2} - \frac{1}{p}\right) i(n-i) \iff \frac{2}{p} \leq 1 - \frac{2\sigma_i(L_\pi)}{i(n-i)}$$

for all $1 \leq i \leq (n-1)/2$. The result follows. □

6.3 Exponents and Parameters

Parameters also have a notion of exponents. Let tempered local parameter ψ_v decompose as a representation of $W_{F_v} \times \mathrm{SL}_2 \times \mathrm{SL}_2$ as

$$\bigoplus_i \tau_i \boxtimes [a_i] \boxtimes [d_i],$$

where the τ_i are unitary supercuspidals of $GL_{T_i}E_w$. Let τ_i have dimension T_i.

Definition 6.7. In the notation above, the coarse exponents $\overline{L}_{\psi_v}$ for ψ_v is the concatenation of the lists

$$\bigsqcup_i ((d_i - 1)/2, (d_i - 3)/3, \cdots, (d_i - 2\lfloor d_i/2 \rfloor)/2),$$

where the result is reordered to be non-increasing.

The exponents L_{ψ_v} for ψ_v are the same except we repeat the ith list $T_i a_i$ times:

$$\bigsqcup_i ((d_i-1)/2, (d_i-3)/3, \cdots, (d_i - 2\lfloor d_i/2 \rfloor)/2)^{(T_i a_i)},$$

with the result ordered to be non-increasing.

Definition 6.8. In the notation above, the extended supercuspidal support of ψ_v is the multiset produced by taking a union of

$$(\tau_i |\det|^l : l \in \langle a_i \rangle \boxplus \langle b_i \rangle)$$

over all i. We use shorthand

$$\langle r \rangle := ((r-1)/2, (r-3)/2, \ldots, (1-r)/2)$$

and define $A \boxplus B$ to be the multiset $(a+b : a \in A, b \in B)$.

We can now state a key input of Mœglin:

Theorem 6.9 ([Mœg0901, Thm 7.2, Prop 4.1])**.** *Let $\psi_v \in \Psi_{G_v}$ and $\pi_v \in \Pi_{\psi_v}$. Then:*

(1) $\overline{L}_{\pi_v} \preceq \overline{L}_{\psi_v}$.

(2) The extended supercuspidal support of π_v is the same as that of ψ_v.

This is not good enough to bound matrix coefficient decay; what we actually desire is:

Conjecture 6.10. *Let $\psi_v \in \Psi_{G_v}$ and $\pi_v \in \Pi_{\psi_v}$. Then $L_{\pi_v} \preceq L_{\psi_v}$.*

Remark 6.11. Conjecture 6.10 would follow from the closure-order conjecture [Xu2405, Conj 2.1] as shown in in [HLLZ24, Thm 4.11(2)].

The closure-order conjecture is known in case of symplectic and orthogonal groups by [HLLZ24]. The argument depends on algorithms computing the set of A-packets containing a representation from [Ato23] in the symplectic/orthogonal case—these are expected to analogize to the unitary case, though the details have not been completed as-of-this-writing.

The closure-order conjecture also holds holds for the ABV packets of [CFM+22] which are conjectured (see Conjecture 8.1 therein) to be the same as the A-packets we use from [Mok15].

However, for our applications, we only need special cases:

Corollary 6.12. *Conjecture 6.10 holds in the following cases:*

(1) π_v is unramified,

(2) $n = 4$.

Proof. Fix a π_v. If the $N_i = T_i a_i$ in (6.1) are all 1, then $L_{\pi_v} = \overline{L}_{\pi_v} \preceq \overline{L}_{\psi_v} \preceq L_{\psi_v}$ by 6.9(1) so 6.10 always holds. This covers case (1) and all π_v for case (2) except those with Langlands data of the form $((\chi, 2, x), 0)$ for some character χ of $E_w^\times$.

Then, such π_v has extended supercuspidal support

$$(\bar{\chi}^\vee |\det|^{1/2+x}, \bar{\chi}^\vee |\det|^{-1/2+x}, \chi |\det|^{1/2-x}, \chi |\det|^{-1/2-x})$$

By 6.9(2), this needs to match that of ψ_v which can only happen if $x = 0, 1/2, 1$.

If $x = 0$, L_{π_v} is all 0's so we are done. If $x = 1/2$, then $\chi = \bar{\chi}^\vee$ and $\psi_v = \chi[1][3] + \chi[1][1]$ (using the natural shorthand). Therefore, $L_{\psi_v} = (1, 0, 0)$ and $L_{\pi_v} = (1/2, 1/2)$ which satisfies the bound. Finally, $x = 1$ would force $\chi = \bar{\chi}^\vee$ and $\psi_v = \chi[1][4]$ which has A-packet containing just characters. This contradicts. □

The case $n = 8$ will be resolved later in Corollary 7.10.

6.4 Bounding Decay by Shape

Note that for global parameter ψ, each L_{ψ_v} only depends on the restriction to the Arthur-SL_2. Therefore, it is constant over $\psi \in \square$.

Definition 6.13. For $\square$ a shape for G, let $L_\square$ be the common value of L_{ψ_v} for non-split, unramified v and $\psi \in \square$.

As a consequence of all the above work and the deep input of Theorem 5.3, we get our final result:

Theorem 6.14. *Let $\square$ be a shape for G and $\psi \in \square$ such that ψ_∞ has regular, integral infinitesimal character.*

Then for all non-split, unramified places v and $\pi \in \Pi_{\psi_v}$ such that conjecture 6.10 holds:

$$\frac{2}{\sigma(\pi)} \geq 1 - \max_{1 \leq i \leq (N-1)/2} \frac{2\sigma_i(L_\square)}{i(n-i)}.$$

Proof. Let $\psi = \bigoplus_i \tau_i \boxtimes [d_i]$ with τ_i cuspidal. Then, all the τ_i have regular, integral infinitesimal character at infinity so by Theorem 5.3, the $\tau_{i,v}$ are all tempered. Then, ψ_v decomposes as $\bigoplus_j \sigma_j \boxtimes [a_j] \boxtimes [d_j]$ for σ_j unitary supercuspidal, so it is in Ψ_v (instead of the larger Ψ_v^+ of [Mok15]).

The result then follows from the equality $L_{\psi_v} = L_\square$, Conjecture 6.10 and Corollary 6.6. $\square$

Remark 6.15. When v is split, analogous notions of exponents for representations π_v of G_v can be defined using the Bernstein-Zelevinsky classification. The analogue of Corollary 6.6 then still holds. If ψ_v is a parameter of G_v then it corresponds to a irreducible representation $\psi_0 \boxtimes \psi_0^\vee$ on $GL_n(F_v \otimes_F E) \cong GL_n(E_w)^2$ where w lies over v. Then Π_{ψ_v} is a singleton containing only the representation π_ψ corresponding to ψ_0 (see e.g. [DGG23, lem 6.1.1]). In particular, $L_\psi = L_{\pi_\psi}$ so Conjecture 6.10 always holds. Therefore, Theorem 6.14 always holds as well.

Motivated by the above:

Definition 6.16. Let $\square$ be a shape for G. Then define $\sigma_\square$ by

$$\frac{2}{\sigma_\square} := 1 - \max_{1 \leq i \leq (N-1)/2} \frac{2\sigma_i(L_\square)}{i(n-i)}.$$

In particular, Theorem 6.14 gives that $\sigma(\pi) \leq \sigma_\square$.

7 Density Hypothesis Proof

Let $G = U_N^{E/F,H}$ be a definite arithmetic unitary group and v_0 a distinguished infinite place. Choose open compact $K' < G^\infty$. Let $\Gamma = G(F) \cap K$ as a subgroup of G_{v_0} so that there is a map

$$\rho^{K'} : G_{v_0} \twoheadrightarrow \Gamma \backslash G_{v_0} \hookrightarrow G(F) \backslash G(\mathbb{A}) / K G_{\infty \backslash v_0}.$$

In our eventual application when K' is golden, $\Gamma = 1$ and the second map will be an bijection.

In this section, we prove the density hypothesis for a variant of the δ-ball family on G_{v_0}: we define functions $f_{v_0}^{\epsilon,Z}$ on G_{v_0} that are approximately indicator functions of balls of radius ϵ and consider families

$$\mathcal{F}_{\epsilon,Z}^{K'}(\pi) := m_\pi \frac{\| \mathbb{P}_{\pi_{v_0}}(f_{v_0}^{\epsilon,Z})\|_2^2}{\|f_{v_0}^{\epsilon,Z}\|_2^2} \mathbf{1}_{\pi_{\infty\setminus v_0} \text{ triv.}} \dim\left((\pi^\infty)^{K'}\right)$$
$$= m_\pi \frac{1}{\|f_\epsilon\|_2^2} \operatorname{tr}_\pi((f_\epsilon \star f_\epsilon^*)\bar{\mathbf{1}}_{G_{\infty\setminus v_0}}\bar{\mathbf{1}}_{K'}) \quad (7.1)$$

in conductor $m(\epsilon) = 1/\epsilon$ and for various choices of Z. We can also consider $f_{v_0}^{\epsilon,Z}$ as an element of $L^2(G(F)\backslash G(\mathbb{A}))^{K'}$ through summing over fibers of $\rho^{K'} : G_{v_0} \twoheadrightarrow \Gamma\backslash G_{v_0}$ to get an alternate interpretation:

$$\mathcal{F}_{\epsilon,Z}^{K'}(\pi) = m_\pi \frac{\| \mathbb{P}_\pi(\rho_*^{K'} f_{v_0}^{\epsilon,Z})\|_2^2}{\|f_{v_0}^{\epsilon,Z}\|_2^2}.$$

The proof is from comparing two bounds: first in Theorem 7.7, we bound

$$\sum_{\pi\in\square} \mathcal{F}_{\epsilon,Z}^{K'}(\pi) = \frac{\| \mathbb{P}_\square(\rho_*^{K'} f_{v_0}^{\epsilon,Z})\|_2^2}{\|f_{v_0}^{\epsilon,Z}\|_2^2}, \quad (7.2)$$

(recalling notation from Definition 5.12). This requires the very serious black-boxed input of the endoscopic classification as used in Theorem 7.1. We then compare this to the matrix-coefficient decay bound Theorem 6.14.

7.1 Input Bound

We now state our black-box input bound.

Fix infinitesimal character λ for G_∞ and let V_λ be the corresponding finite dimensional representation. If $\square$ is a shape and $K' < G^\infty$ is an open compact, define asymptotic automorphic family

$$\mathcal{F}_{\lambda,\square}^{G,K'}(\pi) := m_\pi \mathbf{1}_{\pi\in\square}\mathbf{1}_{\pi_\infty = V_\lambda} \dim\left((\pi_\infty)^{K'}\right).$$

The paper [DGG23, §7-9] used the endoscopic classification of [KMSW14] through an inductive analysis of [Taï17] to upper bound asymptotics of the total mass of $\mathcal{F}_{\lambda,\square}^{G,K'}$ for certain sequences of $K' \to 1$.

The paper [DGG24, §5-6] used the same techniques in the much simpler case of $m(\lambda) \to \infty$. We recall the result:

Theorem 7.1 (Special case of [DGG24, Thm 6.5.1])**.** *In the notation above,*

$$\sum_{\pi\in\mathcal{AR}_{\mathrm{disc}}(G)} \mathcal{F}_{\lambda,\square}^{G,K'}(\pi) \le (\dim_\square \lambda)(\Lambda(G,\square,K') + O_{G,\square,K'}(m(\lambda)^{-1}))$$

for some constant $\Lambda(G,\square,K')$ depending only on the three arguments.

Proof. Recall the definitions of $I_\square^G(\mathrm{EP}_\lambda\bar{\mathbf{1}}_{K'})$ and $S_\square^{G*}(\mathrm{EP}_\lambda\bar{\mathbf{1}}_{K'})$ from [DGG24, §6.1].

Since V_λ is the only representation of compact G_∞ with infinitesimal character λ,

$$\operatorname{tr}_\pi(\mathrm{EP}_\lambda\bar{\mathbf{1}}_{K'}) = \mathbf{1}_{\pi_\infty = V_\lambda} \dim\left((\pi_\infty)^{K'}\right).$$

Therefore,

$$\sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G)} \mathcal{F}^{G,K'}_{\lambda,\square}(\pi) = I^G_\square(\mathrm{EP}_\lambda \bar{\mathbf{1}}_{K'}).$$

The result then follows from the second bound of [DGG24, Thm 6.5.1].

For the reader's convenience, we very roughly sketch the argument of [DGG24, Thm 6.5.1]. By an implementation in [DGG23] of an inductive strategy from [Taï17] inputting the endoscopic classification [KMSW14, Mok15], our count of representations can be upper bounded by a linear combination of terms on the geometric side of Arthur's discrete-at-∞ trace formula from [Art89].

All these [Art89] terms are sums of smaller terms of the form

$$C_\gamma \Phi^H_{\lambda'}(\gamma),$$

where H ranges over groups that are $G_F(\square)$ or smaller, the $\Phi^H_{\lambda'}$ are certain character sums on maximal tori related to traces against the finite dimensional representation on H with infinitesimal character λ' derived from λ, γ ranges over some fixed finite set of rational conjugacy classes of H depending only on $H, \square$ and K', and C_γ are some inexplicit constants that nevertheless depend only on $H, \square, K'$, and γ.

Finally, we apply the analysis of [ST16] to these terms—in particular [ST16, Lem 6.10(ii)] bounds the $\Phi^H_{\lambda'}$, thereby showing that a term for $\gamma = 1$ on $G_F(\square)$ itself dominates. For this term specifically, $\Phi^H_\lambda(\gamma) = \dim_\square \lambda$ and C_γ can be made more precise. $\square$

Remark 7.2. We will in fact only need that

$$(\dim_\square \lambda)^{-1} \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G)} \mathcal{F}^{G,K'}_{\lambda,\square}(\pi) = (\dim_\square \lambda)^{-1} I^G_\square(\mathrm{EP}_\lambda \bar{\mathbf{1}}_{K'})$$

is bounded by a constant independent of λ.

7.2 Indicators of Balls

We now define the functions $f^{\epsilon,Z}_{v_0}$ defining our variant of the f-ball family and bound (7.2). Our main technical tool here is Kirilov's orbit-method character formula; see [Ros78, Ver79] for the full proof and [Kir04, Ch 5] for a textbook summary. Our $f^{\epsilon,Z}_{v_0}$ are close to but not exactly indicator functions, instead chosen specifically to simplify the orbit-method computations.

7.2.1 Modified Indicator Functions

First, consider the case of H a compact, semisimple, and simply connected Lie group. Let $\mathfrak{h}$ be the real Lie algebra for H, $\dim \mathfrak{h} = N$, and $\operatorname{rank} H = r$. Define on $\mathfrak{h}$:

$$j(X) := \det\left(\frac{\sinh(\operatorname{ad} X/2)}{\operatorname{ad} X/2}\right),$$

Consider test functions

$$f^\epsilon \circ \exp := \mathbf{1}_{B_\epsilon(0)} j^{1/2}$$

on H and where balls are defined using the Killing form. Note that f^ϵ is supported on the ball $\exp(B_\epsilon(0))$, is analytic, and takes values close to 1 for small enough ϵ.

We use the Kirilov character formula to compute traces of f^ϵ against the finite dimensional representation V_λ. If $\mathfrak{t}$ is a Cartan for $\mathfrak{h}$, the Killing form gives an embedding $\mathfrak{t}^* \hookrightarrow \mathfrak{h}^*$ so $i\lambda$ can be interpreted as a point in $i\mathfrak{h}^*$. To define a Fourier transform, pick a measure on

$\mathfrak{h}$ that is Plancherel self-dual through the Killing form isomorphisms $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ and associate $x \in \mathfrak{h}^*$ to the multiplicative character $e^{2\pi i x(\cdot)}$ on $\mathfrak{h}$. Then for small enough ϵ:

$$\mathrm{tr}_{\pi_\lambda}(f^\epsilon) = \int_{\mathcal{O}_{\lambda/(2\pi i)}} \widehat{\mathbf{1}_{B_\epsilon(0)}}\, d\omega, \tag{7.3}$$

where the coadjoint orbit $\mathcal{O}_{\lambda/(2\pi i)} \subseteq i\mathfrak{h}^*$ is given its canonical measure as an integral symplectic manifold with total volume $\dim \lambda$.

By a classical result:

$$\widehat{\mathbf{1}_{B_\epsilon(0)}}(\xi/(2\pi)) = \epsilon^N \|\epsilon\xi\|^{-N/2} J_{N/2}(\|\epsilon\xi\|) = \epsilon^{N/2}\|\xi\|^{-N/2} J_{N/2}(\epsilon\|\xi\|),$$

where $J_{N/2}$ is the classical Bessel function of the first kind. Since the adjoint action preserves the Killing form, the integral in (7.3) is constant so:

Lemma 7.3. *For H a compact, semisimple, and simply connected Lie group and in the above notation:*

$$\mathrm{tr}_{V_\lambda}(f^\epsilon) = (\dim \lambda)\epsilon^{N/2}\|\lambda\|^{-N/2} J_{N/2}(\epsilon\|\lambda\|).$$

for small enough ϵ.

Remark 7.4. We can understand the factors in lemma 7.3 through lemma 5.16,

$$\dim \lambda \leq \|\lambda\|^{P_H},$$

and the Bessel function asymptotics:

$$|J_{N/2}(x)| \leq \begin{cases} Cx^{N/2} & x \ll \sqrt{N/2+1} \\ Cx^{-1/2} & x \gg \sqrt{N/2+1} \end{cases}.$$

In particular, this term should be thought of as order

$$\mathrm{tr}_{V_\lambda}(f^\epsilon_\infty) = \begin{cases} O(\epsilon^N \|\lambda\|^{1/2(N-r)}) & \|\lambda\| \ll \epsilon^{-1} \\ O(\epsilon^{(N-1)/2}\|\lambda\|^{-1/2(r+1)}) & \|\lambda\| \gg \epsilon^{-1} \end{cases}.$$

We can generalize this to our G_{v_0} that is compact and (topologically) connected. Then $G_{v_0} = G^{\mathrm{der}}_{v_0} \times Z_{G_{v_0}}/Z_{G^{\mathrm{der}}_{v_0}}$ on points and we have a corresponding canonical factorization on Lie algebras $\mathfrak{g} = \mathfrak{g}^{\mathrm{der}} \times \mathfrak{z}$.

Then for any small enough ϵ and subset $Z \subseteq \mathfrak{z}$ on which exp to G_{v_0} is injective, define

$$f^{\epsilon,Z}_{v_0} \circ \exp := j^{1/2}\mathbf{1}_{B_\epsilon(0)\times Z}. \tag{7.4}$$

The Kirillov character formula a priori computes the trace character of V_λ pulled back to $(G^{\mathrm{der}}_{v_0})_{\mathrm{sc}}$. However, for small enough ϵ, this is the same as its trace against $G^{\mathrm{der}}_{v_0}$.

We can in addition integrate over $Z_{G_{v_0}}$ to compute traces of the pullback to $G^{\mathrm{der}}_{v_0} \times Z_{G_{v_0}}$, noting that V_λ has central character $\lambda|_\mathfrak{z}$ on $\mathfrak{z}$. Since the diagonal embedding of $Z_{G^{\mathrm{der}}_{v_0}}$ intersects $\exp(B_\epsilon(0) \times Z) \subseteq G^{\mathrm{der}}_{v_0} \times Z_{G_{v_0}}$ trivially for small enough ϵ, this is the same as computing traces on G_{v_0}.

In total:

Lemma 7.5. *In our case where G_{v_0} is compact and (topologically) connected:*

$$\mathrm{tr}_{V_\lambda}(f^{\epsilon,Z}_{v_0}) = (\dim \lambda)\epsilon^{N^{\mathrm{der}}/2}\|\lambda\|^{-N^{\mathrm{der}}/2} J_{N^{\mathrm{der}}/2}(\epsilon\|\lambda\|)\widehat{\mathbf{1}_Z}(\lambda|_\mathfrak{z})$$

for small enough ϵ and $Z \subseteq \mathfrak{z}$ on which exp *to G_{v_0} is injective. Here, $\mathfrak{z}$ is given the measure that corresponds to unit Haar measure on $Z_{G_{v_0}}$ and recall that $N^{\mathrm{der}} = \dim \mathfrak{g}^{\mathrm{der}}$.*

We also need to understand traces against $(f_{v_0}^{\epsilon,Z})^* \star f_{v_0}^{\epsilon,Z}$. By Theorem 10 on page 174 of [Kir04],

$$(f_{v_0}^{\epsilon,Z})^* \star_{G_{v_0}} f_{v_0}^{\epsilon,Z} = (\mathbf{1}_{B_\epsilon(0)\times(-Z)} \star_{\mathfrak{g}^{\mathrm{der}}\times\mathfrak{z}} \mathbf{1}_{B_\epsilon(0)\times Z})j^{1/2},$$

so using that abelian Fourier transform takes convolution to product, a similar computation gives:

Lemma 7.6. *In our case where G_{v_0} is compact and (topologically) connected:*

$$\operatorname{tr}_{V_\lambda}((f_{v_0}^{\epsilon,Z})^* \star f_{v_0}^{\epsilon,Z}) = (\dim\lambda)\epsilon^{N^{\mathrm{der}}}\|\lambda\|^{-N^{\mathrm{der}}} J_{N^{\mathrm{der}}/2}(\epsilon\|\lambda\|)^2 \left|\widehat{\mathbf{1}_Z}(\lambda|_{\mathfrak{z}})\right|^2$$

for small enough ϵ and $Z \subseteq \mathfrak{z}$ on which $\exp$ to G_{v_0}. Here, $\mathfrak{z}$ is given the measure that corresponds to unit Haar measure on $Z_{G_{v_0}}$ and recall that $N = \dim\mathfrak{g}^{\mathrm{der}}$.

Finally, note that pulling back to the Lie algebra gives

$$\|f_{v_0}^{\epsilon,Z}\|^2_{G_{v_0}} = \|\mathbf{1}_{B_\epsilon(0)\times Z}\|^2_{\mathfrak{g}} = \operatorname{vol}(Z)\frac{\pi^{N^{\mathrm{der}}/2}}{\Gamma(N^{\mathrm{der}}/2+1)}\epsilon^{N^{\mathrm{der}}}. \tag{7.5}$$

7.2.2 Projection Bounds

With the above Kirilov formula computation, we can now input Theorem 7.1 and bound $\|\mathbb{P}_{L^2_\square} f_{v_0}^{\epsilon,Z}\|_2^2$. We will consider two possible Z: either $Z_\epsilon := (-\epsilon/2, \epsilon/2)$ or $Z_1 := (-1/2, 1/2)$. Here, $\mathfrak{z}$ is parameterized so that Lebesgue measure matches unit Haar measure: i.e. intervals of length 1 exactly cover $Z_{G_{v_0}} = U_1$.

First, since G_{v_0} is compact, we can choose the function EP_λ to be the matrix coefficient of the finite dimensional representation V_λ with infinitesimal character λ. In particular, by Peter-Weyl, any f_{v_0} always has the same orbital integrals as a function of the form

$$\sum_\lambda a_\lambda \mathrm{EP}_\lambda.$$

Since by definition $\operatorname{tr}_{V_\mu}\mathrm{EP}_\lambda = \mathbf{1}_{\mu=\lambda}$ for any two finite-dimensional reps V_μ and V_λ, comparing traces solves for the coefficients and gives:

$$I(f_{v_0}) = I\left(\sum_\lambda (\operatorname{tr}_{V_\lambda} f_{v_0})\mathrm{EP}_\lambda\right)$$

for any invariant distribution I.

In addition, the Plancherel formula gives

$$\|\mathbb{P}_\square f_{v_0}\|_2^2 = \operatorname{tr}_{L^2_\square}(f_{v_0}^* \star f_{v_0}) = I^G_\square((f_{v_0}^* \star f_{v_0})\bar{\mathbf{1}}_{G_{\infty\setminus v_0}}\bar{\mathbf{1}}_{K'}).$$

Therefore:

$$\begin{aligned}
&\| \mathbb{P}_\square(\rho_*^{K'} f_{v_0}^{\epsilon,Z})\|_2^2 \\
&\quad = I_\square^G(((f_{v_0}^{\epsilon,Z})^* \star f_{v_0}^{\epsilon,Z})\bar{\mathbf{1}}_{G_{\infty\setminus v_0}}\bar{\mathbf{1}}_{K'}) \\
&\quad = \sum_{\lambda\in\square} \operatorname{tr}_{V_\lambda}((f_{v_0}^{\epsilon,Z})^* \star f_{v_0}^{\epsilon,Z}) I_\square^G(\mathrm{EP}_\lambda \bar{\mathbf{1}}_{G_{\infty\setminus v_0}}\bar{\mathbf{1}}_{K'}) \\
&\quad \overset{7.6}{=} \sum_{\lambda\in\square}(\dim\lambda)\epsilon^{N^{\mathrm{der}}}\|\lambda\|^{-N^{\mathrm{der}}} J_{N^{\mathrm{der}}/2}(\epsilon\|\lambda\|)^2 \left|\widehat{\mathbf{1}_Z}(\lambda|_{\mathfrak{z}})\right|^2 I_\square^G(\mathrm{EP}_\lambda \bar{\mathbf{1}}_{G_{\infty\setminus v_0}}\bar{\mathbf{1}}_{K'}) \\
&\quad \overset{7.1}{\leq} \epsilon^{N^{\mathrm{der}}}\sum_{\lambda\in\square}(\dim\lambda)\|\lambda\|^{-N^{\mathrm{der}}} J_{N^{\mathrm{der}}/2}(\epsilon\|\lambda\|)^2 \left|\widehat{\mathbf{1}_Z}(\lambda|_{\mathfrak{z}})\right|^2 \dim_\square(\lambda)(\Lambda + O(m(\lambda)^{-1})) \\
&\quad \leq C\epsilon^{N^{\mathrm{der}}}\sum_{\lambda\in\square}(\dim\lambda)\|\lambda\|^{-N^{\mathrm{der}}} J_{N^{\mathrm{der}}/2}(\epsilon\|\lambda\|)^2 \left|\widehat{\mathbf{1}_Z}(\lambda|_{\mathfrak{z}})\right|^2 \dim_\square(\lambda) \\
&\quad \leq C\epsilon^{N_G^{\mathrm{der}}}\sum_{\lambda\in\square}\|\lambda\|^{-N_G^{\mathrm{der}}+P_G+P_{G_F(\square)}-e(\square)} J_{N_G^{\mathrm{der}}/2}(\epsilon\|\lambda\|)^2 \left|\widehat{\mathbf{1}_Z}(\lambda|_{\mathfrak{z}})\right|^2,
\end{aligned} \tag{7.6}$$

for some constant C depending only on G, $\square$, and K' and where the last step uses lemma 5.16 and Corollary 5.18. Recall the convention $\lambda \in \square$ to mean that λ is a possible total infinitesimal character for a parameter of shape $\square$ and also recall formula (5.4) defining $e(\square)$.

Consider first the case $Z = [-\epsilon/2, \epsilon/2]$. Note that $\widehat{\mathbf{1}_Z}$ is zero on any character that sends $\lambda(-1) = -1$, so the lattice of possible λ is of the form $L_G \times L_Z$ where L_Z is a character of $U_1/\pm 1$ and L_G are regular, integral infinitesimal characters for $(G_{v_0})_{\mathrm{ad}}$. Summing over L_Z using Poisson summation on $\mathfrak{z}$ turns the $\left|\widehat{\mathbf{1}_Z}(\lambda|_{\mathfrak{z}})\right|^2$ into $\mathbf{1}_Z \star (\mathbf{1}_Z)^*(0) = \epsilon$ as long as ϵ is small enough. Therefore, our estimate (7.6) becomes:

$$C\epsilon^{N_G} \sum_{\substack{\lambda\in\square \\ \lambda \text{ for } (G_{v_0})_{\mathrm{ad}}}} \|\lambda\|^{-N_G^{\mathrm{der}}+P_G+P_{G_F(\square)}-e(\square)} J_{N_G^{\mathrm{der}}/2}(\epsilon\|\lambda\|)^2. \tag{7.7}$$

In the other case $Z = [-1/2, 1/2]$, note that $\widehat{\mathbf{1}_Z}(\lambda|_{\mathfrak{z}})$ is an indicator function if $\lambda|_{\mathfrak{z}} = 1$, so this simply changes the ϵ^{N_G} coefficient on the sum back into $\epsilon^{N_G^{\mathrm{der}}}$.

Now, we input the asymptotics for $J_{N/2}$ to evaluate the sum. For $\|\lambda\| \ll 1/\epsilon$, the terms in the sum are

$$\ll \epsilon^{N_G^{\mathrm{der}}}\|\lambda\|^{P_G+P_{G_F(\square)}-e(\square)}.$$

The $\lambda \in \square$ that are integral on $(G_{v_0})_{\mathrm{ad}}$ form an $(r_{G_F(\square)} - 1)$-dimensional sublattice of all λ (shifted by a small fixed vector depending only on $\square$ that becomes negligible as $1/\epsilon \to \infty$). Assume $r_{G_F(\square)} \geq 2$. Then, summing over the $1/\epsilon$-ball in this subspace gives something of order

$$\ll \epsilon^{N_G^{\mathrm{der}}-P_G-P_{G_F(\square)}+e(\square)-r_{G_F(\square)}+1} = \epsilon^{N_G-P_G-P_{G_F(\square)}-r_{G_F(\square)}+e(\square)} = \epsilon^{(N_G-P_G)-(N_{G_F(\square)}-P_{G_F(\square)})+e(\square)},$$

after approximating the sum by an integral, which we can do since $r_{G_F(\square)} \geq 2$ means the number of terms in the sum with $\|\lambda\| \ll 1/\epsilon$ goes to infinity as $\epsilon \to 0$.

On the other side, $\|\lambda\| \ll 1/\epsilon$, the terms in the sum are

$$\ll \epsilon^{-1}\|\lambda\|^{-N_G^{\mathrm{der}}+P_G+P_{G_F(\square)}-e(\square)-1},$$

which after summing gives something of the same order

$$\ll \epsilon^{N_G^{\mathrm{der}} - P_G - P_{G_F(\square)} - r_{G_F(\square)} + e(\square) + 1} = \epsilon^{(N_G - P_G) - (N_{G_F(\square)} - P_{G_F(\square)}) + e(\square)}.$$

We substitute this into the sum from (7.7), first in the case of $Z = (-\epsilon/2, \epsilon/2)$. Then, when $r_{G_F(\square)} \geq 2$:

$$\| \mathbb{P}_\square(\rho_*^{K'} f_{v_0}^{\epsilon,Z}) \|_2^2 \ll \epsilon^{N_G + (N_G - P_G) - (N_{G_F(\square)} - P_{G_F(\square)}) + e(\square)}.$$

When $r_{G_F(\square)} = 1$, there is a single $\lambda \in \square$. We can therefore treat $\|\lambda\|$ as a constant and reproduce the same formula (note that for this shape $e(\square) = P_G$).

The case of of Z_1 just removes a power of ϵ. Unifying the two cases by noting that $\|f_{v_0}^{\epsilon,Z_\epsilon}\|_2^2 \asymp \epsilon^{N_G}$ and $\|f_{v_0}^{\epsilon,Z_1}\|_2^2 \asymp \epsilon^{N_G^{\mathrm{der}}}$, we get:

Theorem 7.7. *Normalize $\mathfrak{z}$ so that intervals of length 1 exactly cover $Z_{G_{v_0}} = U_1$. Then, if Z is either $(-\epsilon/2, \epsilon/2)$ or $(-1/2, 1/2)$:*

$$\frac{\| \mathbb{P}_\square(\rho_*^{K'} f_{v_0}^{\epsilon,Z}) \|_2^2}{\|f_{v_0}^{\epsilon,Z}\|_2^2} \ll \epsilon^{(N_G - P_G) - (N_{G_F(\square)} - P_{G_F(\square)}) + e(\square)}.$$

As two special cases, we get ϵ^0 when $\square = (n, 1)$ is the trivial shape and $\epsilon^{N_G - 1}$ when $\square = (1, n)$ is the shape for 1-d representations.

7.3 Density Hypothesis Proof

Now we can put together Theorems 7.7 and 6.14 to prove the density hypothesis.

First,

$$\sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G)} \mathcal{F}_{\epsilon,Z}(\pi) = \frac{\|f_{v_0}^{\epsilon,Z}\|_2^2}{\|f_{v_0}^{\epsilon,Z}\|_2^2} = 1.$$

The automorphic characters restricted to $\Gamma \backslash G_{v_0}$ span $L^2(\Gamma \backslash G_{v_0} / G_{v_0}^{\mathrm{der}})$, so we can calculate projections:

$$\mathbb{P}_{\mathcal{AC}(G)} f(x) = \int_{g \in G_{v_0}^{\mathrm{der}}} f(xg) \, dg.$$

This allows us to compute

$$\sum_{\pi \in \mathcal{AC}(G)} \mathcal{F}_{\epsilon,Z}(\pi) = \frac{\| \mathbb{P}_{\mathcal{AC}(G)}(\rho_*^{K'} f_{v_0}^{\epsilon,Z}) \|_2^2}{\|f_{v_0}^{\epsilon,Z}\|_2^2} \asymp \mathrm{vol}(Z) \epsilon^{N_g - 1}.$$

Since there are finitely many possible $\square$ for each N, Theorem 6.14 gives that the density hypothesis for $\mathcal{F}_{\epsilon,Z_1}$ at some finite, non-split place v would be implied by:

Theorem 7.8.

$$\sum_{\pi \in \square} \mathcal{F}_{\epsilon,Z}(\pi) = \frac{\| \mathbb{P}_\square(\rho_*^{K'} f_{v_0}^{\epsilon,Z}) \|_2^2}{\|f_{v_0}^{\epsilon,Z_1}\|_2^2} \ll \epsilon^{(N_G - 1)\left(1 - \frac{2}{\sigma_\square}\right)}.$$

Proof. The inequality we want to show is that

$$R_G(\square) := (N_G - P_G) - (N_{G_F(\square)} - P_{G_F(\square)}) + e(\square) \geq (N_G - 1)\left(1 - \frac{2}{\sigma_\square}\right) =: S_G(\square).$$

Define the Arthur-SL_2 of a shape $\square$ to be the partition Q determined by the restriction of $\psi \in \square$ to the Arthur-SL_2. Note that the left-hand side of the inequality only depends on the

Arthur-SL_2 so call it $S_G(Q)$. Let $R_G(Q)$ be the minimum of $R_G(\square)$ over $\square$ with Arthur-SL_2 given by Q—this is achieved for the unique such $\square = ((T_i, d_i))$ with all d_i distinct.

It therefore suffices to show that $R_G(Q) \geq S_G(Q)$ for all Q. Let d be the maximum size of a part of Q. Then

$$R_G(Q) \geq R_G(d, 1, \dots, 1) = \frac{n(n+1)}{2} - 1 - \frac{(n-d)(n-d+1)}{2} + \frac{d(d-1)}{2} = nd - 1.$$

Recall the definitions of Q_d and Q'_d from [DGG23, lem 12.4.3], which also gives that if we also have that $Q \neq Q_d$, then

$$S_G(Q) \leq S_G(Q'_d) = (n^2-1)\frac{d-1}{n - \lfloor n/d \rfloor + 1}.$$

By a computer check, this always gives $R_G(Q) \geq S_G(Q)$.

It remains to check the case when $Q = Q_d$. Let $r = \lfloor n/d \rfloor$ and $q = n - rd$. Then

$$R_G(Q_d) = \frac{n(n+1)}{2} - \frac{r(r+1)}{2} - \mathbf{1}_{q \neq 0} + \frac{rd(d-1)}{2} + \frac{q(q-1)}{2}$$

and by [DGG23, 12.3.4],

$$S_G(Q_d) = (n^2-1)\frac{d-1}{n - \lfloor n/d \rfloor}.$$

By computer check again, we always have that $R_G(Q_d) \geq S_G(Q_d)$. □

Summarizing the final result:

Corollary 7.9. *Let $G = U_n^{E,H}$ be a definite unitary group and $K' < G^\infty$ be open compact. Pick infinite place v_0 and define $f_{v_0}^{\epsilon, Z_\epsilon}$ as in (7.4). Then the family*

$$\mathcal{F}_{\epsilon,Z}^{K'}(\pi) := m_\pi \frac{\| \mathbb{P}_{\pi_{v_0}}(f_{v_0}^{\epsilon,Z}) \|_2^2}{\| f_{v_0}^{\epsilon,Z} \|_2^2} \mathbf{1}_{\pi_{\infty \setminus v_0} \ triv.} \dim\left((\pi^\infty)^{K'} \right)$$

satisfies the density hypothesis at unramified finite place v in the following cases:

- $n = 4$,
- *K'_v is hyperspecial,*
- *v is split,*
- *Conjecture 6.10 holds for Arthur-type representations of G_v with a K'_v-fixed vector.*

Proof. This follows from Theorem 7.8 together with Theorem 6.14, remark 6.15 and Corollary 6.12 . □

Corollary 7.10. *Corollary 7.9 holds when $n = 8$*

Proof. For every restriction of a parameter ψ_v to the Arthur and Deligne-SL_2's, we by computer list out all the possible exponents of π_v satisfying the conditions of Theorem 6.9. Bounding these potential $\sigma(\pi_v)$ by 6.6, the only cases of Langlands data that violate the bound in Theorem 7.8 with $\sigma_\square$ replaced by $\sigma(\pi_v)$ are:

$$\pi_v \subseteq [3] \cdot ||^{-1} \rtimes \pi_{\mathrm{temp}} \text{ in packet } \psi_v = [5][1] + [1][3], \tag{7.8}$$

$$\pi_v \subseteq [2] \cdot ||^{-1} \times [2] \cdot ||^{-1} \rtimes 0 \text{ in packet } \psi_v = [4][1] + [1][4]. \tag{7.9}$$

in the natural shorthand describing exponents and Arthur/Deligne-SL_2-pieces.

For the case (7.8), the infinitesimal character of π_v always has a factor of the form $\rho||^0$ or $\rho||^{1/2}$ coming from the choice of 2-dimensional π_{temp}. The $\rho||^{1/2}$ cannot occur since the infinitesimal character of the packet has only integral powers of $||$ and the $\rho||^0$ cannot occur because the two zero powers of $||$ in the infinitesimal character of the packet ψ_v are already accounted for by the $[3]\cdot||^{-1}$.

The case (7.9) cannot occur since it violates [Mœg0901, Thm 6.3]—it corresponds to partition $(2,2,2,2)$ while the packet restricted to the Deligne-SL_2 corresponds to partition $(4,1,1,1,1)$. $\square$

Remark 7.11. The variant of Corollary 7.9 for $\mathcal{F}_{\epsilon,Z_\epsilon}$ does not actually hold: then we would require

$$\frac{\|\mathbb{P}_\square(\rho_*^{K'} f_{v_0}^{\epsilon,Z})\|_2^2}{\|f_{v_0}^{\epsilon,Z_1}\|_2^2} \ll \epsilon^{N_G\left(1-\frac{2}{\sigma_\square}\right)}$$

which reduces to $\epsilon^{N_G-1} \ll \epsilon^{N_G}$ for $\square$ the shape of 1-d representations.

8 Optimal covering

Here we translate the spectral analysis of previously constructed gate sets in $PU(n)$ to the settings of automorphic representation theory and show that the density Theorem 7.9 implies the optimal covering property. Theorem 1.2 from the introduction will then follow from the main result 8.6 of this section and Theorem 1.3 from the main result combined with Proposition 4.18.

We make a technical assumption that $\mathcal{O}_F$ is Euclidean for the approximation property from 4.13 to hold.

Theorem 8.1. *Let $G = U_N^{E/F,H}$ be a definite unitary group and choose distinguished Archimedean place v_0 of F. Let $K^{v_0} = K' G_{\infty\setminus v_0}$ for $K' \le G(\hat{\mathbb{Z}})$ a golden adelic group that is golden at $\mathfrak{p}$ (resp. super-golden satisfying assumption 4.6).*

Recall the definition of the gate set $S_\mathfrak{p} := S_\mathfrak{p}^{K'}$ from 4.4 (resp. ${}^0S_\mathfrak{p}$ and $C_\mathfrak{p}$ from 4.7 and the discussion afterwards). Then $S_\mathfrak{p}$ is a golden gate set (resp. ${}^0S_\mathfrak{p} \cup C_\mathfrak{p}$ is a super-golden gate set) of $G(F_{v_0})/U_1 = PU(n)$.

For convenience, assume $F = \mathbb{Q}$ on the first read so v_0 is the sole infinite place and $K' \le G(\widehat{\mathbb{Z}})$.

8.1 A Hecke Operator

We start by interpreting as a Hecke operator the operation of averaging over translates by the set $S_p^{(\ell)}$ of words in gates that with minimum representation of length ℓ.

First, note that since K' is a golden adelic group, we get the following identifications by lemma 4.2:

$$\begin{aligned} L^2(U(n)) &\cong L^2\left(\Lambda_p\backslash G(F_{v_0}) \times G(F_\mathfrak{p})\right)^{K'_\mathfrak{p}} \\ &\cong L^2(G(\mathbb{Q})\backslash G(\mathbb{A}^\infty) \times G(F_{v_0}))^{K' G_{\infty\setminus v_0}} \cong L^2\left(G(\mathbb{Q})\backslash G(\mathbb{A})\right)^{K' G_{\infty\setminus v_0}}. \quad (8.1)\end{aligned}$$

In particular, we can decompose

$$V = L^2(U(n)) = \bigoplus_{\substack{\pi \in \mathcal{AR}_{\mathrm{disc}}(G) \\ \pi_{\infty\setminus v_0} \text{ trivial}}} \pi_{v_0} \boxtimes (\pi^\infty)^{K'},$$

where the right-translation action corresponds to the action on the left factor of the $\boxtimes$ as representations of $G(F_{v_0}) = U(n)$. We also get corresponding subspaces $V_\square$ and restricted projections $\mathbb{P}_\square : V \to V_\square$.

Through the right factor of the $\boxtimes$, this decomposition also respects an action of Hecke operators:

Definition 8.2. Let $\mathfrak{p}$ be a finite place of F, $\pi_\mathfrak{p}$ a $G_\mathfrak{p}$-representation. Then any finite set $S \subset G_\mathfrak{p}$ defines a Hecke operator $\mathbb{1}_{K_\mathfrak{p} S K_\mathfrak{p}} \in C_c\left(K_\mathfrak{p} \backslash G_\mathfrak{p} / K_\mathfrak{p}\right)$, which acts on $\pi_\mathfrak{p}^{K_\mathfrak{p}}$:

$$\mathbb{1}_{K_\mathfrak{p} S K_\mathfrak{p}}.u := \int_{g \in K_\mathfrak{p} S K_\mathfrak{p}} \pi_v(g).u \, dg = \sum_{s \in K_\mathfrak{p} S K_\mathfrak{p} / K_\mathfrak{p}} \pi_v(s).u,$$

where the integral is normalized by $\mathrm{vol}(K_\mathfrak{p}) = 1$.

Recall the definition of $S_\mathfrak{p}^{(\ell)}$: the set of words in $S_\mathfrak{p}$ of length precisely ℓ in their shortest representation. As a technicality when $G_\mathfrak{p}$ has non-anisotropic center, recall also from Proposition 3.12 their lifts $\widetilde{S}_\mathfrak{p}^{(\ell)}$ and define the following operator $T_{S_\mathfrak{p}^{(\ell)}}$ on V:

$$\left(T_{S_\mathfrak{p}^{(\ell)}} f\right)(g) = \frac{1}{|S_\mathfrak{p}^{(\ell)}|} \sum\nolimits_{s \in \widetilde{S}_\mathfrak{p}^{(\ell)}} f\left(s^{-1} g\right) \qquad \left(f \in L^2(U(n))\ ,\ g \in U(n)\right).$$

In the super-golden case, we replace $\widetilde{S}_\mathfrak{p}^{(\ell)} \mapsto C_\mathfrak{p}{}^0 \widetilde{S}_\mathfrak{p}^{(\ell)}$. This is a Hecke operator:

Lemma 8.3. *For any $\ell \in \mathbb{N}$, there is a bijection between, $K_\mathfrak{p} \widetilde{S}_\mathfrak{p}^{(\ell)} K_\mathfrak{p} / K_\mathfrak{p}$ and $\widetilde{S}_p^{(\ell)}$, and the operator, $T_{S_\mathfrak{p}^{(\ell)}}$, is equal to the normalized Hecke operator $\frac{1}{|S_\mathfrak{p}^{(\ell)}|} \mathbb{1}_{K_\mathfrak{p} \widetilde{S}_\mathfrak{p}^{(\ell)} K_\mathfrak{p}}$ (resp. replacing $\widetilde{S}_\mathfrak{p}^{(\ell)} \mapsto C_\mathfrak{p}{}^0 \widetilde{S}_\mathfrak{p}^{(\ell)}$ in the super-golden case).*

Proof. The second claim follows from the first: thinking of $L^2(U(n)) = L^2(\Lambda_\mathfrak{p} \backslash G_{v_0} \times G_\mathfrak{p})^{K_\mathfrak{p}}$, the s-action on the G_v coordinate is equivalent to a s^{-1} action on the left on the $U(n)$ coordinate.

By Proposition 3.12 as used in Section 4.2, $\widetilde{S}_\mathfrak{p}^{(\ell)}.v_0$ is the set of all $v \in \mathcal{B}$ (or lifts v of depending on $v - v_0 \in X_+(\widehat{A}_H)$ and the choice of $\widetilde{\Sigma}$) such that $\|v - v_0\|' = \ell$. In the super-golden case, we instead look at elements $g\tau$ for $g \in G_\mathfrak{p}$ such that $\|gv_0 - v_0\|' = \ell$ (again possibly lifted depending on $gv_0 - v_0 \in X_+(\widehat{A}_H)$).

Since $K_\mathfrak{p}$ preserves the $v - v_0 \in X_+(\bar{A}_H)$, we get that $K_\mathfrak{p} \widetilde{S}_\mathfrak{p}^{(\ell)}.v_0 = \widetilde{S}_\mathfrak{p}^{(\ell)}.v_0$, hence there is a bijection between $K_\mathfrak{p} \widetilde{S}_\mathfrak{p}^{(\ell)} K_\mathfrak{p} / K_\mathfrak{p}$ and $\widetilde{S}_\mathfrak{p}^{(\ell)}$ (resp. the super-golden analogues). This completes the proof. □

We can now use another key input to bound operator norms of Hecke operators acting on π_v:

Proposition 8.4. [Kam16] *In the above notations (in particular, $K_\mathfrak{p}$ is contained in an Iwahori subgroup), assume that $S \to S / Z_{G_\mathfrak{p}}^{\mathrm{spl}}$ has fibers of size bounded by some fixed M. Then, for any $\sigma \geq 2$, if unitary irreducible representation $\pi_\mathfrak{p}$ of $G_\mathfrak{p}$ is in $L^{\sigma+\epsilon}$ mod center for all $\epsilon > 0$,*

$$\|\mathbb{1}_{K_\mathfrak{p} S K_\mathfrak{p}} \mid \pi_\mathfrak{p}\|_{op} \lesssim \left(|K_\mathfrak{p} S K_\mathfrak{p} / K_\mathfrak{p}|\right)^{\frac{\sigma-1}{\sigma}}$$

asymptotically in $|S|$ for all such S.

Proof. Since K_v is contained in an Iwahori, we can write $\mathbf{1}_{K_\mathfrak{p} S K_\mathfrak{p}}$ as a sum of terms of the form h_w in the notation of [Kam16]. Since the number of double cosets h_w with $l(w) = l_0$ is polynomial in l_0, we have $|K_\mathfrak{p} S K_\mathfrak{p}/K_\mathfrak{p}| \sim q^{l_{\max}}$ where $l_{\max}$ is the maximum of $l(w)$ for w appearing in this sum. This then follows from [Kam16, Thm 1.8] (which can be generalized to non-semisimple $G_\mathfrak{p}$ by noting that the action of $G_\mathfrak{p}$ on $\mathcal{B}$ factors through $G_\mathfrak{p}/Z^{\mathrm{spl}}_{G_\mathfrak{p}}$ and that π_v has unitary central character). □

8.2 Optimal Covering Proof

We can now put everything together to prove Theorem 8.1 using the identifications

$$L^2_\square = V_\square := \bigoplus_{\substack{\pi \in \square \\ \pi_{\infty \setminus v_0} \text{ trivial}}} m_\pi \pi_{v_0} \boxtimes (\pi^\infty)^{K^\infty}.$$

We start with a corollary of the density hypothesis, which allows us to interpolate an inequality from $2/\sigma = 1$ and $2/\sigma = 0$ to all values in between:

Proposition 8.5. *If $n = 4, 8$, for any shape $\square$, any ℓ, and $\epsilon^{N_G - 1} \lesssim |S^{(\ell)}_\mathfrak{p}|^{-1}$,*

$$\|T_{S^{(\ell)}_\mathfrak{p}} \mid V_\square\|^2_{op} \cdot \|\,\mathbb{P}_\square f^{\epsilon, Z_1}_{v_0}\|^2_2 \lesssim |S^{(\ell)}_\mathfrak{p}|^{-1} \cdot \|f^{\epsilon, Z_1}_{v_0}\|^2_2.$$

Proof. This is a reformulation of Corollaries 7.9 and 7.10: it follows from Theorem 6.14, Proposition 8.4, and Theorem 7.8/the extra computations in the proof of Corollary 7.10. □

For convenience, we reindex

$$I_\delta := f^{\epsilon, Z_1}_{v_0}$$

for the ϵ such that it has support of volume δ. In particular $\delta \asymp \epsilon^{N_G - 1}$ so Proposition 8.5 applies to $\delta \lesssim |S^{(\ell)}_p|^{-1}$.

Note also that the projection of this support onto $PU(n)$ has the same volume (normalizing $\mathrm{vol}(U(n)) = \mathrm{vol}(PU(n)) = 1$) and is a ball in an invariant metric. In other words

$$\mathrm{supp}(\bar{f}^{\epsilon, Z_1}_{v_0} : PU(n) \to \mathbb{C}) = B^{PU(n)}(\delta).$$

In addition, since $f^{\epsilon, Z_1}_{v_0}$ is analytic, constant on $U(1)$ orbits, and equal to 1 at the identity,

$$\langle I_\delta, \mathbf{1} \rangle = (\delta + o(\delta)). \tag{8.2}$$

Combining the above propositions, we are now in a position to estimating the covering rate of S_p in terms of the spectrum of the operators $T_{S^{(\ell)}_p}$ evaluated on each subspace $V_\square$ separately.

Proposition 8.6. *If $n = 4, 8$, the optimal covering property (Definition 1.1) for $PU(n)$ holds for $S_\mathfrak{p}$ (resp. $C_\mathfrak{p} \cup {}^0 S_\mathfrak{p}$).*

Proof. For notational simplicity, we argue only in the golden case. The super-golden case follows very similarly.

Our parameter is ℓ, and we take δ as a function of ℓ:

$$\delta := \frac{c_1 (\log |S^{(\ell)}_\mathfrak{p}|)^{c_2}}{|S^{(\ell)}_\mathfrak{p}|} \quad , \quad f_\delta := I_\delta - \langle I_\delta, \mathbf{1} \rangle \mathbf{1}.$$

On the one hand,

$$\|T_{S_{\mathfrak{p}}^{(\ell)}} f_\delta\|_2^2 = \int_{U(n)} \left[T_{S_{\mathfrak{p}}^{(\ell)}} \left(I_\delta - \langle I_\delta, \mathbf{1}\rangle \mathbf{1}\right)(x)\right]^2 \, dx,$$

so since the support of $T_{S_{\mathfrak{p}}^{(\ell)}}(I_\delta)$ is contained in the pullback to $U(n)$ of $B(S_{\mathfrak{p}}^{(\ell)}, \delta) \subseteq PU(n)$,

$$\|T_{S_{\mathfrak{p}}^{(\ell)}} f_\delta\|_2^2 \geq \int_{PU(n)\setminus B(S_{\mathfrak{p}}^{(\ell)},\delta)} \left[\langle I_\delta, \mathbf{1}\rangle \mathbf{1}(x)\right]^2 \, dx = (\delta + o(\delta))^2 \cdot \mu\left(PU(n) \setminus B(S_{\mathfrak{p}}^{(\ell)}, \delta)\right).$$

using (8.2) for the last step.

On the other hand,

$$\|T_{S_{\mathfrak{p}}^{(\ell)}} f_\delta\|_2^2 \leq \|T_{S_{\mathfrak{p}}^{(\ell)}} I_\delta\|_2^2 = \|\sum_{\square} T_{S_{\mathfrak{p}}^{(\ell)}} \mathbb{P}_\square I_\delta\|_2^2 = \sum_{\square} \|T_{S_{\mathfrak{p}}^{(\ell)}} \mathbb{P}_\square I_\delta\|_2^2$$
$$\leq \sum_{\square} \|T_{S_{\mathfrak{p}}^{(\ell)}} \mid V_\square\|_{op}^2 \cdot \|\mathbb{P}_\square I_\delta\|_2^2,$$

so noting that the number of possible $\square$ is a constant depending only on n, Proposition 8.5 gives:

$$\|T_{S_{\mathfrak{p}}^{(\ell)}} f_\delta\|_2^2 \lesssim |S_{\mathfrak{p}}^{(\ell)}|^{-1} \|I_\delta\|_2^2 = |S_{\mathfrak{p}}^{(\ell)}|^{-1}\delta.$$

Combining the two estimates together, we get

$$\mu\left(U(n) \setminus B(S_{\mathfrak{p}}^{(\ell)}, \delta)\right) \lesssim \frac{1}{c_1(\log |S_{\mathfrak{p}}^{(\ell)}|)^{c_2}} \xrightarrow{\ell\to\infty} 0,$$

which proves that S_p has the optimal covering property. $\square$

Finally, we note that Theorem 8.1 follows from Theorem 4.13 and Proposition 8.6.

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