MASTER WORKSHOP

EXERCISES IX (PERMUTATIONS, BROUWER'S FIXED POINT THEOREM, ANALYTIC FUNCTIONS IN THE DISK)

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$|x|$ The Euclidean norm in $\mathbb{R}^n$.

(1) (a) Study the paper

P.D. Lax, Change of Variables in Multiple Integrals, The American Mathematical Monthly 106 (1999), pp. 497-501,

for a very elegant proof of the "change of variables" formula.
Note that the basic identity (2.10) in the paper is Problem 5 in the previous Set VIII.

(b) Fill in all the details in the proof of the
Intermediate Value Theorem: Let φ be a continuous map of the unit ball in $\mathbb{R}^n$ into $\mathbb{R}^n$, such that $\varphi(x) = x$ for $|x| = 1$. Then the image of φ contains the unit ball.

(c) Derive from it the **Brouwer Fixed Point Theorem**.

(2) (a) **Definition.** A permutation $\sigma \in S_n$ is called a "**zigzag permutation**" if for all $2 \leq k \leq n-1$ the triplet $\sigma(k-1), \sigma(k), \sigma(k+1)$ is not monotone.
We say that the permutation begins with a "rise" (resp. "fall") if $\sigma(2) > \sigma(1)$ (resp. $\sigma(2) < \sigma(1)$).

(b) Prove that the number of zigzag permutations beginning with a rise equals the number of zigzag permutations beginning with a fall.
Notation. The number of all zigzag permutations (of n elements) is denoted by $2A_n$.
We adopt the convention (for later notational convenience) $A_0 = A_1 = A_2 = 1$.
For every $0 \leq j \leq n$ we denote $a_j = \frac{A_j}{j!}$.

(c) Prove the recurrence formula

$$2na_n = \sum_{j=0}^{n-1} a_j a_{n-1-j}, \quad n \geq 2.$$

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(d) Prove that the power series
$$g(x)=\sum_{n=0}^{\infty}a_nx^n$$
converges (at least) in the interval $(-1,1)$.

(e) Show that the function $g(x)$ satisfies in $(-1,1)$
$$g'(x)=\frac{1}{2}(1+g(x)^2)$$
and determine $g(x)$ explicitly.

(f) Prove that
$$\tan x=\sum_{k=1}^{\infty}\frac{A_{2k-1}}{(2k-1)!}x^k,$$
and relate the A_k to the Bernoulli numbers (see Problem 3 in Set I)

(3) Let a,b,c,d be positive numbers. Show that
$$\frac{(a^2+a+1)(b^2+b+1)(c^2+c+1)(d^2+d+1)}{abcd}\geq 81.$$

(4) (a) Let $f(z)$ be analytic in the open unit disk D and $|f(z)|\leq 1$ in D. Prove that
$$|f'(z)|\leq\frac{1-|f(z)|^2}{1-|z|^2},\quad z\in D.$$

(b) Prove that
$$\frac{|f(0)|-|z|}{1-|f(0)||z|}\leq|f(z)|\leq\frac{|f(0)|+|z|}{1+|f(0)||z|},\quad z\in D.$$

(c) For every arc $\Gamma\subseteq D$ define a (non-Euclidean) length by $L(\Gamma)=\int_\Gamma\frac{|dz|}{1-|z|^2}$, and show that if $f(z)$ is analytic in D and bounded there by 1 then $L(f(\Gamma))\leq L(\Gamma)$.

(d) Prove the estimates in (b) by considering the length of the segment connecting z to the origin (and using the fact that $\frac{d|f|}{d|z|}\leq|\frac{df}{dz}|$).

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