MASTER WORKSHOP

EXERCISES VIII (GEOMETRY OF CURVES, LAPLACE ASYMPTOTIC METHOD)

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BOOKS

[PM] R.S. Millman and G.D. Parker, Elements of Differential Geometry, Prentice-Hall Inc. 1977.
[E] L. C. Evans, Partial Differential Equations, Graduate Studies in Mathematics Vol. 19, AMS 1998.

$|x|$ The Euclidean norm in $\mathbb{R}^n$.

(1) **THEOREM (THE ISOPERIMETRIC INEQUALITY)**
Let $\boldsymbol{\alpha}$ be a simple closed regular plane curve of length L. Let A be the area of the region bounded by $\boldsymbol{\alpha}$. Then $L^2 \geq 4\pi A$ with equality if and only if $\boldsymbol{\alpha}$ is a circle.
Prove this theorem following Sec. 3-4 of [PM] (where you'll find also the definitions for terms used in the statement).

(2) Give another proof of the isoperimetric inequality as follows.
  (a) Consider the family of straight lines, parametrized by $\beta \in [0, 2\pi]$,
$$x \cos\beta + y \sin\beta - p(\beta) = 0,$$
where $p(\beta)$ is twice continuously differentiable and periodic of period 2π. Let $C : (x(\beta), y(\beta)), \quad \beta \in [0, 2\pi]$, be an envelope to the family. Show that C satisfies the following equations.
$$x'(\beta) = -(p' + p'')\sin\beta, \quad y'(\beta) = (p + p'')\cos\beta.$$
  (b) Show that L and A (length and area, respectively) can be expressed as
$$L = \int_0^{2\pi} p(\beta)d\beta, \qquad A = \frac{1}{2}\int_0^{2\pi}(p(\beta)^2 - p'(\beta)^2)d\beta.$$
  (c) Write the Fourier expansion
$$p(\beta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\beta + b_n \sin n\beta),$$

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and deduce

$$L = \pi a_0, \qquad A = \frac{\pi}{2}\Big[\frac{a_0^2}{2} - \sum_{n=1}^{\infty}(n^2-1)(a_n^2+b_n^2)\Big].$$

(3) (Finding the treasure) A straight gold rod of length 2 is buried (horizontally) in the ground. Its location is not known, but it is known that its (orthogonal) distance from a fixed (known) point does not exceed 1. It is decided to dig a narrow canal that will intersect the rod. What is the minimal length of such a canal, which still ensures that the rod is intersected at some point?

(4) Let $\boldsymbol{\gamma}$ be a curve on the sphere with κ_g (geodesic curvature) constant. Prove that $\boldsymbol{\gamma}$ is a circle.
(See [PM], Problem 4.9 in Sec. 4.4, p. 108).

(5) (The null divergence lemma). Let $\boldsymbol{u}$:$\mathbb{R}^n \to \mathbb{R}^n$ be a smooth function and let $J = D\boldsymbol{u}$ be its Jacobian matrix, whose row(k)=$(\frac{\partial u_k}{\partial x_1}, ..., \frac{\partial u_k}{\partial x_n})$, $\quad k = 1,2,...,n$. Let $cof\, J$ be the corresponding cofactor matrix (so that $det(J)I = J^T cof\, J$). Let Let $\boldsymbol{v}^k$ be the k-th row of $cof\, J$, $\quad k = 1,2,...,n$. Prove that

$$div(\boldsymbol{v}^k) \equiv 0, \quad k = 1,2,...,n.$$

(See [E], Sec. 8.1.4, p. 440. If you know some differential forms, you might find an easy proof).

(6) (a) (Laplace's method for asymptotics). Suppose that $k, l : \mathbb{R} \to \mathbb{R}$ are continuous functions, that l grows at most linearly and that k grows at least quadratically. Assume that there exists a unique point $y_0 \in \mathbb{R}$ such that $k(y_0) = \min_{y\in\mathbb{R}} k(y)$. Prove that

$$\lim_{\varepsilon\to 0^+} \frac{\int_{-\infty}^{\infty} l(y) e^{-\frac{k(y)}{\varepsilon}} dy}{\int_{-\infty}^{\infty} e^{-\frac{k(y)}{\varepsilon}} dy} = l(y_0).$$

(See [E], Sec. 4.5.2, p. 204).

(b) With $k(y)$ as above assume further that it is twice continuously differentiable and that $k''(y_0) > 0$. Show that

$$\lim_{\varepsilon\to 0^+} \frac{\int_{-\infty}^{\infty} e^{-\frac{k(y)}{\varepsilon}} dy}{e^{-\frac{k(y_0)}{\varepsilon}} \int_a^b e^{-\frac{k''(y_0)}{2\varepsilon}(y-y_0)^2} dy} = 1,$$

where (a,b) is any interval containing y_0.

(c) Conclude that

$$\lim_{\varepsilon\to 0^+} \frac{\int_{-\infty}^{\infty} e^{-\frac{k(y)}{\varepsilon}} dy}{\sqrt{\frac{2\pi\varepsilon}{k''(y_0)}}\, e^{-\frac{k(y_0)}{\varepsilon}}} = 1.$$

(d) (Stirling's formula) For any integer n, use the fact that $n! = \int_0^\infty x^n e^{-x} dx$ to get

$$n! = n^{n+1} \int_0^\infty e^{n(\log y - y)} dy,$$

and conclude that

$$\lim_{n\to\infty} \frac{n! e^n}{n^n \sqrt{2\pi n}} = 1.$$

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