MASTER WORKSHOP

EXERCISES VII (COMPLEX FUNCTIONS, GROUPS)

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$|x|$ The Euclidean norm in $\mathbb{R}^n$.

(1) Find the envelopes of the following families of surfaces:
   (a) The family of ellipsoids of constant volume $c > 0$, centered at the origin (in $\mathbb{R}^3$) and having axes along the coordinate axes $(x, y, z.)$.
   (b) The family of planes (in $\mathbb{R}^3$) of the form $ax + by + cz = 1$, with $a^2 + b^2 + c^2 = 1$.
   (c) The family of spheres which are tangent to the three spheres

$$S_1 : (x - \frac{3}{2})^2 + y^2 + z^2 = \frac{9}{4},$$
$$S_2 : x^2 + (y - \frac{3}{2})^2 + z^2 = \frac{9}{4},$$
$$S_3 : x^2 + y^2 + (z - \frac{3}{2})^2 = \frac{9}{4}.$$

(2) How many roots of the equation

$$z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$$

lie in the right half plane?

Reminder. A function v, defined in $\Omega \subseteq \mathbb{C}$ is called **subharmonic** if for any closed disk $\overline{D} \subseteq \Omega$ and any harmonic function u in D which is continuous in $\overline{D}$, the maximum of $v - u$ is obtained on ∂D.

(3) (a) Let $f(z)$ be analytic in Ω. Prove that $|f(z)|$ is subharmonic in Ω.
   (b) Let $f_1(z), f_2(z), ..., f_n(z)$ be analytic in Ω. Prove that $log(\sum_{k=1}^{n} |f_k(z)|^2)$ is subharmonic in Ω.

(4) Does there exist a nonconstant function $f(z)$ which is analytic in $\mathbb{C}$ (i.e., entire), $f(z) \neq 0 \quad \forall z \in \mathbb{C}$, and such that

$$\liminf_{r\to\infty} \frac{\log M(r)}{r^{\frac{1}{3}}} < \infty,$$

where $M(r) = \max_{|z|=r} |f(z)|$?

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(5) Let $M \in GL_2(\mathbb{C})$ (2×2 complex matrices with non-zero determinant). We let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and for $z \in \mathbb{C}$ we let $M(z) = \frac{az+b}{cz+d}$. If $c \neq 0$ we define $M(-\frac{d}{c}) = \infty$.
   (a) Show that $GL_2(\mathbb{C})$ operates on the Riemann sphere $\mathbb{C} \cup \infty$.
   (b) Show that if $M(z)$ is not the identity map $(a = d, b = c = 0)$ then it cannot have more than two fixed points $(M(z) = z)$.
   (c) Assume that $M(z)$ has two fixed points (both different from ∞), and let λ, λ' be the two (complex) eigenvalues of M. Show that if we denote by $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad W' = \begin{pmatrix} w_1' \\ w_2' \end{pmatrix}$ the two corresponding eigenvectors then the two fixed points are given by $w = \frac{w_1}{w_2}, \quad w' = \frac{w_1'}{w_2'}$.
   (d) Assume that $|\lambda| < |\lambda'|$. Show that for every $z \neq w$,
$$\lim_{k\to\infty} M^k(z) = w'.$$
(6) Construct a multiplication table for the group G of rigid motions of a regular pentagon with vertices $1, 2, 3, 4, 5$.
(7) Let G be the group of rigid motions of a cube. Find the order of G.
(8) Find three cyclic subgroups of S_4 that have three different orders.

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