MASTER WORKSHOP

EXERCISES VI (REAL AND COMPLEX FUNCTIONS, SYMMETRIC POLYNOMIALS)

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$|x|$ The Euclidean norm in $\mathbb{R}^n$.

(1) Let $P(x)$ be a real polynomial (in one variable $x \in \mathbb{R}$) of degree n. Assume that $P(x) \geq 0$ for all $x \in \mathbb{R}$, and let $Q(x) = P(x) + P'(x) + P''(x) + ... + P^{(n)}(x)$. Show that $Q(x) \geq 0$ for all $x \in \mathbb{R}$.

(2) Find an example of a real polynomial $p(x, y)$, $(x, y) \in \mathbb{R}^2$, having the following properties.
- The origin $(0, 0)$ is the only critical point of p.
- The origin $(0, 0)$ is a local minimum of p.
- p has no global minimum in $\mathbb{R}^2$.

Reminder. A **critical point** (x_0, y_0) of a differentiable function $g(x, y)$ is a point such that $\nabla g(x_0, y_0) = 0$.

(3) Let $f(z)$ be a meromorphic (i.e., having only poles) function in the whole complex plane $\mathbb{C}$. Assume that there exists a constant $C > 0$ such that

$$|f(z)| \leq \frac{C}{|Im\, z|^{\frac{1}{2}}}, \quad Im\, z \neq 0.$$

Prove that $f(z) = const$.

(4) Let $L \subseteq \mathbb{C}$ be the strip given by $L = \{z; \quad 0 < Re\, z < 1\}$. Let $f(z)$ be analytic in L and continuous and bounded in $\overline{L}$. Prove that

$$\sup_{z \in L} |f(z)| = \max_{k=1,2} \sup_{y \in \mathbb{R}} |f(k + iy)|.$$

Remark. This is one version of the **Phragmen-Lindelöf Theorem.**

(5) A cat walks in a field bounded by two fences that make a right angle. Find all possible trajectories of the cat, if it is known that the tangent line to the trajectory forms a triangle of fixed area with the two fences.

(6) **Definition.** Let $p(x_1, ..., x_n)$ be a real polynomial of n variables. Then p is said to be **symmetric** if for every permutation σ of $\{1, 2, ..., n\}$ we have $p(x_{\sigma(1)}, x_{\sigma(2)}, ...x_{\sigma(n)}) = p(x_1, ..., x_n)$.

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Definition. The $(n+1)$ **basic symmetric polynomials** are,

$$p_0 = 1, \quad p_1(x_1, ..., x_n) = \sum_{i=1}^{n} x_i, ...,$$

$$p_k(x_1, ..., x_n) = \sum_{1 \le i_1 < i_2 ... < i_k \le n} x_{i_1} x_{i_2} ... x_{i_k}, \quad k = 0, ..., n.$$

Let $p(x_1, ..., x_n)$ be a symmetric polynomial of n variables. Prove that it can be expressed as a "polynomial of the basic polynomials", i.e., there exists a real polynomial $g(y_1, ..., y_n)$ such that

$$p(x_1, ..., x_n) = g(p_1, ..., p_n).$$

Furthermore, the coefficients of g are obtained from the coefficients of p by additions and subtractions.

(7) Let $Q(x) = x^n + a_{n-1}x^{n-1} + ... + a_k x^k + ... + a_0$ be a real polynomial, and denote by $\{\alpha_1, ..., \alpha_n\}$ its (complex, in general) roots. Prove that any symmetric polynomial (with real coefficients) of $\{\alpha_1, ..., \alpha_n\}$ can be expressed as a (real) polynomial of $\{a_0, ..., a_{n-1}\}$.

(8) **Definition.** A complex number α is called **algebraic** if it is a zero of a real polynomial with rational coefficients (or integer coefficients, but then the coefficient of x^n is not necessarily one).
Prove that the set of algebraic numbers is a field.

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