MASTER WORKSHOP

EXERCISES V (REAL AND COMPLEX FUNCTIONS)

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$|x|$ The Euclidean norm in $\mathbb{R}^n$.

(1) Let $f : (0,\infty) \to \mathbb{C}$ be in $C_0^1(0,\infty)$ (compactly supported continuously differentiable functions). Show that

$$\int_0^\infty x^{-2}|f(x)|^2 dx \le 4 \int_0^\infty |f'(x)|^2 dx.$$

Suggestion: Use $\frac{d}{dx}(x^{-1}|f(x)|^2) = -x^{-2}|f(x)|^2 + 2x^{-1} Re(f'(x)\overline{f(x)})$.

(2) Let $f : \mathbb{R}^3 \to \mathbb{C}$ be in $C_0^1(\mathbb{R}^3)$ (compactly supported continuously differentiable functions). Show that

$$\int_{\mathbb{R}^3} |x|^{-2}|f(x)|^2 dx \le 4 \int_{\mathbb{R}^3} |\nabla f(x)|^2 dx.$$

Definition. This inequality is called the **Hardy inequality.**

(3) Let $\{f_n(x)\}_{n=1}^\infty$ be a nondecreasing sequence of continuous (real) functions on a compact interval $[a,b] \in \mathbb{R}$. Suppose that

$$(*) \qquad f_n(x) \to f(x), \qquad x \in [a,b],$$

where f is continuous in $[a,b]$. Prove that the convergence (*) is uniform in $[a,b]$.

Remark. This is called **Dini's Theorem.**

(4) Let $\{f_n(x)\}_{n=1}^\infty$ be a sequence of nondecreasing (real) functions (not necessarily continuous) on a compact interval $[a,b] \in \mathbb{R}$. Suppose that

$$(**) \qquad f_n(x) \to f(x), \qquad x \in [a,b],$$

where f is continuous in $[a,b]$. Prove that the convergence (**) is uniform in $[a,b]$.

(5) Let $D = \{z \in \mathbb{C}; \quad |z| < 1\}$ be the open unit disk and let $g : \overline{D} \to \mathbb{C}$ be a continuous function which is analytic in the open disk. Assume further that $|g(z)| = 1$ for $|z| = 1$. Show that g is a rational function.

(6) Let $\Omega \subseteq \mathbb{C}$ be a bounded (open, connected) domain and let f be an analytic function in Ω such that $Re(f) \ge 0$. Is there an analytic function g in Ω such that $Re(f) = |g|$?

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