MASTER WORKSHOP

EXERCISES IV (REARRANGEMENTS OF FUNCTIONS)

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Notation

$|\Omega|, \quad \Omega \subseteq \mathbb{R}^n$ Lebesgue measure of Ω.
$B_R(y)$ The open ball (in $\mathbb{R}^n$) of radius R, centered at y.
ω_n The volume (Lebesgue measure) of $B_1(0)$.
$n\omega_n$ The surface (Lebesgue) measure of $\partial B_1(0)$.
$|x|$ The Euclidean norm in $\mathbb{R}^n$.

$\|\cdot\|_p$ L^p (Lebesgue) norm.

Definition. If $A \subseteq \mathbb{R}^n$ with $|A| < \infty$ we define the **symmetric rearrangement of** A as $A^* = B_r(0)$ such that $\omega_n r^n = |B_r(0)| = |A|$. We define χ_A^* to be the characteristic function of A^*.

Definition. Let $f : \mathbb{R}^n \to \mathbb{C}$ be a measurable function. We say that f **vanishes at infinity** if $|\{x; \quad |f(x)| > t\}| < \infty$ for all $t > 0$.

Definition. Let $f : \mathbb{R}^n \to \mathbb{C}$ vanish at infinity. The **symmetric-decreasing rearrangement** of f is defined by

$$f^*(x) = \int_0^\infty \chi^*_{\{|f|>t\}}(x)dt.$$

(For the following problems you can consult
E.H.Lieb and M. Loss, Analysis, Ch. 3).

(1) Let $f : \mathbb{R}^n \to \mathbb{C}$ vanish at infinity. Show that

$$|f(x)| = \int_0^\infty \chi_{\{|f|>t\}}(x)dt.$$

(2) Prove the following properties of f^*.
   (a) $f^*(x) \geq 0$.
   (b) f^* is radially symmetric (depending only on $|x|$) and nonincreasing (as a function of $|x|$).

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(c) For every $t > 0$,
$$\{x; \quad f^*(x) > t\} = \{x; \quad f(x) > t\}^*,$$
hence
$$|\{x; \quad f^*(x) > t\}| = |\{x; \quad f(x) > t\}|.$$
(d) Let $\varphi \geq 0$ be a nonincreasing function on $[0, \infty)$. Then
$$\int_{\mathbb{R}^n} \varphi(|f(x)|)dx = \int_{\mathbb{R}^n} \varphi(f^*(x))dx,$$
provided that one side is finite.
(e) In particular, for all $1 \leq p \leq \infty$,
$$\|f\|_p = \|f^*\|_p.$$
(f) The rearrangement is "order preserving", i.e., if f, g are two nonnegative functions, vanishing at infinity, such that $f(x) \leq g(x)$ for all $x \in \mathbb{R}^n$, then also $f^*(x) \leq g^*(x)$ for all $x \in \mathbb{R}^n$.

(3) Let f, g be two nonnegative functions on $\mathbb{R}^n$, vanishing at infinity. Prove that
$$\int_{\mathbb{R}^n} f(x)g(x)dx \leq \int_{\mathbb{R}^n} f^*(x)g^*(x)dx,$$
with the understanding that if the left-hand side is infinite, so is the right-hand side.

(4) Let $J : \mathbb{R} \to \mathbb{R}$ be a nonnegative convex function such that $J(0) = 0$. Let f, g be two nonnegative functions on $\mathbb{R}^n$, vanishing at infinity. Prove that
$$\int_{\mathbb{R}^n} J(f^*(x) - g^*(x))dx \leq \int_{\mathbb{R}^n} J(f(x) - g(x))dx,$$
and in particular
$$\|f^* - g^*\|_p \leq \|f - g\|_p, \qquad 1 \leq p \leq \infty.$$

(5) If f, g, h are nonnegative functions on the line, vanishing at infinity, denote
$$I(f, g, h) = (f, g * h)_2 = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)g(x-y)h(y)dxdy.$$
Prove that in this case $I(f, g, h) \leq I(f^*, g^*, h^*)$, with the understanding that the right-hand side is infinite if so is the left-hand side.

(6) Let G be a $p-$group (p prime) and let H be a normal subgroup of order p. Prove that H is contained in the centralizer of G, i.e., every element in H commutes with all elements of G.

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