MASTER WORKSHOP

EXERCISES III (PROPERTIES OF FUNCTIONS)

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(1) Let $f : [0,1] \to \mathbb{R}$ be continuous. Prove that there exists $a \in [0, \frac{2}{3}]$ such that
$$f(a + \frac{1}{3}) - f(a) = \frac{1}{3}(f(1) - f(0)).$$

(2) Let $P(z), \quad z \in \mathcal{C}$, be a complex polynomial of degree $n \geq 2$, such that the coefficient of z^n is 1. Assume that all roots of P are simple and denote them by $z_1, z_2, ..., z_n$. Show that
$$(a) \qquad \sum_{k=1}^{n} P'(z_k)^{-1} = 0,$$
$$(b) \qquad \sum_{k=1}^{n} \Big(\prod_{j=1, j \neq k}^{n} z_j \Big) P'(z_k)^{-1} = (-1)^{n-1}.$$

(3) Let $\alpha : \mathbb{N} \to \mathbb{Q}$ be a bijection and, for $x \in \mathbb{R}$, let N_x be the set $\{k \in \mathbb{N}; \alpha(k) \leq x\}$. Let $\{y_n\}_{n=1}^{\infty}$ a sequence in $(0, \infty)$ such that $\sum\limits_{n=1}^{\infty} y_n < \infty$. Define a function $f : \mathbb{R} \to \mathbb{R}$ by:
$$f(x) = \sum_{k \in N_x} y_k.$$
Prove the following:
   (a) f is strictly monotone.
   (b) f is continuous at all irrational points x.
   (c) At each rational point q, there is a jump discontinuity of size y_n, where $n = \alpha^{-1}(q)$.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly monotone C^∞ function, such that $f(0) = 0$. Define a function $g(x, y), \qquad (x, y) \in \mathbb{R}^2$, by:
$$g(x, y) = \begin{cases} \frac{f(x-y)}{x-y} & \text{if } x \neq y, \\ f'(0), & \text{if } x = y. \end{cases}$$
   (a) Show that $g \in C^\infty(\mathbb{R}^2)$.
   (b) Show that $\big(\frac{\partial^3}{\partial x^2 \partial y} g(x, y)\big)|_{x=y} = -\frac{1}{4} f^{(4)}(0)$.

(5) Find the (planar) quadrilateral with given edges a, b, c, d that includes the greatest area.

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