MASTER WORKSHOP

EXERCISES II (BERNOULLI POLYNOMIALS)

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(1) Let $f(x) = x$ in the interval $0 \le x < 1$. Extend it periodically to the whole real line. Show that $f(x)$ can be expanded in a Fourier series as

$$f(x) = \frac{1}{2} - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k}.$$

(2) The polynomials $B_n(x)$ are defined by the relations:

$$(a) \qquad B_1(x) = x - \frac{1}{2},$$

$$(b) \qquad B_k'(x) = kB_{k-1}(x), \quad k = 2, 3, \ldots.$$

$$(c) \qquad \int_0^1 B_k(x)dx = 0.$$

Show that the sequence $\{B_k(0)\}_{k=1}^{\infty}$ is the sequence of Bernoulli numbers introduced in the previous set of exercises.

Definition. The polynomials $\{B_k(x)\}_{k=1}^{\infty}$ are called **Bernoulli's polynomials**. We define $B_0(x) \equiv 1$.

(3) Prove that in the interval $0 \le x < 1$ the following expansions are valid:

(a)

$$B_1(x) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k},$$

(b)

$$B_2(x) = \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos 2k\pi x}{k^2},$$

(c)

$$B_3(x) = \frac{3}{2\pi^3} \sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k^3},$$

(d)

$$B_4(x) = -\frac{3}{\pi^4} \sum_{k=1}^{\infty} \frac{\cos 2k\pi x}{k^4}.$$

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(4) Let $\mathcal{B} = \{B_{k_j}\}_{j=0}^{\infty}$ be a subsequence of the sequence of all Bernoulli polynomials such that $k_0 = 0, \quad k_1 = 1$ and the sequence $\{k_j\}$ contains infinitely many odd and infinitely many even numbers.

Show that the set of all (finite) linear combinations of elements of $\mathcal{B}$ is dense in $C[0,1]$ (the space of continuous functions on $[0,1]$ normed by max). In other words, $\overline{span\mathcal{B}} = C[0,1]$.

(5) Show that for all $t \in \mathbb{R}$,

$$\frac{xe^{tx}}{e^x - 1} = \sum_{k=0}^{\infty} B_k(t)\frac{x^k}{k!}, \qquad x \quad \text{near} \quad 0.$$

(Formula obtained by Euler, 1738).

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