MASTER WORKSHOP

EXERCISES X (TOPOLOGY, PERMUTATIONS)

MATANIA BEN-ARTZI

BOOKS

[PM] R.S. Millman and G.D. Parker, Elements of Differential Geometry, Prentice-Hall Inc. 1977.
[M] J.R.Munkres, Topology, second edition, Prentice Hall Inc. 2000.

$|x|$ The Euclidean norm in $\mathbb{R}^n$.

(1) Let X be a compact metric space and Y a Hausdorff space. Assume that there is a continuous map $f : X \to Y$ which is onto ($f(X) = Y$). Show that Y is metrizable (i.e., the topology of Y is obtained from a certain metric).

(2) Let X be a compact metric space. Then there exists a continuous map from the Cantor set C onto X.
(See [M] Sec. 27 for the definition of the Cantor set).

(3) Determine the number of permutations $\sigma \in S_n$ such that $\sigma(k) \neq k$ for all $k = 1, 2, ..., n$.

(4) (The Morse Lemma).
Let φ be a C^∞ real function in a neighborhood U of $0 \in \mathbb{R}^n$. Assume that 0 is a nondegenerate critical point of φ, i.e., $\nabla\varphi(0) = 0$ and the Hessian $Q(0) = \big(\frac{\partial^2\varphi}{\partial x_j \partial x_k}\big)_{1\leq j,k\leq n}$ is a nonsingular $(n\times n)$ symmetric matrix. Assume that the signature of $Q(0)$ is $(r, n-r)$, i.e., r (resp. $n-r$) positive (resp. negative) eigenvalues. Prove that there exists a smooth change of variables $x = x(y)$ in a neighborhood V of 0 such that $x(0) = 0$ and

$$\varphi(x(y)) = \varphi(0) + \frac{1}{2}(y_1^2 + ... + y_r^2 - y_{r+1}^2 - ... - y_n^2), \quad y \in V.$$

Suggestion. By a suitable linear change of variables you can assume $Q(0) = diag(1, ...1, -1, ..., -1)$ (with r times 1.)
Write

$$\varphi(x) = \int_0^1 (1-t)\frac{\partial^2}{\partial t^2}(\varphi(tx))dt = \frac{1}{2}(Q(x)x, x)),$$

where $(,)$ is the scalar product and $Q(x)$ depends smoothly on x.
Find a smooth matrix function $A(x)$ such that $A(0) = I$ and $(Q(x)x, x) = (Q(0)A(x)x, A(x)x)$ and define $y = A(x)x$.

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(5) (See [PM], problem 6.3 in Sec. 4-6 and notation there).
Let $X_N = N - (N, n)n$ be the tangential component of the (principal) normal vector N of a unit speed curve γ on a surface M (with unit normal n). Prove that the following are equivalent:
   (a) $X_N = 0$.
   (b) γ is a geodesic.
   (c) X_N is parallel along γ.

(6) Read and fill in all necessary details in the paper:
J. Milnor, Analytic proofs of the "Hairy Ball Theorem" and the Brouwer fixed point theorem, Am. Math. Monthly 85 (1978), pp. 521-524.

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