MASTER WORKSHOP

EXERCISES I (POWER SERIES)

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(1) Let $f(x) = \frac{x}{e^x - 1}$. Show that $f(x)$ can be expanded in a power series for x near 0 (real x) and write

$$f(x) = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k.$$

Compute:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30} ...$$

Show that all coefficients B_k are rational numbers and $B_{2k+1} = 0$ for $k = 1, 2, ...$

Definition. The numbers B_k are called **Bernoulli's numbers**.

(2) Using (notation of previous problem) $f(x) + \frac{x}{2} = \frac{x}{2} \frac{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}$ show that

$$x \coth(x) = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} x^{2k}.$$

(3) By noting that the expansion in the previous problem can be extended to the complex disk $|z| < \pi$ show that:

$$x \cot(x) = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(2k)!} x^{2k},$$

and by using $2\cot(2x) = \cot(x) - \tan(x)$ also:

$$\tan(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k}(2^{2k} - 1)}{(2k)!} B_{2k} x^{2k-1},$$

valid for $|x| < \frac{\pi}{2}$.

(4) (a) Let t be real non-integer, and consider the function $g(x) = \cos(tx)$ in the interval $-\pi < x < \pi$. Show that its Fourier expansion is given by:

$$g(x) = \frac{\sin(t\pi)}{t\pi} + \sum_{k=1}^{\infty} a_k \cos(kx),$$

where $a_k = \frac{(-1)^k 2t}{t^2 - k^2} \frac{\sin(t\pi)}{\pi}$.

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(b) Argue by continuity (as $x \to \pi$) that

$$\cos(t\pi) = \frac{\sin(t\pi)}{t\pi} + \sum_{k=1}^{\infty} \frac{2t}{t^2 - k^2} \frac{\sin(t\pi)}{\pi}.$$

(c) Conclude that, for $0 < |x| < 1$,

$$\cot(\pi x) = \frac{2x}{\pi} \left(\frac{1}{2x^2} + \frac{1}{x^2 - 1^2} + \frac{1}{x^2 - 2^2} + ... \right).$$

(5) By comparing the last expansion with that of Problem 3 show that, for every integer $m \geq 1$,

$$\sum_{k=1}^{\infty} k^{-2m} = (-1)^{m-1} \frac{(2\pi)^{2m}}{2(2m)!} B_{2m},$$

$$\sum_{k=1}^{\infty} (2k-1)^{-2m} = (-1)^{m-1} \frac{(2^{2m} - 1)\pi^{2m}}{2(2m)!} B_{2m}.$$

(6) Show that

$$\int_0^1 \frac{\log x}{1-x} dx = -\frac{\pi^2}{6}.$$

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