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THEORY OF ITERATIONS OF POLYNOMIAL FAMILIES IN THE COMPLEX PLANE

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1. Introduction. 1.1. The significant advances, achieved recently in the theory of one-dimensional dynamical systems, are connected to a great extent with the investigation of concrete families of mappings. In the first place, this refers to the quadratic family:

$$f_c : z \to z^2 - c. \tag{1.1}$$

For complex values of the parameter c one has observed a continual variety in the behavior of the iterations of the mappings f_c: $\mathbf{C} \to \mathbf{C}$. The boundary separating the stable mappings from each other is the boundary of the so-called Mandelbrot set M [1]. It consists of those $c \in \mathbf{C}$, for which the iterations $f_c^n(0) = O(1)$ when $n \to \infty$. A special role is played by the iterations of the point $z = 0$, since this is the unique critical point of the function f_c. One of the properties of M consists in the fact that each point of the boundary of M is a limit point for the superstable values of c (a value of the parameter c is said to be superstable if f_c has a superstable cycle, i.e., a cycle which contains the critical point). It is proved in [2] that the superstable values of c are asymptotically distributed with respect to some measure with support on ∂M. In this paper we give a generalization of this statement to the family

$$z \to z^p - c \, (p \in \mathbf{N},\ p \geqslant 2). \tag{1.2}$$

We give the properties of this measure and we also consider some series connected with the set M and with its generalization to the family (1.2). Finally, we describe a linear algorithm for the computation of the moments of the measure (for p = 2).

1.2. We recall the basic definitions. The successive application of the mapping f: $U \to U$ generates the iterates $f^n : U \to U$; $f^1 \equiv f$, $f^{n+1} = f(f^n)$, $(x_n)_{n\geqslant 0}$, $x_n = f^n(x_0)$ is the orbit of the point x_0; if $x_n = x_0$, then x_0 is a periodic point of period n; the smallest of the periods of x_0 is some $m \geq 1$ and $m|n$; the points $\{x_0, x_1, \ldots, x_{m-1}\}$ form a cycle; if f is differentiable, then

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the multiplier of this cycle is $\lambda = (f^m)'(x_0) = \prod_{i=0}^{m-1} f'(x_i)$; the cycle is said to be stable, neutral, or unstable, according to whether λ lies inside, on the boundary, or outside the unit circle, respectively; a cycle containing the critical point is said to be superstable (for it $\lambda = 0$).

2. Measure on the Bifurcation Set. 2.1. We fix $p \in \mathbf{N}$, $p \geq 2$, and we consider the family

$$f_c : z \to z^p - c. \tag{2.1}$$

We denote by M_p (or simply by M) the set of those $c \in \mathbf{C}$ for which $f_c^n(0) = O(1)$ $(n \to \infty)$. The boundary of M_p consists of those c for which the family (2.1) ceases to be stable [3], [1].

Let $q_n(c) = f_c^n(0)$ be the iterates of zero; q_n is a polynomial of degree p^{n-1}. The set of all of its zeros will be denoted by P_n; $P = \bigcup_{n=1}^{\infty} P_r$ consists of the superstable values of c. We have $P \subset M$ and the set of the limit points of P is ∂M [3]. It is easy to show that M is bounded, closed, perfect, and each component of its interior is simply connected. The Douady–Hubbard theorem [1] asserts that for $p = 2$ M_2 is connected. With obvious modifications, this proof goes through also for $p > 2$.

2.2. We consider a somewhat more general situation. Let $f_c(z) = z^p + a_1(c)z^{p-1} + \ldots + a_p(c)$ be a polynomial of degree $p \geq 2$, whose coefficients $a_i(c)$ are holomorphic functions in a domain $D \subset \mathbf{C}$.

Let $a \in D$ and let q_0 be a critical point of $f_a(z)$. Passing to a local coordinate $\xi : U \to V$, where V is a neighborhood of the point a, $\tau = 0 \in U$, we can assume that q_0 extends to a function q, holomorphic in U. We denote $q_m = f_c^m(q)$.

LEMMA. If for some $m \geq 1$ $q_m \not\equiv q$ and $\tau = 0$ is a zero of multiplicity ℓ of the function $q_m - q$, then 0 is a zero of the same multiplicity of the function $q_{nm} - q$ $(n = 2, 3, \ldots)$.

Proof. We apply induction on n. By assumption, $q_m - q = \tau^l \cdot Q_1(\tau)$, $Q_1(0) \neq 0$. We assume that $q_{nm} - q = \tau^l Q_n(\tau)$, $Q_n(0) \neq 0$. Since $f'(q) = 0$, we have

$$q_{(n+1)m} - q = f^m(q + \tau^l Q_n) - q = (f^m(q) - q) + \sum_{k=2}^{\infty} \frac{1}{k!} \frac{d^k f^m}{dz^k}(q) \cdot (\tau^l Q_n)^k = \tau^l Q_1 + O(\tau^{2l}) = \tau^l Q_{n+1}(\tau),\ Q_{n+1}(0) \neq 0.$$

2.3. We return to the family (2.1) and we introduce the functions

$$\sigma_n(c) = \frac{q_n'(c)}{p^{n-1} q_n(c)},\ n \in \mathbf{N};$$

$\sigma_n(c)$ is holomorphic in $\mathbf{C} \setminus M$ since $q_n(c) \neq 0$, $c \in \mathbf{C} \setminus M$.

LEMMA. There exists $\lim_{n \to \infty} \sigma_n(c) = \sigma(c)$, $c \in \mathbf{C} \setminus M$. The function $\sigma(c)$ is holomorphic in $\mathbf{C} \setminus M$, $\sigma(c) \sim \frac{1}{c}$ $(c \to \infty)$.

Proof. We make use of Bottcher's theorem [1]: if $f(z) = z^p + \ldots + a_p$ is a polynomial, then $(f^n(z))^{p^{-n}} = z + \ldots$ converges uniformly in the neighborhood of $z = \infty$ for $n \to \infty$ to a function that is holomorphic at ∞. From here there follows the convergence of the sequence $(q_n)^{p^{-n}}$ at each point of $\mathbf{C} \setminus M$. This sequence is separated from 0 and ∞ in the neighborhood of each point $c \in \mathbf{C} \setminus M$. Taking the logarithmic derivative, we obtain the required convergence.

2.4. We recall that P_n is the set of the zeros of $q_n(c)$ (taking into account multiplicities).

LEMMA. Assume that the function F(c) is holomorphic in some neighborhood of the compactum M. Then there exists

$$\Phi(F) = \lim_{n \to \infty} \frac{1}{p^{n-1}} \sum_{c \in P_n} F(c) = \frac{1}{2\pi i} \int_\Gamma F(c)\, \sigma(c)\, dc,$$

where Γ is a smooth contour containing M.

Proof. By the residue theorem, we have

$$\frac{1}{2\pi i} \int_\Gamma F(c)\, \sigma(c)\, dc = \lim_{n \to \infty} \frac{1}{2\pi i} \int_\Gamma F \frac{q_n'}{p^{n-1} q_n} dc = \lim_{n \to \infty} \frac{1}{p^{n-1}} \sum_{c \in P_n} F(c).$$

2.5. THEOREM. Let $F: C \to R$ be a continuous function. Then there exists

$$\Phi(F) = \lim_{n\to\infty} \frac{1}{p^{n-1}} \sum_{c\in P_n} F(c).$$

Proof. If F = Re(u), where u is a polynomial, then the assertion follows from Subsection 2.4. If, however, F is an arbitrary continuous function, then by the Walsh–Lebesgue theorem [4], on the boundary of M the function F is the uniform limit of harmonic polynomials. Making use of Subsec. 2.2 and of what has been proved, we obtain the desired result.

2.6. Let μ_n be the probability measure, uniformly distributed at the zeros of the polynomial $q_n(c)$. We have proved that μ_n is weakly convergent for $n \to \infty$ to the measure μ with support on ∂M; μ is defined by a linear functional $\Phi \in C(\mathbb{C})^*$.

3. The Kernel of the Measure. The nonlinearity of the mappings f_c leads to the fast convergence of the functions σ_n at ∞, which allows us, in particular, to describe the kernel of the measure μ (of the functional Φ).

3.1. LEMMA. $\sigma(c) = \sigma_n(c) + O(c^{-p^n}),\ c \to \infty.$

Proof. The relation $q_{n+1} = q_n^p - c$ implies

$$\frac{\ln q_{n+1}}{p^n} = \frac{\ln q_n}{p^{n-1}} + O(c^{-(p^n-1)}),$$

from where by differentiation we obtain the required result.

3.2. LEMMA. Let F(c) be a polynomial of degree ℓ and let $\ell \le p^n - p^{n-1} - 2$. Then

$$\int F q_n \, d\mu = 0.$$

Proof. By virtue of Subsec. 2.4, we have

$$\int F q_n \, d\mu = \frac{1}{2\pi i}\int_\Gamma F q_n \sigma \, dc = \frac{1}{2\pi i}\int_\Gamma F(c)\, q_n(c) \left\{\frac{q_n'(c)}{p^{n-1} q_n(c)} + O(c^{-p^n})\right\} dc =$$

$$= \frac{1}{p^{n-1}} \cdot \frac{1}{2\pi i}\int_\Gamma F(c)\, q_n'(c)\, dc + \frac{1}{2\pi i}\int_\Gamma O(c^{l+p^{n-1}-p^n})\, dc = 0.$$

3.3. COROLLARY. Let $l \ge 0,\ m \ge 1,\ 1 \le r \le p-1,\ 0 \le j \le (p-r)p^{m-1}-2,\ \varepsilon_k \in \{0, 1, \ldots, p-1\}$, $k = 1, 2, \ldots, l,\ \varepsilon_l \ge 1$. Then

$$\int c^j q_m^r q_{m+1}^{\varepsilon_1} \cdot \ldots \cdot q_{m+l}^{\varepsilon_l} \, d\mu = 0.$$

Proof. We make use of Subsec. 3.2 and of the fact that $\deg(c^j q_m^r q_{m+1}^{\varepsilon_1} \cdot \ldots \cdot q_{m+l}^{\varepsilon_{l}-1}) \le p^{m+l} - p^{m+l-1} - 2$.

3.4. LEMMA. For $\ell \ge 0$, $1 \le r \le p-1$, $j \le 0$, and $j + l \le (p-r)p^{m-1} - 2$

$$\int c^j q_m^{lp+r} \, d\mu = 0.$$

Proof. By induction on ℓ we prove the formula $q_m^{\ell p} = c^\ell + \ldots$ (3.1), where the dots indicate a linear combination of terms of the form $c^j q_{m+1}^{\varepsilon_1} \cdot \ldots \cdot q_{m+k}^{\varepsilon_k}$ (3.2) ($\varepsilon_i \in \{0, 1, \ldots, p-1\}$, $0 \le j \le l-1$). Indeed, (3.1) is obvious for $\ell = 0$. Further,

$$q_m^{(l+1)p} = (q_{m+1} + c)\, q_m^{lp} = q_{m+1} q_m^{lp} + c q_m^{lp},$$

and $q_{m+1} c^j q_{m+1}^{\varepsilon_1} \cdot \ldots \cdot q_{m+k}^{\varepsilon_k}$ is a linear combination of terms of the form (3.2) with $j \le \ell$. (3.1) is proved. It remains to multiply both sides by q_m^r and to make use of Subsec. 3.3.

4. Functions Connected with a Bifurcation Set.

We recall that to a given fixed natural number $p \geq 2$ there is associated the bifurcation set M of the family (2.1). For $p = 2$ this is the Mandelbrot set.

4.1. We consider the following functions, defined and holomorphic at ∞: $\varphi_n = (q_n)^{p^{-n}}$, φ_n is uniquely distinguished by the condition $\frac{\varphi_n(c)}{c} \to 1$ $(c \to \infty)$; by Böttcher's theorem, $\varphi = \lim_{n\to\infty} \varphi_n$ is defined. Let $\psi = \varphi^{-1}$ be the inverse mapping. We note that

$$\sigma_n = \frac{\varphi_n'}{\varphi_n}, \quad \sigma = \frac{\varphi'}{\varphi}.$$

We define a sequence of numbers $(C_\ell)_{\ell \geq 0}$ by the Laurent series, converging at ∞:

$$\psi(\omega) = \omega + C_0 + \frac{C_1}{\omega} + \cdots + \frac{C_l}{\omega^l} + \cdots.$$

4.2. Remarks. (1) Of course, the numbers C_ℓ depend also on the fixed p; (2) since M is connected, it follows that ϕ_n and ϕ can be continued in a unique manner into $\mathbf{C}\setminus M$ and, moreover, ϕ maps $\mathbf{C}\setminus M$ univalently onto $D = \{\omega: |\omega| > 1\}$, while $\psi: D \to \mathbf{C}\setminus M$ is the inverse mapping.

4.3. We show that among the numbers C_ℓ there are many zeros. For the proof we need the following

LEMMA. Let $1 \leqslant r \leqslant p-1$, $m \geqslant 1$, $0 \leqslant l \leqslant (p-r)p^{m-1} - 2$. Then,

$$(\varphi(c))^{(lp+r)p^{m-1}} = q_m^{lp+r} + \sum_{j=1}^{l} R_j(c)\, q_m^{(l-j)p+r} + O(c^{-((p-r)p^{m-1}-l-1)}), \quad c \to \infty,$$

where $R_j(c)$ is a polynomial of degree at most j.

Proof. Multiplying the equality

$$\frac{\varphi'}{\varphi} = \frac{q_n'}{p^{n-1} q_n} + O(c^{-p^n})$$

by k and integrating, we obtain

$$\varphi^k = q_n^{kp^{1-n}} + O(c^{-p^n+1+k}).$$

Assume now that $k = (lp+r)p^{m-1}$, $0 \leqslant l \leqslant (p-r)p^{m-1} - 2$, $n - m = \alpha$, where α is such that $p^\alpha \leqslant lp + r < p^{\alpha+1}$. Then

$$\varphi^{(lp+r)p^{m-1}} = q_{m+\alpha}^{(lp+r)p^{-\alpha}} + O(c^{-(p^{m+\alpha}-(lp+r)p^{m-1}-1)}).$$

By induction we conclude that the α-th iterate of (2.1) is

$$f_c^\alpha(z) = z^{p^\alpha} + p_1^{(\alpha)}(c)\, z^{p^\alpha - p} + \cdots + p_i^{(\alpha)}(c)\, z^{p^\alpha - ip} + \cdots,$$

where $p_i^{(\alpha)}$ is a polynomial of degree i. Therefore,

$$(q_{m+\alpha})^{(lp+r)p^{-\alpha}} = (f^\alpha(q_m))^{(lp+r)p^{-\alpha}} = (q_m^{p^\alpha} + \cdots + p_i^{(\alpha)} z^{p^\alpha - ip} + \cdots)^{(lp+r)p^{-\alpha}} =$$

$$= q_m^{lp+r}\left(1 + \cdots + \frac{p_i^{(\alpha)}}{q_m^{ip}} + \cdots\right)^{(lp+r)p^{-\alpha}} = q_m^{lp+r}\left(1 + \sum_{j=1}^{\infty} \frac{R_j(c)}{q_m^{jp}}\right) = q_m^{lp+r} + \sum_{j=1}^{\infty} R_j\, q_m^{(l-j)p+r},$$

where $R_j(c)$ is a polynomial of degree at most j (it does not depend on α, since α is uniquely determined by ℓ, r, p). It remains to note that for $j = \ell + 1$ we have

$$q_m^{r-p} R_{l+1} = O(c^{-(p^{m-1}(p-r)-l-1)}).$$

4.4. THEOREM. $C_{(lp+r)p^{m-1}} = 0$, $r \in \{1, 2, \ldots, p-1\}$, $0 \leqslant l \leqslant (p-r)p^{m-1} - 3$.

Proof. For sufficiently large ρ we have

$$C_k = \frac{1}{2\pi i}\int\limits_{|\omega|=\rho} \omega^{k-1}\psi(\omega)\,d\omega = \frac{1}{2\pi i}\int\limits_{\varphi(|\omega|=\rho)} c\,\varphi^k(c)\,\sigma(c)\,dc.$$

Now we make use of Subsections 4.3 and 3.4.

4.5. Remark. The proved theorem means that for each $p \geq 2$ the zeros in the corresponding sequences $(C_\ell)_{\ell\geq 0}$ are partitioned into $p - 1$ subsequences, the indices in which are almost periodic with almost-period p^n, $n \in \mathbb{N}$.

4.6. In particular, for $p = 2$ one obtains the sequence of numbers $(C_l)_{l>0}$, in which $C_{(2l+1)2^{m-1}} = 0$, $m \geq 3$, $l = 0, 1, \ldots, 2^{m-1} - 3$.

5. Arithmetic Properties of the Coefficients. We consider the case of a quadratic family and we fix $p = 2$.

We recall that $\phi(c)$ maps univalently the exterior of the Mandelbrot set onto the exterior of the unit circle, $\sigma = \frac{\varphi'}{\varphi}$, $\psi = \varphi^{-1}$.

Together with the sequence $(C_\ell)_{\ell\geq 0}$ we consider also two sequences, determined by the expansions of ϕ and σ at ∞:

$$\varphi(c) = c + B_0 + \frac{B_1}{c} + \cdots + \frac{B_l}{c^l} + \cdots,$$

$$\sigma(c) = \frac{1}{c}\left(1 + \frac{A_1}{c} + \frac{A_2}{c^2} + \cdots + \frac{A_l}{c^l} + \cdots\right).$$

The numbers A_ℓ are the moments of the measure μ:

$$c^l\,d\mu = \frac{1}{2\pi i}\int\limits_{\Gamma} c^l\,\sigma(c)\,dc = A_l.$$

Let $P_n = (b_{k,n})_{k=1}^{2^{n-1}}$ be the set of all the zeros of $q_n(c)$. Since

$$\frac{q_n'(c)}{2^{n-1}q_n(c)} = \frac{1}{2^{n-1}}\sum_{l=0}^{\infty}\frac{1}{c^{l+1}}\sum_{k=1}^{2^{n-1}}(b_{k,n})^l,\quad c\to\infty,$$

from the lemma of Subsec. 3.1 for $p = 2$ we obtain

$$A_l = \frac{1}{2^{n-1}}\sum_{k=1}^{2^{n-1}}(b_{k,n})^l,\quad n \geq 1,\ 0 \leq l \leq 2^n - 2.$$

5.1. We compute the exact binary order of the numbers A_ℓ, B_ℓ, and $C_{2\ell+1}$.

By the binary order $\mathrm{ord}_2 m$ of a nonzero integer m we mean the largest integer ℓ such that 2^ℓ divides m; the binary order of a quotient is $\mathrm{ord}_2(m/n) = \mathrm{ord}_2 m - \mathrm{ord}_2 n$; $\mathrm{ord}_2 0 = +\infty$.

5.2. For convenience we set $c = -t$ and let

$$q_n(c) = \sum_{i=1}^{2^{n-1}} a(i, n)\,t^i.$$

The coefficients $a(i, n)$ are natural numbers (for example, $a(1, n) = a(2^{n-1}, n) = 1$). Therefore, also the coefficients of the expansion of $q_n'q_n$ at ∞ are integers; taking into account 3.1, we obtain that A_ℓ, and also B_ℓ and C_ℓ expand into finite binary quotients (for $p = 2$).

5.3. We denote by $S(n)$ the sum of the digits in the binary expansion of the natural number n. It is well known that $S(n) = n - \mathrm{ord}_2(n!)$, $n \in \mathbb{N}$. From here $S(n) + S(m) - S(n+m) = \mathrm{ord}_2 C_{n+m}^n \geq 0$, where $C_{n+m}^n = \frac{(m+n)!}{n!\,m!}$ is the number of combinations of $m + n$ objects taken n at a time.

In the sequel we need several statements, the proofs of which, in view of their elementary nature, are not given.

5.4. LEMMA. Let $\ell \geq 3$. The sum

$$\sum_{k=1}^{\left[\frac{l-1}{2}\right]} C_l^k$$

is odd if ℓ is not a power of two and even in the opposite case.

5.5. LEMMA. For $1 \leqslant l \leqslant 2^{n-1}-2,\ n \geqslant 2,$ the sum

$$\sum_{k=l}^{\left[\frac{l+2^{n-1}-1}{2}\right]} C_{l+2^{n-1}}^k$$

is odd.

5.6. LEMMA. For $2^{n-1} \leqslant l \leqslant 2^n-2,\ n \geqslant 2,$ the sum

$$\sum_{k=1}^{2^{n-1}-1} C_{2^{n-1}+l-k}^l$$

is odd.

5.7. THEOREM. $\mathrm{ord}_2 a(i, n) = S(i) - 1,\ i = 1, 2, 3, \ldots, 2^{n-1}$.

Proof. First we note that if $D = \sum_{k=1}^{l} D_k$ is the sum of ℓ nonzero integers, $\mathrm{ord}_2 D_k = p_k \geq p_0$ for all k and $\mathrm{ord}_2 D_k = p_0$ for an odd number of indices k, then $D \neq 0$ and $\mathrm{ord}_2 D = p_0$.

We shall prove the assertion of the theorem by induction on n. Since $a(1, 1) = a(1, 2) = a(2, 2) = 1$, it follows that the theorem holds for $n = 1, 2$. We assume that $\mathrm{ord}_2 a(\ell, n) = S(\ell) - 1$, $1 \leq \ell \leq 2^{n-1}$, and we prove that $\mathrm{d}_2 a(l, n+1) = S(l) - 1,\ 1 \leqslant l \leqslant 2^n$. We examine the case when 1 is even. In the case of an odd ℓ the proof is similar.

1) $1 \leq \ell \leq 2^{n-1}$.

Since $q_{n+1} = q_n^2 + t$, for $\ell \leq 3$ we have

$$a(l, n+1) = 2 \sum_{k=1}^{\frac{l}{2}-1} a(k, n) a(l-k, n) + \left(a\left(\frac{l}{2}, n\right)\right)^2. \tag{5.1}$$

We have

$$\mathrm{ord}_2\, 2a(k, n) a(l-k, n) = 1 + \mathrm{ord}_2\, a(k, n) +$$

$$+ \mathrm{ord}_2\, a(l-k, n) = 1 + S(k) - 1 + S(l-k) - 1 = S(k) + S(l-k) - 1 \geqslant S(l) - 1$$

by virtue of Subsec. 5.3 and, moreover, equality holds when C_l^k is odd;

$$\mathrm{ord}_2 \left(a\left(\frac{l}{2}, n\right)\right)^2 = 2\left(S\left(\frac{l}{2}\right) - 1\right) = 2 \cdot S\left(\frac{l}{2}\right) - 2 = 2 \cdot S(l) - 2 \geqslant S(l) - 1;$$

equality holds if ℓ is a power of two.

Thus, the number of terms in the sum (5.1) with the least order $p_0 = S(\ell) - 1$ is equal to the number of odd terms in the sum

$$\sum_{k=1}^{\frac{l}{2}-1} C_l^k,$$

if ℓ is not a power of 2, and one unit greater if ℓ is a power of 2. By virtue of 5.4, in both cases this number is odd; this proves the statement for $1 \leq \ell \leq 2^{n-1}$.

2) $2^{n-1}+1 \leqslant l \leqslant 2^n$.

For $l = 2^n$ $\operatorname{ord}_2 a(2^n, n+1) = \operatorname{ord}_2 1 = S(2^n) - 1$.

For $l = 2^{n-1} + r$, $1 \leq r \leq 2^{n-1} - 1$, we reason similarly to 1) and we make use of 5.5.

5.8. Now we can compute the binary orders of the moments A_ℓ. We recall Newton's formula [5]: if δ_ℓ is the sum of the ℓ-th powers of the zeros of the polynomial

$$x^n + a_1 x^{n-1} + \cdots + a_n,$$

then for $\ell \leq n$ we have

$$l < n \quad \delta_l + \delta_{l-1} a_1 + \delta_{l-2} a_2 + \cdots + \delta_1 a_{l-1} + (-1)^l l a_l = 0,$$

while for $\ell > n$ we have

$$\delta_l + \delta_{l-1} a_1 + \delta_{l-2} a_2 + \cdots + \delta_{l-n} a_n = 0.$$

5.9 THEOREM.

$$\operatorname{ord}_2(A_l) = -S(l), \quad l = 1, 2, \ldots .$$

Proof. We denote

$$R(l, n) = \sum_{k=1}^{2^{n-1}} (-b_{k,n})^l, \quad n \in \mathrm{N}, \quad l = 0, 1, 2, \ldots .$$

$(b_{k,n})_{k=1}^{2^{n-1}}$ are the roots of the equation $q_n = 0$).
Then

$$R(l, n) = (-1)^l \cdot 2^{n-1} A_l, \quad 1 \leq l \leq 2^n - 2. \tag{5.2}$$

Therefore, the assertion of the theorem is equivalent to the fact that

$$\operatorname{ord}_2 R(l, n) = n - 1 - S(l), \quad 1 \leq l \leq 2^n - 2 \tag{5.3}$$

We assume that (5.3) has been proved for some $n \geq 2$ and for all $\ell = 1, 2, 3, \ldots, 2^{n-1} - 2$, and we prove it for $\ell = 2^{n-1} - 1, 2^{n-1}, \ldots, 2^n - 2$. Then, by virtue of (5.2), equality (5.3) will be proved for $n + 1$ and $\ell = 1, 2, 3, \ldots, 2^n - 2$. From here, by induction, the validity of the theorem will follow for all ℓ.

Since

$$\frac{q'_2}{2q_2} = \frac{2c - 1}{2(c^2 - c)} = \frac{1}{c}\left(1 + \frac{1}{2c} + \frac{1}{2c^2} + \cdots\right),$$

we have $A_1 = 1/2$, $A_2 = 1/2$, i.e., (5.3) holds for $n = 2, 3$ and $\ell = 1, 2$. Let $\operatorname{ord}_2 R(\ell, n) = n - 1 - S(\ell)$ for some $n \geq 3$ and all $\ell = 1, 2, 3, \ldots, 2^{n-1} - 2$. We show that $\operatorname{ord}_2 R(\ell, n) = n - 1 - S(\ell)$ for $\ell = 2^{n-1} - 1, \ldots, 2^n - 2$.

Since $\deg q_n = 2^{n-1}$ and $q_n(0) = 0$, by Newton's formulas we have

$$R(2^{n-1} - 1, n) = -(R(2^{n-1} - 2, n) a(2^{n-1} - 1, n) + \cdots + R(i - 1, n) a(i, n) + \cdots + R(1, n) a(2, n) + (2^{n-1} - 1) a(1, n)), \tag{5.4}$$

$$R(l, n) = -(R(l - 1, n) a(2^{n-1} - 1, n) + \cdots + R(l - i, n) a(2^{n-1} - i, n) + \cdots + R(l - 2^{n-1} + 1, n) a(1, n)), \tag{5.5}$$

$l = 2^{n-1}, \ldots, 2^n - 2$.

According to what has been proved, $\operatorname{ord}_2 a(i, n) = S(i) - 1$; by the induction hypothesis, $\operatorname{ord}_2 R(l, n) = n - 1 - S(l)$, $1 \leq l \leq 2^{n-1} - 2$.

Let $\ell = 2^{n-1} - 1$. We prove that in the right-hand side of (5.4) all the terms, except the last one, equal to $2^{n-1} - 1$, are even:

$$\operatorname{ord}_2 R(i - 1, n) a(i, n)) = (n - 1 - S(i - 1)) +$$

$$+(S(i)-1) \geqslant 1,$$

since
$$S(i-1) \leqslant n-2, i \leqslant 2^{n-1}-1.$$

Thus, the assertion is proved for $\ell = 2^{n-1} - 1$.

Assume now that the assertion is proved for all indices smaller than ℓ, where $2^{n-1} \leq \ell \leq 2^n - 2$, and we prove it for ℓ. In (5.5) the order of each term is

$$\mathrm{ord}_2(R(l-i,n)a(2^{n-1}-i,n)) = (n-1-S(l-i)) + (S(2^{n-1}-i)-1) \geqslant n-1-S(l),$$

since $S(2^{n-1}-i)+S(l) \geqslant S(l-i)+1$, and, moreover, equality is equivalent to the fact that the number $C^{l}_{2^{n-1}+l-i}$ is odd. It remains to make use of 5.6. The theorem is proved.

5.10. Remark. Making use of the proved theorem and of the expression of the discriminant of a polynomial in terms of its power sums [5], it is easy to show that the discriminant

$$D_n = \prod_{1<i<k<2^{n-1}} (b_{i,n} - b_{k,n})^2$$

of the polynomial q_n for $n \geq 2$ is an odd number.

5.11. We give one more statement concerning the numbers A_ℓ. For this we note that, according to what has been proved, $\mathrm{ord}_2\, 2^\ell A_\ell = \ell - S(\ell) = \mathrm{ord}_2(\ell!) \geq 0$, i.e., $N_\ell = 2^\ell A_\ell$ is an integer ($\ell \in \mathbb{N}$).

THEOREM. For any natural numbers k, r and prime number m > 2 we have

$$N_{km^r} = N_{km^{r-1}} (\mathrm{mod}\, m^r).$$

Remark. Since $N_1 = 1$, it follows, in particular, that $N_m \equiv 1 (\mathrm{mod}\ m)$ for all prime numbers $m > 2$.

The assertion of the theorem follows from the congruence (a particular case of Euler's theorem): $2^{km^r - km^{r-1}} \equiv 1 (\mathrm{mod}\, m^r)$ and from the following statement [6]: if δ_ℓ is the sum of the ℓ-th powers of the roots of the equation $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ with integer coefficients, $a_n \neq 0$, then for any natural k, r and any prime number m we have $\delta_{km^r} \equiv \delta_{km^{r-1}} (\mathrm{mod}\, m^r)$.

5.12. After the model of the proved Theorem 5.9, it is easy to compute the binary orders also of the coefficients B_ℓ of the expansion of the function ϕ, mapping univalently the exterior of the set M onto the exterior of the unit circle.

THEOREM.

$$\mathrm{ord}_2 B_l = -(l+1+\mathrm{ord}_2 (l+1)!) = S(l+1) - 2l - 2,\ l=0,1,2,\ldots.$$

Proof. Since $B_0 = 1/2$, the formula holds for $\ell = 0$. We assume that it holds for all indices less than ℓ and we prove it for ℓ. We rewrite the identity $\sigma = \phi'/\phi$ in the form:

$$1 - \sum_{l=1}^{\infty} \frac{lB_l}{c^{l+1}} = \frac{1}{c} \sum_{l=0}^{\infty} \frac{A_l}{c^l} \left(c + \sum_{l=0}^{\infty} \frac{B_l}{c^l} \right),$$

from where

$$B_l = -\frac{1}{l+1} (A_{l+1} + B_0 A_l + \cdots + B_{l-1} A_1).$$

For $i = 0, 1, \ldots, \ell - 2$ we have:

$$\mathrm{ord}_2 (B_i A_{l-i}) = -(i+1+\mathrm{ord}_2 (i+1)!) - S(l-i) = -(i+1) - \mathrm{ord}_2 (i+1)! - (l-i) + \mathrm{ord}_2 (l-i)! = -(l+1) + \mathrm{ord}_2 (l-i)! - \mathrm{ord}_2 (i+1)! > -(l+1) - \mathrm{ord}_2 (l!) = \mathrm{ord}_2 (A_1 B_{l-1}).$$

Similarly, $\mathrm{ord}_2 A_{l+1} = -(l+1) + \mathrm{ord}_2 (l+1)! > \mathrm{ord}_2 (A_1 B_{l-1})$. Therefore, $\mathrm{ord}_2 B_l = -\mathrm{ord}_2 (l+1) + \mathrm{ord}_2 (A_1 B_{l-1}) = -(l+1+\mathrm{ord}_2 (l+1)!)$. The theorem is proved.

COROLLARY. $B_\ell \neq 0$, $\ell = 0, 1, 2, \ldots$.

5.13. In a similar manner one proves the following

THEOREM.

$$\operatorname{ord}_2 C_{2l+1} = \operatorname{ord}_2 B_{2l+1} = S(l+1) - 4l - 4,$$

$\ell = 0, 1, \ldots$.

In particular, $C_{2\ell+1} \neq 0$. Apparently, all the zero terms of the sequence (C_ℓ) are given by the formula from Subsec. 4.6.

6. A Linear Method for the Computation of the Moments. 6.1. We continue to consider the case $P = 2$. We recall that A_ℓ is the ℓ-th moment of the measure μ.

The sequence $(A_\ell)_{\ell\geq 0}$ can be determined by computing first the coefficients $a(k, n)$ of the polynomial q_n and then making use of the formula from Subsec. 3.1. The corresponding recurrence formulas are nonlinear.

Here we describe a linear algorithm for the computation of A_ℓ, which can be easily implemented on a computer.

6.2. Definition. Let $n \in \mathbb{N}$ and let $n \in N$, $n = \sum_{i=0}^{l} \varepsilon_i 2^i$, $\varepsilon_i \in \{0, 1\}$ be the binary expansion of n; by the n-th elementary polynomial $d_n(c)$ we mean the product

$$d_n(c) = \prod_{\varepsilon_i = 1} f_c^i(c) = \prod_{k=0}^{l} (f_c^k(c))^{\varepsilon_k}. \tag{6.1}$$

In addition, we set $d_0(c) \equiv 1$.

6.3. Since $\deg f_c^k(c) = \deg q_{k+1}(c) = 2^k$, we have $\deg d_n = n$, $n = 0, 1, 2, \ldots$. Therefore, each polynomial $F(c)$ is a finite linear combination of the system $(d_n(c))_{n\geq 0}$. In particular, for each $n \in \mathbb{N}$ there exists a vector $(\alpha(i, m))_{i=0}^{m}$ such that

$$c^m = \sum_{i=0}^{m} \alpha(i,m)\, d_i(c); \tag{6.2}$$

since $d_i(0) = 0\ (i \geq 1)$, we have $\alpha(0, m) = 0$.

$$A_m = \int c^m\, d\mu = \sum_{i=1}^{m} \alpha(i,m) \int d_i(c)\, d\mu. \tag{6.3}$$

6.4. LEMMA. Let $k \in \mathbb{N}$.

(a) If $n \neq 2^k - 1$, then

$$\int d_n(c)\, d\mu = 0.$$

(b) If $n = 2^k - 1$, then

$$\int d_n(c)\, d\mu = 1 - \frac{1}{2^k}.$$

Proof. (a) follows from Subsec. 3.2 (for $p = 2$). (b) Let $n = 2^k - 1$. Then $d_n(c) = q_1(c) \cdot \ldots \cdot q_k(c)$.

By virtue of Subsecs. 2.3, 3.1, we have

$$\int q_1 \cdot \ldots \cdot q_k\, d\mu = \frac{1}{2\pi i} \int_\Gamma q_1 \cdot \ldots \cdot q_k \left(\frac{q_k'}{2^{k-1} q_k} + \frac{D_k}{c^{2k}} + \cdots \right) dc = \frac{1}{2\pi i} \int_\Gamma (c^{2^k - 1} + \ldots) \left(\frac{D_k}{c^{2k}} + \cdots \right) dc = D_k.$$

We find D_k:

$$\frac{\ln q_{k+1}}{2^k} = \frac{\ln q_k}{2^{k-1}} + \frac{1}{2^k}\ln\left(1 - \frac{c}{q_k^2}\right) = \frac{\ln q_k}{2^{k-1}} + \frac{1}{2^k}\left(-\frac{c}{q_k^2} + \frac{1}{2}\frac{c^2}{q_k^4} + \cdots\right) =$$

$$= \frac{\ln q_k}{2^{k-1}} + \frac{1}{2^k}\left(-\frac{1}{c^{2^k-1}} + \cdots\right),$$

$$\frac{q'_{k+1}}{2^k q_{k+1}} = \frac{q'_k}{2^{k-1} q_k} + \frac{2^k - 1}{2^k} \cdot \frac{1}{c^{2^k}} + O\left(\frac{1}{c^{2^{k+1}}}\right),$$

i.e., $D_k = \frac{2^k - 1}{2^k}$.

6.5. Thus, in (6.3) it remains to find the recurrence relations for $\alpha(t, m)$.

6.6. For the convenience of the notations, instead of $d_n(c)$ we shall write sometimes $d(n)$, omitting c.

<u>LEMMA.</u> For $n \geq 0$ we have

$$c \cdot d(n) = \sum_{r=0}^{\operatorname{ord}_2(n+1)} d(n + 2 - 2^r).$$

The <u>proof</u> is carried out by induction on the number $k = \operatorname{ord}_2(n + 1)$, which is the number of ones up to the first zero in the binary expansion of the number n.

6.7. To each sequence $\bar{a} = (a(n))_{n>0}$, $a(n) \in \mathbb{C}$ we associate the formal series

$$\sum a(n) d_n(c).$$

<u>LEMMA.</u> Assume that the series

$$\sum_{k=1}^{\infty} a(2^k - 1)$$

converges. Then

$$c \sum_{n=0}^{\infty} a(n) d_n(c) = \sum_{m=1}^{\infty} b(m) d_m(c),$$

where

$$b(1) = \sum_{k=0}^{\infty} a(2^k - 1),$$

$$b(m) = \sum_{k=0}^{\operatorname{ord}_2(m-1)} a(m + 2^k - 2), \quad m \geqslant 2.$$

<u>Proof.</u> We make use of Subsec. 6.7:

$$\sum_{m=1}^{\infty} b(m) d_m(c) = \sum_{n=0}^{\infty} a(n) \sum_{k=0}^{\operatorname{ord}_2(n+1)} d(n + 2 - 2^k).$$

Let $n + 2 - 2^k = m$, where $0 \leq k \leq \operatorname{ord}_2(n + 1)$. Then $n = m + 2^k - 2$, $n + 1 = (m - 1) + 2^k$, and, therefore, $k \leq \operatorname{ord}_2(m - 1)$. Consequently,

$$b(m) = \sum_{k=0}^{\operatorname{ord}_2(m-1)} a(m + 2^k - 2).$$

It remains to note that for $m = 1$ we have $\operatorname{ord}_2(m - 1) = +\infty$.

<u>6.8. COROLLARY.</u> We set $\alpha(1,1) = 1$, $\alpha(l, 1) = 0$, $l \geqslant 2$.

$$\alpha(l, m+1) = \sum_{k=0}^{\operatorname{ord}_2(l-1)} \alpha(l + 2^k - 2, m); \quad m, l \in \mathbb{N}.$$

Then

$$A_m = \sum_{2^k < m+1} \left(1 - \frac{1}{2^k}\right) \cdot \alpha(2^k - 1, m).$$

6.9. From here there follows at once that $0 < A_m < A_{m+1}$.

The computer calculations allow us to presuppose that in the considered case ($p = 2$) we have, asymptotically,

$$A_m \sim \text{const} \cdot 2^m \cdot m^{-\gamma};\ B_m \sim \text{const} \cdot 2^m \cdot m^{-1-\gamma},\ \gamma \approx 0{,}48.$$

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A NEW PROOF OF DRASIN'S THEOREM ON MEROMORPHIC FUNCTIONS OF FINITE ORDER WITH MAXIMAL DEFICIENCY SUM. I

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1. Introduction. For a function f, meromorphic in the plane $\mathbb{C}$, we make use of the standard notations of the R. Nevanlinna theory: $T(r, f)$, $N(r, a)$, $m(r, a)$, $\bar{N}(r, f)$, $N_1(r)$, $\delta(a)$. In addition, we set $D(z_0, R) = \{z \in \mathbb{C} : |z - z_0| < R\}$. In this paper we investigate meromorphic functions of finite lower order with maximal deficiency sum:

$$\sum_{a \in \bar{\mathbb{C}}} \delta(a) = 2. \tag{1.1}$$

For a function f of finite order, R. Nevanlinna's second fundamental theorem can be formulated in the following form: for each finite collection $a_1, \ldots, a_q$ we have

$$\sum_{j=1}^{q} m(r, a_j) + N_1(r) \leqslant 2T(r, f) + o(T(r, f)),\ r \to \infty.$$

From here and from (1.1) there follows that

$$N_1(r) = o(T(r, f)),\ r \to \infty. \tag{1.2}$$

In order to elucidate what consequences can (1.1) imply, we assume first that a stronger condition than (1.2) is satisfied, namely, $N_1(r) \equiv 0$, i.e., f does not have multiple points. We consider the Schwarzian derivative

$$F = f'''/f' - (3/2)(f''/f')^2. \tag{1.3}$$

A simple computation shows that the Schwarzian derivative has poles only at the multiple points of the function f and, therefore, F is an entire function. Taking into account that f is of

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