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Letter to the Editor

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1. Theorem 2 of [1] is not proved in full generality. Indeed, in the notation of [1], $g_n \to \tilde{g}$ outside a finite set $C \subset J_f$. The case in which the function $\tilde{g}$ is not a constant is impossible; see [1]. If $\tilde{g} \equiv c$, then the case $c \notin C$ is impossible as well, because otherwise, for a large fixed n, the sequence of iterates of g_n is normal at c. The case $c \in C$ is missing in the proof. Consider two subcases.

(A) $J(g)$ is an exceptional Julia set (i.e., J_f is either the Riemann sphere, a circle, or an interval). Then, by the definition of the class $R_d(f)$, all the functions g_n share with f the same measure of maximal entropy μ. If U is a small neighborhood of a point from $J_f \setminus C$, then the sets $u_n = g_n(u)$ tend to the point c. On the other hand, $\mu(u_n) \geq \mu(u) > 0$. This leads to a contradiction. Indeed, there is a k_0 such that for any $k > k_0$ the point $f^k(c)$ is not a critical point of f (otherwise, there would be a periodic orbit in J_f which contained a critical point of f). Therefore, for a fixed k such that $d^{k-k_0} \cdot \mu(u) \geq 1$ and for any n large enough we have the inequalities $\mu(f^k(u_n)) \geq d^{k-k_0} \cdot \mu(u_n) \geq 1$, which leads to a contradiction.

(B) J_f is not exceptional and $c \in C$. This subcase remains open. Thus, for an arbitrary f, the statement of Theorem 2 is a conjecture (call it C1).

2. Let us stress that Theorem 2 is not used in the proofs of other results of [1], and also in the paper [2]. In particular, the main result of [1] (Theorem 1), Theorem 3, Theorem 4, as well as all proofs and results of [2] remain unchanged.

Notice, nevertheless, that in the proof of Theorem 3 (1) of [1] (in the part where $|\lambda_2| = 1$) is assumed) one should define the functions H_1 and H_2 holomorphic at the point a in such a way that $H_1 = g$ and $f \circ H_2 = f^2$, $H_2(a) = a$. Then $H_2'(a)$ is equal to the chosen value $\lambda_2^{1/p}$, where $p \geq 1$ is the multiplicity of the root $x = a$ in the equation $f(x) = b$. Then we consider two possibilities as in the article: either there exists an integer $q \geq 1$ such that $(\lambda_2^{1/p})^q = 1$ or not. Note that $H_2^n \neq \mathrm{id}$ for any $n \geq 1$ because otherwise $f = f^{n+1}$. In the exceptional case $\mu_f = \mu_g = \mu$ we have $\mu(H_1 \circ H_2(A)) \geq d \cdot m \cdot \mu(A)$. Since $|H_1'(a)| = |(H_1 \circ H_2)'(a)| = |\lambda_1| > 1$, we have

$$\lim_{\varepsilon \to 0} \frac{\ln \mu(B(a, \varepsilon))}{\ln \varepsilon}$$

is simultaneously equal to $\ln d / \ln |\lambda_1|$ and bigger than or equal to $\ln(dm)/\ln|\lambda_1|$, a contradiction. The rest of the proof of Theorem 3 remains unchanged.

3. For the following classes of functions f, Theorem 2 (i.e., the conjecture C1) has been proved:

1) J_f is an exceptional Julia set (see the case (A) above).
2) f satisfies the condition of Theorem 3 (2). Then, as follows from Theorem 3 and Remark 2 (see also [2, p. 2186, (A2)]), all the functions g_n share with f the same measure of maximal entropy. As we know, this is impossible (see the case (A) above). In particular, Theorem 2 holds for all hyperbolic rational functions f.

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4. In [3] the following conjecture is stated (call it C2): If J is not an exceptional Julia set of a rational function f, and A is a Möbius transformation such that $A(J) = J$, then $A^q = \mathrm{id}$ for some $q \in \mathbb{N}$. The conjecture C2 is apparently weaker than C1: if C1 holds for a function f, then C2 also holds for the same f (otherwise there would be an infinite set of rational functions $A_0^{-n} f_0 A^n$ with the same Julia set J). Under an additional assumption, C2 is proved in the following proposition, which generalizes the main result of [3] and follows easily from Theorem 1 of [1].

Proposition. *Let A be a Möbius transformation such that $A(J) = J$, and let a fixed point of A be (pre)periodic for f. If J is not exceptional, then $A^q = \mathrm{id}$ for some $q \in \mathbb{N}$.*

Proof.

1) If A is an irrational rotation, the Julia set is locally diffeomorphic to a product of an interval and a Cantor set. This is impossible (see the proof of this fact due to A. Eremenko in [1]).

2) Let A not be a rotation. We can assume (passing to an iterate of f) that $A(0) = 0$, $f(0) = a$, $f(a) = a$. Let $\lambda = f'(a)$. Then $|\lambda| \geq 1$, because $a \in J$. There exists a holomorphic at zero function H_1 such that $f \circ H_1 = f^2$ and $H_1(0) = 0$, $|H_1'(0)| = |\lambda|^{1/p}$, where $p \geq 1$ is the multiplicity of the root $x = 0$ in the equation $f(x) = a$. Denote $H_2 = A$ and $R = H_2^{-1} \circ H_1^{-1} \circ H_2 \circ H_1$ (cf. the proof of Proposition 1 in [1]). If $R = \mathrm{id}$, then H_1 and H_2 commute and $H_i(0) = 0$. Since H_2 is Möbius, H_1 is Möbius too, contrary to $f \circ H_1 = f^2$. If R is not the identity, we have $R'(0) = 1$. Consider two cases:

2a) $A'(0) = 1$. After the change $z \mapsto 1/z$, one can assume $H_2(z) = A(z) = z + 1$, $R(z)$ is holomorphic at ∞ and $R(z) = z + b_0 + b_{1/z} + \cdots$. Pick a point $x \in J$ close enough to ∞. Consider maps $h_m = H_2^{-m} \circ R \circ H_2^m$. Then $h_m(z) \to z + b_0$ as $n \to \infty$, uniformly on z from a neighborhood V of the point $x \in J$. Since J is not exceptional, by Theorem 1, $h_j = h_{j+i}$ for some $i > 0$. Hence, R commutes with H_2^i (which is the shift $z \mapsto z + i$). It follows that R is also a shift, and H_1 is a linear map (in the z-coordinate). But $f \circ H_1 = f^2$, a contradiction.

2b) $|A'(0) \neq 1$. Repeat the proof of Proposition 1 or Lemma 3 to show that $R = \mathrm{id}$. ☐

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