SYMMETRIES ON THE JULIA SET

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INTRODUCTION

Let J_f and μ_f be the Julia set and the maximal entropy measure of the rational function f [1-5]. In this paper it is proved that the class of all rational functions of a fixed degree with common Julia set J_f or, in exceptional cases with common measure μ_f, is finite, with one exception. Also considered is the following question: How are the function of this class connected with f? For polynomials a complete answer was obtained in [6]. Our method leads to a generalization of results of Fatou [7], Julia [8], and Ritt [9] on commuting rational functions in the style of the paper [10].

With the theory of iteration of rational functions one can become acquainted from the surveys [11, 12]. The Julia set J_f is defined as the set of points of the Riemann sphere $\bar{\mathbf{C}}$, in the neighborhood of which the set of iterates $(f^n)_{n\geq 0}$ is not precompact (normal in the sene of Montel [13]). The Julia set coincides with the closure of the repulsive periodic points of f. The measure μ_f is defined to be the unique measure of maximal entropy of the endomorphism f: $\bar{\mathbf{C}} \to \bar{\mathbf{C}}$ [4]; it is characterized by the balancedness property [5]: $\mu_f(f(A)) = m\mu_f(A)$, where $m = \deg f$, for any Borel set A on which f is injective; the support of μ_f coincides with J_f.

The rational f is said to be critically finite if the set P_f of iterates of its critical points is finite. According to Thurston [14, 15], to each such function there corresponds an orbifold $\mathcal{O}$ that is a sphere $\mathbf{C}$ together with the map n: $\bar{\mathbf{C}} \to \mathbf{N} \cup \{\infty\}$, defined as follows. If the point z is not in P_f then n(f) = 1, whereas if $z \in P_f$ then n(z) equals the least-common multiple of the numbers $n(t)\deg_t f$ for all preimages t of the point z: f(t) = z($\deg_t f$ denotes the multiplicity of the function f at the point t). The orbifold is said to be parabolic if $\sum_{z\in P_f}(1-1/n(z)) = 2$. In this case there exist a covering map F: $\mathbf{C} \to \bar{\mathbf{C}}$ and a lift $\tilde{f}$: $z \mapsto az + b$, such that $\deg_z F = n(F(z))$, $z \in \mathbf{C}$, and $f \circ F = F \circ \tilde{f}$ [14, 15]. The measure μ_f is the image $F_* \ell_2$ of the lebesgue measure ℓ_2 on $\mathbf{R}^2$. The parabolic orbifolds and the corresponding covering maps and lifts are described in [15]. We shall use the following assertion, proved in [10]: f is critically finite with parabolic orbifold if and only if the measure μ_f is fibered at some point $z_0 \in J_f$. Here a locally finite Borel measure σ on $\mathbf{R}^2$ is said to be lamellar at the point $z_0 \in \operatorname{supp}\sigma$ [10] if there exists a diffeomorphism ψ of some domain onto a neighborhood of z_0 such that the measure $\psi^*\sigma$ is invariant under translations along the x axis in $\mathbf{R}^2$.

<u>1. Main Results.</u> We term exceptional those cases in which the Julia set is the Riemann sphere $\bar{\mathbf{C}}$, a circle, or a segment (in $\bar{\mathbf{C}}$). Fix a rational function f of degree $m \geq 2$. Let $J = J_f$, $\mu = \mu_f$, and let $H \neq \mathrm{id}$ be a function that is meromorphic in some disc $B(a, r)$ of radius r centered at the point $\in J$.

<u>Definition 1.</u> We call H a symmetry on J if the following conditions are satisfied: 1) $x \in B(a, r) \cap J$ if and only if $H(x) \in H(B(a, r)) \cap J$; 2) in the exceptional cases there exists an $\alpha > 0$ such that $\mu(H(A)) = \alpha\mu(A)$ for any set A on which the map H: $A \to \bar{\mathbf{C}}$ is injective. A family $\mathcal{H}$ of symmetries in the disc $B(a, r)$ is said to be nontrivial if $\mathcal{H}$ is normal in $B(a, r)$ and no limit function for $\mathcal{H}$ is equal to a constant.

Let us state our main result.

<u>THEOREM 1.</u> The function f is critically finite with parabolic orbifold if and only if there exists an infinite nontrivial family of symmetries on J_f.

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We let $R_d(f)$ denote the set of rational functions g of degree d with the property that $J_g = J_f$ and, in the exceptional cases, $\mu_g = \mu_f$. Set

$$R(f) = \bigcup_{d\geq 2} R_d(f).$$

Two rational functions are said to be equivalent if they are conjugate by means of a linear-fractional map. Notice that if f is equivalent to $z^{\pm m}$, then for any $d \geq 2$ the set $R_d(f)$ is isomorphic to the unit circle.

THEOREM 2. If f is not equivalent to $z^{\pm m}$, then $R_d(f)$ is finite for any d.

THEOREM 3. Suppose $g \in R(f)$ and one of the following conditions is satisfied: 1) there is a point a that is preperiodic (i.e., a preimage of a periodic point) for f and periodic and repulsive for g; 2) the limit set P_f' of the iterates of critical points of f is finite and contains no neutral irrational periodic points of f. Then either f and g are critically finite and have a common parabolic orbifold, or $f^{\ell} \circ g^k = f^{2\ell}$ for some positive integers ℓ and k.

Remark 1. Suppose f and g commute. Then, by Theorem 3, either $f^{\ell} = g^k$, or f and g are critically finite with common parabolic orbifold, and so we recover Ritt's theorem [9].

Remark 2. The condition $f^{\ell} \circ g^k = f^{2\ell}$ guarantees that $J_g = J_f$ and $\mu_g = \mu_f$.

THEOREM 4. If J_f is a circle and $g \in R(f)$, then either f is equivalent to $z^{\pm m}$, or there exists a linear-fractional symmetry h and numbers $\ell, k \in \mathbb{N}$ such that $f^{\ell} \circ h = f^{\ell}$ and $g^k = h \circ f^{\ell}$.

2. Auxiliary Propositions.

The following assertions are of independent interest.

LEMMA 1. Let $\lambda \in \mathbb{C}$, $|\lambda| > 1$, and let Φ_n be a sequence of univalent functions in $B(0, \varepsilon)$, such that $\Phi_n(0) \neq 0$ for all $n \in \mathbb{N}$ and $\Phi_n \to \mathrm{id}$ $(n \to \infty)$. Then there exist $\delta \in (0, \varepsilon/2)$, $q \in \mathbb{C}\setminus\{0\}$, and sequences (ℓ_i) and (n_i) of positive integers, such that for any $m \in \mathbb{N} \cup \{0\}$, starting with some number i, the maps R_i: $B(0, \delta) \to B(0, 2\delta)$ given by the formulas

$$R_i(z) = \lambda^{l_i - m}\Phi_{n_i}(\lambda^{-(l_i-m)}\Phi_{n_i}^{-1}(z)) \tag{1}$$

and

$$\lim_{i\to\infty} R_i(z) = z + q\lambda^{-m}, \quad z \in B(0, \delta). \tag{2}$$

Proof. Let $q_n = \Phi_n(0)$. Since $q_n \neq 0$ and $q_n \to 0$, there exist sequences of positive integers (ℓ_i) and (n_i) such that $\lambda^{l_i} q_{n_i} \to q$ $(i \to \infty)$, where $|q| \neq 0$ and is small. For these sequences and small $\delta > 0$ we expand the functions (1) in series and obtain (2).

LEMMA 2. Supose the map R is holomorphic in a neighborhood of the point a, $R(a) = a$, $R'(a) = 1$, and R preserves a finite measure σ such that $\sigma(\{a\}) = 0$ and $\sigma(U) > 0$ for any neighborhood U of the point a. Then $R = \mathrm{id}$.

The proof follows from the description of the local dynamics of R [11].

LEMMA 3. Let $\lambda \in \mathbb{C}$, $|\lambda| > 1$, and suppose in the half-plane $\{z | \operatorname{Re} z > M_0\}$, $M_0 > 0$ there is defined a single-valued analytic function ψ of the form $\psi(z) = 1 + z + O(|z|^{-\gamma})$, $\gamma > 0$, $|z| \to \infty$. Then for any $c > 0$ there exist sequences of positive integers (n_i) and (ℓ_i) and a number $M > M_0$ such that $\lambda^{-n_i}\psi^{l_i}(\lambda^{n_i}z) \to z + c$ $(i \to \infty)$ for all $z \in \Pi = \{z | \operatorname{Re} z > M\}$.

Proof. Choose $M > M_0$ such that $\overline{\psi(\Pi)} \subset \Pi$. It is known [1] that

$$\psi^l(z) = l + z + o(|z|) + o(l) \quad (l \to +\infty, \ z \to \infty).$$

Let $c > 0$ and the sequence (n_i) be such that $\lambda^{n_i} \to 0$ $(i \to \infty)$. Set $l_i = [c\,|\lambda|^{n_i}]$. Then

$$\lambda^{-n_i}\psi^{l_i}(\lambda^{n_i}z) \to z + c \quad (i \to \infty).$$

Using Lemmas 2 and 3 we prove

Proposition 1. If the rational function f and the symmetry H on J_f have a common repulsive fixed point a, then f and H commute.

Proof. In a small neighborhood of a consider the function $R = H \circ f \circ H^{-1} \circ f^{-1}$, where the branches of H^{-1} and f^{-1} are chosen so that $H^{-1}(a) = f^{-1}(a) = a$. We have: $R(a) = a$, $R'(a) = 1$. In the exceptional cases R preserves the measure μ_f and, by Lemma 2, R = id, i.e., $H \circ f = f \circ H$. Now suppose J_f is not the Riemann sphere, a circle, or a segment. Assume $R \neq$ id. Let us show that if the point $z \in J_f$ is close to a, then J_f contains an analytic arc connecting z and a. As shown in [7, 8] (see also [10]), this forces J_f to be $\bar{\mathbb{C}}$, or a circle, or a segment. Thus, let $z \in J_f$ be close to a. Then $\tilde{R}^k(z) \to a$ when $k \to \infty$, where $\tilde{R}$ denotes R or R^{-1}. By a theorem of Schröder [5], there exists a holomorphic change of coordinates in a neighborhood of a which maps J_f into a set that is invariant under the map z z/λ, where $\lambda = f'(a)$. Now subject the new coordinates to the change $z \mapsto A/z^p$ with suitable $A = 0$ and $p \in \mathbb{N}$. and then apply Lemma 3. Proposition 1 is proved.

We shall need the following fact.

Remark 3. A. É. Eremenko showed that there is no neighborhood U such that $U \cap J_f$ is diffeomorphic to the product of an interval and a Cantor set.

Indeed, suppose the ocntrary holds.

One can assume that U is a neighborhood of a repulsive fixed point. Let F be its Poincaré function [1]. Then the full preimage $I = F^{-1}(J_f)$ is the product of a line (say the x axis) and a Cantor set. Now consider some component of the set $\bar{\mathbb{C}} \setminus J_f$ that is periodic for f [11] and let P be a component of the preimage $F^{-1}(D)$; the horizontal strip P is bounded by lines ℓ_1 and ℓ_2 from I. The boundary of D consists of $F(\ell_i)$ (i = 1, 2) and the boundaries of the two periodic cluster sets C_+ and C_- for the meromorphic function $F: P \to \bar{\mathbb{C}}$, obtained when $\operatorname{Re} z \to +\infty$ and $\operatorname{Re} z \to -\infty$. By Iversen's theorem [16], the boundaries of the complete cluster sets C_+ and C_- are contained in the boundary of cluster sets

$$C_+^0 = \bigcap_{M>0} \overline{F(\partial P \cap \{z : \operatorname{Re} z > M\})}$$

and

$$C_-^0 = \bigcap_{M>0} \overline{F(\partial P \cap \{z : \operatorname{Re} z < -M\})}.$$

Since $C_+{}^0$ and $C_-{}^0$ are at most two-connected and lie in ∂D, the domain D is finitely connected. Since the boundary of D contains analytic curves, D cannot be a Siegel disc or a Herman ring [11]. If D is a simply connected domain of direct attraction, then, by a theorem of Fatou [2], J_f is a circle or a segment. We reached a contradiction.

3. Proof of Theorem 1. Suppose $\mathcal{H}$ is an infinite nontrivial family of symmetries on J_f. Let us prove that f is critically finite with parabolic orbifold (the converse is obvious). The proof is broken into steps.

1. By Definition 1, there exist a sequence (H_n), $H \in \mathcal{H}$, a point $a \in J_f$, and a number $\rho_0 > 0$ such that each H_n is univalent in the disc $B_0 = B(a, \rho_0)$ and (H_n) converges in B to a univalent function H. One can assume that: a) H = id (this is achieved by replacing H_n with $H_{n+1}{}^{-1} \circ H_n$); b) a is a repulsive fixed point of f and in B_0 there is defined the branch $f_0{}^{-1}$ of the function f_0, singled out by the condition $f_0{}^{-1}(a) = a$. Set $F_n = H_n{}^{-1} \circ f_0{}^{-1} \circ H_n$. Starting with some n, the maps F_n are defined in a smaller disc $B = B(a, \rho)$, $0 < \rho < \rho_0$. They enjoy the following properties: 1.1) each F_n is univalent in B; 1.2) $F_n \to f_0{}^{-1}$ in B; 1.3) $F_n(B) \subset B$, $F_n(a_n) = a_n$, $F_n{}'(a_n) = \lambda^{-1}$, where $a_n = H_n{}^{-1}(a)$, $\lambda = f'(a)$; 1.4) $x \in J_f \cap B$ if and only if $F_n(x) \in J_f \cap F_n(B)$, n ∈ ; 1.5) in the exceptional cases $\mu(F_n(A)) = m^{-1}\mu(A)$ for any Borelian set $A \subset B$, where $\mu = \mu_f$, $m = \deg f$.

2. Suppose that $a_n = a$ for some n. Consider the function $R = f \circ F_n$. Then $R(a) = a$, $R'(a) = 1$. A verbatim repetition of the proof of Proposition 1 gives R = id. Therefore, $F_n \neq f_0{}^{-1}$ implies $a_n \neq a$.

3. Schröder's theorem and properties 1.1)-1.3) of F_n guarantee the existence of an $\varepsilon > 0$, a sequence of functions (h_n), and a function h, all univalent in $B(0, \varepsilon)$, such that $h_n(0) = a_n$, $h_n{}'(0) = 1$, $h_n \to h$ $(n \to \infty)$, $f_n \circ h_n = h_n(z/\lambda)$, $f_0{}^{-1} \circ h = h(z/\lambda)$, $z \in B(0, \varepsilon)$. Set $\Phi_n = h^{-1} \circ h_n$, $q_n = \Phi_n(0)$. By Sec. 2, either $q_n \neq 0$, or $F_n = f_0{}^{-1}$. Suppose $F_n \neq f_0{}^{-1}$ for large n. Now notice that

$$\lambda^l \Phi_n (\Phi_n^{-1}(z)/\lambda^l) = h^{-1} \circ (f^l \circ F_n^l) \circ h (z),$$

and apply Lemma 1. If $I = h^{-1}(J_1 \cap B)$ and $\nu = h^*\mu$ (the preimage of the measure μ), then we conclude that the set I and the measure ν are invariant under the translations $z \mapsto q + z/\lambda^m$, $q \neq 0$, $m \in \mathbb{N}$. Therefore, the set I is either a full neighborhood of zero, or an interval, or the product of a Cantor set and an interval. The last case is impossible (see Remark 3). In the first two cases the measure μ is lamellar at the point a. It follows that f is critically finite with parabolic orbifold.

4. Thus, we showed that either f has a parabolic orbifold or, starting with some index, $F_n = f_0^{-1}$, i.e.,

$$f_0^{-k} \circ H_n = H_n \circ f_0^{-k}, \quad k \in \mathbb{N}. \tag{3}$$

Now let us carry out the last step: In the disc B choose a small disc B_1 centered at another repulsive fixed point b, $a \neq b$, of some iteration f^p, and let f_1^{-p} be a branch of f^{-p} satisfying $f_1^{-p}(b) = b$, $f_1^{-p}(B_1) \subset B_1$. Repeating the arguments (for the new functions $\tilde{F}_n = H_n^{-1} \circ f_1^{-p} \circ H_n$), we arrive at the equality $f_1^{-p} \circ H_n = H_n \circ f_1^{-p}$. From this and (3) it follows that $H_n(f_0^{-k}(b)) = f_0^{-k}(b)$, $k \in \mathbb{N}$, i.e., $H_n = \mathrm{id}$. The theorem is proved.

4. Functions with Common Julia Set or Common Maximal Entropy Measure: Proofs.

Proof of Theorem 2. Suppose f is not equivalent to $z^{\pm m}$. Find a sequence (g_n) in $R_d(f)$ which converges to a rational function $\tilde{g}$ everywhere but at finitely many points. If $\tilde{g}(z) \equiv c$, then $c \in J_f$. On the other hand, for large n the sequence of iterates of g_n is normal in a neighborhood of c. We reached a contradiction. Therefore, $\tilde{g} \neq$ const and, by Theorem 1, it suffices to consider the case where f has a parabolic orbifold $\mathcal{O}$. Let $g \in R_d(f)$. Since $\mu_g = \mu_f$, $\mathcal{O}$ is also an orbifold for g. One can assume that $J_f = \bar{\mathbb{C}}$. If F_f and F_g are covering maps for f and g, then $F_g^{-1} \circ F_f$ locally preserves the Lebesgue measure on $\mathbf{R}^2$. Consequently, there exists a covering map common for all $g \in R_d(f)$, and only finitely many of lifts, corresponding to a given degree d [15, 10]. Theorem 2 is proved.

Proof of Theorem 3. Suppose $g \in R_d(f)$ and f, g are not critically finite with parabolic orbifold.

1) Passing to iterates one can consider that the points a and $b = f(a)$ are fixed for f and g, respectively, and a is repulsive for g. First, let us prove that b, too, is repulsive for f. Assume the contrary, i.e., $|\lambda_2| = 1$, where $\lambda_2 = f'(b)$. Let p be the multiplicity of the point a in the equation $f(x) = b$. Since $|\lambda_1| > 1$, where $\lambda_1 = g'(a)$ in a neighborhood of b there is defined a holomorphic function H_1 such that $H_1 \circ f = f \circ g$. Set $H_2 = f$. The symmetries H_1 and H_2 satisfy $H_1(b) = H_2(b) = b$, $H_1'(b) = \lambda_1^p$, $H_2'(b) = \lambda_2$. By a holomorphic change of coordinates one can ensure that $H_1(z) = \lambda_1^p z$. If $\lambda_2^q = 1$ for some $q \in \mathbb{N}$ then by Lemma 3 we reach an exceptional case (see the proof of Proposition 1). If, however, $\lambda_2^q \neq 1$ for all $q \in \mathbb{N}$, then expanding H_2 in a series we obtain: $\lambda_1^{p\ell} H_2(z/\lambda_1^{p\ell}) \to \lambda_2 z$, $\ell \to \infty$, and again we arrive at an exceptional case. Therefore, $\mu_f = \mu_g = \mu$. But then $\mu(H_2 \cdot (A)) = m\mu(A)$ and $\mu(H_1 \circ H_2(A)) = md\mu(A)$ for a small neighborhood A of the point b. Moreover, $|H_1'(b)| = |(H_1 \circ H_2)'(b)| = |\lambda_1|^p$. Consequently, $\lim_{\varepsilon \to 0} \ln \mu(B(b, \varepsilon))/\ln \varepsilon$ equals simultaneously $\ln m/\ln |\lambda_1|^p$ and $md/\ln |\lambda_1|^p$: contradiction.

Thus, we proved that $|\lambda_2| \neq 1$. Hence, $|\lambda_2| > 1$. We fix a small neighborhood B of the point a and we shall construct a nontrivial family of symmetries in B. In B there is defined a branch of g_0^{-1} by the condition $g_0^{-1}(a) = b$. Also, in a small neighborhood B_1 of the point b consider the branch of f_0^{-1} specified by the condition $f_0^{-1}(b) = b$. Let h_1 and h_2 be holomorphic changes of coordinates that are defined in neighborhoods of zero and take g_0^{-1} and f_0^{-1} into the maps $z \mapsto z/\lambda_1$ and $z \mapsto z/\lambda_2$, respectively. Set $H_\ell = f^{pk_l} \circ f \circ g_0^{-n_l} = h_2 \circ (\lambda_2^{pk_l} h_2^{-1}) \circ (f \circ h_1 \circ (\lambda_1^{-n_l} h_1^{-1}))$, where (k_ℓ) and (n_ℓ) are chosen so that $\lambda_2^{k_l}/\lambda_1^{n_l} \to 1$, $l \to \infty$, and p is the multiplicity of the point a under f. We have $H_l = h_2 \circ (\lambda_2^{pk_l} \psi) \circ (h_2^{-1}/\lambda_1^{n_l})$, where $\psi = h_2^{-1} \circ f \circ h$, $\psi(u) \sim Cu^p$, $u \to 0$, $C \neq 0$. Expanding ψ in a series, one verifies that the sequence (H_ℓ) converges in B to a holomorphic function $H \neq$ const. Now apply Theorem 1 and conclude that $H_i = H_j$ for some $i \neq j$, $n_i > n_j$. It remains to put $x = g_0^{-n_i}(z)$. Then $f^{pk_i}(x) = f^{pk_j} \circ g^{n_i - n_j}(x)$. This completes the examination of the case 1). In case 2), fix a repelling fixed point a of the function g. Two subcases are possible: a) $\omega_f(a) \subset \bar{P}_f$; b) there exists a point $b \in \omega_f(a) \setminus \bar{P}_f$ [here $\omega_f(a)$ denotes the set of limit points of the sequence $(f^n(a))_{n \geq 0}$]. Since P_f' is finite and contains no neutral irrational cycles, in subcase a) the point a is periodic for f and we arrive at case 1) of the theorem. Now consider subcase b). For some $\delta > 0$ and some sequence $n_k \to \infty$, $f^{n_k}(a) \to b$ and in each disc $B(f^{n_k}(a), 2\delta)$ there is defined a branch $f_k^{-n_k}$ by the condition $f_k^{-n_k}(f^{n_k}(a)) = a$. Fix a small neighborhood $B_0 = B(a, \varepsilon)$ of the point a such that $|g'{*}x(| < 2|\lambda_1|$ for all $x \in B_0$, where $\lambda_1 = g'(a)$. In $B_k = B(f^{n_k}(a), \delta)$ consider the function $\varphi_k = g^{l_k} \circ f_k^{-n_k}$, where ℓ_k is the smallest number for which diam $\varphi_k(B_k) > \varepsilon/4|\lambda_1|$. Then diam $_k(B_k) < \varepsilon/2$ and, by the distortion theorem $C_1 < |\varphi_k'(x) < C_2$ for some C_1, C_2, $0 < C_1 < C_2 < \infty$, and all $k \in \mathbf{N}$, $x \in B_k$. It follows that there exists a disc B centered at a such that $B \subset \varphi_k(B_k)$ for all k. Set $H_k = \varphi_k^{-1}|_B$. Then (H_k) is a nontrivial family of symmetries in B and $H_k = f^{n_k} \circ g_0^{-l_k}$, where the branch g_0^{-1} is defined in B_0 by the condition $g_0^{-1}(a) = a$. By Theorem 1,

$$f^{n_i} = f^{n_j} \circ g^{l_i - l_j}$$

for some $i \neq j$, $\ell_i > \ell_j$. Theorem 3 is proved.

Proof of Theorem 4. Suppose f is not equivalent to $z^{\pm m}$. Since $J_f = S$ is a circle, condition 2) of Theorem 3 is satisfied. Therefore, $f^{2\ell} \circ g^{2k} = f^{4k}$, $g_1 = g^{2k}$. The maps f_1, g_1: $S \to S$ preserve orientation. Let F and G be lifts of these maps to $\mathbf{R}$, and let $\tilde{\mu}$ be a lift of the measure $\mu = \mu_f = \mu_g$ to $\mathbf{R}$ Introduce the homeomorphism

$$\varphi\colon \mathbf{R} \to \mathbf{R},\ \varphi(x) = \mu([0, x]),\ \text{if } x \in [0, 1) \text{ and}$$

$$\varphi(x + n) = \varphi(x) + n,\ n \in \mathbf{Z},\ x \in \mathbf{R}.$$

From the fact that the measure μ is balanced, it follows that the difference $\varphi \circ F(x) - \varphi \circ G(x)$ does not depend on $x \in \mathbf{R}$. From this it follows, upon descending to S, that for some homeomorphism h_0: $S \to S$ and some number α, $|\alpha| = 1$, we have

$$h_0 \circ g_1(z) = \alpha (h_0 \circ f_1)(z),\ z \in S.$$

Thus we proved that $h = g_1 \circ f_1^{-1}$ does not depend on the branch f_1^{-1} on S. Therefore, h is a linear-fractonal function. The theorem is proved.

Remark 4. All assertions of this paper carry over to polynomial-like maps [17] and to RB-domains and maps [18].

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GENERALIZATION OF THE PALEY-WIENER THEOREM IN WEIGHTED SPACES

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1. Introduction

Let X be a linear topological space of complex functions defined on some subset $T \subset \mathbf{R}^n$ $(\mathbf{C}^n)$, and assume that a system of functions $e^{\langle t,z\rangle}$, $z \in \Omega$, is complete in this space. Then the generalized Laplace transform, which takes a linear continuous functional S on X to a function $\hat{S}(z) = (S, \exp(\langle t, x\rangle))$, $z \in \Omega$, establishes an isomorphism between the adjoint space X^* and a linear topological space of functions defined on Ω.

Many mathematicians have devoted their work to the problem of describing the adjoint space in terms of generalized Laplace transform. For example in [1] the projective limit of weighted Banach spaces of the form

$$\{f \in H(D): \|f\| = \sup_z [\,|f(z)|/\exp(-\psi(-\ln d(z)))] < \infty\}$$

was considered, where D is a convex, bounded region in $\mathbf{C}^n$, d(z) is the distance from a point z to ∂D and ψ is a convex function, and a complete description was given of the adjoint space in terms of the generalized Laplace transform. In [3, 4] some generalization of the Paley—Wiener theorem for weighted Hilbert spaces.

The present article is devoted to the problem of describing adjoint spaces in terms of the Laplace transform on the space

$$L^2(I, W) = \left\{f \in L_{\text{loc}}(I): \|f\|^2_{L^2(I,W)} \overset{\text{def}}{=} \int_I |f(t)|^2/W(t)\,dt < \infty\right\},$$

where I is a bounded interval on the real axis and 1/W(t) is a measurable function on I.

THEOREM 1. Let W(t) be a function on I bounded from below by a positive constant and bounded from above on each compact subinterval of I. Let $\tilde{h}(x) = \sup_{t \in I}(xt - \ln\sqrt{W(t)})$— Young's conjugate function of the function $\ln\sqrt{W(t)}$, and define $\rho_{\tilde{h}}(x)$ by the condition

$$\int_{x-\rho_{\tilde{h}}(x)}^{x+\rho_{\tilde{h}}(x)} |\tilde{h}'(x) - \tilde{h}'(t)|\,dt \equiv 1.$$

Then

1. The generalized Laplace transform $\hat{S}(z)$ of the functional S on $L^2(I, W)$ is an entire function satisfying the condition $|\hat{S}|(z)| < C_S \exp(\tilde{h}(x))$,

$$\|\hat{S}\|^2 = \int_{\mathbf{R}}\int_{\mathbf{R}} |\hat{S}(x+iy)|^2 e^{-2\tilde{h}(x)} \rho_{\tilde{h}}(x)\, d\tilde{h}'(x)\, dy \leqslant \pi e \|S\|_{L^2(I,W)}.$$

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