ITERATION OF A POLYNOMIAL, BASIS, AND THE OPERATOR OF MULTIPLICATION BY z

G. M. Levin

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Let $T(z)$ be an arbitrary polynomial of degree $p \geq 2$, and let D be its region of attraction to infinity. In this article, we construct natural bases in the space of functions which are holomorphic on $\mathbb{C} \setminus D$, and in the conjugate space. We describe the set of eigenvalues of the operator of multiplication by z.

1. Thus, let $T(z) = z^p + a_1 z^{p-1} + \ldots + a_p$, let T_k be the kth iteration of T, $T_0(z) \equiv z$, $T^{-k}(B) = \{y \mid T_k(y) \in B\}$, $k \in \mathbb{N}$, $B \subset \mathbb{C}$. The following objects are related to this polynomial [1]: $J = \partial D$, its Julia set; $F = \mathbb{C} \setminus D$; μ, the equilibrium measure (relative to the logarithmic potential) on the compact set F; C, the set of critical points of T; $C_\infty = \bigcup_{k \geq 0} T^{-k}(C)$, $C_* = C_\infty \cap D$.

We define the dual systems of functions E_n, E_n^*, $n \geq 0$.

First, for $k = 1, 2, \ldots, p-1$, we put

$$c_k = \int u^k d\mu(u),$$

$$E_0(z) \equiv E_0^*(z) \equiv 1, \quad E_k(z) = z^k - c_k,$$

$$E_k^*(z) = p\frac{z^{p-k-1} + a_1 z^{p-k-2} + \ldots + a_{p-k-1}}{T'(z)}.$$

Now let $n = \sum_{i=0}^{\infty} \varepsilon(i) p^i$ be the p-adic expansion of the natural number n. We define

$$E_n(z) = \prod_{i=0}^{\infty} E_{\varepsilon(i)}(T_i(z)), \quad E_n^* = \prod_{i=0}^{\infty} E_{\varepsilon(i)}^*(T_i(z)).$$

Clearly $E_n(z)$ is a polynomial of degree n and $E_n^*(z) = O(z^{-n})$, $z \to \infty$. For example, $E_{p^n}(z) = T_n(z) - c_1$.

Proposition 1.

$$\int E_n(z) E_m^*(z)\, d\mu(z) = \delta_{nm}$$

(δ_{nm} is the Kronecker symbol).

Remark 1. If $T'(z) = 0$ at a point $z \in J$, then the integrand might not be summable, and the integral is understood as

$$\lim_{k \to \infty} \frac{1}{p^k} \sum_{T_k(z) = a} E_n(z) E_m^*(z) = \frac{1}{2\pi i} \int_\Gamma E_n(z) E_m^*(z) \sigma(z)\, dz,$$

where a is an arbitrary point of $D \setminus C_*$, Γ is an arbitrary smooth contour which encloses F, and

$$\sigma(z) = \lim_{n \to \infty} T_n'(z)/(p^n T_n(z)), \quad z \in D.$$

2. Let G be the Green's function of the region D with a pole at ∞, $G|_F \equiv 0$. We introduce the function spaces X and Y: $f \in X$ if f is holomorphic in the region $B_\varepsilon(f) = \{z \mid G(z) < \varepsilon\}$, where $\varepsilon > 0$ may be different for each f; $g \in Y$ if g is holomorphic in $(D \setminus C_*) \cup \{\infty\}$ and has poles of order not greater than 1 at points of C_*.

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THEOREM 1. The function f belongs to X (Y, respectively) if and only if there exists a unique sequence $(a_n)_{n\geq 0}$, $a_n \in \mathbb{C}$, such that $\varlimsup_{n\to\infty} |a_n|^{1/n} > 1$ (≤ 1, respectively) and $f(z) = \sum_{n=0}^{\infty} a_n E_n(z)$ ($E_n^*(z)$, respectively), where $z \in B_\varepsilon(f)$ ($D \setminus C_*$, respectively).

Proof. The proof is based on the following expansion of the Cauchy kernel:

$$\frac{1}{\lambda - z} = \sigma(\lambda) \sum_{n=0}^{\infty} E_n(z) E_n^*(\lambda),$$

which holds for $G(z) < G(\lambda)$, $\lambda \in D \setminus C_*$, and the estimates

$$\lim_{n\to\infty} \frac{\ln |E_n(\lambda)|}{n} = -\lim_{n\to\infty} \frac{\ln |E_n^*(\lambda)|}{n} = G(\lambda), \quad \lambda \in D \setminus C_*.$$

Remark 2. The estimate for E_n can be strengthened on the set F: for some constant $C = C(F)$

$$|E_n(z)| \leq C^{S_p(n)}, \quad z \in F, \quad n \in \mathbb{N},$$

where $S_p(n)$ is the sum of the digits in the p-adic expansion for n.

Remark 3. From Proposition 1 and Theorem 1 there follow criteria for the expansion of functions in X into series in the iterations $(T_n)_{n\geq 0}$.

Remark 4. If $g \in Y$, then the function σg is holomorphic in D and equal to zero at infinity. We introduce the topology on Y induced by uniform convergence on compact sets in the space of functions $\{g_1 = \sigma g, \ g \in Y\}$, and on X the standard topology of functions which are locally analytic on F [2]. Then the conjugate of X is isomorphic to Y. The isomorphism is given by the relation:

$$g \in Y \mapsto \beta \in X^*, \quad \beta = \frac{1}{2\pi i} \int_\Gamma f(\lambda) g(\lambda) \sigma(\lambda) \, d\lambda.$$

3. We define the matrix $Z = (\alpha_{i,n})_{i,n\geq 0}$ by the expansion

$$zE_n(z) = \sum_{i=0}^{\infty} \alpha_{i,n} E_i(z), \quad n = 0, 1, \ldots \quad (\alpha_{i,n} = 0 \quad \text{for} \quad i > n+1).$$

The transposed matrix corresponds to the identity

$$zE_n^* = \sum_{i=n-1}^{p^l - 1} \alpha_{n,i} E_i^*(z), \quad l \geq 1, \quad 1 \leq n \leq p^l - 1, \quad z \in D \setminus C_*.$$

Let A be the set of eigenvalues of Z.

THEOREM 2. The set A consists of those λ for which $\lambda \in J$ and

$$\exists m \geq 0 : \lim_{n\to\infty} \frac{T_n'(T_m(\lambda))}{p^n} = \infty.$$

Exactly one eigenvector $\bar{a}(\lambda) = (a_n(\lambda))_{n\geq 0}$ corresponds to each $\lambda \in A$.

Remark 5. If, for example, $\lambda \in A \setminus C_\infty$, then $a_n(\lambda) = E_n^*(\lambda)$.

Proposition 2. Either A is dense in J or A is empty; J is a circle and T(z) is linearly conjugate to z^p.

Remark 6, From the ergodic theorem it follows that $\mu(A) = 1$ if J is not connected. If J is connected, then we apply the iterated logarithm law [3], from which $\mu(A) = 0$.

4. In the case $T(z) = z^2 - t$, the matrix Z is given by

$$zE_n(z) = E_{n+1}(z) + t \sum_{k=1}^{\operatorname{ord}(n+1)} E_{n+1-2^k}(z),$$

where ord (m) is the number of zeros up to the first unity in the binary expansion of m (the binary order of m). A knowledge of Z gives a linear algorithm for the calculation of the moments of μ.

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SPECIAL VECTORS IN VERMA MODULES OVER AFFINE ALGEBRAS

F. G. Malikov

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1. Let $\hat{\mathfrak{g}}$ be an affine Lie algebra, $\hat{\mathfrak{h}}$ its Cartan subalgebra, Δ_+ the set of positive roots, Δ_+^{re} (Δ_+^{im}) the set of positive real (imaginary) roots, $\hat{\mathfrak{g}}_\alpha$ the root subspace correspond-to the root α. Consider the Verma module $M(\lambda)$, $\lambda \in (\hat{\mathfrak{h}})^*$.

A vector $w \in M(\lambda)$ is called special if for each $\alpha \in \Delta_+$ $\hat{\mathfrak{g}}_\alpha \cdot w = 0$.

A family of hyperplanes $H_{n,\alpha} \subset (\hat{\mathfrak{h}})^*$ indexed by pairs (n, α), where $n \in N$, $\alpha \in \Delta_+$, has been constructed in [1]; it has been shown there that $M(\lambda)$ is reducible if and only if $\lambda \in \bigcup_{n\in N, \alpha\in\Delta_+} H_{n,\alpha}$. If $\alpha \in \Delta_+^{re}$ and λ is a common position point of the hyperplane $H_{n,\alpha}$, then $M(\lambda)$ has a unique special vector. The formula for it has been found in [2].

If $\alpha \in \Delta_+^{im}$, then the hyperplane $H_{n,\alpha}$ does not depend on n and α, this fact allows us to abbreviate the notation to H_{im}; it is defined by the equation

$$\lambda(c) + g = 0, \tag{1}$$

where c is a generator of the center of $\hat{\mathfrak{g}}$ and g is a number (if $\hat{\mathfrak{g}}$ is an extended algebra of currents and c a standard generator of the center, then g is Coxeter's dual number of the corresponding finite-dimensional algebra). In this note we exhibit a construction of a family of special vectors in $M(\lambda)$ for $\lambda \in H_{im}$ whose existence was conjectured in [2]. As a corollary, we obtain a formula for the character of the irreducible module $L(\lambda)$ if λ is a common position point of the hyperplane H_{im}. This formula was conjectured in [1].

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, σ an automorphism of order d of the algebra $\mathfrak{g}$ induced by an automorphism of its Dynkin diagram. Each affine algebra has the form $\bigoplus_{i\in Z} (\mathfrak{g}^{(\mathrm{res}_d i)} \otimes t^i) \oplus C\cdot c$ for some $\mathfrak{g}$ and σ, where $\mathrm{res}_d i$ is the residue of the number i modulo d, $\mathfrak{g}^{(\mathrm{res}_d i)}$ is the eigensubspace relative to σ corresponding to the eigenvalue $\exp(2\pi\sqrt{-1}i/d)$. For all n, $k \in N$ consider series of the form

$$\sum_{x_1+\ldots+x_k=n,\ x_1\leqslant\ldots\leqslant x_k} e_{x,1} \otimes t^{x_1} \cdot \ldots \cdot e_{x,k} \otimes t^{x_k},$$

where $x = (x_1, \ldots, x_k)$, $e_{x,i} \in \mathfrak{g}^{(\mathrm{res}_d x_i)}$. Such series define operators in $M(\lambda)$ and generate the completion $\tilde{U}(\hat{\mathfrak{g}})$ of the universal enveloping algebra $U(\hat{\mathfrak{g}})$.

We denote $e_1 \otimes t^{x_1} \cdot \ldots \cdot e_k \otimes t^{x_k} := e_{\tau(1)} \otimes t^{x_{\tau(1)}} \cdot \ldots \cdot e_{\tau(k)} \otimes t^{x_{\tau(k)}}$, where τ is a permutation of minimal length satisfying the condition $x_{\tau(i)} \le x_{\tau(j)}$, if $i < j$. Let $T(\mathfrak{g}) = \bigoplus_{0\leqslant k\leqslant d-1} T(\mathfrak{g})^{(k)}$ be the tensor algebra of the space $\mathfrak{g}$ written as a sum of eigensubspaces relative to σ. For each $n \in N$ we will define a linear map $\Psi_n: T(\mathfrak{g})^{(\mathrm{res}_d n)} \to \tilde{U}(\hat{\mathfrak{g}})$. To this end, we choose in $T(\mathfrak{g})^{(\mathrm{res}_d n)}$ a basis consisting of tensors of the form $e_1 \otimes \ldots \otimes e_k$, where $e_i \in \mathfrak{g}^{(j_i)}$ and $j_1 + \ldots + j_k \equiv n$. Put

$$\Psi_n(e_1 \otimes \ldots \otimes e_k) = \sum_{x_1+\ldots+x_k=n,\ x_i\equiv j_i} (k!)^{-1} : e_1 \otimes t^{x_1} \cdot \ldots \cdot e_k \otimes t^{x_k}:$$

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