$$\forall \beta \in W^{1}_{b,0}(A_n') : \beta x \in W^1(H) \subset \mathscr{D}_0.$$

By choosing a suitable sequence of random elements $\{\beta_n x;\ n \geq 1\}$, we see that Lemma 3 holds.

Proof of the Assertions in Example 2. Using the random variable α we can construct a sequence of random variables $\{\beta_n = \varphi_n(\alpha),\ \varphi_n \in C^{1,b};\ n \geq 1\}$ such that

1) $\beta_n = \frac{1}{\alpha}$ for $\alpha > \frac{1}{n}$; $n \geqslant 1$;

2) $n+1 \geqslant \beta_n > 0 \pmod P$, $n \geqslant 1$;

3) $\beta_n \in W^1$, $\|D\beta_n\| \in L_\infty(\Omega, \mathscr{F}, P)$.

By considering the sequence $\{\beta_n \alpha x;\ n \geq 1\}$ we have $x \in \mathscr{D}$. In addition, for each $n \geq 1$, integrating by parts we obtain

$$\langle \beta_n \alpha x; \xi \rangle = \beta_n \langle \alpha x; \xi \rangle + \langle \alpha x; D\beta_n \rangle.$$

On the set $\left\{\alpha > \frac{1}{n}\right\} \in \mathscr{E} : D\beta_n = -\frac{1}{\alpha^2} D\alpha$. Consequently,

$$\langle x; \xi \rangle = \lim_{n\to\infty} \langle \beta_n \alpha x; \xi \rangle = \frac{1}{\alpha} \langle \alpha x; \xi \rangle - \frac{1}{\alpha} \langle x; D\alpha \rangle \pmod P.$$

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PERIODIC POINTS OF POLYNOMIALS

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We recall the most important facts of the Julia–Fatou theory as related to the iteration of polynomials (cf. [1-3]). Let P be a polynomial of degree $m \geq 2$, and P^n its n-th iteration. A point z is called periodic if $P^n z = z$ for some $n \in \mathbb{N}$. The set $\{p^k z\}_{k=1}^{n}$ is then called a cycle, and its cardinality is called the order of the cycle. The number $\lambda = (P^n)'(z)$, where $z \neq \infty$, is called a multiplicator of a cycle of order n. A cycle is called repulsive if $|\lambda| > 1$. Let $D_\infty = \{z \in \overline{\mathbb{C}} : P^n z \to \infty,\ n \to \infty\}$. It is easy to see that D_∞ is a region and that $\infty \in D_\infty$. The boundary of this region is called the Julia set $J = J(P)$. An equivalent definition is as follows. Let N(P) be the largest open set in $\mathbb{C}$ on which the family $\{P^n\}$ is normal. Then $J(P) = \mathbb{C} \setminus N(P)$. The Julia set is perfect and fully invariant, i.e., $P^{-1}(J) = J$. Furthermore, $J(P^n) = J$, $n \in \mathbb{N}$. The polynomial P has no more than $m - 1$ nonrepulsive cycles [2]. On the other hand, the number of repulsive cycles is infinite; their union is a dense subset of J.

Let D be a region, and $z_0 \in \partial D$. A point z_0 is called attainable (from D) if there exists a curve $\Gamma \subset D$ which ends on z_0.

THEOREM 1. Repulsive periodic points of the polynomial P are attainable from D_∞.

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In the case where the set J(P) is connected, Theorem 1 was announced by Douady [2]. As far as we know, the proof was not published.

Denote by T_m a polynomial defined by the functional equation $\cos m\omega = T_m(\cos\omega)$. The Julia set $J(T_m)$ is the set $[-1, 1]$. If $R_m(z) = z^m$, then $J(R^m)$ is the unit circle. The polynomials T_m and R_m play an extremely important role in iteration theory [1, 3].

THEOREM 2. Suppose the set J(P) is connected. Then the equation

$$|\lambda| \leqslant m^{2n}, \quad m = \deg P \tag{1}$$

for the multiplicator λ holds for any cycle of order n. Equality is achieved in Eq. (1) if and only if P is conjugate to T_m by a linear transformation, and λ is the multiplicator of the endpoint of the interval J(P) (which is a fixed point).

Note that if Theorem 1 is proved, then Eq. (1) (without the case of equality) follows from Theorem 3 of Pommerenke's paper [4].

THEOREM 3. For any polynomial P of degree m, one of the following is true:

1) there exists a cycle of order n with multiplicator λ, such that $|\lambda| > m^n$;

2) P is conjugate to R_m by a linear transformation.

Note that it is enough to prove Theorems 1 and 2 for fixed points, i.e., cycles of order 1.

The proof of Theorems 1 and 2 is cased on the study of the entire function introduced by Poincaré. Suppose $P(z_0) = z_0$, $P'(z_0) = \lambda$, $|\lambda| > 1$. From Poincaré's theorem [1], the functional equation

$$f(\lambda z) = P(f(z)) \tag{2}$$

has an entire solution f; moreover, this solution is uniquely determined by the conditions

$$f(0) = z_0, \quad f'(0) = 1. \tag{3}$$

[The simplest proof of these facts (it belongs to Poincaré) goes as follows: we first determine the formal power series $f(z) = z_0 + z + c_2z^2 + \ldots$ satisfying Eq. (2), and then show, by a direct analysis of the coefficients, that the series converges in some neighborhood of zero; finally, we extend the function f into $\mathbb{C}$ with the help of Eq. (2), taking into account that $|\lambda| > 1$.]

Denote by I the set of points on which the family $\{f(\lambda^n z) : n \in \mathbb{N}\}$ is not normal. It is clear that $I = f^{-1}(J)$. Set $D = f^{-1}(D_\infty)$. It follows from Eq. (2) and the full invariance of J and D_∞ that

$$\lambda I = I, \quad \lambda D = D. \tag{4}$$

Let G be the Green's function for the region D_∞ with a pole in ∞ which is continued by the zero function on $\mathbb{C}\setminus D_\infty$. The function G is continuous and subharmonic in $\mathbb{C}$ and obeys the functional equation found in [2]:

$$G(P(z)) = mG(z), \quad z \in \mathbb{C}. \tag{5}$$

[This property follows immediately from the evident relation $G(z) = \lim_{n\to\infty} m^{-n}\ln|P^n(z)|$.]

The function $u(z) = G(f(z))$ is continuous and subharmonic in $\mathbb{C}$. It follows from Eqs. (2) and (5) that

$$u(\lambda z) = mu(z), \; z \in \mathbb{C}. \tag{6}$$

Subharmonic functions satisfying Eq. (6) play an important role in the theory of entire functions (see, for example, [5]).

The order of a subharmonic function u in $\mathbb{C}$ is determined by the formula $\rho = \overline{\lim_{r\to\infty}} \ln B(r, u)/\ln r$, where $B(r, u) = \max\{u(z) : |z| = r\}$. It follows from (6) that

$$\rho = \ln m/\ln|\lambda|. \tag{7}$$

According to the "subharmonic version of the Denjoy—Carleman—Ahlfors theorem" [6, Theorem 4.16], the number of connected components of the set $\{z : u(z) > 0\}$ does not exceed $\max\{2\rho, 1\}$. We denote these components by $D_1, \ldots, D_p$. From Eq. (4) there exists an N such that $\lambda^N D_1 = D_1$. Choose a point $\omega_0 \in D_1$ and connect it by a curve $\lambda_0 \subset D_1$ with the point $\lambda^{-N}\omega_0 \in D_1$. Then $\Gamma = \bigcup_{n=0}^{\infty} \lambda^{-Nn}\Gamma_0 \subset D_1$ is a curve which goes to zero. Its image $f(\Gamma) \subset D_\infty$ is a curve which goes to z_0 by virtue of Eq. (3). This proves Theorem 1.

To prove Theorem 2, we note that if the Julia set is connected, then the set $I = f^{-1}(J)$ contains the continuum K connecting 0 and ∞. Indeed, if this is not true, then there exists a closed Jordan curve γ which separates 0 and ∞; moreover, $\gamma \cap I = \phi$. Let V be a neighborhood of zero on which f is bijective [this exists by virtue of Eq. (3)]. Choose V small enough so that J is not contained in f(V). Let M be a number large enough so that $\lambda^{-M}\gamma \subset V$. Taking Eq. (4) into account, we find $\lambda_{-M}\gamma \cap I = \phi$. Thus $f(\lambda^{-M}\gamma) \cap J = \phi$ and the curve $f(\lambda^{-M}\gamma)$ separates z_0 and ∞. This contradicts the fact that the set J is connected.

A classical theorem of Wiman (see, for example, [5]) states that for a harmonic function v of order $\rho < 1$, the inequality

$$\overline{\lim_{r\to\infty}} A(r, v)/B(r, v) \geqslant \cos \pi\rho,$$

holds, where $A(r, v) = \inf\{v(z) : |z| = r\}$.

Since $u(z) = 0$, $z \in K$, we have that $A(r, u) \equiv 0$, and hence $\rho \geq 0.5$. The inequality (1) (with $n = 1$) now follows from Eq. (7).

Suppose now that equality holds in Eq. (1). Then $\rho = 0.5$. We show that the subharmonic function $u \geq 0$ of order 0.5 satisfying conditions (6) and $A(r, v) \equiv 0$ necessarily takes the form

$$u(re^{i\theta}) = cr^{1/2}\cos\frac{1}{2}(\theta - \theta_0), \quad |\theta| \leqslant \pi, \tag{8}$$

where $c > 0$ and $\theta_0 \in [-\pi, \pi]$ are some constants. This result may be derived from [7, 8]; nonetheless, we present an independent simple proof.

The function u can be represented as [9]

$$u(z) = \int_{\mathbb{C}} \ln\left|1 - \frac{z}{\zeta}\right| d\mu_\zeta,$$

where μ is some Borel measure. Let $n(t) = \mu\{\zeta : |\zeta| \leq t\}$. Then $\rho = \lim_{t\to\infty} \ln n(t)/\ln t$. Let

$$u^*(z) = \int_0^\infty \ln\left|1 - \frac{z}{t}\right| dn(t).$$

The subharmonic function u^* is of order ρ. We show that the measure μ is concentrated on the ray $\ell = \{\zeta : \arg\zeta = \theta_0\}$. Suppose this is not so. Taking into account the fact that for fixed $r > 0$ the quantity $\ln|1 - re^{i\theta}|$ has a strict minimum for $\theta = 0$, we obtain

$$u^*(r) = A(r, u^*) < A(r, u) = 0, \quad r > 0.$$

Then it follows from Eq. (6) that $u^*(|\lambda|z) = mu^*(z)$, and hence

$$\overline{\lim_{r\to\infty}} A(r, u^*)/B(r, u^*) < 0.$$

This contradicts Wiman's theorem.

Thus the measure μ is concentrated on some ray ℓ, and the function u is harmonic in $\mathbb{C}\setminus\ell$. Since $u \geq 0$, the inequality $u > 0$ holds in $\mathbb{C}\setminus\ell$. Moreover, $u = 0$ on ℓ, since $A(r, u) \equiv 0$. Thus, u has the form of Eq. (8).

From this it follows that I is a ray. Since $I = f^{-1}(J)$, there exists a circle V such that $J \cap V$ is an analytic curve. Then it follows from Fatou's theorem [1, p. 225] that the polynomial P is conjugate to either T_m or R_m. The latter case is eliminated by direct checking. Theorem 2 is proved.

Remark. Let G be a region, and let $z_0 \in \partial G$ be an attainable boundary point. Two curves Γ_1, $\Gamma_2 \subset G$ ending on the point z_0 are called equivalent if there exists a sequence of curves $\gamma_n \subset G$, $\gamma_n \to z_0$ connecting Γ_1 and Γ_2. From the results of Douady [2, Sec. 6, Lemma 1], it follows that if the Julia set is connected, then there exists a finite number of classes of equivalent curves in D_∞ which end on the periodic point z_0. It is possible to show that the number p of these classes is equal to the number of connected components of the set $D = \{z : u(z) > 0\}$. Applying the Denjoy–Carleman–Ahlfors theorem, we find $p \le 2\rho$. From Eq. (7) we find that $|\lambda| \le m^{2/p}$ for a fixed point, or $|\lambda| \le m^{2n/p}$ for a cycle of order n. More delicate arguments show that the equality in these estimates is possible in only two cases: 1) p = 1; P is conjugate to T_m and z_0 is the endpoint of the interval J(P); 2) p = 2; P is conjugate to T_m and z_0 is an interior point of the interval J(P).

We now go to the proof of Theorem 3. Without loss of generality, we can assume that the leading coefficient of the polynomial P is equal to 1. This is always possible to achieve by conjugating with a linear function which does not change the multiplicators. We will need the following lemmas.

LEMMA 1. Let

$$c = c(A) = \sum_{P(z)=A} P'(z).$$

Then c does not depend on A. Moreover,

$$\sum_{P^n(z)=A} (P^n)'(z) = c^n, \quad n \in \mathbb{N}, \tag{9}$$

$$\sum_{P^n(z)=z} (P^n)'(z) = m^n(m^n - 1) + c^n, \quad n \in \mathbb{N}, \tag{10}$$

$$m = \deg P.$$

Proof. From the residue theorem

$$c(A) - c(B) = \int_{|z|=r} \left\{\frac{(P')^2}{P-A} - \frac{(P')^2}{P-B}\right\} dz,$$

where r is sufficiently large. The expression in the integral is $O(z^{-2})$, $z \to \infty$; hence c(A) = c(B). We prove Eq. (9) by induction:

$$\sum_{P^{k+1}(z)=A} (P^{k+1})'(z) = \sum_{P^k(\omega)=A} (P^k)'(\omega) \sum_{P(z)=\omega} P'(z) = c \sum_{P^k(\omega)=A} (P^k)'(\omega).$$

In view of Eq. (9), it is enough to prove Eq. (10) for n = 1. Then Eq. (10) follows from the fact that the residue of the function

$$\frac{P'(z)(P(z)-z)'}{P(z)-z} - \frac{(P')^2(z)}{P(z)}$$

in the point ∞ is equal to $m(m-1)$. The lemma is proved.

LEMMA 2. Suppose all the fixed points of the polynomial P, with the possible exception of one, have multiplicators equal to $m = \deg P$. Then P is conjugate to R_m.

Proof. By conjugating with the function $z + a$, we make the exceptional point be equal to 0. From the hypotheses of the lemma, it follows that $P(z) - z = z/m(P'(z) - m)$. Solving this differential equation with the boundary condition $P(0) = 0$, we find that $P = R_m$.

LEMMA 3. Let $c, \lambda \in \mathbb{C}$. If $\mathrm{Re}(\lambda^n) > \mathrm{Re}(c^n)$ for all $n \in \mathbb{N}$, then $\lambda > 0$.

Proof. The cases $c\lambda = 0$, $\arg c = \pm\pi/2$, $\arg \lambda = \pm\pi/2$ are easily eliminated. There exists a $\delta > 0$ such that infinitely many points c^n lie within the angle $\{z : |\arg z| \le \pi/2 - \delta\}$.

This implies that $|\lambda| \geq |c|$. If $\arg\lambda \neq 0$, then infinitely many points λ^n lie within some angle of the form $\{z : |\arg z - \pi| \leq \pi/2 - \delta\}$. Thus $|\lambda| = |c|$. Setting $\theta_1 = \arg\lambda$, $\theta_2 = \arg c$, we obtain

$$\cos n\theta_1 > \cos n\theta_2, \quad n \in \mathbb{N}. \tag{11}$$

In particular, $\cos 2n\theta_1 > \cos 2n\theta_2$, which implies

$$\cos^2 n\theta_1 > \cos^2 n\theta_2, \quad n \in \mathbb{N}. \tag{12}$$

It follows from Eqs. (11) and (12) that $\cos n\theta_1 > 0$, $n \in \mathbb{N}$. Thus $\theta_1 = 0$, which is shat was needed.

We now finish the proof of Theorem 3. Suppose that the moduli of the multiplicators of all cycles do not exceed m^n, where n is the order of the cycle.

Let λ be the multiplicator of any fixed point. Then, by assumption,

$$\operatorname{Re} \sum_{P^n(z)=z} (P^n)'(z) \leqslant m^n (m^n - 1) + \operatorname{Re}(\lambda^n); \tag{13}$$

moreover, the equality holds only if the multiplicators of all the fixed points, with the exception of one, are equal to m. Then from Lemma 2 we find that P is conjugate to R_m. Suppose that the inequality (13) is strict for all $n \in \mathbb{N}$. Comparing Eqs. (13) and (10), we find that $\operatorname{Re}(\lambda^n) > \operatorname{Re}(c^n)$, $n \in \mathbb{N}$. From Lemma 3, $\lambda > 0$. This holds for all fixed points. If all their multiplicators are equal to m, we apply Lemma 2 again. Suppose that the equation $\lambda_i < m - \varepsilon$, $i = 1, 2$, $\varepsilon > 0$ holds for two multiplicators. Choose a sequence n_k such that $\operatorname{Re}(c^{n_k}) \geq 0$, $k \in \mathbb{N}$ is satisfied. By virtue of Eq. (10), we find that either $m^{n_k}(m^{n_k} - 1 \leqslant \sum_{P^{n_k}(z)=z} \operatorname{Re}(P^{n_k})'(z) \leqslant m^{n_k}(m^{n_k} - 2) + 2(m - \varepsilon)^{n_k}$ or $m^{n_k} \leq 2(m - \varepsilon)^{n_k}$, which is impossible. Theorem 3 is proved.

<u>Remark.</u> Suppose P(z) is a polynomial of degree $m \geq 2$ whose Julia set is connected. Denote by K(P) the lower bound of those $x > 0$, for which the inequality $|\lambda| \leq m^{nx}$ is satisfied for multiplicators λ of all cycles of order $n \in \mathbb{N}$. We have proved that $1 \leq K(P) \leq 2$. By the method of extremal lengths it is possible to prove the strict inequality $K(P) < 2$ for the case in which the mapping $P : J(P) \to J(P)$ is hyperbolic [3].

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