Let $\mathbb{F}_q$ be a finite field. I'll remind some defintions from "examples ".

For any multiplicative character $\lambda : \mathbb{F}_q^\star \to \mathbb{C}^\star$ we define

$$L_\lambda := \{f \in \mathbb{C}[\mathbb{F}_q] | f(ax) = \lambda(a(f(x) \forall a \in \mathbb{F}_q^\star, x \in \mathbb{F}_q$$

a) Find $dim(L_\lambda)$.

We fix a non-trivial additive character $\psi : \mathbb{F}_q \to \mathbb{C}^\star$ and define the Fourier transform $\mathbb{F} : \mathbb{C}[\mathbb{F}_q] \to \mathbb{C}[\mathbb{F}_q]$ by

$$\mathcal{F}(f)(x) = \frac{1}{\sqrt{q}} \sum_{y \in \mathbb{F}_q} \psi(-xy) f(y)$$

As follows from Proposition 1.5 in "examples "the Fourier transform is unitary.

b) Show that $\mathbb{F}(L_\lambda) = L_{\lambda^{-1}}$.

c) For any multiplicative character $\lambda : \mathbb{F}_q^\star \to \mathbb{C}^\star$ we define

$$g(\lambda) := \sum_{a \in \mathbb{F}_q^\star} \psi(a)\lambda(a)$$

Show that $|g(\lambda)| = \sqrt{q}$ if λ is a non-trivial multiplicative character of $\mathbb{F}_q$.

Let $G = GL_2(\mathbb{F}_q)$ be the group of invertible 2×2-matrices over $\mathbb{F}_q$, $B \subset G$ be subgroup of matrices of the form

$$b_{x,y,z} = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \quad , x, y \in \mathbb{F}_q^\star, z \in \mathbb{F}_q$$

For any multiplicative characters λ, μ of of $\mathbb{F}_q$ we define $\lambda \otimes \mu : B \to \mathbb{C}^\star$ by

$$\lambda \otimes \mu(b_{x,y,z}) = \lambda(x)\mu(y)$$

It is clear that $\lambda \otimes \mu$ is a character [1-dimensional representation] of the group B. We denote by $\rho_{\lambda,\mu} : B \to Gl(V_{\lambda,\mu})$ the induced representation $ind_B^G(\lambda \otimes \mu)$.

d) Show that $G = B \cup BwB, w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

e) Show that the representation $\rho_{\lambda,\mu}$ is irreducible if $\lambda \neq \mu$ and $dim(\rho_{\lambda,\mu}) = q + 1$.

f) The representation $\rho_{\lambda,\lambda}$ is a direct sum of two irreducible representations one of dimension 1 and the other dimension q.

g) Representations $\rho_{\lambda,\mu}$ and $\rho_{\lambda',\mu'}$ are equivalent iff either $(\lambda,\mu) = (\lambda',\mu')$ or $(\lambda,\mu) = (\mu',\lambda')$.

Let $SL_2(\mathbb{F}_q) \subset GL_2(\mathbb{F}_q)$ be the subgroup of matricies of determinant $1, B_1 \subset B$ be the subgroup of elements of the form $b_{x,X^{-1},z}$. For any character ω of $\mathbb{F}_q^\star$ we denote by $\tilde{\omega} : B_1 \to \mathbb{C}^\star$ the character given by $\tilde{\omega}(b_{x,X^{-1},z}) = \omega(x)$.

h) Fine for which characters ω of $\mathbb{F}_q^\star$ the representation $ind_{B_1}^{SL_2(\mathbb{F}_q)}\tilde{\omega}$ is irreducible.