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# A VARIATIONAL PROBLEM ARISING IN ECONOMICS: APPROXIMATE SOLUTIONS AND THE LAW OF LARGE NUMBERS[1]

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## 1. INTRODUCTION

Let $\{f_t\}_{t\in T}$ be a collection of real functions defined on the non-negative orthant of a Euclidean space (say, $R^n_+$), which is parametrized by an arbitrary (measurable) space T. Let $a$ in $R^n_+$ be fixed, and choose $t_1, t_2, \ldots, t_m$ in T. We consider then the following optimization problem:

$$\text{Maximize } \frac{1}{m}\sum_{i=1}^{m} f_{t_i}(x_i)$$

$$\text{subject to } \frac{1}{m}\sum_{i=1}^{m} x_i = a \quad \text{and} \quad x_i\in R^n_+ \quad \text{for all} \quad 1 \leq i \leq m.$$

Assume now that $t_1, t_2, \ldots$ are actually independently drawn from T according to a fixed distribution $\mu$. As $m$ increases, we obtain better and better samples of T, and the optimal value of the above problem should converge to the optimal value of the limit (variational) problem, namely,

$$\text{Maximize } \int_T f_t(\underset{\sim}{x}(t))d\mu(t)$$

$$\text{subject to } \int_T \underset{\sim}{x}(t)d\mu(t) = a \quad \text{and} \quad \underset{\sim}{x}(t)\in R^n_+ \quad \text{for all} \quad t\in T.$$

This is the Law of Large Numbers we are seeking.

The purpose of this paper is to prove it using the least necessary assumptions. In particular, we are able to dispose of any topological structure (i.e., relations on $f_t$'s for "close" values of t) and also of the condition of Aumann & Perles [3] (which guarantees the existence of an optimal solution to the variational problem, by "compactifying" it in the appropriate sense). This is done in Section 4.

In order to prove our results, we first obtain in Section 3 a complete characterization of the approximate solutions to the variational problem, which is of independent interest. Section 2 contains the precise mathematical model and assumptions we use.

The problem we study here arises in many applications, especially in the so-called "allocation processes". E.g., T is a set of economic agents and $f_t$ is the production function of t (for n inputs and one output); or, $f_t$ is the utility function of trader t (for n commodities); or, t is "time", and so on. For a recent application of the Law of Large Numbers we obtain here, the reader is referred to Groves & Hart [5]. Moreover, in some instances in the study of games and economies with a continuum of participants, one may be able to obtain results by our methods without using the Aumann & Perles [3] condition (e.g., in value theory, cf. Aumann & Shapley [4] and others). For a generalization of the Law of Large Numbers (to "random sets", or correspondences), see the forthcoming paper of Artstein & Hart.

## 2. THE MODEL

We start with some notations. The n-dimensional Euclidean space will be denoted by $R^n$, its non-negative and positive orthants by $R^n_+$ and $(R^n_+)^0$, respectively. Superscripts will denote coordinates; for x and y in $R^n$, the scalar product $x\cdot y$ is $\sum_{i=1}^n x^i y^i$. As a convenient norm, we will use $\|x\| = \sum_{i=1}^n |x^i|$ (note that on $R^n_+$, this norm is a linear function). Given a topological space $X$, $\mathcal{B}(X)$ will be its Borel $\sigma$-field (generated by all open subsets of $X$).

The mathematical model of our problem is as follows. $(T,\mathcal{C},\mu)$ is a fixed probability space, which we decompose into its non-atomic part $T_0$ and a countable number of atoms $T_1$ (cf. Hildenbrand [7], D.I(12)). $f(\cdot,\cdot)$ is a real function on $R^n_+ \times T$, satisfying the following assumptions:

(2.1) $f$ is $(\mathcal{B}(R^n_+) \otimes \mathcal{C})$-measurable.

(2.2) There exist a real constant $c_1$ and an integrable real function $\underset{\sim}{c}_2$ on $T$ such that $f(x,t) \geq c_1\|x\| + \underset{\sim}{c}_2(t)$, for all $x \in R^n_+$ and $\mu$-a.e. $t \in T$.

($c_1$ and $\underset{\sim}{c}_2$ may be negative).

Note that (2.1) implies the measurability of $f(\underset{\sim}{x}(t),t)$ whenever $\underset{\sim}{x}: T \to R^n_+$ is measurable; if $\underset{\sim}{x}$ is integrable, then $\int f(\underset{\sim}{x}(t),t)d\mu(t)$ is well defined (possibly, equal to $+\infty$). In the various economic models, both assumptions are usually satisfied. Note that (2.2) holds in each of the following cases: $f$ is non-negative; $f(\cdot,t)$ is a non-decreasing function on $R^n_+$ for all $t \in T$, and $f(0,t)$ is integrable; $\inf\{f(x,t) \mid x \in R^n_+\}$ is integrable; and others.

We can now define the "optimum function" F. For every $a$ in $R^n_+$,

$$F(a) = \sup\{\int_T f(\underset{\sim}{x}(t),t)d\mu(t) \mid \int_T \underset{\sim}{x}(t)d\mu(t) = a,\ \underset{\sim}{x}(t) \in R^n_+ \text{ for all } t \in T\}.$$

In view of (2.1) and (2.2), $F(a)$ is either finite or $+\infty$.

At this point it will be useful to introduce some notations (given the fixed space $(T,\mathcal{C},\mu)$): $\int_S \underset{\sim}{z}$ for $\int_S \underset{\sim}{z}(t)d\mu(t)$; $\int \underset{\sim}{z}$ for $\int_T \underset{\sim}{z}$; $f(\underset{\sim}{z})$ for the function (on $T$) $f(\underset{\sim}{z}(t),t)$, and $\int f(\underset{\sim}{z})$ for $\int f(\underset{\sim}{z}(t),t)d\mu(t)$. "Almost everywhere" in $T$ is meant with respect to $\mu$.

In order for $F$ to be a concave function, it is sufficient to assume that all the functions $f(\cdot,t)$ are concave. However, due to the convexification effect of integration in the non-atomic part $T_0$ of $T$, we assume only

(2.3) For every $t \in T_1$, $f(\cdot,t)$ is a concave function on $R^n_+$.

Proposition 2.4: Assume (2.1), (2.2) and (2.3). Then $F$ is a concave function on $R^n_+$.

Proof: Let $\phi(t) = \{(x,\alpha) \in R^n_+ \times R \mid \alpha < f(x,t)\}$, then $\phi$ is a correspondence with measurable graph (by (2.1)). It is easy to check that

$$\{(a,\alpha) \in R^n_+ \times R \mid \alpha < F(a)\} = \int_T \phi(t)d\mu(t)$$

(see Hildenbrand ([ 7 ], D.II) for the definition of $\int\phi$).

By Lyapunov's Theorem (e.g., see Hildenbrand ([7], D.II.4, Theorem 3), $\int_{T_0}\phi$ is a convex set; since $\phi(t)$ is convex for all $t \in T_1$ by (2.3), $\int_T\phi$ is convex, and $F$ is a convex function. □

Our last assumption is that $F$ is not infinite (otherwise, there will be no interesting results).

(2.5) There is $a_0$ in $(R^n_+)^0$ such that $F(a_0)$ is finite.

Proposition 2.6: Assume (2.1), (2.2), (2.3) and (2.5). Then $F$ is finite on all $R^n_+$.

Proof: Otherwise $F$ would be an "improper" concave function, hence infinite throughout $(R^n_+)^0$ (cf. Rockafellar [9], Theorem 7.2), contradicting (2.5). □

Remark 2.7: If $F$ is infinite on all $(R^n_+)^0$, but finite for some $a$ on the boundary of $R^n_+$, then the whole problem should be projected on this subspace (since $a^i = 0$ and $\int\underset{\sim}{x} = a$ imply $\underset{\sim}{x}^i(t) = 0$ for a.e. $t \in T$).

In the next section (Corollary 3.7), we will see that, in the presence of (2.1)-(2.3), Assumption (2.5) is equivalent to the stronger requirement (3.6)--which is also easier to verify in the various models.

Next, we define the notion of a "supporting hyperplane." Let $g: R^n_+ \to (-\infty,+\infty]$, $x \in R^n_+$ and $\varepsilon \geq 0$. A vector $\lambda \in R^n$ is an $\varepsilon$-super-gradient of $g$ at $x$ if

$$g(y) \leq g(x) + \lambda\cdot(y - x) + \varepsilon ,$$

for all $y$ in $R^n_+$. The set of all such $\lambda$'s will be denoted by $\partial_\varepsilon g(x)$, and $\partial g(x) \equiv \partial_0 g(x)$ is the set of super-gradients (cf. Rockafellar [9], p. 219).

Proposition 2.8: Let $g$ be a concave function from $R^n_+$ to $(-\infty,+\infty]$. Then, for every $x \in (R^n_+)^0$, $g$ is continuous at $x$, and $\partial g(x) \neq \emptyset$.

Proof: Theorems 10.2 and 23.4 in Rockafellar [9].

## 3. APPROXIMATE SOLUTIONS

In this section we give a characterization of all approximate solutions to the variational problem - namely, those feasible functions that come within $\varepsilon$ of the optimal value. We denote by $\partial_\varepsilon f(x,t)$ the set of $\varepsilon$-super-gradients of $f(\cdot,t)$ at $x$.

Theorem 3.1: Assume (2.1), (2.2), (2.3) and (2.5). Let $\underset{\sim}{x}$ be an integrable function from $T$ to $R^n_+$, with $\int_T \underset{\sim}{x} = a \in (R^n_+)^0$, and let $\varepsilon \geq 0$. Then a necessary and sufficient condition for

$$(3.2) \qquad \int_T f(\underset{\sim}{x}(t),t)d\mu(t) \geq F(a) - \varepsilon$$

is that there exist a $\lambda$ in $R^n_+$ and an integrable function $\underset{\sim}{\delta}$ from $T$ to $[0,\infty)$, satisfying

$$(3.3) \qquad \lambda \in \partial_{\underset{\sim}{\delta}(t)} f(\underset{\sim}{x}(t),t) \text{ , for } \mu\text{-a.e. } t \text{ in } T ,$$

and

$$(3.4) \qquad \int_T \underset{\sim}{\delta}(t)d\mu(t) \leq \varepsilon .$$

Furthermore: in the necessity part, $\lambda$ can be chosen independently of $\underset{\sim}{x}$ and $\varepsilon$, and such that $\lambda \in \partial F(a)$; (3.3) and (3.4) imply $\lambda \in \partial_\varepsilon F(a)$; as for the sufficiency, only (2.1), (2.2) and $a \in R^n_+$ (not $(R^n_+)^0$) are needed.

A particular case is $\varepsilon = 0$. We then obtain essentially[2] Theorem 5.1 in Aumann & Perles [3].

Corollary 3.5: Assume (2.1), (2.2), (2.3) and (2.5), and let $a \in (R^n_+)^0$. Then $F(a)$ is attained at a non-negative $\underset{\sim}{x}$ with $\int \underset{\sim}{x} = a$ if and only if there is a $\lambda$ in $R^n$ such that $\lambda \in \partial f(\underset{\sim}{x}(t),t)$ for $\mu$-a.e. $t$ in $T$. Furthermore, $\lambda \in \partial F(a)$, and only (2.1), (2.2) and $a \in R^n_+$ are needed for the "if" part.

Proof of Theorem 3.1: Sufficiency is easy: integrating the super-gradient inequality given by (3.3) for $y = \underset{\sim}{y}(t)$, we get

$$\int f(\underset{\sim}{y}) \leq \int f(\underset{\sim}{x}) + \lambda \cdot \left(\int \underset{\sim}{y} - a\right) + \varepsilon .$$

Taking the supremum of the left-hand side over all $\underset{\sim}{y}$ with $\int \underset{\sim}{y} = a$ gives (3.2); and over all integrable $\underset{\sim}{y}$ -- it implies $\lambda \in \partial_\varepsilon F(a)$.

To prove necessity, let $\lambda \in \partial F(a)$; the existence of such $\lambda$ is guaranteed by Proposition 2.8 (here is the only place where $a \in (R^n_+)^0$ is used). Define, for $y \in R^n_+$ and $t \in T$,

$$g(y,t) = f(y,t) - f(\underset{\sim}{x}(t),t) - \lambda \cdot (y - \underset{\sim}{x}(t)) ,$$

and

$$\underset{\sim}{\delta}(t) = \sup_{y \in R^n_+} g(y,t) .$$

Since $g$ is $(\mathcal{B}(R^n_+) \otimes C)$-measurable, $\underset{\sim}{\delta}$ is $\overline{C}_\mu$ - measurable, where $\overline{C}_\mu$ is the completion of $C$ with respect to $\mu$ (cf. the Projection Theorem in Hildenbrand

[7], D.I. (11)). Therefore, $\delta$ differs from a C-measurable function on a set of $\mu$-measure zero; since we have to prove (3.3) only a.e., we can sssume without loss of generality that $\delta$ is C-measurable.

We have to show that (3.4) holds. For every $m = 1,2,\ldots,$ let

$$\delta_m(t) = \begin{cases} \max\{\delta(t) - \frac{1}{m}, 0\} & , \text{ if } \delta(t) < \infty , \\ m & , \text{ if } \delta(t) = \infty , \end{cases}$$

and choose $y_m(t) \in R^n_+$ such that

$$g(y_m(t),t) \geq \delta_m(t)$$

for a.e. $t$ in $T$, and $y_m$ is measurable (we use the Measurable Selection Theorem - see Hildenbrand [7], D.II, Theorem 1). Now, let $k = 1,2,\ldots,$ and define

$$y^k_m(t) = \begin{cases} y_m(t) & , \text{ if } \|y_m(t)\| \leq k , \\ x(t) & , \text{ otherwise} . \end{cases}$$

Since $y^k_m$ is integrable, $\lambda \in \partial F(a)$, and (3.2) is satisfied, we get

$$\int f(y^k_m) \leq F(\int y^k_m) \leq F(a) + \lambda\cdot(\int y^k_m - a)$$
$$\leq \int f(x) + \lambda\cdot\int (y^k_m - x) + \varepsilon ,$$

or $\int g(y^k_m) \leq \varepsilon$. A repeated application of Lebesgue's Monotone Convergence Theorem, first for $k \to \infty$ and then for $m \to \infty$, finally gives us $\int \delta_m \leq \varepsilon$ and $\int \delta \leq \varepsilon$. □

We will next show that (2.5) is actually equivalent to the following (stronger) assumption[3]:

(3.6) There exist a real constant $c_3$ and an integrable real function $c_4$ on $T$ such that $f(x,t) \leq c_3\|x\| + c_4(t)$, for all $x \in R^n_+$ and $\mu$-a.e. $t \in T$.

<u>Corollary 3.7:</u> Assume (2.1), (2.2) and (2.3). Then (2.5) is equivalent to (3.6).

<u>Proof:</u> Assume (2.5), and let $a \in (R^n_+)^0$. Choose $\varepsilon > 0$, and find $x$ with $\int x = a$ satisfying (3.2). Then (3.3) implies

$$f(y,t) \leq \lambda\cdot y + [f(x(t),t) - \lambda\cdot x(t) + \delta(t)],$$

for all $y \in R^n_+$ and $\mu$-a.e. $t \in T$. The expression in the square brackets is integrable (recall (2.2) and $\int f(x) \leq F(a) < \infty$), proving (3.6). The other direction is trivial. □

<u>Remark 3.8:</u> Since (2.2) and (3.6) can be joined into:

(3.9) There exist a real constant $c_0$ and an integrable real function $c$ on $T$ such that $|f(x,t)| < c_0\|x\| + c(t)$, for all $x \in R^n_+$ and $\mu$-a.e. $t \in T$, one may always replace "(2.1), (2.2), (2.3) and (2.5)" with "(2.1), (2.3) and (3.9)."

It is easy to check that (3.9) is equivalent to:

(3.10) There exist a real constant $c$ and an integrable function $\underset{\sim}{\eta}$ from T to $R^n_+$, such that $|f(x,t)| \leq c\|x\|$, for all $x$ in $R^n_+$ with $\|x\| \geq \underset{\sim}{\eta}(t)$, and $\mu$-a.e. $t$ in T.

In contrast to the "small o" condition of Aumann & Perles [3], we have here a "big O" condition; thus, we will call (3.10): "$f(x,t) = O(\|x\|)$ as $\|x\| \to \infty$, integrably in $t$".

## 4. LAW OF LARGE NUMBERS

Let $a \in R^n_+$ be fixed. For a finite set of points $t_1, t_2, \ldots, t_m$ in T, consider the optimization problem of maximizing $1/m \sum_{k=1}^{m} f(x_k, t_k)$ subject to $1/m \sum_{k=1}^{m} x_k = a$ and $x_k \in R^n_+$ for all $k = 1,2,\ldots,m$. Now assume that the points $t_k$ are drawn at random from T, independently and according to the given probability measure $\mu$. As $m$ increases, we get a "good sample" of T. The problem we address here is whether the optimal value of the finite problem converges to that of the limit problem--namely, F(a). In case f is of "finite type," i.e., there are only finitely many distinct functions in $\{f(\cdot,t)\}_{t\in T}$, the answer is positive, using a straightforward application of the Strong Law of Large Numbers (T is decomposed into finitely many subsets, on each one $f(\cdot,t)$ being constant in $t$). This can be extended, by making some topological assumptions on T and f ($t$ and $t'$ are "close" implies $f(\cdot,t)$ and $f(\cdot,t')$ are "close"--e.g., see Assumption 4 in Arrow & Radner [1]).

In this section we will show that no additional assumptions (to the basic ones in Section 2) are needed. We will use again the super-gradient approach and the results in Section 3.

We start by describing the model precisely. Let $(\Omega, F, P)$ be a probability space; since we are interested in almost sure statements, we will assume without loss of generality that it is complete. Let $\{\tau_k\}_{k=1}^{\infty}$ be a sequence of independent and identically distributed T-valued random variables, and let $\mu$ be their common distribution; namely, $\tau_k$ is a function from $\Omega$ to T that is measurable (i.e., $\tau_k^{-1}(C) \in F$ for all $C \in C$), and $P \circ \tau_k^{-1} = \mu$. Given $m = 1,2,\ldots$ and $\omega \in \Omega$, let

$$F_{m,\omega}(a) = \sup\left\{\frac{1}{m}\sum_{k=1}^{m} f(x_k, \tau_k(\omega)) \;\middle|\; \frac{1}{m}\sum_{k=1}^{m} x_k = a,\ x_k \in R^n_+ \text{ for all } k = 1,2,\ldots,m\right\}.$$

We assume that f satisfies (2.1) and (3.9), but not necessarily (2.3). Therefore, we define $\bar{f}: R^n_+ \times T \to R$ as follows: for $t \in T_0$, $\bar{f}(\cdot,t) \equiv f(\cdot,t)$; for ($\mu$-a.e.) $t \in T_1$, $\bar{f}(\cdot,t)$ is the concavification[4] of $f(\cdot,t)$. Since the number of atoms is at most countable, measurable mappings are almost everywhere constant on each atom, and concave functions on $R^n_+$ are Borel-measurable[5], it follows that $\bar{f}$ may be taken to be $(\mathcal{B}(R^n_+)\otimes C)$-measurable. Thus, $\bar{f}$ satisfies (2.1), (2.3) and (3.9), and we will denote by $\bar{F}$ the optimum function it generates.

Remark 4.1: If we replace each atom in $T_1$ by a non-atomic continuum of the same measure and with the same (identical) $f(\cdot,t)$, the resulting optimum function will again be $\bar{F}$. This follows easily from Lyapunov's Theorem, in a similar way to our proof of Proposition 2.4. (see Hildenbrand [7], D.II.4, Corollary to Theorem 3).

Theorem 4.2: Assume (2.1) and (3.9). Then

$$(4.3) \qquad P(\{\omega \in \Omega \mid \lim_{m\to\infty} F_{m,\omega}(a) = \bar{F}(a) \text{ for all } a \in R^n_+\}) = 1 .$$

Moreover, for every compact subset $C$ of $R^n_+$,

$$(4.4) \qquad \sup_{a\in C} | F_{m,\omega}(a) - \bar{F}(a) | \xrightarrow[m\to\infty]{P\text{-a.s.}} 0 .$$

Remarks: (i) For every $a$ in $R^n_+$, we have a strong Law of Large Numbers: the sequence $\{F_{m,\omega}(a)\}_{m=1}^{\infty}$ converges P-almost surely to (the constant) $\bar{F}(a)$. Furthermore, the exceptional set is independent of $a$, and the convergence is uniform on compact sets.

(ii) We do not use any concavity assumptions (not even (2.3))--therefore the limit is $\bar{F}$ and not $F$.

Before proving our result, we state the general form of the Strong Law of Large Numbers we need.

Theorem 4.5 (SLLN) : Let $\xi$ be an integrable real function on $T$. Then

$$\frac{1}{m}\sum_{k=1}^{m} \xi(\tau_k) \xrightarrow[m\to\infty]{P\text{-a.s.}} \int_T \xi(t)d\mu(t) .$$

Proof: Loève ([8]. 17.3b), or Halmos ([6], §47, Exercise (7)).

The proof of Theorem 4.2 will be in a sequence of Propositions--some of them are of independent interest. To shorten notation, we will write [...] for the set $\{\omega | \dots\}$.

Proposition 4.6: Assume (2.1) and (3.9). Let $a \in (R^n_+)^0$ and $\lambda \in \partial\bar{F}(a)$. Then

$$P[\limsup_{m\to\infty} F_{m,\omega}(b) \le \bar{F}(a) + \lambda\cdot(b - a) \text{ for all } b \in R^n_+] = 1 .$$

Proof: Let $\varepsilon > 0$, and obtain (for $\bar{f}$) $x$, $\lambda$ and $\delta$, satisfying (3.2), (3.3), (3.4) and $\lambda \in \partial\bar{F}(a)$ (since $\bar{f}$ satisfies (2.1), (2.3) and (3.9), we use Theorem 3.1 and Remark 3.8). Using (3.3) for[6] $t = \tau_k(\omega)$ and averaging, we get for any $y_1, y_2, \dots, y_m$ in $R^n_+$

$$\frac{1}{m}\sum_{k=1}^{m} \bar{f}(y_k,\tau_k(\omega)) \le \frac{1}{m}\sum_{k=1}^{m} \bar{f}(x(\tau_k(\omega)),\tau_k(\omega))$$

$$+ \lambda \cdot \frac{1}{m}\sum_{k=1}^{m} (y_k - x(\tau_k(\omega))) + \frac{1}{m}\sum_{k=1}^{m} \delta(\tau_k(\omega)),$$

hence, for all $b \in R^n_+$ (recall that $f \le \bar{f}$)

$$F_{m,\omega}(b) \leq \frac{1}{m}\sum_{k=1}^{m} f(\underset{\sim}{x}(\tau_k(\omega)),\tau_k(\omega)) + \lambda\cdot(b - \frac{1}{m}\sum_{k=1}^{m}\underset{\sim}{x}(\tau_k(\omega)))$$

$$+ \frac{1}{m}\sum_{k=1}^{m} \underset{\sim}{\delta}(\tau_k(\omega)).$$

Now SLLN applied to the integrable functions $\bar{f}(\underset{\sim}{x})$, $\underset{\sim}{x}^i$ (for all $i = 1,2,\ldots,n$) and $\underset{\sim}{\delta}$ gives (see (3.4)) for all $b \in R^n_+$

$$\limsup_{m\to\infty} F_{m,\omega}(b) \leq \bar{F}(a) + \lambda\cdot(b - a) + \varepsilon ,$$

P-a.s. □

Remark: In particular, this proves that P-a.s.,

$$\limsup_{m\to\infty} F_{m,\omega}(a) \leq \bar{F}(a) ,$$

which is the result needed in Groves & Hart [5].

For $a \in R^n$ and $\delta > 0$, let $B(a;\delta)$ be the open ball of radius $\delta$ around a; i.e., $B(a;\delta) = \{b \in R^n \mid \|a - b\| < \delta\}$.

Proposition 4.7: Assume (2.1) and (3.9), and let $a \in (R^n_+)^0$. Then for every $\varepsilon > 0$ there is $\delta \equiv \delta(a,\varepsilon) > 0$ such that P-a.s.

$$\liminf_{m\to\infty} \left( \inf_{b\in B(a;\delta)} F_{m,\omega}(b)\right) > \bar{F}(a) - \varepsilon.$$

Proof: First, we will make the additional assumption that $\mu$ is a non-atomic measure (i.e., $T = T_0$ and $\bar{f} \equiv f$). Let $c_0$ and $\underset{\sim}{c}$ be given by (3.9), fix $\varepsilon > 0$, and put $\varepsilon_1 = \varepsilon/(4 + 3c_0)$. By Propositions 2.4 and 2.8, $\bar{F}$ is continuous on $(R^n_+)^0$; let $\delta_1 > 0$, $\delta_1 \leq \varepsilon_1$ be such that $|\bar{F}(a) - \bar{F}(b)| < \varepsilon_1$ whenever $\|b - a\| \leq \delta_1$. Let $\delta = \delta_1/(2n)$, $a_1 = a - (\delta,\delta,\ldots,\delta)$ and $a_2 = a-(2\delta,2\delta,\ldots,2\delta)$; then $\|b - a\| < \delta$ implies $b^i \geq a_1^i = a_2^i + \delta$ for all $1 \leq i \leq n$. Let $\underset{\sim}{x} \geq 0$ satisfy $\int\underset{\sim}{x} = a_2$ and $|\int f(\underset{\sim}{x}) - \bar{F}(a_2)| < \varepsilon_1$; since $\|a - a_2\| = \delta_1$, we have $|\int f(\underset{\sim}{x}) - \bar{F}(a)| < 2\varepsilon_1$. Finally, let $T' \subset T$ be such that $\mu(T') > 0$, $\|\int_{T'}\underset{\sim}{x}\| < \varepsilon_1$, $|\int_{T'}f(\underset{\sim}{x})| < \varepsilon_1$ and $|\int_{T'}\underset{\sim}{c}| < \varepsilon_1$; put $T'' = T\setminus T'$.

For P-a.e. $\omega \in \Omega$, we have (by SLLN), for all $m$ large enough, $K' \equiv K'_{m,\omega} = \{k \leq m \mid \tau_k(\omega) \in T'\} \neq \emptyset$ (since $\mu(T') > 0$) and $1/m \sum_{k=1}^{m}\underset{\sim}{x}(\tau_k(\omega)) \leq a_1$ (since $\int\underset{\sim}{x} = a_2 \ll a_1$); denote $K'' = \{1,2,\ldots,m\}\setminus K'$. For each such $m$, consider the following allocation for $F_{m,\omega}(b)$ with $b \in B(a;\delta)$ : $y_k = \underset{\sim}{x}(\tau_k(\omega))$ for all $k \in K''$, and the remaining $mb - \sum_{k\in K''} y_k$ is distributed (arbitrarily) among all $k \in K'$ (here we use $b \geq a_1$ for all $\|b - a\| < \delta$). Using (3.9) for $k \in K'$, we obtain

$$F_{m,\omega}(b) \geq \frac{1}{m}\sum_{k=1}^{m} f(y_k,\tau_k(\omega))$$

$$\leq \int f(\underset{\sim}{x}) + [\frac{1}{m} \sum_{k\in K''} f(y_k, \tau_k(\omega)) - \int_{T''} f(\underset{\sim}{x})]$$

$$- \int_{T'} f(\underset{\sim}{x}) - \frac{1}{m}[c_0 \| mb - \sum_{k\in K''} y_k \| + \sum_{k\in K''} \underset{\sim}{c}(\tau_k(\omega))] .$$

As $m \to \infty$, SLLN implies P-a.s. (recall that $y_k = \underset{\sim}{x}(\tau_k)$ for $k \in K''$)

$$\frac{1}{m} \sum_{k\in K''} f(y_k, \tau_k(\omega)) \to \int_{T''} f(\underset{\sim}{x}) ,$$

$$\frac{1}{m} \sum_{k\in K''} y_k \to \int_{T''} \underset{\sim}{x} = a_2 - \int_{T'} \underset{\sim}{x} ,$$

$$\frac{1}{m} \sum_{k\in K'} \underset{\sim}{c}(\tau_k(\omega)) \to \int_{T'} \underset{\sim}{c} .$$

Therefore, we have a.s. in $\Omega$

$$\liminf_{m\to\infty} ( \inf_{b\in B(a;\delta)} F_{m,\omega}(b))$$

$$\geq \int f(\underset{\sim}{x}) - |\int_{T'} f(\underset{\sim}{x})| - c_0( \sup_{b\in B(a;\delta)} \| b - a_2 \| + \|\int_{T'} \underset{\sim}{x}\|) - |\int_{T'} \underset{\sim}{c}|$$

$$> \bar{F}(a) - \varepsilon_1 - \varepsilon_1 - \varepsilon_1 - c_0(\delta + \delta_1 + \varepsilon_1) - \varepsilon_1$$

$$\geq \bar{F}(a) - \varepsilon .$$

Finally, in the general case when $\mu$ has an atomic part, we replace each atom by a non-atomic continuum. In view of Remark 4.1 and the fact that $F_m$ will not change (it depends only on the distribution of the functions $f(\cdot,t)$ over $t\in T$), this completes the proof. □

The last two propositions together imply

<u>Corollary 4.8:</u> Assume (2.1) and (3.9), and let $a \in (R^n_+)^0$. Then for every $\varepsilon>0$ there is $\delta \equiv \delta(a,\varepsilon) > 0$ such that P-a.s.,

$$\limsup_{m\to\infty} ( \sup_{b\in B(a;\delta)} |\bar{F}(a) - F_{m,\omega}(b)|) < \varepsilon .$$

<u>Proof of Theorem 4.2</u>: Let $a \in (R^n_+)^0$, $\varepsilon > 0$, and $\delta \equiv \delta(a;\varepsilon)$ be given by Corollary 4.8. Propositions 2.4 and 2.8 imply that $\bar{F}$ is continuous on $(R^n_+)^0$; therefore, we assume that $\delta$ also satisfies

$$\sup_{b\in B(a;\delta)} |\bar{F}(a) - \bar{F}(b)| < \varepsilon \quad .$$

Therefore, we obtain P-a.s.

$$\limsup_{m\to\infty} \left( \sup_{b\in B(a;\delta)} |\bar{F}(b) - F_{m,\omega}(b)| \right) < 2\varepsilon \ .$$

Let $C \subset (R^n_+)^0$ be a compact set. It has a finite cover by open balls $B(a;\delta(a,\varepsilon))$ therefore P-a.s.

$$\limsup_{m\to\infty} \left( \sup_{b\in C} |\bar{F}(b) - F_{m,\omega}(b)| \right) < 2\varepsilon \quad .$$

Since $\varepsilon$ is arbitrary, (4.4) is proved for this C. The same holds for each $R^k_+$ of the boundary $R^n_+$.

As for (4.3), it now follows easily from (4.4) by the $\sigma$-compactness of $R^n_+$. □

NOTES

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2] Since we do not assume monotonicity, $\lambda$ need not be non-negative.

3] A similar result has been obtained by Artstein [2], p. 919.

4] The concavification $\hat{g}$ of a function $g: R^n_+ \longrightarrow R$ is the smallest concave function that is greater or equal to $g$ throughout $R^n_+$. If $g$ is bounded from above by a linear function, then $\hat{g}$ is finite on all $R^n_+$ (and bounded by the same linear function). In our case, this bound is given by (3.9) (actually, (3.6)).

5] A concave function on $R^n_+$ is lower-semi-continuous (cf. Rockafellar [9], Theorem 10.2).

6] Although (3.3) holds only for $\mu$-a.e. t in T, SLLN implies that P-a.s., (3.3) is satisfied at $t = \tau_k(\omega)$ for all $k = 1,2,\ldots$ .

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