Convergence in a Lattice: A Counterexample

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We will exhibit a subset X of $[0,1]^{\mathbb{N}}$ with the following properties:

(1) X is closed (relative to coordinatewise convergence).

(2) X is a lattice: every $x, y \in X$ have a least upper bound and a greatest lower bound[1] in X, denoted $x \vee y$ and $x \wedge y$, respectively (i.e., $x \vee y \in X$, $x \vee y \geq x, y$, and if $z \in X$ satisfies $z \geq x, y$ then $z \geq x \vee y$; similarly for $x \wedge y$).

(3) The join and meet operations $\vee, \wedge$ are continuous.

(4) There exists a sequence $a_n \in X$ such that $\lim_{n\to\infty} a_n = a$ but $\lim_{n\to\infty} \lim_{N\to\infty} \vee_{m=n}^{N} a_m = b \neq a$.

This settles (in the negative) a question posed by Phil Reny.

The construction is as follows. Define the functions f and f_k for each $k \in \mathbb{N}$ from $\{0,1\}^{\mathbb{N}}$ into $[0,1]$ by:

$$f_k(x) := \frac{\log s_k(x)}{\log k}, \text{ and}$$
$$f(x) := \sup_{k\in\mathbb{N}} f_k(x).$$

where $s_k(x) := \max\{\sum_{i=k+1}^{2k} x^{(i)}, 1\}$ (we write $x^{(i)}$ for the i-th coordinate of x). Let

$$X := \{(\xi, x) \in [0,1] \times \{0,1\}^{\mathbb{N}} : \xi \geq f(x)\}.$$

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[1]Relative to coordinatewise inequalities.

We will now prove our claims.

(1) For every k, the function $f_k(x)$ is continuous in x (it depends only on the k coordinates of x from $k+1$ to $2k$), and therefore the supremum of these functions is lower semicontinuous: $\lim x_n = x$ implies $\liminf f(x_n) \geq f(x)$. Therefore X is closed.

(2) $(\xi, x) \vee (\eta, y) = (\zeta, z)$, where $z^{(i)} = \max\{x^{(i)}, y^{(i)}\}$ for all i and $\zeta = \max\{\xi, \eta, f(z)\}$, and $(\xi, x) \wedge (\eta, y) = (\omega, w)$, where $w^{(i)} = \min\{x^{(i)}, y^{(i)}\}$ for all i and $\omega = \min\{\xi, \eta\}$ (note that $\omega = \min\{\xi, \eta\} \geq \min\{f(x), f(y)\} \geq f(w)$ since f is monotonic).

(3) The meet operations is just the coordinatewise minimum, and thus it is continuous. As for the join, let $(\xi_n, x_n) \vee (\eta_n, y_n) = (\zeta_n, z_n)$ with $(\xi_n, x_n) \to (\xi, x)$ and $(\eta_n, y_n) \to (\eta, y)$, and put $(\zeta, z) := (\xi, x) \vee (\eta, y)$. We have to show that $(\zeta_n, z_n) \to (\zeta, z)$. We have $z_n^{(i)} = \max\{x_n^{(i)}, y_n^{(i)}\} \to \max\{x^{(i)}, y^{(i)}\} = z^{(i)}$ for all i, and, in the ζ-coordinate, $\zeta_n = \max\{\xi_n, \eta_n, f(z_n)\}$ and $\zeta = \max\{\xi, \eta, f(z)\}$. Now $\xi_n \to \xi$, $\eta_n \to \eta$ and $\liminf f(z_n) \geq f(z)$ (see **(1)**) imply $\liminf \zeta_n \geq \zeta$. To complete the proof we will show that $\limsup f(z_n) \leq \zeta$.

Indeed, fix $\varepsilon > 0$. For every n, let k_n be such that $f(z_n) \leq f_{k_n}(z_n) + \varepsilon$. We distinguish two cases.

Case 1: There exists a finite k that appears infinitely often in the sequence k_n. Taking that subsequence, we have $k_n = k$ for all n, and so

$$f(z_n) \leq f_k(z_n) + \varepsilon \to_n f_k(z) + \varepsilon \leq f(z) + \varepsilon \leq \zeta + \varepsilon,$$

or $\limsup f(z_n) \leq \zeta + \varepsilon$.

Case 2: $k_n \to_n \infty$. For each k we have

$$s_k(z_n) \leq s_k(x_n) + s_k(y_n) \leq 2 \max\{s_k(x_n), s_k(y_n)\}$$

(since $z_n^{(i)} = \max\{x_n^{(i)}, y_n^{(i)}\} \leq x_n^{(i)} + y_n^{(i)}$ for all i), and so

$$f_k(z_n) \leq \max\{f_k(x_n), f_k(y_n)\} + \frac{\log 2}{\log k}.$$

Now $f_k(x_n) \leq f(x_n) \leq \xi_n$ (since $(\xi_n, x_n) \in X$) and similarly $f_k(y_n) \leq \eta_n$, which implies

$$f(z_n) - \varepsilon \leq f_{k_n}(z_n) \leq \max\{\xi_n, \eta_n\} + \frac{\log 2}{\log k_n} \to_n \max\{\xi, \eta\} \leq \zeta$$

(here we use $k_n \to \infty$). Thus, again, $\limsup f(z_n) \leq \zeta + \varepsilon$.

In both cases we got $\limsup f(z_n) \leq \zeta + \varepsilon$. Since ε is arbitrary, the proof is complete.

(4) Let $a_n = (0, e_n)$, where $e_n^{(i)} = 1$ for $i = n$ and $e_n^{(i)} = 0$ otherwise, then $a_n \to (0, \mathbf{0})$ (we write $\mathbf{0}$ for the "all-0" sequence $(0, 0, ..., 0, ...)$), but

$$\vee_{m=n}^{\infty} a_m \geq \vee_{m=n+1}^{2n} a_m = \vee_{m=n+1}^{2n} (0, e_m) = (1, d_n) \to_n (1, \mathbf{0}) \neq (0, \mathbf{0}),$$

where $d_n^{(i)} = 1$ for $n+1 \leq i \leq 2n$ and $d_n^{(i)} = 0$ otherwise (so $f(d_n) = f_n(d_n) = 1$).