“Calibeating”: Beating Forecasters at Their Own Game

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Joint work with

Dean P. Foster

University of Pennsylvania &
Amazon Research NY

Papers

Papers

- Sergiu Hart
  "Calibration: The Minimax Proof", 1995
  Matching, Dynamics and Games for the Allocation of Resources 2025

  www.ma.huji.ac.il/hart/publ.html#calib-minmax

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- Dean P. Foster and Sergiu Hart
  "Smooth Calibration, Leaky Forecasts, Finite Recall, and Nash Dynamics"
  Games and Economic Behavior 2018

  www.ma.huji.ac.il/hart/publ.html#calib-eq

Papers

Papers

- Dean P. Foster and Sergiu Hart
  "Forecast Hedging and Calibration"
  Journal of Political Economy 2021
  www.ma.huji.ac.il/hart/publ.html#calib-int

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- Dean P. Foster and Sergiu Hart
  " 'Calibeating': Beating Forecasters at Their Own Game"
  Theoretical Economics 2023
  - Addendum 2024; Errata 2026 (arXiv)

  www.ma.huji.ac.il/hart/publ.html#calib-beat

Papers

Papers

- Dean P. Foster and Sergiu Hart
  "Proper Calibeating"
  2025
  www.ma.huji.ac.il/hart/publ.html#calib-proper

Calibration

Calibration

- Forecaster says: "***The probability of rain tomorrow is*** p"

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  - for every forecast p:
    in the days when the forecast was p, the proportion of rainy days equals p

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- Forecaster is **CALIBRATED** if
  - for every forecast p:
    in the days when the forecast was p, the proportion of rainy days equals p
    (or: is close to p in the long run)

Calibration

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CALIBRATION *can be guaranteed*
(no matter what the weather will be)

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- Foster 1999: simple procedure
- Foster and Hart 2016 [publ 2021]: simplest procedure, by "Forecast Hedging"

Forecast-Hedging (FH)

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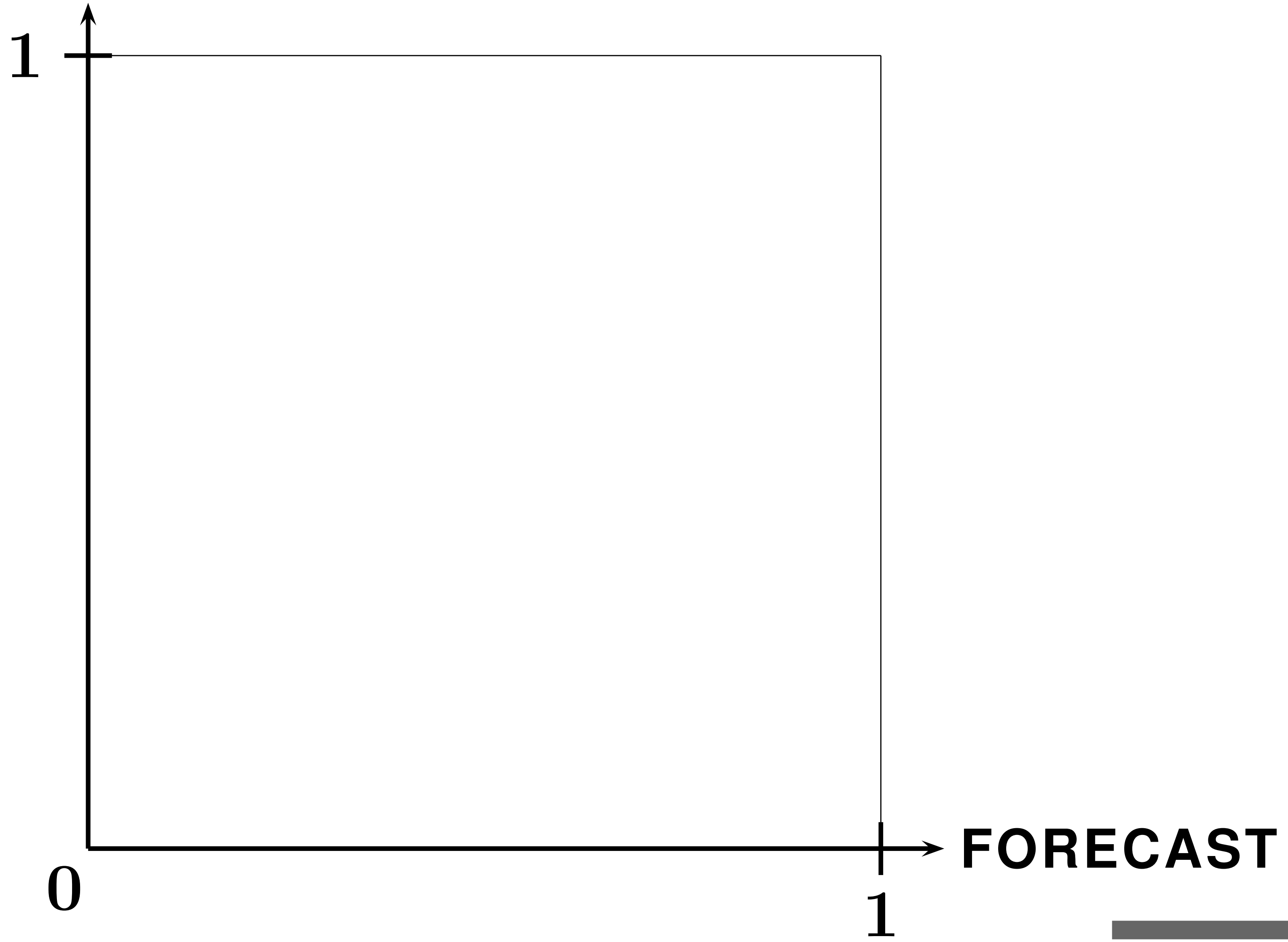
AVERAGE ACTION (= frequency of rain)



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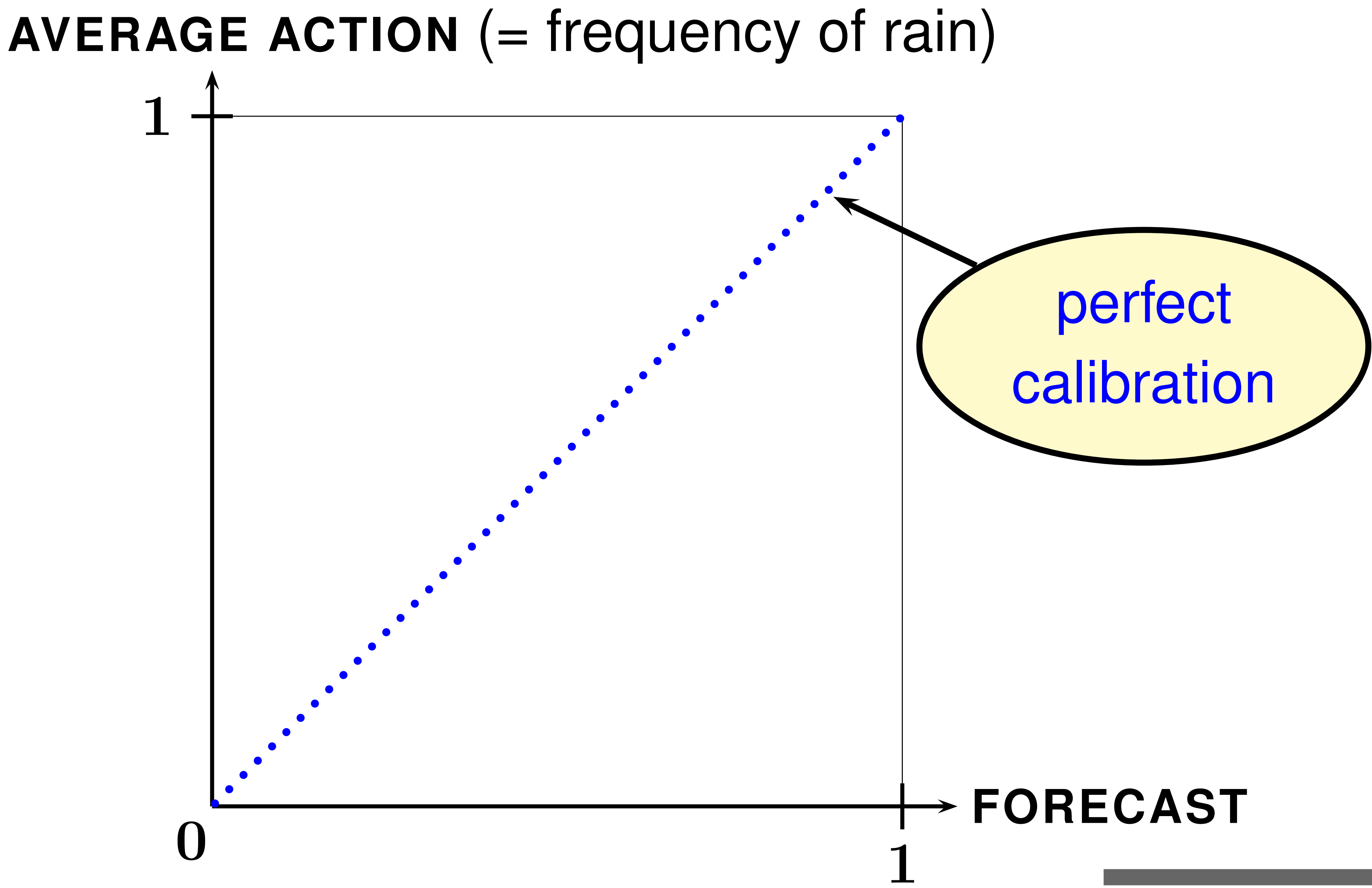

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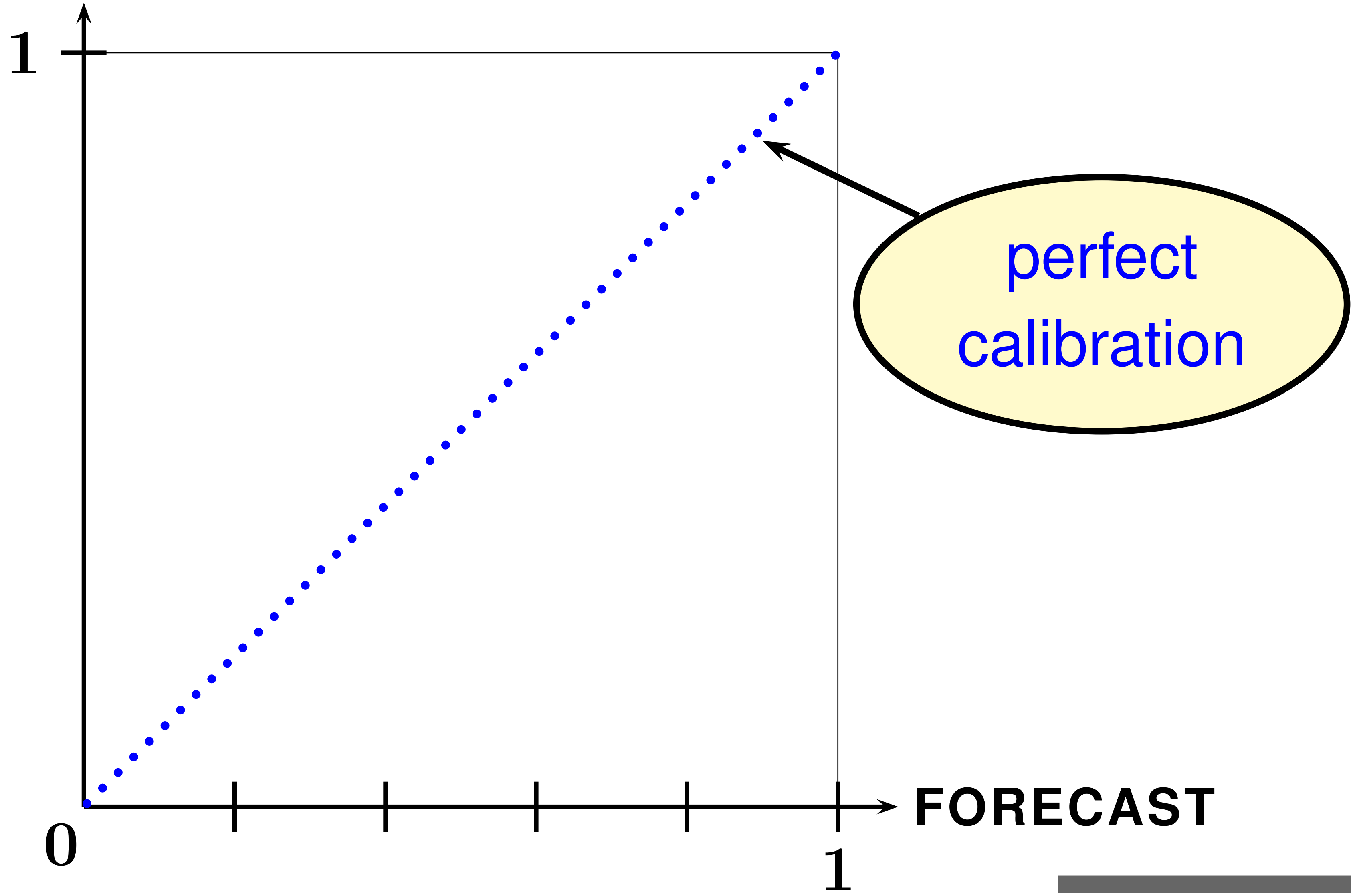

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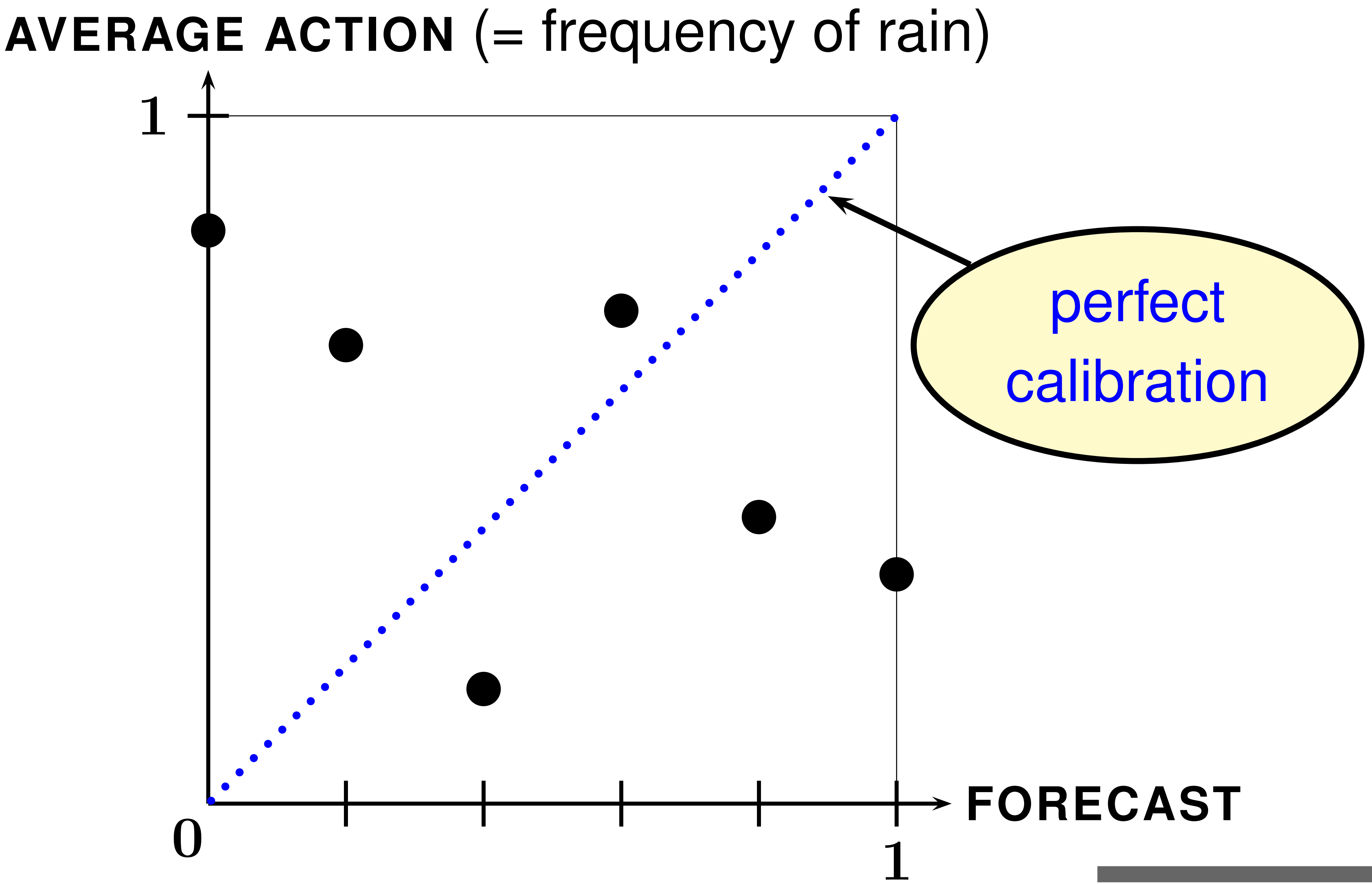

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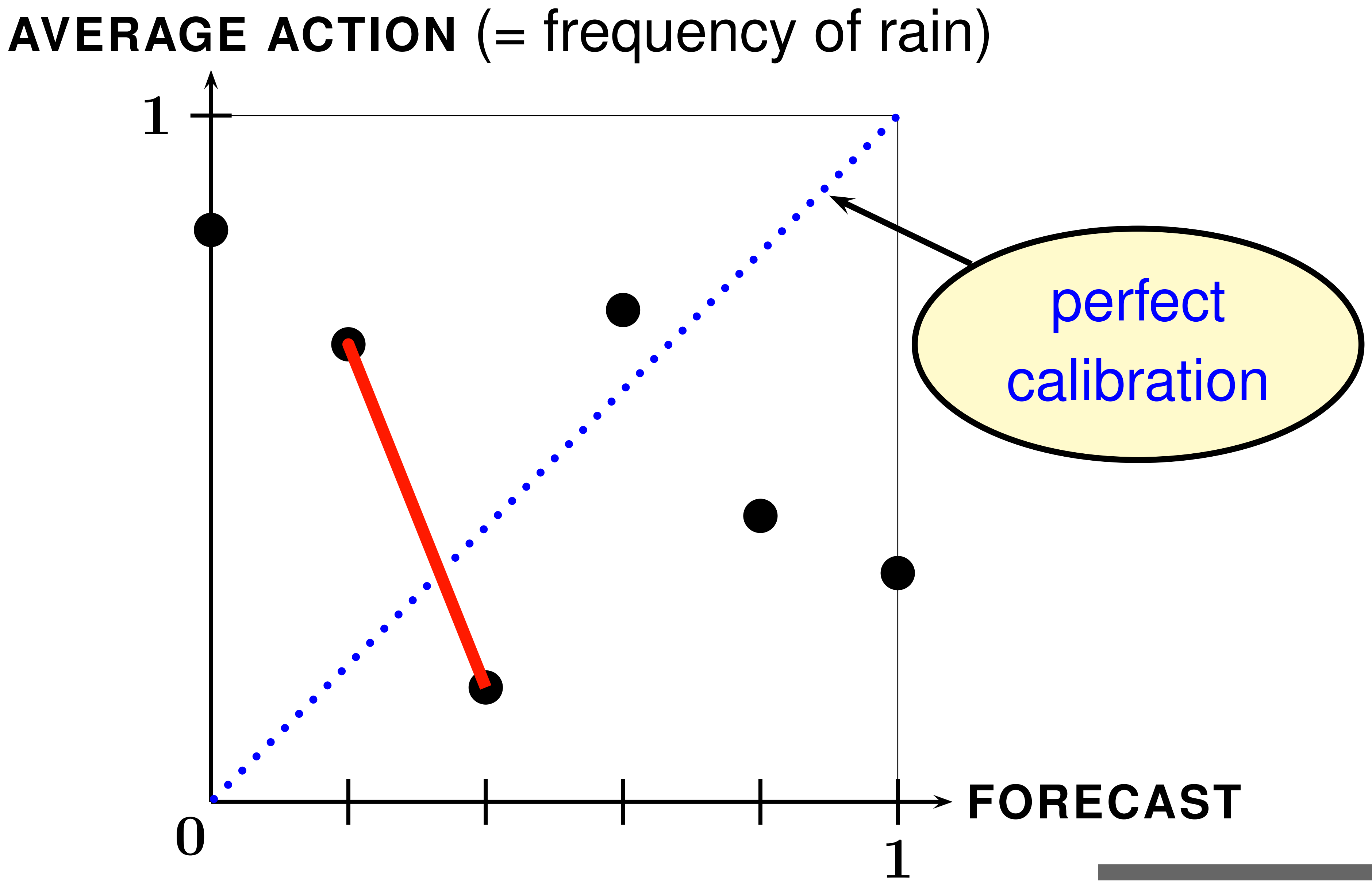

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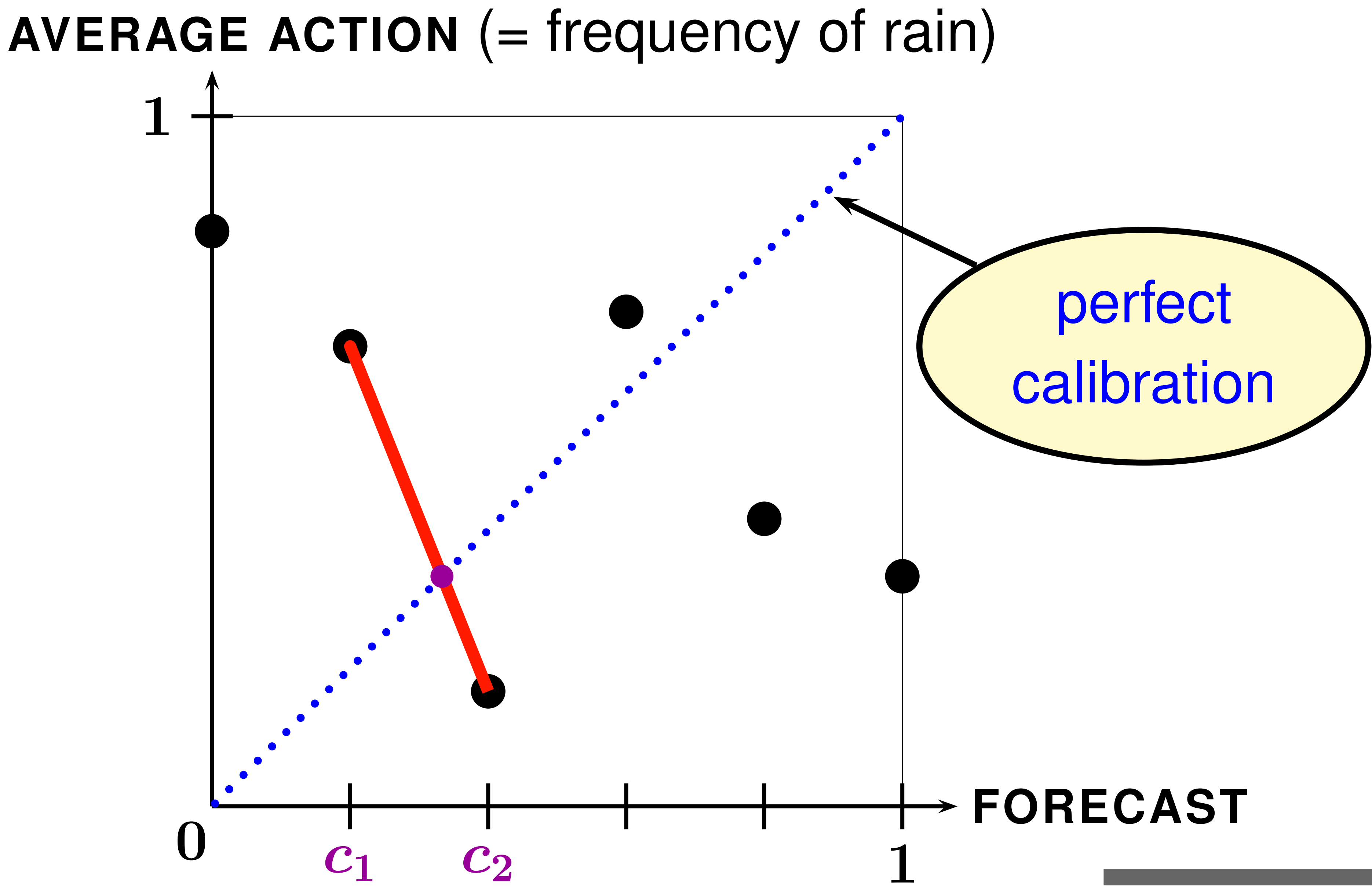

Forecasting

Forecasting ?

Forecasting ?

BACK-casting !

Forecasting ?

BACK-casting !

(not fore-casting)

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... prophet looking backwards ...

Forecasting ?

BACK-casting !

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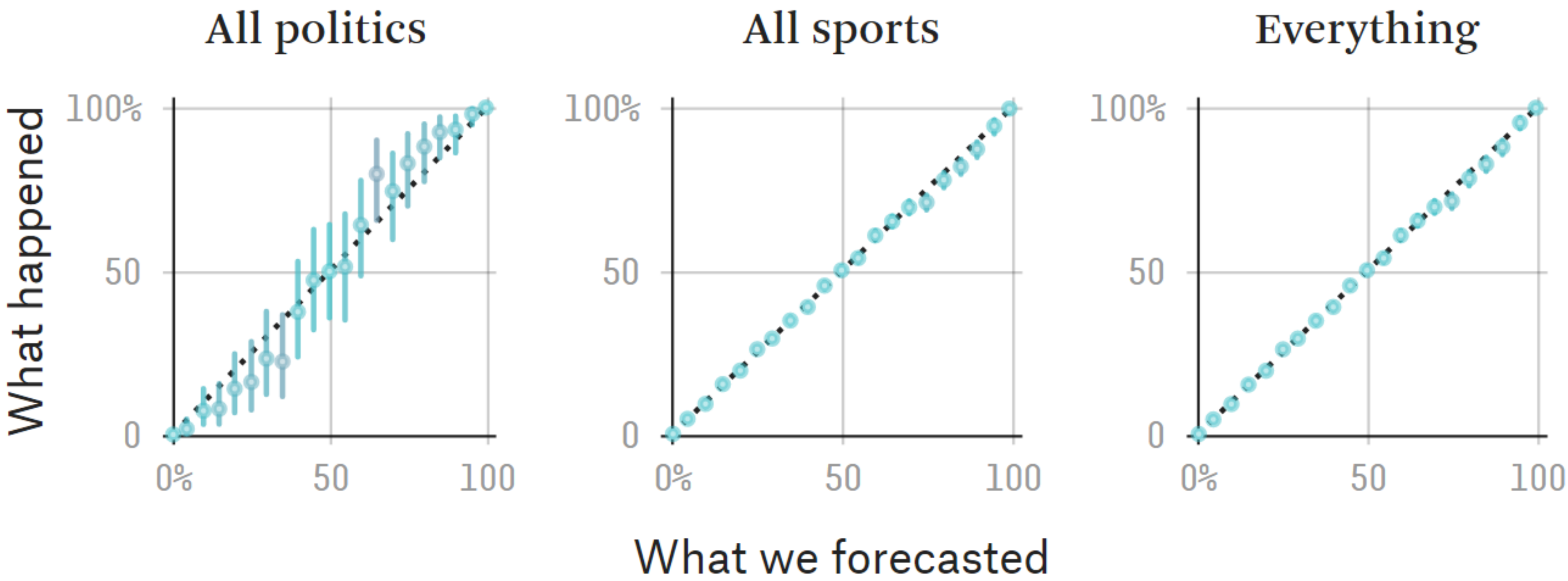
Friedrich von Schlegel (1797)
Walter Benjamin

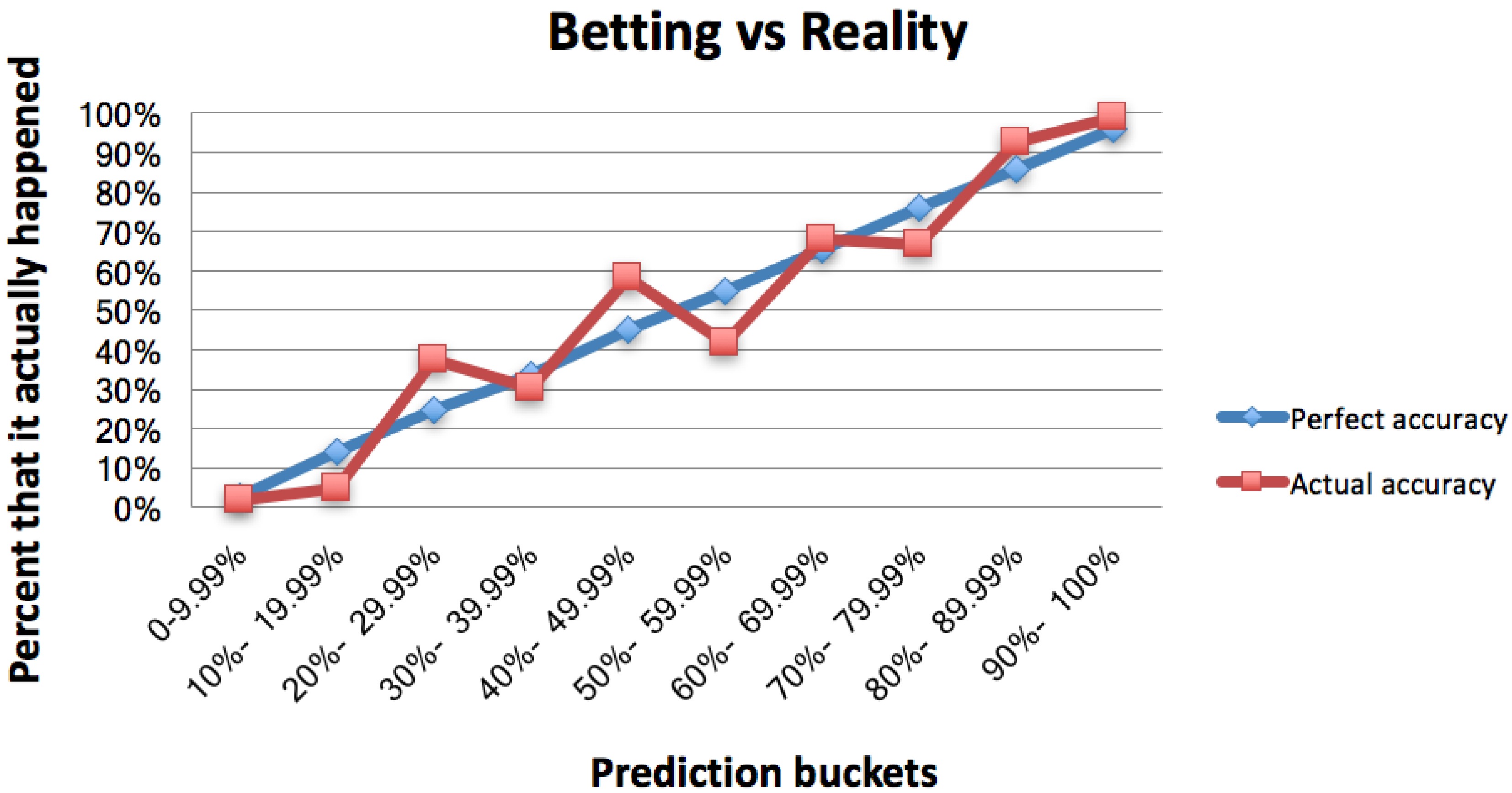
Calibration in Practice

Calibration in Practice



Calibration plots of FiveThirtyEight.com
(as of June 2019)

Calibration in Practice



Calibration plot of ElectionBettingOdds.com
(2016 – 2018)

Example

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time	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	...

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rain	1	0	1	0	1	0	

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F1	100%	0%	100%	0%	100%	0%	

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time	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	...
rain	1	0	1	0	1	0	
F1	100%	0%	100%	0%	100%	0%	
F2	50%	50%	50%	50%	50%	50%	

Example

time	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	...
rain	1	0	1	0	1	0	
F1	100%	0%	100%	0%	100%	0%	
F2	50%	50%	50%	50%	50%	50%	

F1: **CALIBRATION** $= \mathbf{0}$

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time	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	...
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F1: **CALIBRATION** $= \mathbf{0}$

F2: **CALIBRATION** $= \mathbf{0}$

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F1	100%	0%	100%	0%	100%	0%	
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F1: **CALIBRATION** $= 0$ **IN-BIN VARIANCE** $= 0$

F2: **CALIBRATION** $= 0$

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F1: **CALIBRATION** $= \mathbf{0}$ **IN-BIN VARIANCE** $= \mathbf{0}$

F2: **CALIBRATION** $= \mathbf{0}$ **IN-BIN VARIANCE** $= \mathbf{\frac{1}{4}}$

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$$\bar{a}(x) = \frac{\sum_{t=1}^{T} \mathbf{1}_x(c_t)\, a_t}{\sum_{t=1}^{T} \mathbf{1}_x(c_t)}$$

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Proof.

$$\mathbb{E}[(X - c)^2] = \mathbb{V}ar(X) + (\bar{X} - c)^2$$

where c is a constant and X is a random variable with $\bar{X} = \mathbb{E}[X]$

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F1: $\mathcal{K} = 0 \quad \mathcal{R} = 0$

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“Experts”

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✗ CALIBRATION score

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$\longleftrightarrow$ ***LOW*** **REFINEMENT SCORE**

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Can one **GAIN CALIBRATION**

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- Can one decrease $\mathcal{B} = \mathcal{R} + \mathcal{K}$ by $\mathcal{K}$?

“Expertise” and Calibration

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Question:

Can one do this **ONLINE** ?

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$$\mathcal{B}^{\mathrm{c}}_T \leq \mathcal{B}^{\mathrm{b}}_T - \mathcal{K}^{\mathrm{b}}_T + \mathrm{o}(1) \quad \text{as } T \to \infty$$

for ALL sequences a_t and b_t (uniformly)

- Consider a forecasting sequence b_t (in a [finite] set B)
- At each time t generate a forecast c_t
  - **ONLINE**: use only b_t and history
  - such that

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"Calibeating"

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c "BEATS" b by b's CALIBRATION score

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c "BEATS" b by b's CALIBRATION score

- GUARANTEED for ALL sequences of actions and forecasts

Example

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time	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	...
rain	1	0	1	0	1	0	
b	80%	40%	80%	40%	80%	40%	

Example

time	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	...
rain	1	0	1	0	1	0	
$\boldsymbol{b}$	80%	40%	80%	40%	80%	40%	

$\boldsymbol{b}$: $\mathcal{K}^{\mathrm{b}} = \mathbf{0.1}$ $\mathcal{R}^{\mathrm{b}} = \mathbf{0}$ $\mathcal{B}^{\mathrm{b}} = \mathbf{0.1}$

Example

time	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	...
rain	1	0	1	0	1	0	
$\boldsymbol{b}$	80%	40%	80%	40%	80%	40%	
$\boldsymbol{c}$	100%	0%	100%	0%	100%	0%	

$\boldsymbol{b}$: $\mathcal{K}^{\mathrm{b}} = 0.1$ $\quad \mathcal{R}^{\mathrm{b}} = 0$ $\quad \mathcal{B}^{\mathrm{b}} = 0.1$

Example

time	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	...
rain	1	0	1	0	1	0	
b	80%	40%	80%	40%	80%	40%	
c	100%	0%	100%	0%	100%	0%	

b: $\mathcal{K}^{\mathrm{b}} = 0.1$ $\quad \mathcal{R}^{\mathrm{b}} = 0$ $\quad \mathcal{B}^{\mathrm{b}} = 0.1$

c: $\mathcal{K}^{\mathrm{c}} = 0$ $\quad \mathcal{R}^{\mathrm{c}} = 0$ $\quad \mathcal{B}^{\mathrm{c}} = 0$

Calibeating

time	1	2	3	4	5	6	...
rain	1	0	1	0	1	0	
b	80%	40%	80%	40%	80%	40%	
c	100%	0%	100%	0%	100%	0%	

b: $\mathcal{K}^{\text{b}} = 0.1$ $\mathcal{R}^{\text{b}} = 0$ $\mathcal{B}^{\text{b}} = 0.1$

c: $\mathcal{K}^{\text{c}} = 0$ $\mathcal{R}^{\text{c}} = 0$ $\mathcal{B}^{\text{c}} = 0$

c ***calibeats*** b: $\mathcal{B}^{\text{c}} \leq \mathcal{B}^{\text{b}} - \mathcal{K}^{\text{b}}$

Calibeating

time	1	2	3	4	5	6	...
rain	1	0	1	0	1	0	
b	80%	40%	80%	40%	80%	40%	
c	100%	0%	100%	0%	100%	0%	

b: $\mathcal{K}^{\mathrm{b}} = 0.1$ $\mathcal{R}^{\mathrm{b}} = 0$ $\mathcal{B}^{\mathrm{b}} = 0.1$

c: $\mathcal{K}^{\mathrm{c}} = 0$ $\mathcal{R}^{\mathrm{c}} = 0$ $\mathcal{B}^{\mathrm{c}} = 0$

c ***calibeats*** b: $\mathcal{B}^{\mathrm{c}} \leq \mathcal{B}^{\mathrm{b}} - \mathcal{K}^{\mathrm{b}} = \mathcal{R}^{\mathrm{b}}$

Calibeating

Calibeating

(that was easy ...)

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Can one **CALIBEAT** ***in general, non-stationary, situations ?***

Calibeating

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Can one* CALIBEAT *in general, non-stationary, situations ?

- **Weather** is arbitrary and not stationary

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- **Bin averages** do not converge
- ONLINE
- GUARANTEED (even against adversary)

Calibeating

Calibeating

Theorem

There exists a **CALIBEATING** procedure

A Way to Calibeat

A Way to Calibeat

Theorem

The procedure

$$c_t = \bar{a}^{\mathrm{b}}_{t-1}(b_t)$$

GUARANTEES b-CALIBEATING

A Simple Way to Calibeat

Theorem

The procedure

$$c_t = \bar{a}^{\mathrm{b}}_{t-1}(b_t)$$

GUARANTEES b-**CALIBEATING**

Forecast the average action of the current b-forecast

Proof

Proof

$$\mathbb{V}\mathbf{ar} \;=\; \frac{1}{T}\sum_{t=1}^{T}\left\|x_t - \bar{x}_T\right\|^2$$

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$$\mathbb{V}\text{ar} = \frac{1}{T}\sum_{t=1}^{T}\|x_t - \bar{x}_T\|^2$$

$$= \frac{1}{T}\sum_{t=1}^{T}\left(1-\frac{1}{t}\right)\|x_t - \bar{x}_{t-1}\|^2$$

Proof

$$\mathbb{V}\text{ar} = \frac{1}{T}\sum_{t=1}^{T} \|x_t - \bar{x}_{\color{red}T}\|^2$$

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$$
\begin{aligned}
\mathbb{V}\mathrm{ar} &= \frac{1}{T}\sum_{t=1}^{T} \left\| x_t - \bar{x}_T \right\|^2 \\
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&= \frac{1}{T}\sum_{t=1}^{T} \left\| x_t - \bar{x}_{t-1} \right\|^2 - \mathrm{o}(1)
\end{aligned}
$$

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(*) $\mathrm{o}(1) = \mathrm{O}\left(\frac{1}{T}\sum_{t=1}^{T}\frac{1}{t}\right) = \mathrm{O}\left(\frac{\log T}{T}\right)$

Proof

$$
\begin{aligned}
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\end{aligned}
$$

Proof: "Online Variance"

$$\begin{aligned}
\mathbb{V}\text{ar} &= \frac{1}{T}\sum_{t=1}^{T} \|x_t - \bar{x}_T\|^2 \\
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&= \underbrace{\frac{1}{T}\sum_{t=1}^{T} \|x_t - \bar{x}_{t-1}\|^2}_{\widetilde{\mathbb{V}\text{ar}}} - \text{o}(1) \\
&= \widetilde{\mathbb{V}\text{ar}} - \text{o}(1)
\end{aligned}$$

Proof: “Online Variance”

$$\mathbb{V}\mathrm{ar} \;=\; \widetilde{\mathbb{V}\mathrm{ar}} - o(1)$$

Proof: “Online Refinement”

$$\mathbb{V}\mathbf{ar} = \widetilde{\mathbb{V}\mathbf{ar}} - \mathbf{o}(1)$$

$$\mathcal{R}^{\mathbf{b}} = \widetilde{\mathcal{R}}^{\mathbf{b}} - \mathbf{o}(1)$$

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&= \qquad\qquad \mathcal{B}^{\mathrm{c}} \qquad\qquad - \mathrm{o}(1)
\end{aligned}$$

$c_t = \bar{a}_{t-1}(b_t)$

Calibeating

Calibeating

Theorem

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GUARANTEES b-CALIBEATING:

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$$\Leftrightarrow \quad \mathcal{K}^{\mathrm{c}} = 0$$

Self-Calibeating = Calibrating

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- $D \subset C$ finite δ-grid of C (for $\delta > 0$)
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“Wrong” Proof !

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"Right" Proof. Use the **MINIMAX** theorem

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"Right" Proof. Use the MINIMAX theorem

- $h(x, v) := \|v - x\|^2 - \|v - f(x)\|^2$

Stochastic "Fixed Point"

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Stochastic "Fixed Point"

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- For every distribution Q on $v \in C$:
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- $\Rightarrow$ [**MINIMAX**]
  There exists a distribution P on $x \in D$ s.t.

  $$h(P, v) \leq \delta^2 \quad \text{for every } v \in C$$

Stochastic “Fixed Point”

Theorem. There exists a probability distribution P on the δ-grid D of C such that:

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- **Obtained by solving a MINIMAX problem (LP)**

Outgoing Minimax (FH)

Theorem. There exists a probability distribution P on the δ-grid D of C such that:

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- **Obtained by solving a MINIMAX problem (LP)**
- Moreover: solving a FIXED POINT problem yields a probability distribution P that is ALMOST DETERMINISTIC: its support is included in a ball of size δ

Calibrating

Calibrating

Theorem

There is a stochastic procedure that **GUARANTEES CALIBRATION**

Calibrating

Theorem

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Proof. Self-calibeating + Stochastic Fixed Point

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Note. δ-**CALIBRATION**

Calibrated Calibeating

Calibrated Calibeating

Theorem

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Calibrated Calibeating

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There is a stochastic procedure that **GUARANTEES CALIBEATING** and **CALIBRATION**

Calibrated Calibeating

Theorem

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Proof. Calibeat the **joint** binning of $\boldsymbol{b}$ and $\boldsymbol{c}$, by applying Stochastic Fixed Point

Calibrated Calibeating

Theorem

There is a stochastic procedure that **GUARANTEES CALIBEATING** and **CALIBRATION**

Proof. Calibeat the **joint** binning of b and c, by applying Stochastic Fixed Point

---

STRONG CALIBEATING:

$$\mathcal{R}(\mathbf{c}) \leq \mathcal{R}(\mathbf{b}) \quad \text{and} \quad \mathcal{K}(\mathbf{c}) = \mathbf{0}$$

Continuous Calibration

Continuous Calibration

- **CONTINUOUS CALIBRATION**: combine the days when the forecast was ***close to*** p

Continuous Calibration

- **CONTINUOUS CALIBRATION**: combine the days when the forecast was ***close to*** p (smooth out the calibration score)

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- **Theorem**:

> There exists a ***deterministic*** procedure that is **CONTINUOUSLY CALIBRATED**.

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> There exists a ***deterministic*** procedure that is **CONTINUOUSLY CALIBRATED**.

Deterministic $\Rightarrow$ holds also when the forecasts are ***"leaked"***

Continuous Calibration

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Foster and Kakade (2004, 2006)
Foster and Hart (2018, **2021**)

Continuous-Calibrated Calibeating

Continuous-Calibrated Calibeating

Theorem

There is a ***deterministic*** procedure that **GUARANTEES CALIBEATING**

Continuous-Calibrated Calibeating

Theorem

There is a ***deterministic*** procedure that **GUARANTEES** **CALIBEATING** and **CONTINUOUS CALIBRATION**

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Proof. Calibeat the **joint** binning of $\boldsymbol{b}$ and $\boldsymbol{c}$, by a Fixed Point result (Brouwer)

Multi-Calibeating

Multi-Calibeating

Theorem

There is a ***deterministic*** procedure
that **GUARANTEES**

simultaneous CALIBEATING
of several forecasters

Multi-Calibeating

Theorem

There is a ***stochastic*** procedure
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and **CALIBRATION**

Multi-Calibeating

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Proof. Calibeat the **joint** binning

In all the results above:

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	CALIBRATION	
Obtained by	***Minimax***	
Procedure	***stochastic***	

... and Continuous Calibration

In all the results above:

	CALIBRATION	CONTINUOUS CALIBRATION
Obtained by	***Minimax***	***Fixed Point***
Procedure	***stochastic***	***deterministic***

Refinement Score and Brier Score

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Claim. The REFINEMENT score is the ***minimal*** BRIER score over all ***relabelings of the bins***:

Refinement Score and Brier Score

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$$\mathcal{R}_T(\mathrm{b}) = \min_{\phi} \mathcal{B}_T(\phi(\mathrm{b}))$$

Refinement Score and Brier Score

Claim. The REFINEMENT score is the ***minimal*** BRIER score over all ***relabelings of the bins***:

$$\mathcal{R}_T(\mathrm{b}) = \min_{\phi} \mathcal{B}_T(\phi(\mathrm{b}))$$

where the minimum is taken over all

$$\phi : B \longrightarrow \Delta(A)$$

and

$$\phi(\mathrm{b}) \equiv (\phi(b_1), ..., \phi(b_T))$$

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Proof. Label each bin b with $\phi(b) = \bar{a}_T(b)$

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Calibeating

- c CALIBEATS b:

$$\mathcal{B}_T(\mathbf{c}) \leq \mathcal{R}_T(\mathbf{b}) + o(1)$$

Calibeating

- c **CALIBEATS** b:

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Calibeating

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Calibeating

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where $\phi : B \rightarrow \Delta(A)$

- c **MULTI-CALIBEATS** $\mathrm{b}_1, ..., \mathrm{b}_N$:

$$\mathcal{B}(\mathrm{c}) \leq \min_{1 \leq n \leq N} \min_{\phi^n} \mathcal{B}(\phi^n(\mathrm{b}^n))$$

Calibeating

- c **CALIBEATS** b:

$$\mathcal{B}(c) \le \mathcal{R}(b)$$

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    (replaces $\|\cdot\|^2$)

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(as $T \to \infty$)

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Results

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1. Calibration $\Longrightarrow$ **PROPER-CALIBRATION**

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- $u : A \times X \to \mathbb{R}$
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$\Rightarrow$ L^u is a **PROPER SCORING RULE**

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- **REGRET OF (BEST REPLYING TO) FORECASTS** $\mathbf{c}$

$$\text{REG}_T^u(\mathbf{c}) := \max_{\xi:\Delta(A)\to X} \frac{1}{T}\sum_{t=1}^{T} u(a_t, \xi(c_t)) - \mathcal{U}_T(\mathbf{c})$$

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EVERY bounded decision maker

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$$\mathcal{U}_T(\mathrm{c}) \geq \mathcal{U}_T(\mathrm{b}) + \textbf{REG}_T^u(\mathrm{b}) - o(1)$$

for **EVERY** bounded u

What does every forecaster deserve ?

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A PROPER CALIBEATING

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