“Calibeating”: Beating Forecasters at Their Own Game

Sergiu Hart

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Joint work with

Dean P. Foster

University of Pennsylvania &
Amazon Research NY

Papers

Papers

- Dean P. Foster and Sergiu Hart
  “Forecast Hedging and Calibration”
  - First version: 2016
  - *Journal of Political Economy*, 2021

www.ma.huji.ac.il/hart/publ.html#calib-int

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- Dean P. Foster and Sergiu Hart
  “ ‘Calibeating’: Beating Forecasters at Their Own Game”
  - First version: 2020
  - Center for Rationality DP-743, 2021

  www.ma.huji.ac.il/hart/publ.html#calib-beat

Calibration

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    in the days when the forecast was p, the proportion of rainy days equals p

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- Forecaster is **CALIBRATED** if
  - for every forecast $\boldsymbol{p}$:
    in the days when the forecast was $\boldsymbol{p}$, the proportion of rainy days equals $\boldsymbol{p}$
    (or: is close to $\boldsymbol{p}$ in the long run)

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CALIBRATION ***can be guaranteed***

(no matter what the weather will be)

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- Foster 1999: simple procedure
- Foster and Hart 2016 [publ 2021]: simplest procedure, by "Forecast Hedging"

Forecast-Hedging

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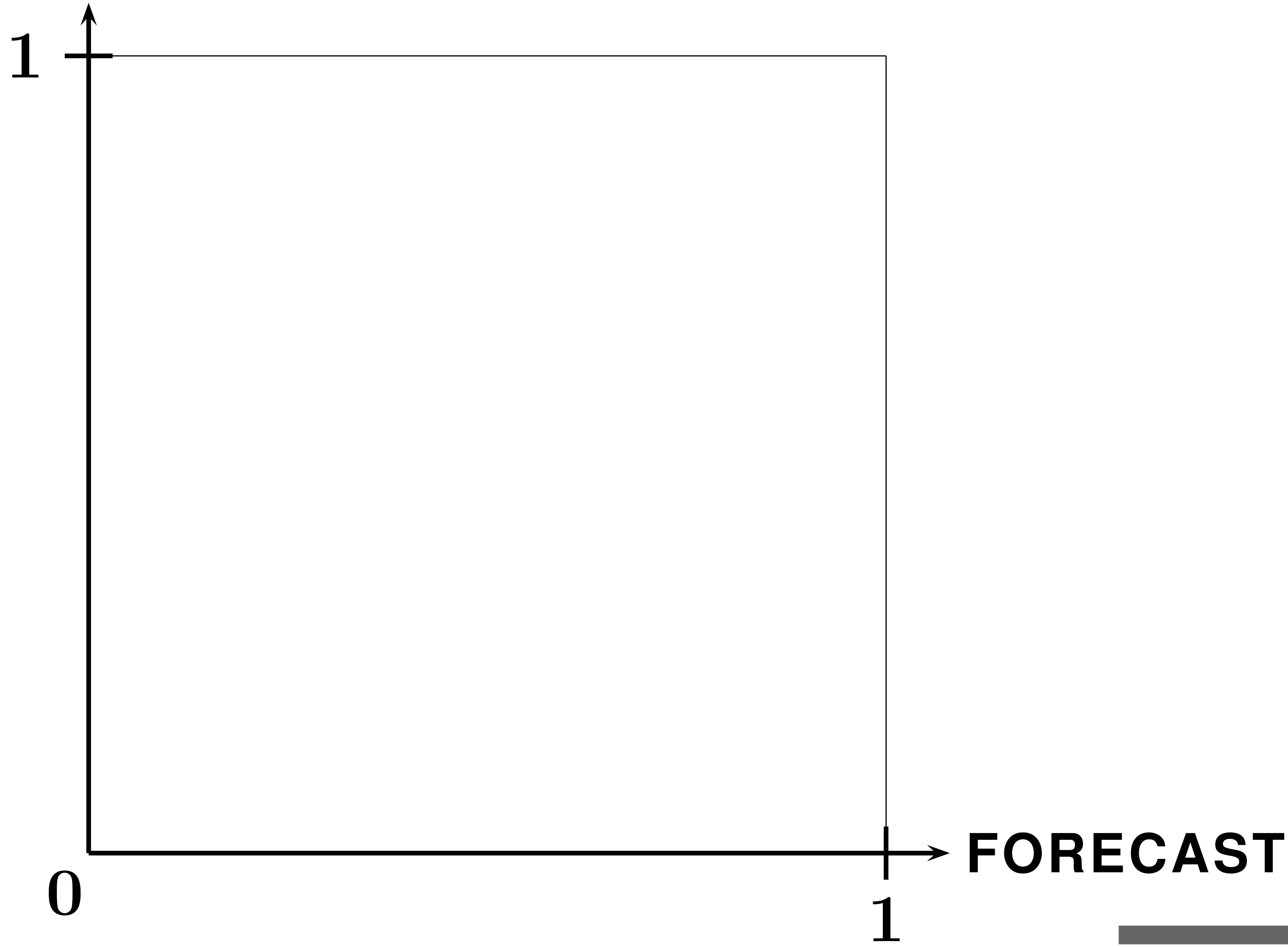
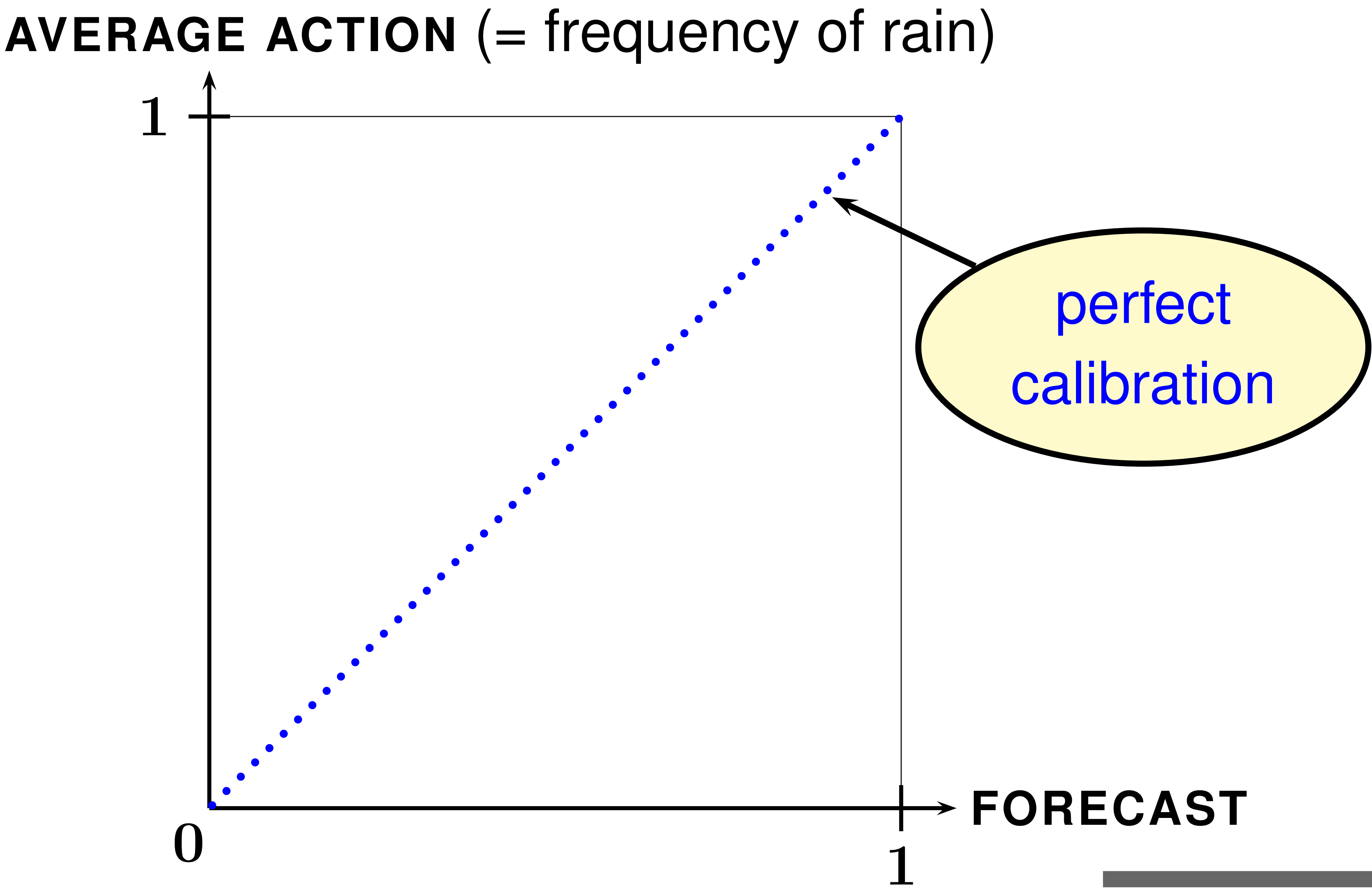
AVERAGE ACTION (= frequency of rain)



Forecast-Hedging



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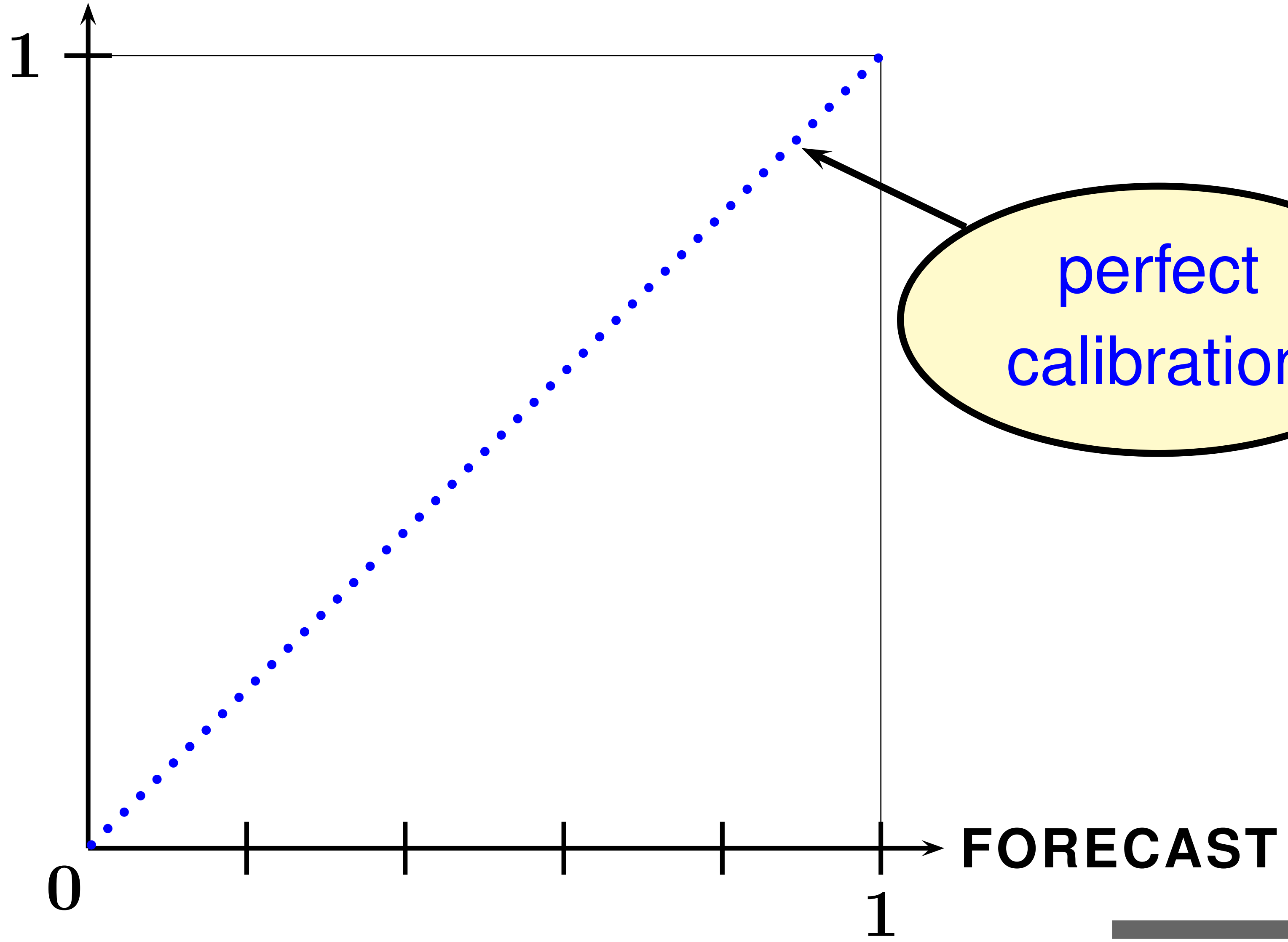

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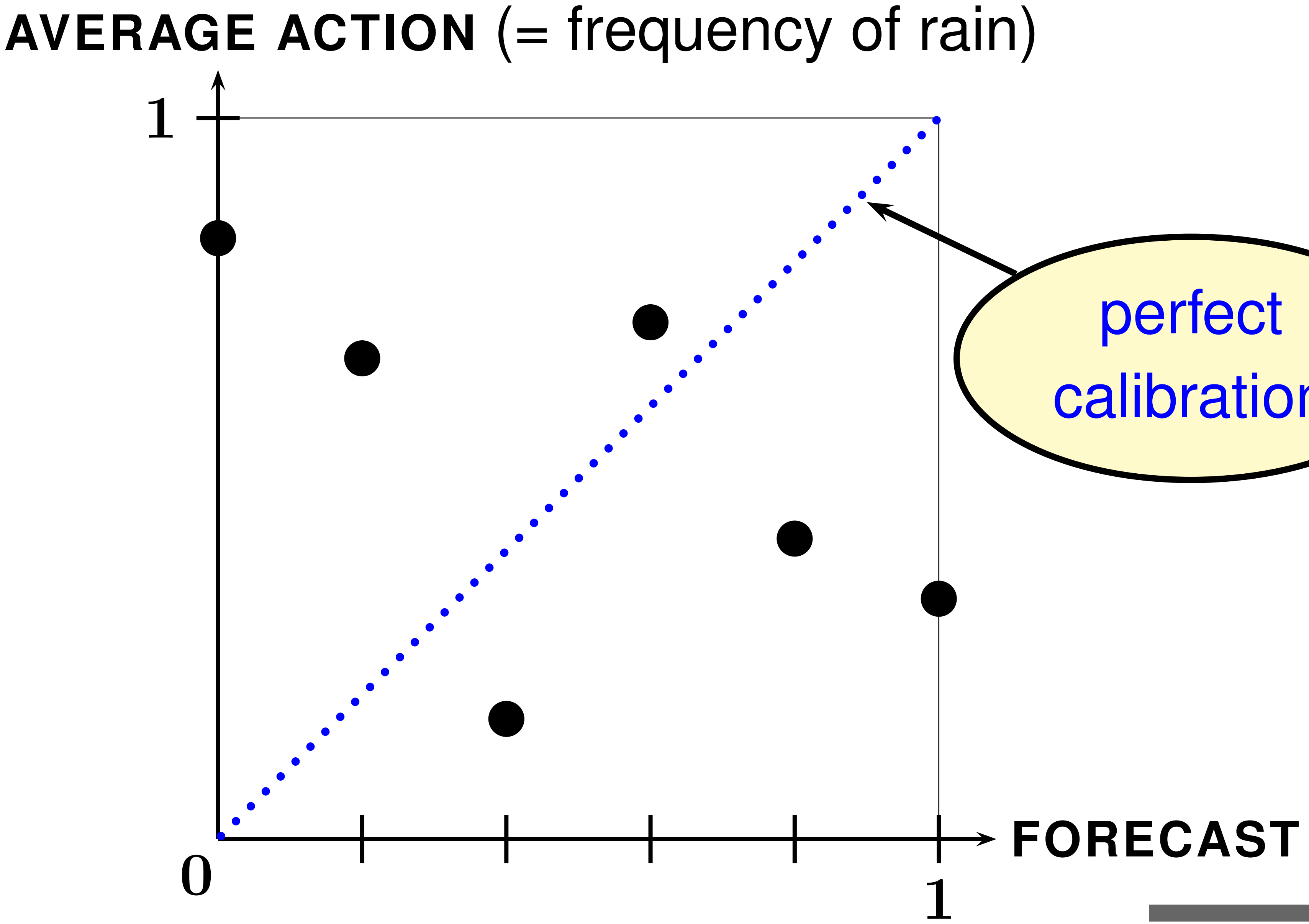

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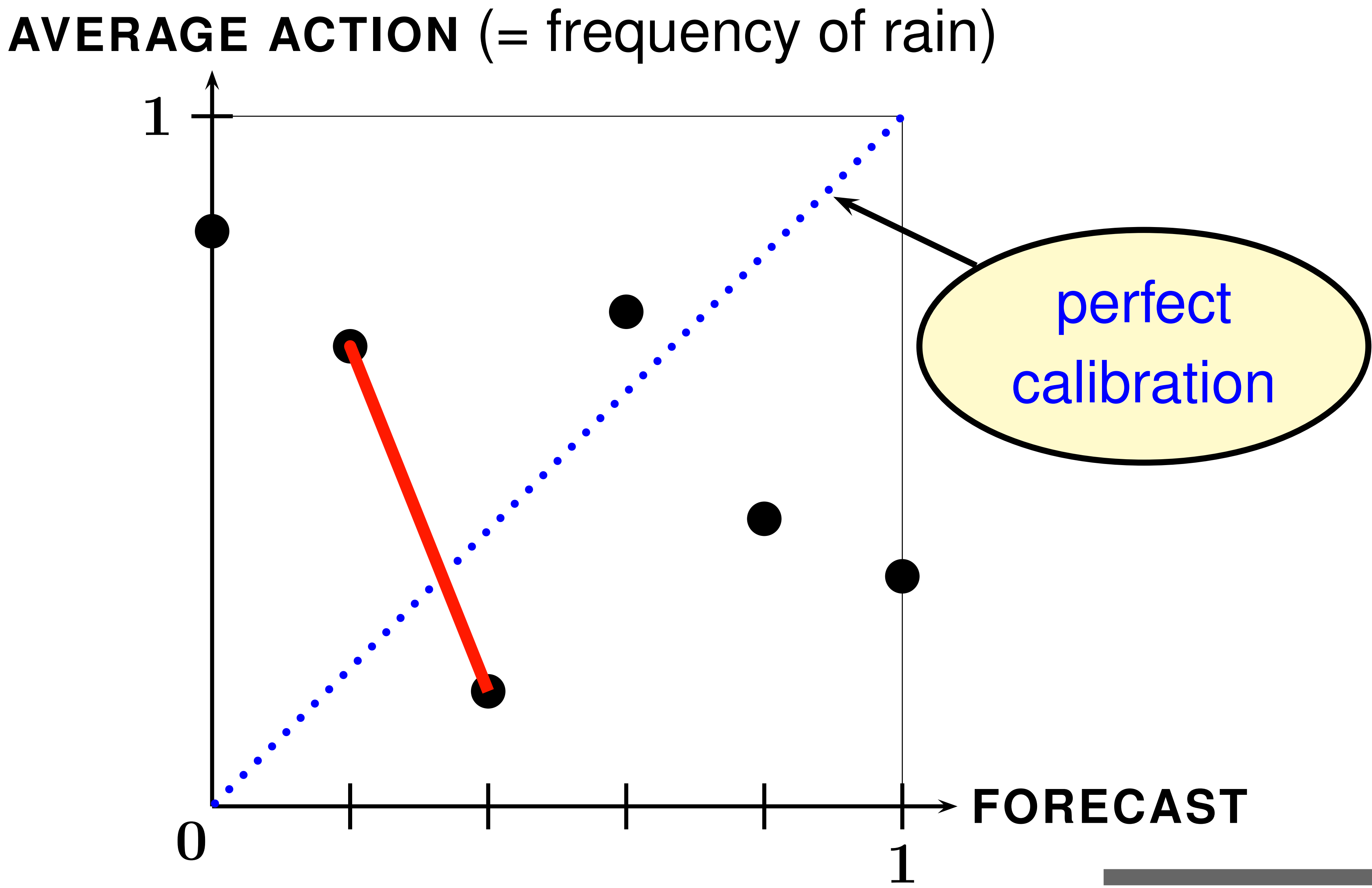

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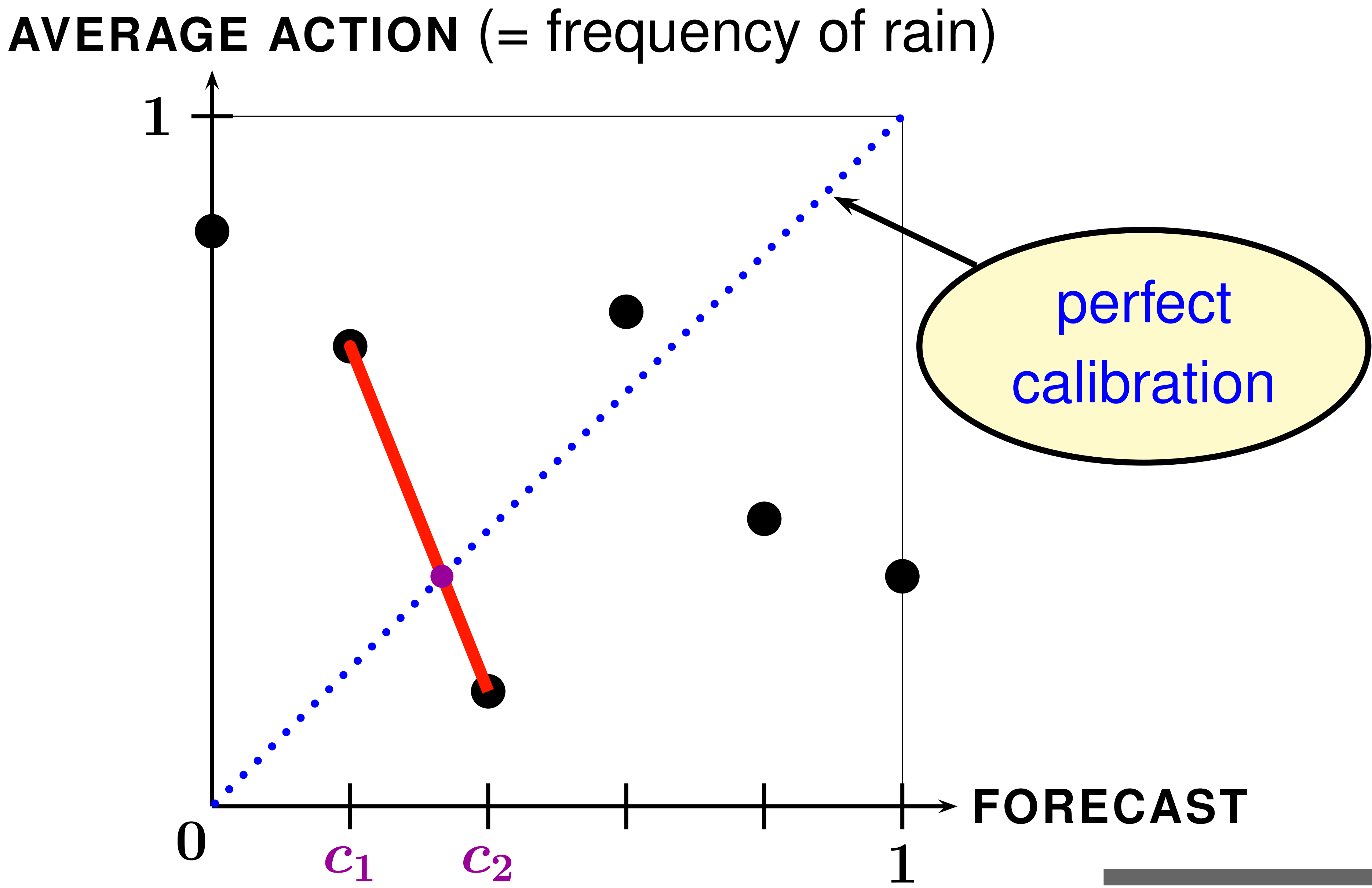

Calibration in Practice

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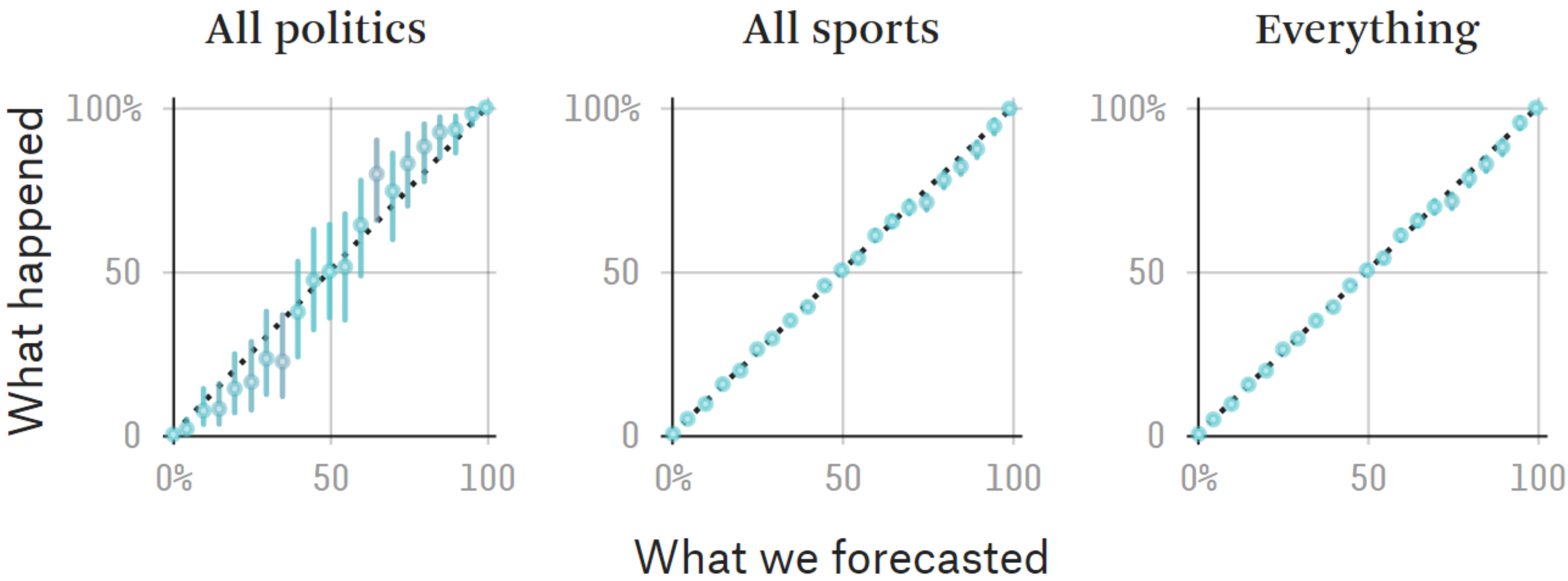
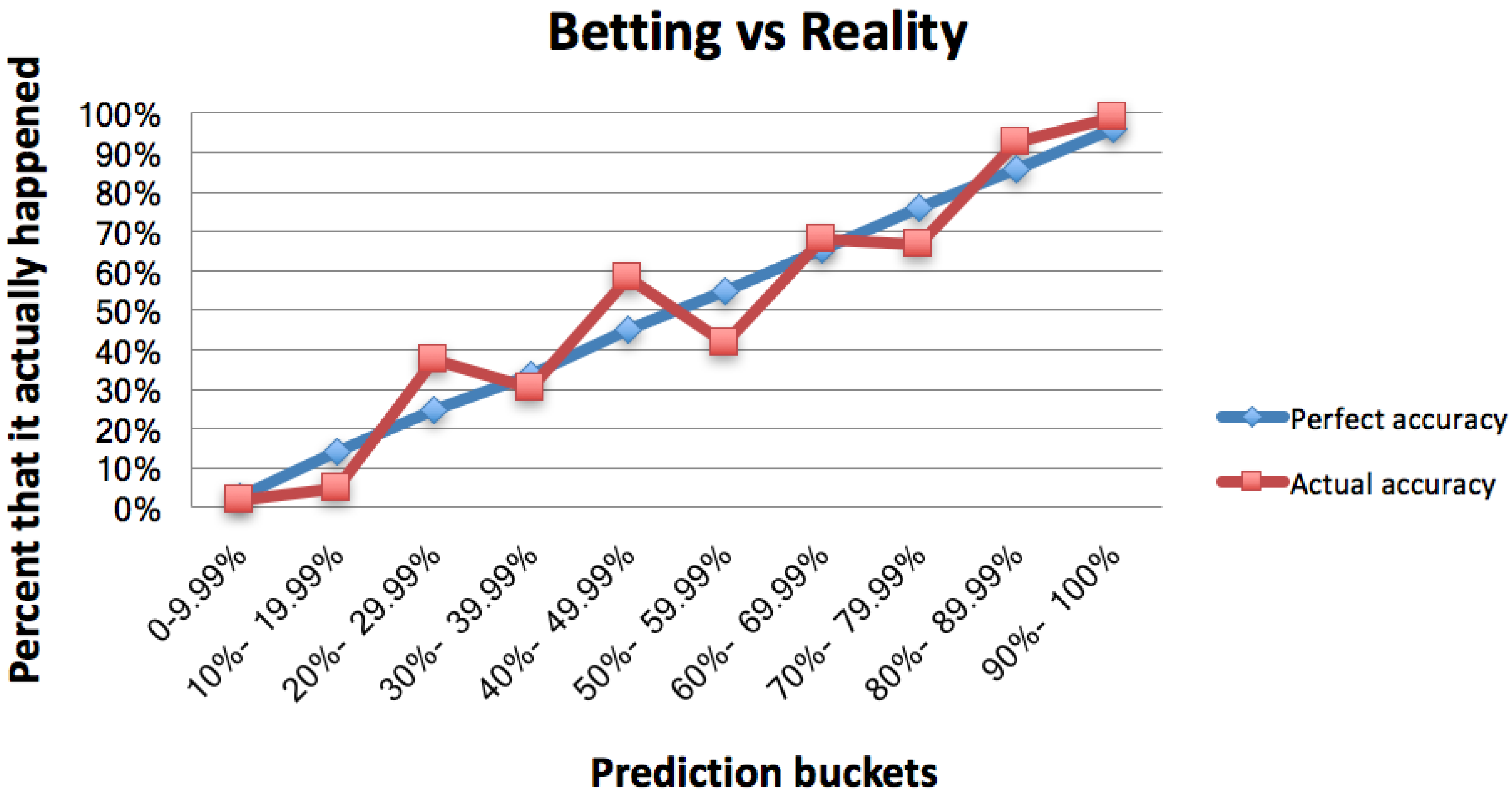


Calibration plots of FiveThirtyEight.com
(as of June 2019)

Calibration in Practice



Calibration plot of ElectionBettingOdds.com (2016 – 2018)

Example

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time	1	2	3	4	5	6	...

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F2	50%	50%	50%	50%	50%	50%	

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rain	1	0	1	0	1	0	
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F1: **CALIBRATION** $= \mathbf{0}$

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F2: **CALIBRATION** $= \mathbf{0}$

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F1: **CALIBRATION** $= 0$ **IN-BIN VARIANCE** $= 0$

F2: **CALIBRATION** $= 0$

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time	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	$\mathbf{\ldots}$
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F1: **CALIBRATION** $= \mathbf{0}$ **IN-BIN VARIANCE** $= \mathbf{0}$

F2: **CALIBRATION** $= \mathbf{0}$ **IN-BIN VARIANCE** $= \frac{\mathbf{1}}{\mathbf{4}}$

Notations

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$$\bar{a}(x) = \frac{\sum_{t=1}^{T} \mathbf{1}_x(c_t)\, a_t}{\sum_{t=1}^{T} \mathbf{1}_x(c_t)}$$

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Proof.

$$\mathbb{E}[(X - c)^2] = \mathbb{V}ar(X) + (\bar{X} - c)^2$$

where c is a constant and X is a random variable with $\bar{X} = \mathbb{E}[X]$

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“Experts”

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Testing experts:

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$\Leftrightarrow$ ***LOW*** **REFINEMENT SCORE**

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Can one **GAIN CALIBRATION**

without **LOSING “EXPERTISE”** ?

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- Can one decrease $\mathcal{B} = \mathcal{R} + \mathcal{K}$ by $\mathcal{K}$?

“Expertise” and Calibration

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Question:

Can one do this **ONLINE** ?

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$$\mathcal{B}_T^{\mathrm{c}} \leq \mathcal{B}_T^{\mathrm{b}} - \mathcal{K}_T^{\mathrm{b}} + \mathrm{o}(1) \quad \text{as } T \to \infty$$

for **ALL** sequences a_t and b_t (uniformly)

- Consider a forecasting sequence b_t (in a [finite] set B)
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"Calibeating"

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c "BEATS" b by b's CALIBRATION score

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- GUARANTEED for ALL sequences of actions and forecasts

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time	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	...
rain	1	0	1	0	1	0	
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$\boldsymbol{b}$: $\mathcal{K}^{\mathrm{b}} = 0.1$ $\mathcal{R}^{\mathrm{b}} = 0$ $\mathcal{B}^{\mathrm{b}} = 0.1$

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Calibeating

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$\boldsymbol{b}$: $\mathcal{K}^{\mathrm{b}} = 0.1 \quad \mathcal{R}^{\mathrm{b}} = 0 \quad \mathcal{B}^{\mathrm{b}} = 0.1$

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b: $\mathcal{K}^{\mathrm{b}} = 0.1$ $\mathcal{R}^{\mathrm{b}} = 0$ $\mathcal{B}^{\mathrm{b}} = 0.1$

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Calibeating

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(that was easy ...)

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Can one* CALIBEAT *in general, non-stationary, situations ?

Calibeating

(that was easy ...)

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- **Weather** is arbitrary and not stationary

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- **ONLINE**
- **GUARANTEED** (even against adversary)

A Simple Way to Calibeat

A Simple Way to Calibeat

Theorem

The procedure

$$c_t = \bar{a}^{\mathrm{b}}_{t-1}(b_t)$$

GUARANTEES b-**CALIBEATING**

A Simple Way to Calibeat

Theorem

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GUARANTEES b-**CALIBEATING**

Forecast the average action of the current b-forecast

Proof

Proof

$$\mathbb{V}\text{ar} = \frac{1}{T}\sum_{t=1}^{T}\|x_t - \bar{x}_T\|^2$$

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$$
\begin{aligned}
\mathbb{V}\text{ar} &= \frac{1}{T}\sum_{t=1}^{T} \|x_t - \bar{x}_T\|^2 \\
&= \frac{1}{T}\sum_{t=1}^{T} \left(1 - \frac{1}{t}\right) \|x_t - \bar{x}_{t-1}\|^2
\end{aligned}
$$

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(*) $\mathrm{o}(1) = \mathrm{O}\left(\frac{1}{T}\sum_{t=1}^{T}\frac{1}{t}\right) = \mathrm{O}\left(\frac{\log T}{T}\right)$

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Proof: “Online Variance"

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&= \underbrace{\frac{1}{T}\sum_{t=1}^{T} \|x_t - \bar{x}_{t-1}\|^2}_{\widetilde{\mathbb{V}\text{ar}}} - \mathrm{o}(1) \\
&= \widetilde{\mathbb{V}\text{ar}} - \mathrm{o}(1)
\end{aligned}
$$

Proof: "Online Variance"

$$\mathbb{V}\mathbf{ar} \;=\; \widetilde{\mathbb{V}\mathbf{ar}} - \mathbf{o}(1)$$

Proof: "Online Refinement"

$$\mathbb{V}\mathbf{ar} = \widetilde{\mathbb{V}\mathbf{ar}} - \mathrm{o}(1)$$

$$\mathcal{R}^{\mathrm{b}} = \widetilde{\mathcal{R}}^{\mathrm{b}} - \mathrm{o}(1)$$

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\mathbb{V}\mathrm{ar} &= \widetilde{\mathbb{V}\mathrm{ar}} - \mathrm{o}(1) \\
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Proof: “Online Refinement”

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$c_t = \bar{a}_{t-1}(b_t)$

Calibeating

Calibeating

Theorem

$$c_t = \bar{a}^{\mathrm{b}}_{t-1}(b_t)$$

GUARANTEES b-CALIBEATING:

$$\mathcal{B}^{\mathrm{c}} \leq \mathcal{B}^{\mathrm{b}} - \mathcal{K}^{\mathrm{b}}$$

Self-Calibeating

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Self-Calibeating = Calibrating

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“Fixed Point”

How do we get c_t “close to” $\bar{a}_{t-1}(c_t)$?

Stochastic "Fixed Point"

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Theorem There exists a probability distribution on (a δ-grid D of) C such that for every $x \in C$

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- $C \subset \mathbb{R}^m$ compact convex
- $D \subset C$ finite δ-grid of C for $\delta > 0$
- $g : D \to \mathbb{R}^m$ arbitrary function

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Obtained by solving a Minimax problem (LP)

Outgoing Minimax (FH)

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- **Obtained by solving a Minimax problem (LP)**
- Moreover, **solving a Fixed Point problem** yields a probability distribution that is **ALMOST DETERMINISTIC**: its support is included in a ball of size δ

Calibrating

Calibrating

Theorem

There is a stochastic procedure that **GUARANTEES CALIBRATION**

Theorem

There is a stochastic procedure that **GUARANTEES** **CALIBRATION**

Proof. Self-calibeating + Outgoing Minimax

Calibrating

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Proof. Self-calibeating + Outgoing Minimax

Note. δ-**CALIBRATION**

Calibrated Calibeating

Calibrated Calibeating

Theorem

There is a stochastic procedure that **GUARANTEES** **CALIBEATING**

Calibrated Calibeating

Theorem

There is a stochastic procedure that **GUARANTEES CALIBEATING** and **CALIBRATION**

Calibrated Calibeating

Theorem

There is a stochastic procedure that **GUARANTEES CALIBEATING** and **CALIBRATION**

Proof. Calibeat the joint binning of b and c, by the Outgoing Minimax theorem

Multi-Calibeating

Multi-Calibeating

Theorem

There is a ***deterministic*** procedure that **GUARANTEES**

simultaneous CALIBEATING
of several forecasters

Multi-Calibeating

Theorem

There is a ***stochastic*** procedure that **GUARANTEES**

simultaneous CALIBEATING of several forecasters

and **CALIBRATION**

Multi-Calibeating

Theorem

There is a ***stochastic*** procedure
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Proof. Calibeat the joint binning

In all the results above:

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	CALIBRATION	
Obtained by	***Minimax***	
Procedure	***stochastic***	

... and Continuous Calibration

In all the results above:

	CALIBRATION	CONTINUOUS CALIBRATION
Obtained by	***Minimax***	***Fixed Point***
Procedure	***stochastic***	***deterministic***

Successful Economic Forecasting

Successful Economic Forecasting

TAKING PRIDE IN OUR RECORD

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"We have correctly forecasted 8 of the last 5 recessions"

Successful Economic Forecasting

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