Advances in Value Theory

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Value, introduced by Lloyd Shapley in 1953, is a central concept in game theory and its applications. The value of a game to a player is best viewed as an *a priori* evaluation of his expected payoffs (in a similar way that the "utility" of a "gamble" or "lottery" is a measure of the prospective outcomes). The Shapley value and its generalizations have been extensively studied, and continue to be the focus of much research, both theoretically (leading to interesting and deep mathematical problems), as well as in applications (in economics, political sciences, etc.).

This paper surveys recent advances in the study of the various value concepts. It covers mainly the last decade; the survey of Aumann (1978) is an excellent reference up to that date. We will deal here mostly with *theory,* the applications of value (which are becoming more and more abundant) being the subject of another survey.

It should be immediately pointed out that this is in no way a comprehensive survey (the allotted time will hardly suffice just to write down the references of papers dealing with value!). It is rather a collection of some results which seem most significant, either in opening up new paths, or in closing up (difficult) gaps left from previous research. Moreover, the presentation will be as informal as possible, the aim being only to focus on the main ideas; for precise and detailed formulations one should of course refer to the original papers.

A two-way standard classification will be used: on one hand, *finite* versus *large* games (the former referring to general games with finitely many players, and the latter to asymptotic results—when the number of players increases—and to games with infinitely many players); and on the other hand, *transferable utility* (TU) versus *nontransferable utility* (NTU) games (also known as "with/without side payments").

1. FINITE GAMES WITH TRANSFERABLE UTILITY

The main solution concept here, from which the whole theory has developed, is the *Shapley value* (Shapley, 1953). It may be surprising that after so many years there are still new and interesting things to say about the Shapley value, even in finite TU games!

GAME THEORY AND APPLICATIONS

First, the standard notations[1]: A *game* (N, v) consists of a finite set N of "players" together with a "characteristic function" $v: 2^N \to \mathbb{R}$ that associates to every "coalition" $S \subset N$ its "worth" $v(S)$; it is assumed that $v(\emptyset) = 0$. A *solution function* ϕ assigns to each game (N, v) a "payoff vector" $\phi(N, v) = (\phi^i(N, v))_{i \in N} \in \mathbb{R}^N$.

1.1 Marginal Contributions

As it is well known, the standard axioms determining the Shapley value are: *Efficiency* (which means that the values add up to $v(N)$, the worth of the grand coalition); *symmetry* or *equal treatment* (which means that identical players have equal values); *additivity* (the value of the sum of two games is the sum of the values of the two games[2]); and finally, *null player* (the value of a player who never contributes anything is zero).

In a recent paper, Young (1985) has shown that one may replace the last two axioms—"additivity" and "null player"—with the following requirement: *The value of a player depends only on his marginal contributions;* i.e., given two games (N, v) and (N, w) (with the same set of players) and a player $i \in N$, if $v(S) - v(S \setminus i) = w(S) - w(S \setminus i)$ for all coalitions S, then $\phi^i(N, v) = \phi^i(N, w)$.[3] We will refer to this as the "*marginal contributions*" axiom.

Young's result is: A solution function satisfies "efficiency", "symmetry" and "marginal contributions", if and only if it is the Shapley value.[4]

It is easy to derive the "null player" axiom: the marginal contributions of a null player are the same as in the "null game" w (defined by $w(S) = 0$ for all S); "efficiency" and "symmetry" imply that the value of w is zero for all players. The surprising fact is that the "additivity" axiom is also implied by these other three axioms.[5] The proof (by induction on the number of players) makes use of "symmetry" in an essential way; it seems therefore interesting to drop "symmetry" and characterize other solutions that satisfy the "marginal contributions" axiom (e.g., the "weighted Shapley values").

1.2 Potential

Paying players according to their marginal contributions is a principle of long standing, in particular in economics. If it is applied in a straightforward manner

[1] $\mathbb{R}$ denotes the real line, $\emptyset$ is the empty set, and all set inclusions $\subset$ are weak.

[2] One may use "average" instead of "sum": if one of two equally probable games will be played, then the "total" (*a priori*) value is the average of the values of the two games.

[3] $\setminus$ denotes set subtraction; we will write $S \setminus i$ instead of the correct but more cumbersome $S \setminus \{i\}$.

[4] Young (1985) actually uses a stronger axiom, which clearly implies the "marginal contributions" axiom: given two games (N, v) and (N, w) and a player $i \in N$, if $v(S) - v(S \setminus i) \geq w(S) - w(S \setminus i)$ for all S, then $\phi^i(N, v) \geq \phi^i(N, w)$.

[5] Note that a *linear* operator that satisfies "marginal contributions" and "null player" must be a linear combination of the marginal contributions, since it vanishes whenever they all vanish (this remark is due to Dov Monderer); however, here the value is *not* assumed to be linear (which implies "additivity").

it leads, however, to infeasible or inefficient allocations (since, in general, $\Sigma_{i \in N} [v(N) - v(N \setminus i)]$ differs from $v(N)$).

The following approach has therefore been suggested by Hart and Mas-Colell (1989): Associate to each game (N, v) just *one* real number $P(N, v)$, called the *potential* of the game, and compute marginal contributions according to these numbers. It is now required that these marginal contributions satisfy "efficiency," that is[6]

$$\sum_{i \in N} [P(N, v) - P(N \setminus i, v)] = v(N).$$

The main result is that there exists a unique[7] real function on games—the *potential function*—such that its marginal contributions are always efficient (i.e., the equation above is satisfied); moreover, the resulting payoff vector is precisely the Shapley value.

It is remarkable that just one requirement, namely the equation, suffices. Furthermore, it turns out that, for any given game, one needs to apply this equation only to this one game and its subgames in order to determine the potential—and thus, *a fortiori,* the Shapley value—uniquely (this is in contrast to the usual axiomatizations, where large classes of games have to be considered together).

The potential function approach has led to a number of new results. In particular, a formulation of "internal consistency" for the Shapley value has been obtained, which parallels similar developments for other solution concepts (core, kernel, nucleolus, bargaining solutions, etc.). Finally, it should be pointed out that the potential approach can be extended, for instance, to "weighted values" and to NTU games.

2. Finite Games with Nontransferable Utility

A value for NTU games usually satisfies the following requirements (these are *minimal* desiderata): For TU games, it coincides with the Shapley value; for two-person games, it coincides with the Nash bargaining solution; and, finally, it is "NTU invariant" (also referred to as "scale covariant"; it means that rescaling the payoffs, *independently* among the players, leads to the same rescaling of the values). The two most studied NTU values are due to Harsanyi (1963) and Shapley (1969).

2.1 A Debate

The papers of Roth (1980) and Shafer (1980) have exhibited some examples where the NTU values seem to behave somewhat unintuitively. The ensuing

[6]The (sub)game $(N \setminus i, v)$ is the restriction of the game (N, v) to $N \setminus i$, and $P(N \setminus i, v)$ is its potential.

[7]Up to an additive constant, which of course does not change the marginal contributions.

debate (see Aumann, 1985b, 1986; Hart, 1985b; Roth, 1986) led to a renewed interest in this subject.

I will not get into this controversy here (the "truth"—if there is such a thing—lies, as usual, somewhere in between; see also the paragraph one before last of Section 2.2). I will just point out that some of these papers deal with fundamental issues and should be of interest to the large community of researchers in game theory and its applications (an extensive bibliography on the theory and applications of the NTU value may be found in Aumann, 1985b).

2.2 Axiomatizations

Partly, perhaps, as an outcome of the above debate, the NTU solutions have been recently axiomatized (in the same way that the Shapley value and the Nash bargaining solution have each been shown to be uniquely determined by appropriate sets of plausible axioms).

This approach has been pioneered by Aumann (1985a), who provided an axiomatization for the Shapley NTU value (also known as the "λ-transfer value"). Let ϕ stand now for a set-valued function that associates to an NTU game (N, V) a *set* of payoff vectors in $\mathbb{R}^N$. Fix the set of players N, and consider the following axioms (required for all games in an appropriate class of games Γ):

Axiom 1. *Nonemptiness (i.e., $\phi(V) \neq \emptyset$);*

Axiom 2. *Efficiency (i.e., $\phi(V)$ contains only Pareto-efficient vectors);*

Axiom 3. *Scale covariance (i.e., $\phi(\lambda V) = \lambda\phi(V)$ for all strictly positive vectors $\lambda \in \mathbb{R}^N$);*

Axiom 4. *Conditional additivity (i.e., $\phi(V + W)$ contains all elements of $\phi(V) + \phi(W)$ that are Pareto-efficient for $V + W$);*

Axiom 5. *Independence of irrelevant alternatives (i.e., if $V \subset W$, then $\phi(V)$ contains all elements of $\phi(W)$ that are feasible in V);*

Axiom 6. *Unanimity games (i.e., if V corresponds to a TU unanimity game v, then $\phi(V)$ consists of a unique element: the Shapley value of v).*

Aumann's result can now be stated:

Theorem 1. *ϕ satisfies Axioms 1 to 6 on Γ if and only if $\phi(V)$ is the set of Shapley NTU values of the game V for every $V \in \Gamma$.*

Following this, Hart (1985a) has provided an axiomatization for the Harsanyi NTU value:

THEOREM 2. *ϕ satisfies Axioms 1 to 6 on Γ if and only if $\phi(V)$ is the set of Harsanyi NTU solutions of the game V for every $V \in \Gamma$.*

The two theorems are indeed formally identical. There are two differences: A minor one is that the set Γ of games changes. A major one is that, in Theorem 1, one deals with payoff *vectors,* and in Theorem 2, with payoff *configurations.* A "payoff configuration" $\mathbf{x}$ is a list of payoff vectors for all coalitions; i.e., $\mathbf{x} = (x_S)_{S \subset N}$ with $x_S = (x_S^i)_{i \in S} \in \mathbb{R}^S$ for all $S \subset N$. Thus, $\phi(V)$ is a set of payoff vectors (i.e., elements of $\mathbb{R}^N$) in Theorem 1, and it is a set of payoff configurations in Theorem 2 (the axioms have to be interpreted accordingly in each case).

On one hand, it is quite remarkable that essentially identical requirements characterize the two NTU values. On the other hand, the main difference between the two axiomatizations suggests that the grand coalition N plays a more prominent role in the Shapley NTU value than in the Harsanyi one, where intermediate coalitions are taken into account no less "seriously." This further leads to the conjecture that the Shapley NTU value may be better suited for large games (see the discussion at the end of Section 5 in Hart, 1985a).

In a parallel development, Kalai and Samet (1985) have studied and axiomatized the class of *egalitarian solutions.* These solutions are not NTU invariant: only when the players' payoffs are *equally* (rather than independently) rescaled they change accordingly. One may thus call these *CU solutions* (for "comparable utilities") since they apply when, for instance, the ratios between the individual utility scales are given.

3. Large Games with Transferable Utility

It is now assumed that the number of players is infinite, and some (if not all) are individually insignificant; such a game is called a *large game.* For example, the generalization of a weighted majority game is a *scalar measure game* $v = f \circ \mu$, where μ is a probability measure (the "fraction of the vote") and f is a real function defined on the closed interval $[0, 1]$. Let N denote the support of μ; a weighted majority game results if the set N is finite and f is the indicator function of an interval $[q, 1]$; a large game corresponds to an infinite N.

There are two main approaches to the study of value in games with infinitely many players: the *asymptotic* approach, where the given game is approximated by sequences of finite games, and the value is defined as the limit of the corresponding Shapley values, and the *axiomatic* approach, where the value is determined by a set of axioms, similar to those used for finite games.

3.1 The Asymptotic Approach

A game v with infinitely many players has an *asymptotic value* if, for any approximating sequence of finite games, the limit of their Shapley values exists

TABLE I

Asymptotic Value of Scalar Measure Games $f \circ \mu$

	f	
Support of μ	"Nice"	General
Countable	Artstein (1972)	Berbee (1981)[a]
Nonatomic continuum	Aumann and Shapley (1974); Kannai (1966)	Neyman (1981)[b]
Finitely many atoms and a nonatomic continuum	Fogelman and Quinzii (1980)	Neyman (1979)[b]
General[c]	—[d]	Neyman (1988)

[a]Berbee does not actually prove the existence of the asymptotic value in this case; however, his paper solves the main difficulty here (see Neyman, 1986).

[b]Neyman (1981) actually predates Neyman (1979).

[c]That is, countably many atoms and a nonatomic continuum.

[d]The result of Fogelman and Quinzii (1980) may be extended to this case as well as by appropriate approximations, which is not so in the general ("non-nice") case (this remark is due to Abraham Neyman).

and is moreover independent of the particular sequence chosen. The study of the asymptotic value of scalar measure games, in particular, has required very deep mathematical advances (some of these problems were open for a long time), mainly in probability theory (cf. the work of Neyman and of Berbee). The final result of Neyman (1988) (that includes all previous ones) is that the game $v = f \circ \mu$ has an asymptotic value for any function f in bv' (the space of functions of bounded variation that vanish at 0 and are continuous at 0 and 1) and any probability measure μ.

Table I summarizes the development of these results. It is organized according to the support of the measure μ; moreover, the case where the function f is "nice" (for instance, differentiable or absolutely continuous) is shown separately (it corresponds to the spaces of games pNA, pFL, etc.).

3.2 The Axiomatic Approach

We consider now nonatomic games, which are games with infinitely many players that are individually insignificant. Aumann and Shapley (1974) define a *value* ϕ as an operator on a space Q of nonatomic games that satisfies a number of axioms. The *value existence problem* consists of finding spaces Q, as large as possible, on which a value exists.[8]

A most important contribution to this problem has been made by Mertens (1988). The space Q he has obtained contains, in particular, all scalar measure

[8]The restriction of a value on a subspace is a value on the subspace as well.

games $v = f \circ \mu$ where μ is a nonatomic probability measure and $f \in bv'$ (see Section 3.1 above), all sums and products of such games, all games of the form $v = \min\{\mu_1, \ldots, \mu_n\}$ where the μ_i's are nonatomic probability measures and n is an arbitrary integer (these include the so-called "n-glove markets"), and others.

To explain the key insight that led to this result, one must first recall the well known "diagonal formula," which says that the value of a player is the average of his marginal contributions to coalitions that are perfect samples of the grand coalition (these coalitions are referred to as "diagonal coalitions"). This formula—which is just the "Law of Large Numbers" in this framework—works well for "differentiable" games, where the marginal contribution is the corresponding (directional) derivative.

What happens however when the game is not "differentiable" along the "diagonal"? The idea is then to average the marginal contributions as one moves in a *neighborhood* of the diagonal (rather than *on* the diagonal itself). For this, an appropriate probability measure P needs to be defined on coalitions (actually, "ideal coalitions"). A first approach, using normal distributions (by the "Central Limit Theorem") may however be shown to fail to generate a value.

Mertens observed that the "symmetry" axiom of the value indicates that P should be invariant under all automorphisms of the underlying space of players. Therefore, if S is a "random coalition," chosen according to P, and μ_0, μ_1 are any two nonatomic probability measures, then $\mu_0(S)$ and $\mu_1(S)$ have the same distribution.[9] If, in addition, μ_0 and μ_1 are mutually singular, then $\mu_0(S)$ and $\mu_1(S)$ are independent; moreover, $\mu_\alpha(S) = (1 - \alpha)\mu_0(S) + \alpha\mu_1(S)$ are identically distributed for all $0 \leq \alpha \leq 1$ (since each μ_α is a nonatomic probability measure). But this is just strict stability of index 1, which characterizes the Cauchy distribution. Mertens shows that the probability P is well defined (by these Cauchy "marginals"), and moreover unique, and that a value is generated (it should be noted that carrying out this program required very intricate and complex arguments).

3.3 Measure-based Values

The symmetry axiom requires the value to be independent of the "names of the players"; i.e., to be covariant with all "permutations" of the players. When the number of players is infinite, the space of players is endowed with a σ-field (the "coalitions"), and the "permutations" become "automorphisms": one-to-one mappings of the space onto itself that are measurable in both directions.

There are however many models where only an appropriate subgroup of auto-

[9] The space of players is assumed to be "standard," i.e. isomorphic to the unit interval [0, 1] with the Borel σ-field. This implies (see Aumann and Shapley, 1974, Lemma 6.2) that for any two nonatomic probability measures there exists an automorphism that maps one to the other.

morphisms should be considered. For instance, when an underlying probability measure μ is given (the "population measure"[10]), the value is required to be covariant only with *μ-preserving* automorphisms. This axiom is called *μ-symmetry,* and an operator satisfying the standard value axioms, with "symmetry" replaced by "μ-symmetry," is a *μ-value.* Since any value is also a μ-value, the problem now is usually one of *uniqueness:* Given a space of games Q, on which a unique value ϕ exists, is ϕ also the unique μ-value there?[11]

Even on the simplest such space, pNA(μ), which is generated by the scalar measure games of the form $v = f \circ \nu$, where f is "nice" (i.e., differentiable or absolutely continuous) and ν is a probability measure (absolutely continuous with respect to μ[12]), the uniqueness of the μ-value turns out to be a quite nontrivial matter.[13] This result has been recently proved by Monderer (1986).

3.4 Equivalence to the Core

The well known "Value Equivalence Principle" states that, in a perfectly competitive economy, all value allocations are competitive (and, moreover, the two sets coincide if there is sufficient differentiability); see Aumann (1978, Section 6) for some discussion and references. In game theoretic terms, this means that the values of large market games lie (approximately) in the core (since the core is just the set of competitive allocations).

Recently, Wooders and Zame (1987a) have proved a new quite general result of this type. They consider games satisfying only two conditions:

(1) superadditivity; and
(2) uniformly bounded individual marginal contributions (i.e., there exists a constant M such that $v(S) - v(S \setminus i) \leq M$ for all S and i).

Their result is essentially the following:[14] Given $\varepsilon > 0$, if each player has enough "substitutes,"[15] then the value belongs to the (weak) ε-core.[16]

[10] The interpretation being that $\mu(S)$ is the fraction of the total population that is contained in the coalition S.

[11] Two technical points: First, all the games in Q are absolutely continuous with respect to μ; and second, an additional requirement is needed to rule out the always existent "degenerate μ-value," which assigns to each game v an appropriate multiple of μ.

[12] μ is a fixed nonatomic probability measure.

[13] As a general rule, value uniqueness results are much harder than existence ones; there are indeed relatively few spaces on which the value has been shown to be unique.

[14] For simplicity of exposition, the result is presented here in terms of games rather than "attributes" and "pregames" as in the original paper.

[15] I.e., players of the same type; actually, "near substitutes" suffice.

[16] The "weak ε-core" is the set of all efficient payoff vectors x satisfying

$$\Sigma_{i \in S}\, x^i \geq v(S) - \varepsilon|S|,$$

for all coalitions S, where $|S|$ denotes the number of elements of S.

It may seem surprising at first glance that such a result would hold without assuming the essential property of market games, namely, "positive homogeneity of degree 1" (which means, for instance, that $v(T) = kv(S)$ whenever T is a "k-replica" of S, i.e., $|T| = k|S|$ and for each player i in S there are exactly k players identical to i in T). However, the two basic assumptions (1) and (2) above do imply a sort of homogeneity in the limit (note that the proof of Wooders and Zame goes along different lines). Indeed, for a simple example, let f be a nonnegative function defined on the positive integers that satisfies, for all n, m:

(1) $f(n) + f(m) \leq f(n + m)$; and
(2) there exists M such that $f(n + 1) - f(n) \leq M$.

Then it can be easily shown that $f(n)/n$ converges to a finite limit. Clearly, this argument (that corresponds to the case where there is only one type of player) is readily generalized to functions of several variables.

4. Large Games with Nontransferable Utility

There has been little research done here. A possible reason may be that the difficult problems in the TU case need to be settled first (the NTU values being usually defined via some auxiliary TU games). Also, most of the results for large NTU games are usually in applications, especially in economics (e.g., the "Value Equivalence" theorems, etc.).

Two exceptions should be noted: First, the work of Wooders and Zame (1987b), which extends their results in the TU case (see Section 3.4 above) to the NTU case, using the Shapley NTU value.

Second, an approach to the "Value Equivalence Principle" for the Harsanyi NTU value, due to Imai (1983) (there is however a drawback, since the "diagonal principle" is assumed rather than proved; without this, the problem appears to be quite difficult).

Acknowledgments

I would like to thank Robert J. Aumann, Jean-François Mertens and Abraham Neyman for useful discussions on some of these topics. Partial financial support by the National Science Foundation (Grant SES-85-10123) and by the U.S.-Israel Binational Science Foundation (Grant 85-00342) is gratefully acknowledged.

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