HIPPOPOTOMONSTROSESQUIPEDALIOPHOBIA (THE FEAR OF LONG WORDS)

Definition. Given a word $w(x_1, ..., x_r)$ in a free group, and a group G, we define a word map $w_G : G^r \to G$, $(g_1, ..., g_r) \mapsto w(g_1, ..., g_r)$. Given $w_1 \in F_{r_1}$ and $w_2 \in F_{r_2}$ we write $w_1 * w_2 \in F_{r_1+r_2}$ for their *concatenation. We write* $w^{*t} = w * ... * w$. E.g. $w = [x, y]$. *Then* $w * w = [x, y] \cdot [z, w]$.

We would like to study the solution set $\{x, y \in G : [x, y] = g\}$, for $g \in G$.

Definition. Let G be a compact group, with Haar measure μ_G. Let $w \in F_r$. Then $\tau_{w,G} := (w_G)_* \mu_G^r$ is the *word measure* associated to w and G. If G is finite $\tau_{w,G}(g) = \frac{|w^{-1}(g)|}{|G|^r}$.

w induces a natural random walk on G. Suppose G finite.

1^{st} **step:** randomly choose $g_1, ..., g_r$ and move to $w(g_1, ..., g_r)$. Probability to reach $g \in G$ is $\tau_{w,G}(g)$.

2^{nd} **step:** randomly choose $h_1, ..., h_r$ and move to $w(g_1, ..., g_r) \cdot w(h_1, ..., h_r)$, and so on. Probability to reach $g \in G$ at the t-th step is

$$\tau^{*t}_{w,G}(g) = \tau_{w^{*t},G}(g) = \frac{|(w^{*t})^{-1}(g)|}{|G|^{rt}}.$$

Slogan: "word measures admit uniform behavior in high complexity".

Fix $w \in F_r$, and consider $w_G : G^r \to G$ for $G = \mathrm{SL}_2(\mathbb{F}_5)$ and $G = \mathrm{SL}_{10^9}(\mathbb{F}_{4001})$.

Theorem (Larsen-Shalev-Tiep). *For any $w \in F_r$, there exist $N(w), \epsilon(w) > 0$ such that:*

(1) ***LST '11**. $(w * w)_G : G^{2r} \to G$ is surjective on every finite simple group G, $|G| > N(w)$.*
(2) ***LS '12**. $|\tau_{w,G}(g)| < |G|^{-\epsilon(w)}$ $\forall g$ and for every finite simple group G, $|G| > N(w)$.*
(3) ***LST '19**. (Uniform L^∞-mixing time) $\exists t(w) > 0$, such that*

$$\lim_{Gfsg, |G| \to \infty} \left(\max_{g \in G} \left| \tau^{*t(w)}_{w,G}(g) \, |G| - 1 \right| \right) = 0.$$

Explicitly, $t(w) \sim \ell(w)^4$, $\epsilon(w) \sim \ell(w)^{-4}$.

Question. *Do we have similar phenomena for:*

- *compact p-adic groups, $\{\mathrm{SL}_n(\mathbb{Z}_p)\}_{n,p}$ (i.e the collection $\{\mathrm{SL}_n(\mathbb{Z}/p^k\mathbb{Z})\}_{n,p,k}$)*
- *Compact simple Lie groups, e.g. $\{\mathrm{SU}_n\}_n$.*

- (**Hui-Larsen-Shalev '13**) $w * w : \mathrm{SU}_n^{2r} \to \mathrm{SU}_n$ is surjective for $n \gg_w 1$.
- (**Avni-Gelander-Kassabov-Shalev '13**) w^{*3} is surjective on $\mathrm{SL}_n(\mathbb{Z}/p^k\mathbb{Z})$, for $p \gg 1$ and k.

Approach: word maps on $\mathrm{SL}_n(\mathbb{F}_p)$, $\mathrm{SL}_n(\mathbb{Z}_p)$, SU_n all come from one polynomial map $w_n : \mathrm{SL}_n^r \to \mathrm{SL}_n$. We will see that the singularities of w_n provide probabilistic information on $\mathrm{SL}_n(\mathbb{Z}_p)$ and SU_n, even when they are completely different groups.

0.1. **Singularities and pushforward measures.** $w_n : \mathrm{SL}_n^r(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$ is a special case of a polynomial map $f : X \to Y$ between smooth varieties. For simplicity, $X = \mathbb{C}^n$ and $Y = \mathbb{C}^m$.

A polynomial map $f : \mathbb{C}^n \to \mathbb{C}^m$ is **smooth** if it is a submersion (Df_x is onto $\forall x \in \mathbb{C}^n$).

Example. Let $f = x^2$. Then $f^{(1)} = 2xx'$, $f^{(2)} = 2(x')^2 + 2xx''$. Hence

$$J_2(f)(x, x', x'') = (x^2, 2xx', 2(x')^2 + 2xx'').$$

Definition. Given $f = (f_1, ..., f_m) : \mathbb{C}^n \to \mathbb{C}^m$, its k-th jet is the map $J_k(f) : \mathbb{C}^{n(k+1)} \to \mathbb{C}^{m(k+1)}$ given by $(f, f^{(1)}, ..., f^{(k)})$.

Definition. Let $f : \mathbb{C}^n \to \mathbb{C}^m$. $0 < \epsilon \le 1$.

(1) f is ϵ-flat if all its fibers are of dimension $\le n - \epsilon m$.
(2) f is ϵ-jet-flat, if $J_k(f)$ is ϵ-flat for all $k \in \mathbb{N}$.

Example. $f : \mathbb{C} \to \mathbb{C}$ $x \mapsto x^d$ is $\frac{1}{d}$-jet flat.

Theorem (**Clucker-G.-Hendel**)**.** *Let $f_1, ..., f_m \in \mathbb{Z}[x_1, ..., x_n]$, where $f_{p,k} : (\mathbb{Z}/p^k\mathbb{Z})^n \to (\mathbb{Z}/p^k\mathbb{Z})^m$ is the induced map, $\tau_{p,k}(y) := \frac{\left|f_{p,k}^{-1}(y)\right|}{p^{kn}}$. Then $f = (f_1, ..., f_m)$ is ϵ-jet flat iff $\exists C, M > 0$ s.t for every $k \in \mathbb{N}$, prime p, and $y \in (\mathbb{Z}/p^k\mathbb{Z})^m$:*

$$\tau_{p,k}(y) < C \cdot k^M p^{-\epsilon k m} \sim \left|(\mathbb{Z}/p^k\mathbb{Z})^m\right|^{-\epsilon}.$$

0.2. **Word maps on simple algebraic groups and Lie algebras.** We can now make sense of LST for word maps on geometric objects.

Definition. A **Lie algebra word** $w(X_1, ..., X_r)$ is a $\mathbb{Q}$-linear combination of iterated commutators. E.g $X + [X, Y]$ and $[[X, Y], Y]$. For each Lie algebra $\mathfrak{g}$, w induces a **Lie algebra word map** $w_{\mathfrak{g}} : \mathfrak{g}^r \to \mathfrak{g}$. If $w = [X, Y]$ then we set $w * w := [X, Y] + [Z, W]$.

Theorem 1 (G.-Hendel)**.** *Let w be a Lie algebra word of degree d. Then $\exists C > 0$, s.t for every simple Lie algebra $\mathfrak{g}$, where $0 \neq w_{\mathfrak{g}}$:*

(1) *$w_{\mathfrak{g}} : \mathfrak{g}^r \to \mathfrak{g}$ is $\frac{1}{Cd^3}$-jet-flat.*
(2) *The map $w_{\mathfrak{g}}^{*t}$ is 1-jet-flat for $t \ge Cd^4$.*

Theorem 2 (G.-Hendel)**.** *Let $w \in F_r$ of length ℓ. Then $w_n : \mathrm{SL}_n(\mathbb{C})^r \to \mathrm{SL}_n(\mathbb{C})$ is $\frac{1}{10^{17}\ell^9}$-jet-flat $\forall n$.*

0.3. **Probabilistic applications.**

Theorem 3 (G.-Hendel)**.** *Let $w \in F_r$. For every $n, k \in \mathbb{N}$, $p \gg_n 1$, and $A \in \mathrm{SL}_n(\mathbb{Z}/p^k\mathbb{Z})$:*

$$\left|\tau_{w,\mathrm{SL}_n(\mathbb{Z}/p^k\mathbb{Z})}(A)\right| < \left|\mathrm{SL}_n(\mathbb{Z}/p^k\mathbb{Z})\right|^{-\frac{1}{2\cdot 10^{17}\ell(w)^9}}.$$

Let $\mathrm{SL}_n^1(\mathbb{Z}_p)$ be the kernel of $\mathrm{SL}_n(\mathbb{Z}_p) \to \mathrm{SL}_n(\mathbb{F}_p)$ (first congruence subgroup).

Theorem 4 (**G-Hendel**)**.** *Let* $w \in F_r$*.* $\exists C > 0$*, s.t for every* $t \geq C\ell(w)^4$*,* $n \in \mathbb{N}$*,* $p \gg_n 1$*,* $\tau_{w,\mathrm{SL}_n^1(\mathbb{Z}_p)}^{*t}$ *has bounded density.*