INVARIANT SETS AND MEASURES OF NONEXPANSIVE GROUP AUTOMORPHISMS

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Abstract. We prove that the restriction of a probability measure invariant under a nonhyperbolic, ergodic and totally irreducible automorphism of a compact connected abelian group to the leaves of the central foliation is severely restricted. We also prove a topological analogue of this result: the intersection of every proper closed invariant subset with each central leaf is compact.

1. Introduction

A continuous automorphism α of an additive compact abelian group X is *expansive* if there exists a neighbourhood $N(0)$ of the identity $0 \in X$ with $\bigcap_{n\in\mathbb{Z}} \alpha^n(N(0)) = \{0\}$, *irreducible* if every closed α-invariant subgroup $Y \subsetneq X$ is finite, *totally irreducible* if every nonzero power of α is irreducible, and *ergodic* if it is topologically transitive (and hence ergodic with respect to the normalized Haar measure λ_X of X).

In this paper we study the collection of invariant measures of a nonhyperbolic, ergodic and totally irreducible automorphism of the n-torus $\mathbb{T}^n$ or, more generally, of a nonexpansive, ergodic and totally irreducible automorphism α of a compact connected abelian group X. Every nonhyperbolic, ergodic and irreducible automorphism α of $\mathbb{T}^n$ is partially hyperbolic in the usual sense[1] with the additional property that its derivative $D\alpha$ preserves the length of vectors in E_c. For an arbitrary nonexpansive, ergodic and totally irreducible continuous automorphisms α of a compact connected abelian group X these 'sub-bundles' can be more complicated objects (due to the fact that the group need not be locally connected), but an analogue of this strong form of partial hyperbolicity also holds in this more general situation.

2000 *Mathematics Subject Classification.* 37A05, 37A45, 37B05, 37C40.

Key words and phrases. Nonexpansive group automorphisms, nonhyperbolic toral automorphisms, invariant measures.

[1] A C^2 diffeomorphism f of a Riemannian manifold M is *partially hyperbolic* if there is a Df-invariant splitting $TM = E_s \oplus E_c \oplus E_u$ of the tangent manifold TM of M in which at least two of the sub-bundles are nontrivial, so that Df uniformly expands all vectors in E_u, uniformly contracts all vectors in E_s, and the vectors in E_c are neither expanded as strongly as any vector in E_u nor contracted as strongly as any vector in E_s.

Let α be a nonexpansive, ergodic and totally irreducible automorphism of a compact connected abelian group X. The normalized Haar measure λ_X of X is obviously invariant under α, and Y. Katznelson [6] proved that the measure-preserving system $(X, \alpha, \mathcal{B}_X, \lambda_X)$ (where $\mathcal{B}_X$ denotes the Borel sigma-algebra of X) is measure-theoretically isomorphic to a Bernoulli shift.

There is another family of — admittedly not very interesting — α-invariant ergodic probability measures on X: let $X^{(0)} \subset X$ be the dense central subgroup of α defined in (3.3), on which α acts isometrically. Then the closure of the α-orbit of any element $x \in X^{(0)}$ is a compact α-invariant subset of $X^{(0)}$ (and hence of X) on which α acts with a unique α-invariant measure denoted by $\tilde{\lambda}_x$.

It is not immediate how to construct other invariant measures; in fact, the main result in this paper shows that *all* α-invariant probability measures $\mu \neq \lambda_X$ on X satisfy a somewhat surprising rigidity phenomenon related to the scarcity of invariant measures under a multidimensional abelian semigroup of toral endomorphisms. This scarcity of invariant measures was conjectured by H. Furstenberg and is still open, though there are important partial results by several authors including D. Rudolph [9] for the one-dimensional case and A. Katok and R. Spatzier [5] in the higher-dimensional case.

In order to describe this rigidity property we use a construction from [5] to define a system of 'conditional' measures on the leaves of the central foliation induced by an α-invariant measure μ on X. In general, if we start with an α-invariant probability measure μ on X, these leaf measures will only be sigma-finite. Indeed, for $\mu = \lambda_X$, the induced measure on each central leaf is the (infinite) Haar measure on the leaf. Our main result is that the leaf measures are finite for any α-invariant probability measure μ on X which does not contain a copy of λ_X in its ergodic decomposition.

Theorem (Theorem 5.1). *Let α be a nonexpansive, ergodic and totally irreducible automorphism of a compact connected abelian group X with normalized Haar measure λ_X, and let μ be an α-invariant probability measure on X which is singular with respect to λ_X. Then the conditional measure ρ_x on the central leaf through x (defined in (4.20)) is finite for almost every $x \in X$.*

Both the statement and the proof of Theorem 5.1 are modelled on Host's proof of Rudolph's Theorem in [3] and its generalization in [4].

The following two definitions can easily be adapted to the general setting of partially hyperbolic maps.

Definition 1.1. Two α-invariant probability measures μ_1, μ_2 on X are *centrally equivalent* if they have an invariant joining ν (i.e. an $(\alpha \times \alpha)$-invariant measure ν on $X \times X$ which projects to μ_1 and μ_2, respectively) so that, for

ν-a.e. $(x, y) \in X \times X$, x and y lie on the same central leaf; in other words,

$$x - y \in X^{(0)} \text{ for } \nu\text{-a.e. } (x, y) \in X \times X,$$

where $X^{(0)} \subset X$ is the central subgroup of α defined in (3.3).

Definition 1.2. An α invariant probability measure μ on X is *virtually hyperbolic* if there exists an α-invariant Borel set $Z \subset X$ with $\mu(Z) = 1$ which intersects every central leaf in at most one point, i.e. with $Z \cap (x+Z) = \varnothing$ for every $x \in X^{(0)}$.

In Section 6 we prove that Theorem 5.1 implies the following result.

Theorem 1.3. *Let α be a nonexpansive, ergodic and totally irreducible automorphism of a compact connected abelian group X with normalized Haar measure λ_X, and let μ be an α-invariant probability measure on X which is singular with respect to λ_X. Then the following conditions are satisfied.*

(1) *There is a virtually hyperbolic α-invariant probability measure μ' on X which is centrally equivalent to μ;*
(2) *If μ is weakly mixing (or, more generally, if the point spectrum of the action of α on $L^2(X, \mathcal{S}, \mu)$ contains no eigenvalue of α of absolute value 1), then μ is virtually hyperbolic;*
(3) *If μ is ergodic, but not necessarily weakly mixing, we write, for every $x \in X^{(0)}$, $\tilde{\lambda}_x$ for the unique α-invariant probability measure on $X^{(0)}$ — and hence on X — concentrated on the compact orbit closure $\overline{\{\alpha^n x : n \in \mathbb{Z}\}}$ of x under α. Then μ is an ergodic component of $\mu' * \tilde{\lambda}_{x_0}$ for some $x_0 \in X^{(0)}$.*[2]

Finally, in Section 7 we prove the following topological analogue of the Theorems 1.3 and 5.1.

Theorem (Theorem 7.1). *Let α be a nonexpansive, ergodic and totally irreducible automorphism of a compact connected abelian group X. Then any closed α-invariant subset $Y \subsetneq X$ intersects every central leaf in a compact subset of the leaf.*

Acknowledgements. This research has been supported in part by NSF grant DMS 0140497 (E.L.) and FWF Project P16004-N05 (K.S.). During part of this work, both authors received support from the American Institute of Mathematics and NSF grant DMS 0222452. We would furthermore like to express our gratitude to the Mathematics Department, University of Washington, Seattle, the Newton Institute, Cambridge and ETH Zürich for

[2]More precisely: for every x_0, the ergodic decomposition of $\mu' * \tilde{\lambda}_{x_0}$ gives a probability measure ν on the space $M_1(X)$ of probability measures on X, where we consider $M_1(X)$ as a compact metric space using the usual weak* topology. Then for an appropriate choice of x_0, the measure μ is in the support of ν.

hospitality during parts of this work, and Doug Lind for some interesting and useful discussions.

2. Irreducible group automorphisms

Let α and β be continuous automorphisms of compact abelian groups X and Y, respectively. Then α and β are *conjugate* if there exists a continuous group isomorphism $\phi\colon X \longrightarrow Y$ with

$$\beta \circ \phi = \phi \circ \alpha, \tag{2.1}$$

and β is a *factor* of α if there exists a continuous surjective group homomorphism $\phi\colon X \longrightarrow Y$ satisfying (2.1). The map ϕ in (2.1) is called an (*algebraic*) *conjugacy* or an (*algebraic*) *factor map*. The automorphisms α and β are *weakly conjugate* if each of them is a factor of the other, and *finitely equivalent* if each of them is a factor of the other with a finite-to-one factor map.

We recall a few basic facts about irreducible ergodic automorphisms of compact abelian groups. Let $R_1 = \mathbb{Z}[u^{\pm 1}]$ be the ring of Laurent polynomials with integral coefficients. We write every $h \in R_1$ as

$$h = \sum_{m \in \mathbb{Z}} h_m u^m \tag{2.2}$$

with $h_m \in \mathbb{Z}$ for every $m \in \mathbb{Z}$ and $h_m = 0$ for all but finitely many m.

Let α be an automorphism (always assumed to be continuous) of a compact abelian group X with (additive) dual group $\hat{X}$, and let $\hat{\alpha}$ be the dual automorphism of $\hat{X}$ defined by

$$\langle \hat{\alpha} a, x \rangle = \langle a, \alpha x \rangle$$

for every $x \in X$ and $a \in \hat{X}$, where $\langle a, x \rangle$ denotes the value of $a \in \hat{X}$ at $x \in X$. For every $h = \sum_{n \in \mathbb{Z}} h_n u^n \in R_1$, $x \in X$ and $a \in \hat{X}$ we set

$$h(\alpha)(x) = \sum_{n \in \mathbb{Z}} h_n \alpha^n x, \qquad h(\hat{\alpha})(a) = \sum_{n \in \mathbb{Z}} h_n \hat{\alpha}^n a, \tag{2.3}$$

and note that

$$\langle h(\hat{\alpha})(a), x \rangle = \langle \widehat{h(\alpha)}(a), x \rangle = \langle a, h(\alpha)(x) \rangle. \tag{2.4}$$

The dual group $\hat{X}$ is a module over the ring R_1 with operation

$$h \cdot a = h(\hat{\alpha})(a) \tag{2.5}$$

for $h \in R_1$ and $a \in \hat{X}$. In particular,

$$u^m \cdot a = \hat{\alpha}^m a \tag{2.6}$$

for $m \in \mathbb{Z}$ and $a \in \hat{X}$. This module is called the *dual module* $M = \hat{X}$ of α. Conversely, if M is an R_1-module, we obtain an automorphism α_M on the compact abelian group

$$X_M = \widehat{M} \tag{2.7}$$

whose dual automorphism is defined by

$$\hat{\alpha}_M a = u \cdot a \tag{2.8}$$

for every $a \in M$.

Examples 2.1 ([11])**.** (1) Let $M = R_1$. Since R_1 is isomorphic to the direct sum $\sum_{\mathbb{Z}} \mathbb{Z}$ of copies of $\mathbb{Z}$, indexed by $\mathbb{Z}$, the dual group $X = \widehat{R_1}$ is isomorphic to the cartesian product $\mathbb{T}^{\mathbb{Z}}$ of copies of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We write a typical element $x \in \mathbb{T}^{\mathbb{Z}}$ as $x = (x_n)$ with $x_n \in \mathbb{T}$ for every $n \in \mathbb{Z}$ and choose the following identification of $X_{R_1} = \widehat{R_1}$ and $\mathbb{T}^{\mathbb{Z}}$: for every $x = (x_n)$ in $\mathbb{T}^{\mathbb{Z}}$ and $h = \sum_{n\in\mathbb{Z}} h_n u^n \in R_1$,

$$\langle x, h \rangle = e^{2\pi i \sum_{n\in\mathbb{Z}} h_n x_n}. \tag{2.9}$$

Under this identification the automorphism α_{R_1} on $X_{R_1} = \mathbb{T}^{\mathbb{Z}}$ becomes the shift

$$(\tau x)_m = x_{m+1} \tag{2.10}$$

with $m \in \mathbb{Z}$ and $x = (x_m) \in X_{R_1} = \mathbb{T}^{\mathbb{Z}}$.

(2) Let $I \subset R_1$ be an ideal, and let $M = R_1/I$. Since M is a quotient of the additive group R_1 by an $\hat{\alpha}_{R_1}$-invariant subgroup, the dual group X_M is the α_{R_1}-invariant subgroup

$$\begin{aligned} X_{R_1/I} = I^{\perp} &= \{x \in X_{R_1} = \mathbb{T}^{\mathbb{Z}} : \langle x, h \rangle = 1 \text{ for every } h \in I\} \\ &= \Big\{x \in \mathbb{T}^{\mathbb{Z}} : \sum_{n\in\mathbb{Z}} h_n x_{m+n} = 0 \pmod 1 \\ &\qquad\qquad \text{for every } h \in I \text{ and } m \in \mathbb{Z}\Big\} \\ &= \{x \in \mathbb{T}^{\mathbb{Z}} : h(\tau)(x) = 0 \text{ for every } h \in I\}, \end{aligned} \tag{2.11}$$

and $\alpha_{R_1/I}$ is the restriction of $\tau = \alpha_{R_1}$ to $X_{R_1/I} \subset \mathbb{T}^{\mathbb{Z}} = X_{R_1}$.

We can express (2.11) as

$$X_{R_1/I} = \widehat{X/I} = I^{\perp} = \bigcap_{h\in I} \ker(h(\tau)). \tag{2.12}$$

If $I = (f) = fR_1$ is the principal ideal generated by some $f \in R_1$, then (2.12) becomes

$$X_{R_1/(f)} = \widehat{X/(f)} = (f)^{\perp} = \ker(f(\tau)). \tag{2.13}$$

(3) Let α be the automorphism of the m-torus $X = \mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ defined by a matrix $A \in \mathrm{GL}(m, \mathbb{Z})$. Then the dual module $M = \hat{X}$ is equal to $\mathbb{Z}^m$ with operation $f \cdot \mathbf{m} = f(A^{\top})(\mathbf{m})$ for every $f \in R_1$ and $\mathbf{m} \in \mathbb{Z}^m$ (cf. (2.5)), where $A^{\top} \in \mathrm{GL}(m, \mathbb{Z})$ is the transpose matrix of A.

The automorphism α is irreducible if and only if the characteristic polynomial $f = f_0 + \dots + f_{m-1}u^{m-1} + u^m$ of A is irreducible, and α is conjugate to $\alpha_{R_1/(f)}$ if and only if A is conjugate in $\mathrm{GL}(m, \mathbb{Z})$ to the companion matrix

$$C_f = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -f_0 & -f_1 & \cdots & -f_{m-2} & -f_{m-1} \end{pmatrix} \in \mathrm{GL}(m, \mathbb{Z}). \tag{2.14}$$

Theorem 2.2. *Let α be an irreducible automorphism of an infinite compact connected abelian group X. Then there exists a unique irreducible polynomial $f = f_0 + \dots + f_n u^n \in R_1$ with the following properties.*

(1) *$n \geq 1$, $f_n > 0$ and $f_0 \neq 0$;*
(2) *α is finitely equivalent to $\alpha_{R_1/(f)}$, where $(f) = fR_1 \subset R_1$ is the ideal generated by f (cf. Example 2.1 (2));*
(3) *α is ergodic if and only if f is not cyclotomic (i.e. f does not divide $u^m - 1$ for any $m \geq 1$);*
(4) *α is expansive if and only if f has no roots of absolute value 1.*
(5) *α is totally irreducible if and only if f has no two distinct roots whose ratio is a root of unity.*

Conversely, if $f = f_0 + \dots + f_n u^n \in R_1$ is an irreducible polynomial satisfying condition (1) above, then the group $X_{R_1/(f)}$ in (2.11) is connected and the automorphism $\alpha_{R_1/(f)}$ of $X_{R_1/(f)}$ is irreducible.

Proof. The statements (1)–(4) and the converse follow from [11, Proposition 2.7 and Theorem 29.2].

For the proof of (5) we note that α is totally irreducible if and only if $\alpha_{R_1/(f)}$ is totally irreducible. Since $\alpha_{R_1/(f)}^m$ is dual to multiplication by u^m on $\hat{X} = R_1/(f) = M$, $\alpha_{R_1/(f)}^m$ is irreducible if and only if the subgroup

$$N = \{h(u^m) : h \in R_1\}/(f) \subset R_1/(f) = M$$

has finite index in $R_1/(f)$. As the group M is torsion-free, the latter condition is equivalent to the statement that $N \otimes_{\mathbb{Z}} \mathbb{Q} = M \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^n$, and hence to the condition that the elements $\{(1+(f)), (u^m+(f)), \dots, (u^{m(n-1)}+(f))\}$ in M are rationally independent. In other words, $\alpha_{R_1/(f)}^m$ is reducible if and only if one can find a nonzero element $(k_0, \dots, k_{n-1}) \in \mathbb{Z}^n$ with

$$g(u^m) = k_0 + k_1 u^m + \dots + k_{n-1} u^{m(n-1)} \in (f), \tag{2.15}$$

where we may assume without loss of generality that the resulting polynomial $g \in R_1$ is irreducible. By evaluating (2.15) on any root θ of f we obtain that $g(\theta^m) = 0$ for every root θ of f, and Galois theory shows that the degree of g is equal to the number of distinct elements in the set $\Omega_f^{(m)} = \{\theta^m : \theta \text{ is a root of } f\}$. This proves that $\alpha_{R_1/(f)}^m$ is irreducible if and only if the cardinality of $\Omega_f^{(m)}$ is equal to n, which implies (5). □

Example 2.1 (2) gives an explicit representation — up to finite equivalence — of every irreducible automorphism of a compact connected abelian group X. For an alternative description we follow [2] (for background see [10], [11, Section 7] and [12]).

Let α be an irreducible automorphism of an infinite compact connected abelian group X, and let $f \in R_1$ be the irreducible polynomial appearing in Theorem 2.2. We fix a root $\theta \in \bar{\mathbb{Q}}$ of f, denote by $K = \mathbb{Q}(\theta)$ the algebraic number field generated by θ, and write $P^{(K)}$, $P_f^{(K)}$, and $P_\infty^{(K)}$, for the sets of places (= equivalence classes of valuations), finite places and infinite places of K. For every place v of K and every valuation $\phi \in v$, the *v-adic completion* K_v of K (i.e. the completion of K with respect to metric $\delta(a,b) = \phi(a-b)^\gamma$ for some suitable $\gamma > 0$) is a locally compact, metrizable field and hence a locally compact additive group. We fix a Haar measure λ_v on the additive group K_v and denote by $\operatorname{mod}_{K_v} \colon K_v \longrightarrow \mathbb{R}$ the map satisfying

$$\lambda_v(aB) = \operatorname{mod}_{K_v}(a)\lambda_v(B) \tag{2.16}$$

for every $a \in K_v$ and every Borel set $B \subset K_v$. The restriction of mod_{K_v} to K is a valuation in v, denoted by $|\cdot|_v$.

Let

$$P = \{v \in P_f^{(K)} : |\theta|_v \neq 1\}, \quad S = P_\infty^{(K)} \cup P. \tag{2.17}$$

For every infinite place $v \in P_\infty^{(K)}$, the v-adic completion K_v is either equal to $\mathbb{R}$ or to $\mathbb{C}$ (in particular, $K_v = \mathbb{C}$ for any $v \in S^{(0)}$). We write

$$\iota_v \colon K \longrightarrow K_v (= \mathbb{R} \text{ or } \mathbb{C}) \tag{2.18}$$

for the embedding of K in its completion K_v and use the same symbol ι_v to denote the corresponding identification of K_v with $\mathbb{R}$ or $\mathbb{C}$.

The set

$$W = \prod_{v \in S} K_v \tag{2.19}$$

is a locally compact algebra over K with respect to coordinate-wise addition, multiplication and scalar multiplication (with scalars in K). We write every $w \in W$ as $w = (w_v) = (w_v,\, v \in S)$ with $w_v \in K_v$ for every $v \in S$ and define

$$\|w\| = \max_{v \in S} |w_v|_v. \tag{2.20}$$

Let $\bar{\beta}$ be the automorphism of W given by

$$\bar{\beta} w = (\theta w_v) \tag{2.21}$$

for every $w = (w_v) \in W$.

We put

$$\mathcal{R} = \{a \in K : |a|_v \le 1 \text{ for every } v \in P^{(K)} \smallsetminus S\} \supset \mathfrak{o}_K, \tag{2.22}$$

where $\mathfrak{o}_K$ is the ring of integers in K, and denote by

$$\iota \colon K \longrightarrow W \tag{2.23}$$

the diagonal embedding $a \mapsto \iota(a) = (a, \dots, a)$, $a \in K$. By abuse of notation we identify each K_v, $v \in S$, with the subgroup

$$\{w \in W : w_{v'} = 0 \text{ for every } v' \neq v\} \subset W.$$

Theorem 2.3. *Suppose that α is an automorphism of an infinite compact connected abelian group X. Then α is irreducible if and only if there exist an element $\theta \in \bar{\mathbb{Q}}^\times = \bar{\mathbb{Q}} \smallsetminus \{0\}$ and a finitely generated $\mathbb{Z}[\theta^{\pm 1}]$-submodule $L \subset K = \mathbb{Q}(\theta)$ such that α is algebraically conjugate to the automorphism $\beta_{(\theta,L)}$ on the quotient group*

$$Y_L = W/\iota(L) \tag{2.24}$$

induced by $\bar{\beta}$ (cf. (2.17)–(2.21)).

(1) *The following conditions are equivalent.*
   (a) *α is ergodic,*
   (b) *θ is not a root of unity.*

(2) *The following conditions are equivalent.*
   (a) *α is expansive,*
   (b) *The orbit of θ under the action of the Galois group* $\mathrm{Gal}[\bar{\mathbb{Q}} : \mathbb{Q}]$ *does not intersect $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$.*

(3) *The following conditions are equivalent.*
   (a) *$X \cong \mathbb{T}^n$ for some $n \ge 1$,*
   (b) *$S = P^{(K)}_\infty$,*
   (c) *θ is an algebraic unit.*

(4) *The following conditions are equivalent.*
   (a) *α is totally irreducible,*
   (b) *The orbit of θ under the action of the Galois group* $\mathrm{Gal}[\bar{\mathbb{Q}} : \mathbb{Q}]$ *does not contain two distinct elements whose ratio is a root of unity.*

Proof. [2, Corollary 3.5], [11, Theorem 7.1 and Propositions 7.2–7.3] and Theorem 2.2 in this paper. □

Corollary 2.4. *In the notations of Theorem 2.3, for any $\theta \in \bar{\mathbb{Q}}^\times$ and finitely generated $\mathbb{Z}[\theta^{\pm 1}]$-submodule $L \subset K = \mathbb{Q}(\theta)$, and any $S' \subsetneq S$ there is an automorphism ψ of Y_L commuting with $\beta_{(\theta,L)}$ so that ψ expands the subgroup*

$$W_{S'} = \left\{w \in W : w_v = 0 \textit{ for every } v \in S \setminus S'\right\}.$$

Proof. Let $\mathcal{R} \supset \mathfrak{o}_K$ be the ring of S-integers as in (2.22). We claim that there is a $\tau \in \mathcal{R}$ so that

$$|\tau|_v > 1 \qquad \text{for all } v \in S'. \tag{2.25}$$

Indeed, if θ a unit in $\mathfrak{o}_K$, or more generally if $S' \subsetneq P_\infty^{(K)}$, (2.25) is a direct consequence of the Dirichlet unit theorem, and the general case follows from the extension of this theorem to the S-arithmetic context that can be proved in the same way [12, Chap. 4, Th. 9].

Since L is a finitely generated $\mathbb{Z}[\theta^{\pm 1}]$-submodule, and θ generates K it follows that L is commensurable to $\mathcal{R}$, i.e. $[L : L \cap \mathcal{R}], [\mathcal{R} : L \cap \mathcal{R}] < \infty$. Define $\bar{\tau} : W \to W$ using τ as in (2.21); since $\tau \in \mathcal{R}$ the map $\bar{\tau}$ preserves $\mathcal{R}$, and so a suitable power of $\bar{\tau}$, which without loss of generality we can assume to be $\bar{\tau}$ itself, preserves L.

It follows that $\psi = \beta_{(\tau,L)}$ is an automorphism of Y_L commuting with $\beta_{(\theta,L)}$ with the required properties. □

3. Structure and examples of nonexpansive automorphisms

Let α be a nonexpansive irreducible ergodic automorphism of a compact connected abelian group X. We apply the Theorem 2.3 and assume that

$$\alpha = \beta_{(\theta,L)}, \qquad X = W/\iota(L), \tag{3.1}$$

for some $\theta \in \bar{\mathbb{Q}}^\times$ and some finitely generated $\mathbb{Z}[\theta^{\pm 1}]$-submodule $L \subset K = \mathbb{Q}(\theta)$. Denote by λ_X the normalized Haar measure of X and write

$$\pi\colon W \longrightarrow X = W/\iota(L) \tag{3.2}$$

for the quotient map (cf. (2.17)–(2.24)). In the notation of (2.17) and (2.19) we set

$$\begin{aligned} S^{(0)} &= \{v \in S : |\theta|_v = 1\} \subset P_\infty^{(K)}, \\ W^{(0)} &= \{w = (w_v) \in W : w_v = 0 \text{ for every } v \in S \smallsetminus S^{(0)}\} \\ &\cong \prod_{v \in S^{(0)}} K_v \cong \mathbb{C}^{|S^{(0)}|}, \\ X^{(0)} &= \pi(W^{(0)}). \end{aligned} \tag{3.3}$$

The *central subgroup* group $X^{(0)} \subset X$ is α-invariant and dense by irreducibility. Furthermore, since $|L/\iota(\mathcal{R})| < \infty$ (cf. (2.22) and [2]) and $\iota(\mathcal{R}) \cap W^{(0)} = \{0\}$ by the product formula ([1, Theorem 10.2.1]), $L \cap W^{(0)} = \{0\}$.

Examples 3.1. (1) Let α be a nonexpansive irreducible ergodic automorphism of $X = \mathbb{T}^m$ defined by a matrix $A \in \mathrm{GL}(m, \mathbb{Z})$ with real eigenvalues $\theta_1, \dots, \theta_{m_1}$ and complex eigenvalues $\theta_{m_1+1}, \bar{\theta}_{m_1+1}, \dots, \theta_{m_1+m_2}, \bar{\theta}_{m_1+m_2}$, where $m = m_1 + 2m_2$, and where $\bar{\theta}_i$ is the complex conjugate of θ_i for

$i = m_1 + 1, \dots, m_1 + m_2$. We fix an eigenvalue θ of A, set $K = \mathbb{Q}(\theta)$, and obtain that $S = P_\infty^{(K)}$, $W \cong \mathbb{R}^{m_1} \times \mathbb{C}^{m_2}$, and that

$$W^{(0)} = \bigoplus_{\substack{j=m_1+1,\dots,m_1+m_2\\ |\theta_j|=1}} \mathbb{C}$$

is the subspace of $W \cong \mathbb{R}^m$ on which A acts isometrically. Since α is ergodic, $\dim_{\mathbb{R}}(W^{(0)}) \leq \dim_{\mathbb{R}}(W) - 2 = m - 2$.

Take, for example, the irreducible ergodic and nonexpansive automorphism α of $X = \mathbb{T}^4$ determined by the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \in \mathrm{GL}(4, \mathbb{Z}).$$

If $\theta > 1$ is the dominant eigenvalue of A, then the algebraic number field $K = \mathbb{Q}[\theta]$ has two real places v_1, v_2 (corresponding to the real roots $\theta_1 = \theta$ and $\theta_2 = \theta^{-1}$ of the characteristic polynomial $f = u^4 - u^3 - u^2 - u + 1$ of A) and one complex place v_3 (corresponding to the two complex roots θ_3 and $\bar{\theta}_3$ of f of absolute value 1). Then $S^{(0)} = \{v_3\}$, $W^{(0)} \cong K_{v_3} = \mathbb{C}$, and the central subgroup $X^{(0)} \subset X$ of α is a densely embedded copy of $\mathbb{C}$.

For another example of this form we take the automorphism α of $X = \mathbb{T}^6$ defined by the matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \in \mathrm{GL}(6, \mathbb{Z})$$

with dominant eigenvalue $\theta > 1$. The algebraic number field $K = \mathbb{Q}[\theta]$ has two real places v_1, v_2 (corresponding to the real roots $\theta_1 = \theta$ and $\theta_2 = \theta^{-1}$ of the characteristic polynomial $f = u^6 - u^5 - u^4 - u^3 - u^2 - u + 1$ of B) and two complex places v_3, v_4 (corresponding to the four complex roots θ_3, θ_4 and $\bar{\theta}_3, \bar{\theta}_4$ of f of absolute value 1). Then $S^{(0)} = \{v_3, v_4\}$, $W^{(0)} \cong K_{v_3} \oplus K_{v_4} = \mathbb{C}^2$, and the central subgroup $X^{(0)} \subset X$ of α is a densely embedded copy of $\mathbb{C}^2$.

(2) Let $f = 5u^2 - 6u + 5 \in R_1$, and let $\alpha = \alpha_{R_1/(f)}$ be the automorphism of the compact connected abelian group $X = X_{R_1/(f)}$ defined in (2.11). Since f is irreducible and all roots of f have absolute value 1 (they are of the form $\theta = \frac{3}{5} \pm i \cdot \frac{4}{5}$), α is ergodic and nonexpansive by Theorem 2.2. If θ is a root of f and $K = \mathbb{Q}(\theta)$, then $P \subset P_f^{(K)}$, $S^{(0)} = P_\infty^{(K)}$, $W = W^{(0)} \times \prod_{v \in P} K_v$, where $W^{(0)} \cong \mathbb{C}$ and $\prod_{v \in P} K_v$ is zero-dimensional, and $X \cong W/L$ for some discrete co-compact β-invariant subgroup $L \subset W$. In this example the central subgroup $X^{(0)} \subset X$ is a densely embedded copy of $\mathbb{C}$.

(3) Let $f = 6u^4 + 3u^3 + 10u^2 + 6u + 6 \in R_1$, and let $\alpha = \alpha_{R_1/(f)}$ be the automorphism of the compact connected abelian group $X = X_{R_1/(f)}$ defined in (2.11). Again f is irreducible, all roots of f have absolute value 1, and α is

ergodic and nonexpansive by Theorem 2.2. If θ is a root of f and $K = \mathbb{Q}(\theta)$, then $P \subset P_f^{(K)}$, $S^{(0)} = P_\infty^{(K)}$, $W = W^{(0)} \times \prod_{v \in P} K_v$, where $W^{(0)} \cong \mathbb{C}^2$ and $\prod_{v \in P} K_v$ is zero-dimensional, and $X \cong W/L$ for some discrete co-compact $\bar{\beta}$-invariant subgroup $L \subset W$. Here the central subgroup $X^{(0)} \subset X$ is a densely embedded copy of $\mathbb{C}^2$.

The group $W^{(0)} \cong \mathbb{C}^{|S^{(0)}|}$ in (3.3) is an algebra with respect to coordinate-wise addition and multiplication. We define a map $\iota_0 \colon K \longrightarrow W^{(0)}$ by setting

$$\iota(a)_v = \begin{cases} \iota_v(a) & \text{if } v \in S^{(0)}, \\ 0 & \text{if } v \in S \smallsetminus S^{(0)} \end{cases}$$

for every $a \in K$ (cf. (2.18)), set

$$\xi = \iota_0(\theta), \tag{3.4}$$

where θ is the algebraic number appearing in Theorem 2.3 and (3.1), and denote by

$$\Gamma = \overline{\{\xi^m : m \in \mathbb{Z}\}} \tag{3.5}$$

the closure of the multiplicative subgroup $\{\xi^m : m \in \mathbb{Z}\} \subset W^{(0)}$. Then Γ is a compact abelian multiplicative subgroup of $W^{(0)}$. For every $\gamma = (\gamma_v) \in \Gamma$ we denote by $M_\gamma \colon W^{(0)} \longrightarrow W^{(0)}$ multiplication by γ, i.e.

$$M_\gamma w = (\gamma_v w_v) \tag{3.6}$$

for every $w = (w_v) \in W^{(0)}$.

Proposition 3.2. *If α is totally irreducible, then for any two distinct elements $v, v' \in S^{(0)}$, the natural projection of Γ to $K_v \oplus K_{v'}$ is surjective.*

Proof. Let $\xi_{vv'}$ be defined by

$$(\xi_{vv'})_\nu = \begin{cases} \theta & \text{for } \nu = v, v', \\ 0 & \text{otherwise.} \end{cases}$$

Clearly the projection of Γ to $K_v \oplus K_{v'}$ is equal to $\overline{\{\xi_{vv'}^m : m \in \mathbb{Z}\}}$. Let $\xi_v = \iota_v(\theta) \in \mathbb{C}$ and $\xi_{v'} = \iota_{v'}(\theta) \in \mathbb{C}$. Since $v, v' \in S^{(0)}$ we know that $|\xi_v| = |\xi_{v'}| = 1$ (cf. (2.18)).

In order to prove our claim it suffices to show that, for any nonzero element $(m, m') \in \mathbb{Z}^2$,

$$\xi_v^m \xi_{v'}^{m'} \neq 1. \tag{3.7}$$

That $\xi_v^m \neq 1$ for $m \neq 0$ follows from ergodicity (ξ_v is a root of an irreducible polynomial with integer coefficients, which is noncylotomic if α is to be ergodic). To prove (3.7) for the case where both $m, m' \neq 0$, we note that since ξ_v and $\xi_{v'}$ are conjugate under the Galois group of the splitting field of the polynomial f, we also have that $\xi_{v'}^m = \xi_3^{m'}$ for some $\xi_3 \in \mathbb{C}$ with $f(\xi_3) = 0$ (it could be that $\xi_3 = \xi_v$). We can now apply the same argument for ξ_3 and obtain that $\xi_3^m = \xi_4^{m'}$ for some root $\xi_4 \in \mathbb{C}$ of f, etc. Since f has finitely many roots, we will eventually get an equation of the form $\xi_j^{m^k} = \xi_j^{(-m')^k}$ for some positive integer k and some root ξ_j of f. As all roots of f are conjugate under the Galois group, this shows that

$$\theta^{m^k} = \theta^{(-m')^k}.$$

If $m^k \neq (-m')^k$ then θ is a root of unity, which is a contradiction to ergodicity. Otherwise $m = \pm m'$, and either $\xi_v^m = \xi_{v'}^m$ or $\xi_v^m = \xi_{v'}^{-m}$.

First suppose that $\xi_v^m = \xi_{v'}^m$. Since v and v' are inequivalent valuations, $\xi_v \neq \xi_{v'}$, and hence $\xi_v \xi_{v'}^{-1}$ is a nontrivial root of unity, contrary to the hypothesis that α is totally irreducible (cf. Theorem 2.2).

If $\xi_v^m = \xi_{v'}^{-m}$, then the complex conjugate $\xi' = \overline{\xi_{v'}}$ of $\xi_{v'}$ is again a root of f satisfying that $\xi_v^m = \xi'^m$, and the same argument as above shows that $\xi_v \xi'^{-1}$ is a nontrivial root of unity. Again this violates the total irreducibility of α. □

4. Conditional measures on the leaves of the central foliation

We assume that α and X are of the form (3.1) and use the notation of (2.17)–(2.24). Write $\mathcal{F}$ for the foliation of X by the cosets of the central subgroup $X^{(0)} = \pi(W^{(0)}) \subset X$ (cf. (3.3)), and fix a nonatomic α-invariant probability measure μ on the Borel field $\mathcal{S} = \mathcal{B}_X$ of X. Note that we do not make any assumptions regarding ergodicity of μ.

Since the central subgroup $X^{(0)}$ is dense by irreducibility, the foliation of X into cosets of $X^{(0)}$ has no Borel cross-section, and one cannot generally decompose μ directly into a family of measures supported on the individual leaves of $\mathcal{F}$. In order to overcome this difficulty we break up each of these leaves into countably many atoms of an appropriate sub-sigma-algebra $\mathcal{A} \subset \mathcal{S}$, decompose the measure μ with respect to this sigma-algebra, and recombine the conditional measures supported by the individual atoms on each leaf into a leaf-measure.

It will be necessary to work not just with one such sigma-algebra $\mathcal{A}$ but with a sequence $(\mathcal{A}^{(k)}, k \geq 1)$ of sigma-algebras whose atoms consist of larger and larger pieces of leaves of $\mathcal{F}$. In order to describe these sigma-algebras we fix an integer $q > 1$ with $|q|_v = 1$ for every $v \in P$ and set $\Lambda = \frac{1}{q} L \subset K$. Then $\iota(\Lambda)$ is a discrete co-compact subgroup of W (cf. (2.23)), and we choose a Borel set $\Delta \subset W$ with compact closure such that

$$\begin{gathered} \Delta \cap (\Delta + \iota(a)) = \varnothing \text{ for every nonzero } a \in \Lambda, \\ \bigcup_{a \in \Lambda} \Delta + \iota(a) = W. \end{gathered} \tag{4.1}$$

The first equation in (4.1) implies that the restriction to Δ of the map $\pi\colon W \longrightarrow X$ in (3.2) is injective, and that $\pi(\Delta)$ is therefore a Borel subset of X. After replacing Δ by $\Delta + w_0$ for some $w_0 \in W$, if necessary, we may take it that the sets

$$\mathcal{Q} = \{\pi(\Delta + \iota(a)) : a \in \Lambda\} \tag{4.2}$$

form a Borel partition of X into $N = |L/qL|$ sets with the following properties.

(i) $\mu(\partial Q) = 0$ for every $Q \in \mathcal{Q}$;
(ii) For every $a \in \Lambda$, the restriction of the map π in (3.2) to $\Delta + \iota(a)$ is injective, and $\pi(\Delta + \iota(a)) = \pi(\Delta + \iota(a'))$ if and only if $a - a' \in L$;
(iii) For every $Q \in \mathcal{Q}$ and $w \in W$, the set $W^{(0)} \cap (\pi^{-1}(Q) - w)$ is a countable union of sets with disjoint and compact closures.

Let T_y denote the map

$$T_y x = x + y \tag{4.3}$$

for every $x, y \in X$. We denote by $B_{W^{(0)}}(w, r)$ the ball of radius $r > 0$ around w in $W^{(0)}$; while it will not be important for us which norm we use in $W^{(0)}$, the natural norm to take is

$$\|w\| = \max_{v \in S^{(0)}} |w_v| \tag{4.4}$$

(cf. (2.20)). Finally we write $B_{\mathcal{F}}(x, r)$ for the ball of radius r around x in the leaf $x + \pi(W^{(0)})$ of $\mathcal{F}$, i.e.

$$B_{\mathcal{F}}(x, r) = x + \pi(B_{W^{(0)}}(0, r)) = T_x \circ \pi(B_{W^{(0)}}(0, r)). \tag{4.5}$$

Proposition 4.1. *There exist a sequence of fundamental domains* ($\Delta^{(n)}$, $n \geq 1$) *for* $\iota(\Lambda)$ *and a corresponding sequence of partitions* ($\mathcal{Q}^{(n)}$, $n \geq 1$) *of* X *in* (4.2) *with the properties* (i)–(iii) *above, such that*

$$\sum_{Q \in \mathcal{Q}^{(n)}} \mu\bigl((Q + \pi(B_{W^{(0)}}(0, n))) \bigtriangleup Q\bigr) \leq 2^{-n} \tag{4.6}$$

for every $n \geq 1$.

Proof. Let $\Delta \subset W$ be a fundamental domain for Λ such that the corresponding partition $\mathcal{Q}$ satisfies conditions (ii)–(iii) above and such that

$$\lambda_X(\partial Q) = 0 \qquad \text{for every } Q \in \mathcal{Q}. \tag{4.7}$$

Let $\beta_{(\tau,L)}$ be an automorphism of X commuting with α expanding $W^{(0)}$ as in Corollary 2.4. Set $\Delta_n = \beta_{(\tau,L)}^{k_n}(\Delta)$ with corresponding partitions $\mathcal{Q}_n$. Clearly these also satisfy (ii)–(iii); furthermore, since $\mathcal{Q}$ satisfied (4.7), if k_n increases fast enough we can guarantee that

$$\sum_{Q \in \mathcal{Q}_n} \lambda_X\bigl((Q + \pi(B_{W^{(0)}}(0, n))) \bigtriangleup Q\bigr) \leq 2^{-n}$$

for every $n \geq 1$. Since for any two Borel sets $Q, B \subset X$,

$$\begin{aligned}\int \mu\big((Q+s+B) \triangle (Q+s)\big)\, d\lambda_X(s) \\ &= \iint \big|1_{Q+s+B}(x) - 1_{Q+s}(x)\big|\, d\mu(x)\, d\lambda_X(s) \\ &= \lambda_X\big((Q+B) \triangle Q\big),\end{aligned}$$

where 1_{Q+s+B} and 1_{Q+s} are the indicator function of the sets $Q+s+B$ and $Q+s$, there is a sequence $(x_n,\, n \geq 1)$ so that the translated partitions $\mathcal{Q}^{(n)} = \mathcal{Q}_n + x_n$ satisfy (4.6) and the conditions (i)–(iii) for every $n \geq 1$. For later use we choose a bounded sequence $(w_n,\, n \geq 1)$ in W with $\pi(w_n) = x_n$ for every $n \geq 1$ and set $\Delta^{(n)} = \Delta_n + w_n$, $n \geq 1$. □

Definition 4.2. Let $\mathcal{A} \subset \mathcal{S}$ be a countably generated sigma-algebra, and let $\mathcal{C} \subset \mathcal{A}$ be a countable algebra which generates $\mathcal{A}$. The *atom* $[x]_{\mathcal{A}}$ of a point $x \in X$ in $\mathcal{A}$ is defined as

$$[x]_{\mathcal{A}} = \bigcap_{C \in \mathcal{C} : x \in C} C = \bigcap_{A \in \mathcal{A} : x \in A} A.$$

Lemma 4.3. *For every $n \geq 1$, let $\Delta^{(n)}$ be the fundamental domain for $\iota(\Lambda) \subset W$ and $\mathcal{Q}^{(n)}$ the partition of X described in Proposition 4.1. Then there exist a countably generated sigma-algebra $\mathcal{A}^{(n)} \subset \mathcal{S}$ with*

$$[x]_{\mathcal{A}^{(n)}} = \pi\big((\Delta^{(n)} + \iota(a)) \cap (W^{(0)} + w)\big) \tag{4.8}$$

for every $x \in X$, where $[x]_{\mathcal{A}_{\mathcal{Q}^{(n)}}}$ is the atom of $\mathcal{A}^{(n)}$ containing x and $a \in \Lambda$ and $w \in W$ satisfy that $\pi(w) = x$ and $w \in \Delta^{(n)} + \iota(a)$.

Proof. Fix $a \in \Lambda$ for the moment. We set $W' = \big(\prod_{v \in S \smallsetminus S^{(0)}} K_v\big)$, denote by $\kappa \colon W = W^{(0)} \times W' \longrightarrow W'$ the second coordinate projection (cf. (3.3)), and write $\mathcal{B}_{W'}$ for the (countably generated) Borel field of W'. The sigma-algebra $\mathcal{A} = \{\kappa^{-1}(B) \cap (\Delta + \iota(a)) : B \in \mathcal{B}_{W'}\}$ of subsets of $\Delta + \iota(a)$ is again countably generated, and its atoms are of the form $(\Delta+\iota(a)) \cap (W^{(0)}+w)$, $w \in \Delta+\iota(a)$. Since the restriction of π to $\Delta + \iota(a)$ is injective, π maps $\mathcal{A}$ to a countably generated sigma-algebra $\mathcal{A}_Q$ of subsets of $Q = \pi(\Delta + \iota(a)) \in \mathcal{Q}^{(n)}$ whose atoms are of the required form. The sigma-algebra $\mathcal{A}^{(n)}$ is defined as the unique sub-sigma-algebra of $\mathcal{S} = \mathcal{B}_X$ which contains the partition $\mathcal{Q}^{(n)}$ and induces $\mathcal{A}_Q$ on each $Q \in \mathcal{Q}^{(n)}$. □

For any countably generated sigma-algebra $\mathcal{A} \subset \mathcal{S}$ we consider the decomposition of μ with respect to the sigma-algebra $\mathcal{A}$, i.e. a set of probability measures $\{\mu_x^{\mathcal{A}} : x \in X\}$ on X with the following properties.

(1) For all $x, x' \in X$ with $[x]_{\mathcal{A}} = [x']_{\mathcal{A}}$,

$$\mu_x^{\mathcal{A}} = \mu_{x'}^{\mathcal{A}} \text{ and } \mu_x^{\mathcal{A}}([x]_{\mathcal{A}}) = 1,$$

(2) For every $B \in \mathcal{S}$, the map $x \mapsto \mu_x^{\mathcal{A}}(B)$ is Borel (and hence $\mathcal{A}$-measurable),

(3) For every bounded Borel map $f\colon X \longrightarrow \mathbb{R}$,

$$\int f \, d\mu_x^{\mathcal{A}} = E_\mu(f|\mathcal{A})(x)$$

for μ-a.e. $x \in X$, where $E_\mu(\cdot|\cdot)$ denotes conditional expectation.

In order to make notation less cumbersome we set, for every $n \in \mathbb{Z}$ and $k \geq 1$,

$$\mathcal{A}_n^{(k)} = \alpha^{-n}(\mathcal{A}_{\mathcal{Q}^{(k)}}), \qquad \mathcal{A}^{(k)} = \mathcal{A}_0^{(k)} = \mathcal{A}_{\mathcal{Q}^{(k)}}, \tag{4.9}$$

and denote by $\{\mu_x^{\mathcal{A}_n^{(k)}} : x \in X\}$ and $\{\mu_x^{\mathcal{A}^{(k)} \vee \mathcal{A}_n^{(k)}} : x \in X\}$ the decompositions of μ with respect to the sigma-algebras $\mathcal{A}_n^{(k)}$ and $\mathcal{A}^{(k)} \vee \mathcal{A}_n^{(k)}$, respectively.

Definition 4.4. A Borel measure ρ on $W^{(0)}$ is *locally finite* if $\rho(C) < \infty$ for every compact set $C \subset W^{(0)}$. Let $M_\infty(W^{(0)})$ be the set of all locally finite (and hence sigma-finite) Borel measures on $W^{(0)}$, furnished with the smallest topology in which the map $\rho \mapsto \int f \, d\rho$ from $M_\infty(W^{(0)})$ to $\mathbb{R}$ is continuous for every continuous map $f\colon W^{(0)} \longrightarrow \mathbb{R}$ with compact support. In this topology $M_\infty(W^{(0)})$ is a separable metrizable space.

For every $w \in W^{(0)}$ we denote by

$$\bar{T}_w v = v + w, \tag{4.10}$$

the translation by w on $W^{(0)}$. The maps $\rho \mapsto \rho\bar{T}_w$ and $\rho \mapsto \rho\bar{\beta}$ are homeomorphisms of $M_\infty(W^{(0)})$ for every $w \in W^{(0)}$, where $\bar{\beta}$ is defined in (2.21).

For the next theorem, we take r_0 to be large enough so that

$$[x]_{\mathcal{A}^{(1)}} \subset B_{\mathcal{F}}(x, r_0) \tag{4.11}$$

for all $x \in X$ (cf. (4.5)).

Proposition 4.5. *There is a Borel map $x \mapsto \rho_x$ from X to $M_\infty(W^{(0)})$ and an α-invariant Borel set X' of full μ-measure with the following properties (for notation we refer to* (2.21), (4.9) *and* (4.10)*).*

(1) *For every $x \in X'$, every bounded Borel set $B \subset W^{(0)}$ and every sufficiently large k,*

$$\rho_x(B) = \frac{1}{\mu_x^{\mathcal{A}^{(k)}}(B_{\mathcal{F}}(x, r_0))} \mu_x^{\mathcal{A}^{(k)}}(T_x \circ \pi(B)); \tag{4.12}$$

(2) *For every $x \in X$,*

$$\rho_x = \rho_{\alpha x} \bar{\beta}; \tag{4.13}$$

(3) *There exists a Borel map* $K_\mu\colon X \times W^{(0)} \longrightarrow \mathbb{R}$ *so that, for every* $x \in X'$ *and every* $w \in W^{(0)}$ *with* $x + \pi(w) \in X'$,

$$e^{K_\mu(x,w)} \rho_{x-\pi(w)} = \rho_x \bar{T}_w. \tag{4.14}$$

We begin the proof of Proposition 4.5 with a lemma.

Lemma 4.6. *There exists a Borel set* $X' \subset X$ *with* $\mu(X') = 1$*, which is invariant under* $\bar{T}_w$ *for every* $w \in W^{(0)}$*, so that for all* $x \in X'$ *and* $r > 0$,

$$B_{\mathcal{F}}(x,r) \subset [x]_{\mathcal{A}^{(k)}} \tag{4.15}$$

for every sufficiently large k.

Proof. The set

$$N_{r,k} = \{x \in X : B_{\mathcal{F}}(x,r) \not\subset [x]_{\mathcal{A}^{(k)}}\}.$$

is equal to

$$\begin{aligned}\bigcup_{\substack{Q,Q' \in \mathcal{Q}^{(k)}\\ Q \neq Q'}} \big(Q \cap (Q' + \pi(B_{W^{(0)}}(0,r)))\big) &= \bigcup_{Q' \in \mathcal{Q}^{(k)}} \big((Q' + \pi(B_{W^{(0)}}(0,r))) \smallsetminus Q'\big)\\ &\subset \bigcup_{Q' \in \mathcal{Q}^{(k)}} \big((Q' + \pi(B_{W^{(0)}}(0,r))) \bigtriangleup Q'\big).\end{aligned}$$

and hence Borel. Since $\sum_{k \geq 1} \mu(N_{r,k}) < \infty$ for every $r > 0$ by (4.6), it follows that

$$X' = X \smallsetminus \bigcup_{r>0} \bigcap_{n \geq 1} \bigcup_{k \geq n} N_{r,k}$$

is a Borel set of full measure. From the definition of X' it is also clear that any $x \in X'$ satisfies (4.15), and that X' consists of a union of full $\mathcal{F}$ leaves, i.e. that it is invariant under $\bar{T}_w$ for any $w \in W^{(0)}$. □

Proof of Proposition 4.5. We take X' to be the set of all $x \in X$ with the following properties.

(1) For every $r > 0$ and $n \in \mathbb{Z}$, and for every sufficiently large $k \geq 1$ (depending on r and n),

$$B_{\mathcal{F}}(x,r) \subset [x]_{\mathcal{A}_n^{(k)}}; \tag{4.16}$$

(2) For every $k, l \geq 1$ and $n \in \mathbb{Z}$,

$$\mu_x^{\mathcal{A}^{(k)}}([x]_{\mathcal{A}^{(k)} \vee \mathcal{A}_n^{(l)}}) > 0, \qquad \mu_x^{\mathcal{A}_n^{(l)}}([x]_{\mathcal{A}^{(k)} \vee \mathcal{A}_n^{(l)}}) > 0, \tag{4.17}$$

$$\begin{aligned}\mu_x^{\mathcal{A}^{(k)} \vee \mathcal{A}_n^{(l)}} &= \frac{1}{\mu_x^{\mathcal{A}^{(k)}}([x]_{\mathcal{A}^{(k)} \vee \mathcal{A}_n^{(l)}})} \cdot \mu_x^{\mathcal{A}^{(k)}}\big|_{[x]_{\mathcal{A}^{(k)} \vee \mathcal{A}_n^{(l)}}}\\ &= \frac{1}{\mu_x^{\mathcal{A}_n^{(l)}}([x]_{\mathcal{A}^{(k)} \vee \mathcal{A}_n^{(l)}})} \cdot \mu_x^{\mathcal{A}_n^{(l)}}\big|_{[x]_{\mathcal{A}^{(k)} \vee \mathcal{A}_n^{(l)}}},\end{aligned} \tag{4.18}$$

where $\mu_x^{\mathcal{A}^{(k)}}\big|_{[x]_{\mathcal{A}^{(k)}\vee\mathcal{A}_n^{(l)}}}$ and $\mu_x^{\mathcal{A}_n^{(l)}}\big|_{[x]_{\mathcal{A}^{(k)}\vee\mathcal{A}_n^{(l)}}}$ are the restrictions of $\mu_x^{\mathcal{A}^{(k)}}$ and $\mu_x^{\mathcal{A}_n^{(l)}}$ to the atom $[x]_{\mathcal{A}^{(k)}\vee\mathcal{A}_n^{(l)}}$ of x in $\mathcal{A}^{(k)}\vee\mathcal{A}_n^{(l)}$;

(3) For every $k\ge 1$ and $n\in\mathbb{Z}$,

$$\mu_x^{\mathcal{A}_n^{(k)}} = \mu_{\alpha^n x}^{\mathcal{A}^{(k)}}\alpha^n. \tag{4.19}$$

Note that by (4.11) and (4.17) (with $l=1$ and $n=0$),

$$\mu_x^{\mathcal{A}^{(k)}}(B_{\mathcal{F}}(x,r_0))>0$$

for every $k\ge 1$ and $x\in X'$. Furthermore, by (4.16) and (4.18) (again with $n=0$),

$$\frac{1}{\mu_x^{\mathcal{A}^{(k)}}(B_{\mathcal{F}}(x,r_0))}\mu_x^{\mathcal{A}^{(k)}}\big|_{[x]_{\mathcal{A}^{(k)}\vee\mathcal{A}^{(l)}}} = \frac{1}{\mu_x^{\mathcal{A}^{(l)}}(B_{\mathcal{F}}(x,r_0))}\mu_x^{\mathcal{A}^{(l)}}\big|_{[x]_{\mathcal{A}^{(k)}\vee\mathcal{A}^{(l)}}}$$

for all $x\in X'$ and all sufficiently large k,l, so that

$$\rho_x(B)=\lim_{k\to\infty}\frac{1}{\mu_x^{\mathcal{A}^{(k)}}(B_{\mathcal{F}}(x,r_0))}\mu_x^{\mathcal{A}^{(k)}}(T_x\circ\pi(B)) \tag{4.20}$$

exists for every $x\in X'$ and every Borel set $B\subset W^{(0)}$.

Equation (4.14) easily follows from the fact that, for every $x\in X'$, every $w\in W^{(0)}$ with $x-\pi(w)\in X'$, and every sufficiently large k,

$$y=x-\pi(w)\in[x]_{\mathcal{A}^{(k)}},$$

and hence

$$\mu_y^{\mathcal{A}^{(k)}}=\mu_x^{\mathcal{A}^{(k)}}.$$

The sequence

$$\log\frac{\mu_{x-\pi(w)}^{\mathcal{A}^{(k)}}(B_{\mathcal{F}}(x-\pi(w),r_0))}{\mu_x^{\mathcal{A}^{(k)}}(B_{\mathcal{F}}(x,r_0))}=\log\frac{\mu_x^{\mathcal{A}^{(k)}}(B_{\mathcal{F}}(x-\pi(w),r_0))}{\mu_x^{\mathcal{A}^{(k)}}(B_{\mathcal{F}}(x,r_0))}$$

is eventually constant, and we set

$$K_\mu(x,w)=\begin{cases}\lim\limits_{k\to\infty}\log\frac{\mu_x^{\mathcal{A}^{(k)}}(B_{\mathcal{F}}(x-\pi(w),r_0))}{\mu_x^{\mathcal{A}^{(k)}}(B_{\mathcal{F}}(x,r_0))} & \text{for } x\in X' \text{ and } w\in W^{(0)} \\ & \qquad\text{with } x-\pi(w)\in X', \\ 0 & \text{otherwise.}\end{cases}$$

Equation (4.13) is immediate from (4.16), (4.18) and (4.19) (with $k=l$ and $n=1$).

Finally we extend the map $x\mapsto\rho_x$ to X by setting $\rho_x=0$ for every $x\in X\smallsetminus X'$ and note that the resulting map from X to $M_\infty(W^{(0)})$ is Borel. □

5. Finiteness of the central leaf measures

Theorem 5.1. *Let α be a nonexpansive, ergodic and totally irreducible automorphism of a compact connected abelian group X with normalized Haar measure λ_X, and let μ be an α-invariant probability measure on X which is singular with respect to λ_X. Then there exists a Borel set $X' \subset X$ with $\mu(X') = 1$ such that $\rho_x(W^{(0)}) < \infty$ for every $x \in X'$ (cf. (4.12)).*

We begin the proof of Theorem 5.1 with a series of lemmas in which we denote the l-th derivative of a map f by $f^{(l)}$.

Lemma 5.2. *For every $s \geq 1$ we can find a constant $A_s > 0$ such that, for every polynomial $p(x) = \sum_{l=0}^{2s-1} a_l x^l$ of degree $\leq 2s-1$ and every $\varepsilon > 0$,*

$$\sup_{t\in(-\varepsilon,\varepsilon)} |p(t)| \geq A_s \cdot \max_{0\leq l\leq 2s-1} (\varepsilon^l |a_l|).$$

Proof. The statement of the lemma is clearly unchanged by rescaling p and ε, so that we may assume that $\varepsilon = 1$ and $\max_l |a_l| = 1$. We can now set

$$A_s = \inf\left\{ \sup_{t\in(-1,1)} |p(t)| : p(x) = \sum_{l=0}^{2s-1} a_l x^l \text{ with } \max_l |a_l| = 1 \right\} > 0. \qquad \square$$

Lemma 5.3. *Let $\varepsilon > 0$, $s \geq 1$, and let $A_s > 0$ be the constant appearing in Lemma 5.2. Then*

$$\sup_{t\in(-\varepsilon,\varepsilon)} |f(t)| \geq \frac{A_s B}{2(2s-1)!}. \tag{5.1}$$

for every $B > 0$ and every map $f\colon (-\varepsilon,\varepsilon) \longrightarrow \mathbb{R}$ with $2s$ derivatives at every point such that

$$\max_{0\leq l\leq 2s-1} |f^{(l)}(t)| \geq \frac{B}{\varepsilon^l} \quad \text{for every } t \in (-\varepsilon,\varepsilon) \tag{5.2}$$

and

$$\sup_{t\in(-\varepsilon,\varepsilon)} |f^{(2s)}(t)| < \frac{A_s B}{\varepsilon^{2s}}. \tag{5.3}$$

Proof. Consider the Taylor expansion

$$p(x) = \sum_{l=0}^{2s-1} \frac{f^{(l)}(0)}{l!} x^l$$

of f of degree $2s-1$. From Lemma 5.2 we know that there is some $t \in (-\varepsilon,\varepsilon)$ with $|p(t)| \geq \frac{A_s B}{(2s-1)!}$, and Taylor's Theorem allows us to find a $\xi \in [0,1]$ with

$$f(t) = p(t) + \frac{f^{(2s)}(\xi t)}{(2s)!} t^{2s}.$$

Thus

$$|f(t)| \geq |p(t)| - \varepsilon^{2s} \cdot \left(\sup_{t\in(-\varepsilon,\varepsilon)} |f^{(2s)}(t)|\right) / (2s)!$$

$$\geq \frac{A_s B}{(2s-1)!} - \frac{A_s B}{(2s)!} \geq \frac{A_s B}{2(2s-1)!}. \qquad \square$$

Lemma 5.4. *Let* $p(t) = \sum_{k=1}^{s}(a_k \cos(2\pi m_k t) + b_k \sin(2\pi m_k t))$ *be a trigonometric polynomial, where the* m_k*,* $k = 1, \dots, s$*, are distinct positive integers. Let* $\|p\| = \max_{k=1,\dots,s}(|a_k + ib_k|)$*. Then there exists a constant* $c_2 > 0$*, which depends on* s *and* $M = \max_{k=1,\dots,s}|m_k|$*, but not on the coefficients* a_k, b_k*,* $k = 1, \dots, s$*, such that*

$$\left|\int_0^1 e^{ip(t)}\, dt\right| \leq c_2 \cdot \|p\|^{-1/2s}. \tag{5.4}$$

Proof. We first claim that, unless all coefficients a_k and b_k are 0, the derivative p' of p does not have zeros of order $> 2s-1$. Indeed,

$$p^{(2l)}(t) = \sum_{k=1}^{s}(-1)^l (2\pi m_k)^{2l}(a_k \cos(2\pi m_k t) + b_k \sin(2\pi m_k t))$$

for every $l \geq 0$. If $p^{(2l)}(t_0) = 0$ for $l = 1, \dots, s$, the nonsingularity of the Vandermonde matrix (due to our hypothesis that the m_k are all distinct) implies that

$$a_k \cos(2\pi m_k t_0) + b_k \sin(2\pi m_k t_0) = 0$$

for $k = 1, \dots, s$. Similarly, if $p^{(2l-1)}(t_0) = 0$ for $l = 1, \dots, s$, then

$$-a_k \sin(2\pi m_k t_0) + b_k \cos(2\pi m_k t_0) = 0$$

for $k = 1, \dots, s$, and by combining these statements we get that $a_k = b_k = 0$ for $k = 1, \dots, s$. In fact, this argument gives more: since one can bound the norm of the inverse of the Vandermonde matrix for all choices $0 < m_1 < \cdots < m_s \leq M$ by some function of M, there exists a constant $c'_M > 0$ depending only on M, such that

$$\max_{1 \leq l \leq 2s} |p^{(l)}(t)| \geq c'_M \|p\|$$

for every $t \in \mathbb{R}$ and every choice of the coefficients a_k, b_k in p.

Trivially there exists, for every $l \geq 0$ and $M \geq 1$, a constant $c'_{l,M} > 0$ such that

$$|p^{(l)}(t)| \leq c'_{l,M} \|p\|$$

for every $l \geq 0$ and $t \in \mathbb{R}$.

In order to complete the proof of Lemma 5.4 we recall the van der Corput Lemma in [7, p. 220]: if ϕ is a real-valued function on an interval $[a, b] \subset \mathbb{R}$ with a monotonic derivative satisfying that $\phi'(t) > A > 0$ for every $t \in [a, b]$, then

$$\left|\int_a^b e^{i\phi(t)}\, dt\right| \leq \frac{4}{A}.$$

Since a trigonometric polynomial of degree M such as $p''(t)$ can have at most $2M$ roots in the interval $[0,1)$, the interval $[0,1]$ can be divided into at most $2M+1$ subintervals $I_1, I_2, \ldots$, on each of which p' is monotonic. By applying the van der Corput Lemma on each of these subintervals separately we have that, for any $A > 0$,

$$\left| \int_0^1 e^{ip(t)}\, dt \right| \le \frac{8M+4}{A} + \lambda(\{0 \le t \le 1 : |p'(t)| < A\}), \tag{5.5}$$

where λ is the Lebesgue measure on $\mathbb{R}$.

It remains to estimate $\lambda(\{0 \le t \le 1 : |p'(t)| < A\})$. In fact, we claim that there exists a constant $c'' > 0$ with

$$\lambda(\{0 \le t \le 1 : |p'(t)| < A\}) \le c'' \left(\frac{A}{\|p\|} \right)^{\frac{1}{2s-1}} \tag{5.6}$$

for every $A > 0$. Estimates of this kind can be found e.g. in [8]; for completeness we provide a proof below.

As p' is monotonic on every subinterval I_k,

$$I'_k = I_k \cap \{0 \le t \le 1 : |p'(t)| < A\}$$

is connected and hence an interval, and we apply Lemma 5.3 with $f = p'$ on some sufficiently small subinterval $I''_k \subset I'_k$. The conditions (5.2) and (5.3) are clearly satisfied for some $B = \|p\| \cdot \lambda(I''_k)^{2s-1}$, and Lemma 5.3 guarantees the existence of a constant $c_1 > 0$ with

$$A \ge \sup_{t \in I'_k} p'(t) \ge c_1 \|p\| \lambda(I'_k)^{2s-1}$$

or

$$\lambda(I'_k) \le c_1^{-\frac{1}{2s-1}} \cdot \left(\frac{A}{\|p\|} \right)^{\frac{1}{2s-1}}.$$

By summing over k we obtain (5.6).

According to (5.5) and (5.6),

$$\left| \int_0^{2\pi} e^{2\pi i p(t)}\, dt \right| \le \frac{2}{A} + c'' \cdot \left(\frac{A}{\|p\|} \right)^{\frac{1}{2s-1}},$$

and by taking $A = \|p\|^{1/2s}$ we get (5.4). □

We derive from this the following estimate.

Lemma 5.5. *For every nontrivial character $a \in \hat{X}$ there exists a constant $c_a > 0$ with*

$$\int_\Gamma \langle a, \pi(M_\gamma w) \rangle \, d\gamma \le c_a \cdot \min(1, \|w\|^{-1/2s})$$

for every $w \in W^{(0)}$, where $s = |S^{(0)}|$, and where Γ and M_γ are defined in (3.5) and (3.6).

Proof. We recall that $W = \prod_{v\in S} K_v$, consider each K_v (by abuse of notation) as an additive subgroup of W, and identify X with $W/\iota(L)$ as in (2.24). Let a be a nontrivial character of $X = W/\iota(L)$, and let $f\colon w \mapsto f(w) = \langle a, \pi(w)\rangle$ be the corresponding character of W. We write $f_0 = f|_{W^{(0)}}$ for the restriction of f to $W^{(0)}$. The isomorphisms $\iota_v\colon K_v \longrightarrow \mathbb{C}$, $v \in S^{(0)}$, in (2.18) allow us to write $f_0\colon W^{(0)} \longrightarrow \mathbb{C}$ as the map

$$w \mapsto f_0(w) = e^{2\pi i \sum_{v\in S^{(0)}} \Re(a_v \iota_v(w_v))},$$

where $a_v \in \mathbb{C}$ for every $v \in S^{(0)}$, and where $\Re$ denotes the real part. Since the image $\pi(K_v)$ of K_v is dense in X for every v by irreducibility, $f_0|_{K_v}$ is a nontrivial character, hence $a_v \neq 0$ for every $v \in S^{(0)}$.

Let v, v' be distinct elements of $S^{(0)}$. By Proposition 3.2, the projection of Γ to $K_v \oplus K_{v'} \subset W^{(0)}$ maps Γ onto the set

$$\{w \in W : |w_v| = |w_{v'}| = 1 \text{ and all other coordinates are } 0\}.$$

Hence there exists a closed one-dimensional subgroup

$$\Gamma_0 = \bigl\{\bigl(\iota_v^{-1}(z^{m_v})\bigr)_{v\in S^{(0)}} : |z| = 1\bigr\} \subset \Gamma \subset W^{(0)}$$

such that the integers m_v, $v \in S^{(0)}$, are all distinct.

Now we use Lemma 5.4 to check that there exists a constant c with

$$\left|\int_{\Gamma_0} f(M_\gamma w)\, d\gamma\right| = \left|\int_0^1 e^{2\pi i \Re\left(\sum_{v\in S^{(0)}} a_v \iota_v(w_v) e^{2\pi i m_v t}\right)} dt\right| \le c \cdot \|w\|^{-1/2s}$$

for every $w \in W^{(0)}$, and by integrating over Γ we see that

$$\begin{aligned}\left|\int_\Gamma f(M_\gamma w)\, d\gamma\right| &= \left|\int_\Gamma \int_{\Gamma_0} f(M_{\gamma_0} M_\gamma w)\, d\gamma_0\, d\gamma\right| \\ &\le c \cdot \int_\Gamma \|M_\gamma w\|^{-1/2s} d\gamma = c \cdot \|w\|^{-1/2s}. \qquad \square\end{aligned}$$

Lemma 5.6. *For every nontrivial character $a \in \hat{X}$ there exists a constant $c_a > 0$ such that*

$$\begin{aligned}\int_\Gamma \left|\int_{W^{(0)}} \langle a, \pi(M_\gamma w)\rangle\, d\bar{\tau}(w)\right|^2 d\gamma \\ \le c_a \cdot \int \min(1, \|w - w'\|^{-1/2s})\, d\bar{\tau}(w)\, d\bar{\tau}(w')\end{aligned} \tag{5.7}$$

for every probability measure $\bar{\tau}$ on $W^{(0)}$, where $s = |S^{(0)}|$.

Proof. By Fubini's theorem,

$$\begin{aligned}&\int_\Gamma \left|\int_{W^{(0)}} \langle a, \pi(M_\gamma w)\rangle\, d\bar{\tau}(w)\right|^2 d\gamma \\ &\quad = \int_\Gamma \int_{W^{(0)}} \int_{W^{(0)}} \langle a, \pi(M_\gamma w)\rangle \overline{\langle a, \pi(M_\gamma w')\rangle}\, d\bar{\tau}(w)\, d\bar{\tau}(w')\, d\gamma\end{aligned}$$

$$= \int_{W^{(0)}} \int_{W^{(0)}} \int_{\Gamma} \langle a, \pi(M_\gamma(w - w')) \rangle \, d\gamma \, d\bar{\tau}(w) \, d\bar{\tau}(w').$$

From Lemma 5.5 we know that there exists a constant $c_a > 0$ with

$$\int_{\Gamma} \langle a, \pi(M_\gamma(w - w')) \rangle \, d\gamma \leq c_a \min(1, \|w - w'\|^{-1/2s})$$

for every $w \neq w'$ in $W^{(0)}$, and by integrating we obtain (5.7). ☐

Corollary 5.7. *Let $\bar{\tau}$ be a probability measure on $W^{(0)}$, $x_0 \in X$, and let $\rho = (\bar{\tau}\pi^{-1})T_{-x_0}$ (so ρ is supported on the central leaf through x_0). For every $N \in \mathbb{N}$ we set*

$$\rho_N = \frac{1}{N} \sum_{i=0}^{N-1} \rho\alpha^i$$

Then for every nontrivial character $a \in \hat{X}$

$$\limsup_{N \to \infty} \left| \int \langle a, x \rangle \, d\rho_N(x) \right|^2 \leq c_a \cdot \int \min(1, \left\|w - w'\right\|^{-1/2s}) \, d\bar{\tau}(w) \, d\bar{\tau}(w'),$$

where c_a is as in Lemma 5.6.

Proof. By Cauchy-Schwarz,

$$\left| \int \langle a, x \rangle \, d\rho_N(x) \right|^2 = \left| \frac{1}{N} \sum_{i=0}^{N-1} \int \langle a, x \rangle \, d\rho\alpha^i \right|^2 \leq \frac{1}{N} \sum_{i=0}^{N-1} \left| \int \langle a, x \rangle \, d\rho a^i(x) \right|^2$$

$$= \frac{1}{N} \sum_{i=0}^{N-1} \left| \int_{W^{(0)}} \langle a, \pi(w) \rangle \, d\bar{\tau}\bar{\beta}^i(w) \right|^2$$

The map

$$\gamma \mapsto \left| \int_{W^{(0)}} \langle a, \pi(w) \rangle \, d(\bar{\tau} M_\gamma)(w) \right|$$

from Γ to $\mathbb{R}^+$ is continuous and bounded, so by the unique ergodicity of the action of $\bar{\beta}$ on Γ

$$\frac{1}{N} \sum_{i=0}^{N-1} \left| \int_{W^{(0)}} \langle a, \pi(w) \rangle \, d\bar{\tau}\bar{\beta}^i(w) \right|^2 \to \int_{\Gamma} \left| \int_{W^{(0)}} \langle a, \pi(w) \rangle \, d\bar{\tau} M_\gamma \right|^2 \, d\gamma.$$

We can now apply Lemma 5.6 to conclude the proof of this corollary. ☐

Proof of Theorem 5.1. Consider the α-invariant Borel set

$$B = \{x : \rho_x(W^{(0)}) = \infty\}.$$

We will show that

$$\mu' = \frac{1}{\mu(B)} \mu\big|_B = \lambda_X \tag{5.8}$$

whenever $\mu(B) > 0$.

Assume therefore that $\mu(B) > 0$, and let $X' \subset X$ be the α-invariant Borel set of full measure described in Proposition 4.5. From (4.14) it follows that, if $x \in B \cap X'$, then any other point in $X' \cap (x - \pi(W^{(0)}))$ also lies in $B \cap X'$. Hence

$$\mu_x^{\mathcal{A}^{(k)}}(B) \in \{0, 1\}$$

for μ-a.e. $x \in X$ and every $k \geq 1$. We can thus choose, for every $k \geq 1$, a set $B^{(k)} \in \mathcal{A}^{(k)}$ with

$$\mu(B^{(k)} \triangle B) = 0.$$

We fix temporarily a large number $r > 0$ and a small $\varepsilon > 0$. According to (4.15) there exist an increasing sequence $(n_k,\, k \geq 1)$ of natural numbers and a Borel set $D \subset B$ with $\mu'(D) > 1 - \varepsilon$ so that, for any $x \in D$ and $k \geq 1$,

$$[x]_{\mathcal{A}^{(n_k)}} + \pi(B_{W^{(0)}}(0, r)) \subset [x]_{\mathcal{A}^{(n_{k+1})}},$$
$$0 < \mu_x^{\mathcal{A}^{(n_{k+1})}}([x]_{\mathcal{A}^{(n_k)}} + \pi(B_{W^{(0)}}(0, r))) < \varepsilon$$

(in the second of these conditions we use the fact that $\rho_x(W^{(0)}) = \infty$ for every $x \in B$). For every $K \geq 1$ and $x \in X$ we set

$$\tau_x^K = \frac{1}{K} \sum_{k=1}^{K} \mu_x^{\mathcal{A}^{(n_k)}}.$$

Since $B^{(n_k)} \in \mathcal{A}^{(n_k)}$ is equal to B (mod μ) and hence also (mod μ'), and since μ' is α-invariant, we have that

$$\mu' = \int (\mu_x^{\mathcal{A}^{(n_k)}} \alpha^n)\, d\mu'(x)$$

for every $k \geq 1$ and $n \in \mathbb{Z}$, and hence that

$$\mu' = \int (\tau_x^K \alpha^n) d\mu'(x) \tag{5.9}$$

for every $K \geq 1$ and $n \in \mathbb{Z}$. We define $\bar{\tau}_x^K \in M_\infty(W^{(0)})$ by

$$\bar{\tau}_x^K(C) = \tau_x^K(T_x \circ \pi(C))$$

for every Borel set $C \subset W^{(0)}$, where π and T_x are taken from (3.2) and (4.3). For any $x \in D$,

$$\begin{aligned} M(x, r) &= (\bar{\tau}_x^K \times \bar{\tau}_x^K)\big(\{(w_1, w_2) \in (W^{(0)})^2 : \|w_1 - w_2\| < r\}\big) \\ &= \frac{1}{K^2} \cdot \sum_{k=1}^{K} \sum_{k'=1}^{K} (\mu_x^{\mathcal{A}^{(n_k)}} \times \mu_x^{\mathcal{A}^{(n_{k'})}})\big(\{(x + \pi(w_1), x + \pi(w_2)) : \\ &\qquad\qquad (w_1, w_1) \in (W^{(0)})^2 \text{ and } \|w_1 - w_2\| < r\}\big) \\ &\leq \frac{1}{K^2} \cdot \sum_{k=1}^{K} (\mu_x^{\mathcal{A}^{(n_k)}} \times \mu_x^{\mathcal{A}^{(n_k)}})(X \times X) \\ &\quad + \frac{2}{K^2} \cdot \sum_{1 \leq k < k' \leq K} (\mu_x^{\mathcal{A}^{(n_k)}} \times \mu_x^{\mathcal{A}^{(n_{k'})}})\big(\{(y, y + \pi(w)) : \\ &\qquad\qquad y \in [x]_{\mathcal{A}^{(n_k)}}, \|w\| < r\}\big) \end{aligned}$$

$$\leq \frac{1}{K} + \frac{2}{K^2} \cdot \sum_{k<k'} \mu_x^{\mathcal{A}^{(n_{k'})}} \big([x]_{\mathcal{A}^{(n_k)}} + \pi(B_{W^{(0)}}(0,r))\big) < \frac{1}{K} + \varepsilon.$$

Using this, we see that for any $x \in D$,

$$\begin{aligned}\int_{W^{(0)}} \int_{W^{(0)}} \min(1, \|w - w'\|^{-1/2s})\, d\bar{\tau}_x^K(w)\, d\bar{\tau}_x^K(w') \\ \leq M(x,r) + r^{-1/2s}(1 - M(x,r)) \\ \leq K^{-1} + \varepsilon + r^{-1/2s}.\end{aligned} \tag{5.10}$$

Equation (5.9) shows that, for an arbitrary positive integer K,

$$\begin{aligned}\left| \int_X \langle a, x\rangle\, d\mu' \right|^2 &= \lim_{N\to\infty} \left| \int_X \int_X \langle a, y\rangle\, d\left[\frac{1}{N}\sum_{n=0}^{N-1} \bar{\tau}_x^K \alpha^n\right](y)\, d\mu'(x)\right|^2 \\ &\leq \int_X \limsup_{N\to\infty} \left| \int_X \langle a, y\rangle\, d\left[\frac{1}{N}\sum_{n=0}^{N-1} \bar{\tau}_x^K \alpha^n\right](y)\right|^2 d\mu'(x),\end{aligned} \tag{5.11}$$

where the first limit is actually over a constant sequence. By Corollary 5.7, (5.11) and (5.10),

$$\begin{aligned}\left| \int_X \langle a, x\rangle\, d\mu'(x) \right| &\leq \int_{X \smallsetminus D} \limsup_{N\to\infty} \left| \int_X \langle a, y\rangle\, d\left[\frac{1}{N}\sum_{n=0}^{N-1} \bar{\tau}_x^K \alpha^n\right](y)\right|^2 d\mu'(x) \\ &\quad + \int_D \limsup_{N\to\infty} \left| \int_X \langle a, y\rangle\, d\left[\frac{1}{N}\sum_{n=0}^{N-1} \bar{\tau}_x^K \alpha^n\right](y)\right|^2 d\mu'(x) \\ &\leq \mu'(X \smallsetminus D) + c_a \cdot \left(K^{-1} + \varepsilon + r^{-1/2s}\right) \\ &\leq \varepsilon + c_a \cdot \left(K^{-1} + \varepsilon + r^{-1/2s}\right).\end{aligned}$$

Since ε, K, r were arbitrary we see that $\int_X \langle a, x\rangle\, d\mu'(x) = 0$ for every $a \in \hat{X}$, and that μ' is therefore equal to λ_X. □

6. Virtually hyperbolic measures and central equivalence

In this section, we deduce Theorem 1.3 from Theorem 5.1. For any locally compact metric space Y, we let $M_f(Y) \subset M_\infty(Y)$ denote the finite Borel measures on Y.

Lemma 6.1. *There is a Borel map $\mathsf{c}_m : M_f(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$ which commutes with the action of the isometry group of $\mathbb{R}^d$ and is invariant under scalar multiplication: that is, if $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is an isometry of $\mathbb{R}^d$ and $t > 0$, then*

$$\mathsf{c}_m(\rho) = F \circ \mathsf{c}_m(t\rho F). \tag{6.1}$$

Remark 6.2. For measures $\rho \in M_f(\mathbb{R}^d)$ which have finite first moments, the vector of moments

$$\mathsf{c}_m(\rho) = \left(\int x_1 \, d\rho(x_1, \dots, x_d), \dots, \int x_d \, d\rho(x_1, \dots, x_d) \right) \in \mathbb{R}^d$$

would satisfy all these requirements. Unfortunately, there are measures for which this naive definition of center of mass does not make sense; the lemma should be interpreted as an alternative, generalized notion of a center of mass which works for any measure in $M_f(\mathbb{R}^d)$.

Proof. For a given $r, \varepsilon > 0$, and for every $\rho \in M_f(\mathbb{R}^d)$, let

$$S_{r,\varepsilon}(\rho) = \{x \in \mathbb{R}^d : \rho(B(x,r)) \geq \varepsilon\},$$
$$a_{r,\varepsilon}(\rho) = \int_{S_{r,\varepsilon}(\rho)} x \, d\rho(x), \qquad m_{r,\varepsilon}(\rho) = \rho(S_{r,\varepsilon}(\rho)).$$

Note that the maps $\rho \mapsto m_{r,\varepsilon}(\rho)$ and $\rho \mapsto \frac{a_{r,\varepsilon}(\rho)}{m_{r,\varepsilon}(\rho)}$, the latter when defined, are invariant under the action of isometry group of $\mathbb{R}^d$ on $M_f(\mathbb{R}^d)$. In order to get a map c_m which is defined everywhere we arbitrarily fix $r > 0$, set

$$n_r(\rho) = \min\{n : m_{r,1/n}(\rho) > 0\}, \quad a_r(\rho) = a_{r,1/n_r}(\rho), \quad m_r(\rho) = m_{r,1/n_r}(\rho),$$

and put $\mathsf{c}_m(\rho) = \frac{a_r(\rho)}{m_r(\rho)}$. □

Proof of Theorem 1.3. We first show that for every α-invariant measure μ which is singular with respect to the Haar measure λ_X there is a virtually hyperbolic measure μ' which is centrally equivalent to it. Indeed, consider the map $\tau : X \longrightarrow X$ defined by

$$\tau(x) = \pi \circ \mathsf{c}_m(\rho_x) + x.$$

Let X' be the subset of full measure of X in Proposition 4.5. Then for any $x \in X'$ we have that

$$\tau \circ \alpha(x) = \pi \circ \mathsf{c}_m(\mu_{\alpha x}) + \alpha x = \alpha(\pi \circ \mathsf{c}_m(\rho_x)) + \alpha x = \alpha \circ \tau(x)$$

where the second equality follows from (4.13) and (6.1). Similarly, by (4.14),

$$\begin{aligned}\tau(x) &= \pi \circ \mathsf{c}_m(\rho_x) + x = \pi \circ \mathsf{c}_m(\rho_{y-\pi(w)}) + x \\ &= \pi \circ \mathsf{c}_m(e^{K_\mu(y,w)} \cdot \rho_{y-\pi(w)}) + x \\ &= \pi \circ \mathsf{c}_m(\rho_y \bar{T}_w) + x = \pi \circ \mathsf{c}_m(\rho_y) + x + \pi(w) = \tau(y)\end{aligned} \tag{6.2}$$

for any $x, y \in X'$ with $y - x = \pi(w) \in X^{(0)}$. By setting $\mu' = \mu\tau^{-1}$ we get a new α-invariant probability measure on X which is clearly centrally equivalent to μ. The α-invariant set $Y = \tau(X') \subset X$ is analytic and has full μ'-measure, and (6.2) implies that Y intersects each central leaf in at most one point. By choosing an α-invariant Borel subset $Z \subset Y$ with $\mu'(Z) = 1$ we see that μ' is virtually hyperbolic.

We now specialize to the case where μ is ergodic. The map $\tau^*\colon X' \longrightarrow W^{(0)}$ defined by $\tau^*(x) = -\mathsf{c}_m(\rho_x)$ (so $x = \tau(x) + \pi \circ \tau^*(x)$) satisfies that $\tau^*(\alpha x) = \bar{\beta}\tau^*(x) \subset \Gamma\tau^*(x)$ for every $x \in X'$ (where Γ is defined in (3.5)), and the ergodicity of μ and the compactness of Γ together imply that there exist an element $w^* \in W^{(0)}$ so that for μ-almost every $x \in X$ it holds that $\tau^*(x) \in \Gamma w^* = \{M_\gamma w^* : \gamma \in \Gamma\}$.

For every $\gamma \in \Gamma$ we define a map $\tau_\gamma : X' \to X$ by

$$\tau_\gamma(x) = \tau(x) + M_\gamma \tau^*(x),$$

and let $\mu_\gamma = \mu\tau_\gamma^{-1}$. Then $\mu_e = \mu$ with $e \in \Gamma$ the identity, and

$$\mu' * \tilde{\lambda}_{\pi(w^*)} = \int_\Gamma \mu_\gamma d\gamma,$$

with $\tilde{\lambda}_{\pi(w^*)}$ as in Theorem 1.3. Since the identity is in the support of Haar measure on Γ, and since the map $\gamma \mapsto \mu_\gamma$ is continuous with respect to the weak* topology, μ an ergodic component of $\mu' * \tilde{\lambda}_{\pi(w^*)}$ in the sense of Theorem 1.3.

For every $\gamma \in \Gamma$ we write $M_\gamma w^*$ as $M_\gamma w^* = (\gamma_v w_v^*,\, v \in S)$ with $w_v^* = 0$ for every $v \in S \smallsetminus S^{(0)}$ (cf. (3.3)). similarly we set $\tau^*(x) = (\tau^*(x)_v)$ for every $x \in X''$. Then there exists, for every $v \in S^{(0)}$ with $w_v^* \neq 0$, a well-defined map $f_v \colon X' \longrightarrow \mathbb{C}$ with

$$\tau^*(x)_v = f_v(x) w_v^*$$

for every $x \in X''$, where we are identifying K_v with $\mathbb{C}$ (cf. (2.18)). Since f_v is obviously an eigenfunction of α for every $v \in S^{(0)}$ with $w_v^* \neq 0$, we have arrived at the following alternative: either the map $x \mapsto \pi \circ \mathsf{c}_m(x) = x - \tau(x)$ is zero almost everywhere, which implies that $\mu = \mu'$, hence virtually hyperbolic, or μ is not weakly mixing; indeed, this argument shows that the point spectrum of μ (more precisely: the point spectrum of the action of α on $L^2(X, \mathcal{S}, \mu)$) contains some eigenvalue of α of absolute value 1. □

7. Central leaves and closed invariant subsets

This section is devoted to proving the following topological analogue to Theorem 5.1.

Theorem 7.1. *Let α be a nonexpansive, ergodic and totally irreducible automorphism of a compact connected abelian group X. Then any closed α-invariant subset $Y \subsetneq X$ intersects every central leaf in a compact subset of the leaf.*

The key to this theorem is the following lemma in which we call a subset $A \subset W^{(0)}$ *R-separated* if $\|x - y\| \geq R$ for any two distinct elements $x, y \in A$.

Lemma 7.2. *Let α and X be as in Theorem 7.1. Then for any $\varepsilon > 0$ there exist positive integers R, K so that for any R-separated subset $A \subset W^{(0)}$ with at least K elements, the set*

$$\tilde{A} = \bigcup_{n=1}^{\infty} \alpha^{-n}(\pi(A) + x_0)$$

is ε-dense in X.

Proof. Let $\{f_1, \dots, f_k\}$ be a partition of unity of X (i.e. a set of nonnegative continuous functions so that $\sum_{i=1}^{k} f_i \equiv 1$) so that the support of each f_i has diameter at most ε. Clearly, to show that $\tilde{A}$ is ε-dense it is sufficient to find some probability measure ρ supported on $\tilde{A}$ so that

$$\int_X f_i d\rho > 0$$

for every $i = 1, \dots, k$. Since the linear span of $\hat{X}$ is dense in $C(X)$, there exists a finite subset $\Xi \subset \hat{X}$ containing the identity element $0 \in \hat{X}$ so that for each i we can find an approximation

$$\tilde{f}_i(x) = \sum_{a \in \Xi} u_{i,a} \langle a, x \rangle$$

to f_i in the linear span of Ξ so that

$$\|f_i - \tilde{f}_i\|_\infty < \|f_i\|_1 / 100.$$

Let $\Xi' = \Xi \setminus \{0\}$.

We denote by c_a the constant in Lemma 5.6 and Corollary 5.7 and define R, K by

$$R^{2s} = K = 100 \max_i \left(\frac{\sum_{a \in \Xi} |u_{i,a} c_a|}{\|f_i\|_1} \right).$$

Now suppose that $A \subset W^{(0)}$ is an R-separated set of cardinality $\geq K$ and $x_0 \in X$ is arbitrary. We define

$$\bar{\tau} = \frac{1}{|A|} \sum_{w \in A} \delta_w, \quad \rho = (\bar{\tau}\pi^{-1}) T_{-x_0}, \quad \rho_N = \frac{1}{N} \sum_{i=0}^{N-1} \rho \alpha^i,$$

where δ_w is the point-mass at w. For every N, ρ_N is supported on $\tilde{A}$, and if N is large enough, then

$$\begin{aligned} \left| \int \langle a, x \rangle \, d\rho_N(x) \right|^2 &\leq 2c_a \iint \min(1, \left\| w - w' \right\|^{-1/2s}) \, d\bar{\tau}(w) \, d\bar{\tau}(w') \\ &\leq 2c_a (K^{-1} + R^{-1/2s}) = 4c_a K^{-1} \end{aligned}$$

for every $a \in \Xi$.

For N sufficiently large we obtain that

$$\left|\int \tilde{f}_i d\rho_N - \int \tilde{f}_i dx\right| \leq \sum_{a\in\Xi'} \left|u_{i,a}\int \langle a,x\rangle d\rho_N\right|$$
$$\leq 4K^{-1}\sum_{a\in\Xi'} |u_{i,a}c_a| \leq \|f_i\|_1/25.$$

But then

$$\left|\int f_i d\rho_N\right| \geq \left|\int \tilde{f}_i d\rho_N\right| - \|f_i\|_1/100 \geq \left|\int \tilde{f}_i dx\right| - \|f_i\|_1/20 \geq \|f_i\|_1/2 > 0$$

for $i = 1,\dots,k$, and we are done. □

Proof of Theorem 7.1. Suppose that $Y \subsetneq X$ is α-invariant and closed, and that the intersection of Y with some central leaf $X^{(0)} + x_0$ is not compact. Fix a $w_0 \in \pi^{-1}(x_0)$ and take $C = [\pi^{-1}(Y) - w_0] \cap W^{(0)}$.

By our assumptions, C is a closed unbounded subset of $W^{(0)}$. Let $\varepsilon > 0$ be arbitrary, and let K, R be as in Lemma 7.2. Take A to be a finite R-separated subset of C of cardinality $\geq K$. Then the set $\tilde{A} \subset X$ defined in that lemma is a subset of Y and is ε-dense. So Y is ε-dense, and since ε was arbitrary, $Y = X$. □

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