ON THE PROJECTIONS OF MEASURES INVARIANT UNDER THE GEODESIC FLOW

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1. Introduction

Let M be a compact Riemannian surface (a two-dimensional Riemannian manifold),with μ a probability measure on the unit tangent bundle SM invariant under the geodesic flow.

We are interested in understanding the image, and especially the dimension of the image, of μ under the natural projection $SM \to M$. It will expire that the properties of interest of this specific projection are immediate consequences of general methods used to study properties of a typical member of a family of projections. Many authors contributed in this directions, and we make no attempt to be thorough. We refer the reader to [9],[6],[5], [10],[11], [4], [14] and the recent [13], where additional references can also be found regarding the dimension of projections.

There are many different possible ways to define the dimension of a measure μ. In what follows, we shall use the **information dimension**, which for the measures we will consider, namely measures invariant under the geodesic flow on the unit tangent bundle of a surface, is closely related through a theorem of L-S. Young [17] to the entropy (we also mention the related [3] by M. Brin and A. Katok). It is defined as follows:

$$\begin{aligned} \underline{\dim}_x \mu &:= \underline{\lim}_{\varepsilon \to 0} \frac{\log \mu\,[B(x,\varepsilon)]}{\log \varepsilon} \\ \underline{\dim} \mu &:= \text{ess-inf}\, \underline{\dim}_x \mu, \end{aligned} \tag{1.1}$$

where as usual $B(x,\epsilon)$ denotes a ball of radius ϵ around x (according to the implicitly given metric on the space containing x; in the case of a geodesic flow this space is SM).

Date: November 21, 2002.

E. L. supported in part by NSF grant DMS-0140497.

For an invariant measure μ, the Lyapunov exponent $\lambda(x)$ is defined at μ-a.e. point x by

$$\lambda(x) = \lim_{t\to\infty} \frac{1}{t} \log \|D_x g_t\|,$$

where g_t is the geodesic flow. If $\lambda(x) > 0$ for μ-a.e. x, then the dimension is a true limit $\delta(x)$ for μ-a.e. x, and, by Young's formula [17]:

$$\delta(x) = 1 + 2\frac{h(x)}{\lambda(x)},$$

where $h(x)$ is the entropy of the ergodic component of the point x. Of course, if μ is not ergodic, $\delta(x)$ can be a nontrivial function of x.

Theorem 1.1. *Let M be a compact Riemannian surface, and μ a probability measure on the a unit tangent bundle SM invariant under the geodesic flow. Let $\Pi : SM \to M$ be the natural projection. Then the following holds:*

(1) *if $\underline{\dim}\,\mu \leq 2$ then*
$$\underline{\dim}\,\mu = \underline{\dim}\,\Pi\,(\mu)$$
(2) *if $\underline{\dim}\,\mu > 2$ then $\Pi(\mu)$ is absolutely continuous with respect to Lebesgue measure vol_M on M.*

By the variational principle [7]-[8] if $h(x) = \lambda(x) > 0$ for μ-a.e. x, then the measure μ is absolutely continuous with respect to the Liouville measure SM. Theorem 1.1 implies

Corollary 1.2. *With the above notations, if $h(x) > \frac{1}{2}\lambda(x) > 0$ for μ-a.e. x, then the measure $\Pi(\mu)$ is absolutely continuous with respect to the Lebesgue measure on M.*

In fact, one can get a substantially stronger result regarding the projected measure if one assume that μ has locally-finite s-energy for $s > 2$. In this case, the Radon-Nikodym derivative

$$\frac{d\Pi\mu}{d\,vol_M} \in L^2(M).$$

Possibly, using the methods of [13] one may even show this Radon-Nikodym derivative has fractional derivative in the Sobolev sense. Unfortunately, their result is not directly applicable in this case, and while we have not pursued this direction this seems an interesting direction for future research.

In higher dimensions, the direct extension of our approach yields results which are likely not sharp. We remark that the question we study here has some relation to the theory of Kakeya sets: for example

Proposition 2.5 can be used to derive the well-known fact that planar Kakeya sets have full dimension, and transversality is key in both cases. Establishing sharp lower bound on the dimension of Kakeya sets in dimensions $d > 2$ is a notoriously difficult question; so finding a sharp version of Theorem 1.1 for higher dimensions is a very interesting question, though it might well be in view of the additional structure at hand that this second question is more tractable. We refer the reader to [2], [16] and [15] for more information regarding Kakeya sets.

Note that here and throughout this note, all measures are assumed to be Radon measures.

2. Transversality and projections

Let $\gamma_0(t)$ denote a geodesic through $\Pi(\xi)$ satisfying $\gamma_0'(0) = \xi$. Instead of working on the whole manifold, we will restrict our attention to a small compact subset $\tilde{U} \subset SM$ which is the closure of its interior which we choose as follows:

First we choose local coordinates on a neighborhood U of $\Pi(\xi) \in M$, i.e. a map $\Phi : U \to \mathbb{R}^2$, with the following properties:

(1) $[-1, 1]^2 \subset \Phi(U)$
(2) $\Phi \circ \Pi(\xi) = (0, 0)$
(3) The metric tensor g_{ij} expressed in the local coordinates satisfies

$$g_{ij}(0, 0) = \lambda^{-2}\delta_{ij}.$$

(4) For any point in $\mathcal{C}_1 := \Phi^{-1}([-1, 1] \times \{-1\})$ and another point in $\mathcal{C}_2 := \Phi^{-1}([-1, 1] \times \{1\})$, there is a unique shortest geodesics $\gamma_1 : [0, t_0] \to M$ connecting them. Furthermore, for any $t \in [0, t_0]$, we have that $\gamma_1(t) \in U$ and $\mathbf{x}(t) = (x^1(t), x^2(t)) = \Phi(\gamma_1(t))$ satisfies

$$\left|\dot{x^1}(t)\right| < 0.1\lambda \tag{2.1}$$

$$0.9\lambda < \dot{x^2}(t) < 1.1\lambda \qquad \forall t \in [0, t_0] \tag{2.2}$$

(note that the factor of λ is the correct normalization to use since $\gamma_1'(t) \in SM$, i.e. $g_{ij}(\mathbf{x})\dot{x}^i\dot{x}^j = 1$ at any t.)

(5) The coordinate chart Φ distorts distances by at most a factor of two: by 3, this implies that for any $p, q \in U$, we have that

$$|\Phi(p) - \Phi(q)|/2 < \lambda d_M(p, q) < 2|\Phi(p) - \Phi(q)|, \tag{2.3}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^2$.

For any $p_1, p_2 \in M$, we let γ_{p_1,p_2} denote the shortest geodesics connecting them with $\gamma_{p_1,p_2}(0) = p_1$ (since we will only use this notation for p_1, p_2 sufficiently close to one another, there will be a unique such geodesics) parametrized by (Riemannian) arc length; thus

$$\gamma_{p_1,p_2}(d_M(p_1,p_2)) = p_2.$$

We now define a map Ψ defined on triplets $x_1, x_2 \in (-1-\epsilon, 1+\epsilon), t \in (-\epsilon, 1+\epsilon)$ with image in SM in the following way: set $p_1 = \Phi^{-1}(x_1, -1)$ and $p_2 = \Phi^{-1}(x_2, 1)$ and set

$$\Psi(x_1, x_2, t) = \gamma'_{p_1,p_2}(d_M(p_1,p_2)t) \tag{2.4}$$

Clearly, the conditions above imply that Ψ is a diffeomorphism.

Finally, we are in a position to define $\tilde{U}$:

$$\tilde{U} = \Psi([-1,1] \times [-1,1] \times [0,1]), \tag{2.5}$$

Set $\tilde{\mu} = \frac{1}{\mu(\tilde{U})}\mu|_{\tilde{U}}$ and $\mu' = \Pi(\tilde{\mu})$. Since $\tilde{U}$ does not necessarily consist of full fibers of $SM \to M$, in general μ' is NOT the restriction of $\Pi(\mu)$ to some open set. Nevertheless, the following proposition clearly implies Theorem 1.1, as easily follows from taking a cover of SM by the interior of finitely many such sets:

Proposition 2.1. *With the notations above,*

1. *if* $\underline{\dim}\,\tilde{\mu} \leq 2$ *then*
$$\underline{\dim}\,\mu' = \underline{\dim}\,\tilde{\mu}$$
2. *if* $\underline{\dim}\,\tilde{\mu} > 2$ *then* μ' *is absolutely continuous with respect to Lebesgue measure.*

The proof of this proposition, as do the proofs of many results about dimensions of projections, uses the closely related notion of energy of a measure. The α-**energy** of a measure ν on a metric space X, denoted by $\mathcal{E}_\alpha(\nu)$, is defined as follows:

$$\begin{aligned}\mathcal{E}_\alpha(\nu) &= \int_{X\times X} \frac{d\nu(x)\,d\nu(y)}{d(x,y)^\alpha} \\ &= \alpha^{-1}\int_X\int_0^\infty \frac{\nu(B(x,r))}{r^{\alpha+1}} dr d\nu(x)\end{aligned} \tag{2.6}$$

The notions of energy and dimension are related; for example, for any probability measure μ on a metric space (X,d), if $\mathcal{E}_\alpha(\mu) < \infty$, then $\underline{\dim}\mu \geq \alpha$. We need the following proposition, which is slightly more refined information in this direction:

Proposition 2.2. *Let* μ *be a probability measure on a Riemannian manifold* X. *Let* f_n *be a sequence of measurable functions on* X, *with values in* $[0,1]$, *and such that for every* x, *the sequence* $f_n(x) \uparrow 1$. *Let*

μ_n be defined by $d\mu_n(x) = f_n(x)d\mu(x)$, and assume that for some α they all have finite α-energy. Then $\underline{\dim}\mu \geq \alpha$.

Proof. First we make the following easy observation: if m_r is monotone nondecreasing in r, with $\underline{\lim}_{r\to 0} \log(m_r)/\log(r) < \alpha$ then

$$\int_0^1 \frac{m_r}{r^{\alpha+1}} dr = \infty. \tag{2.7}$$

Indeed, take α' in the range $\underline{\lim} \log(m_r)/\log(r) < \alpha' < \alpha$ and let $1 = r_0 > r_1 > \dots$ be a decreasing sequence with $r_{i+1} < r_i/2$, such that

$$m_{r_i} > {r_i}^{\alpha'}.$$

Then

$$\begin{aligned} \int \frac{m_r}{r^{\alpha+1}} dr &= \sum_i \int_{r_{i+1}}^{r_i} \frac{m_{r_{i+1}}}{r^{\alpha+1}} dr \\ &\geq C \sum_i \frac{m_{r_{i+1}}}{{r_{i+1}}^{\alpha}} \\ &\geq C \sum_i \frac{{r_{i+1}}^{\alpha'}}{{r_{i+1}}^{\alpha}} = \infty \end{aligned}$$

By taking $m_r = \nu(B(x,r))$ and applying (2.7) for every x for which $\underline{\dim}_x(\nu) < \alpha$ one immediately gets that if $\underline{\dim}\,\nu < \alpha$ then $\mathcal{E}_\alpha(\nu) = \infty$, or equivalently that if $\mathcal{E}_\alpha(\nu) < \infty$ then $\underline{\dim}\,\nu \geq \alpha$.

Suppose now that we are only given that $\mathcal{E}_\alpha(\nu_i) < \infty$, with ν_i as in the statement of the proposition, but that $\mathcal{E}_\alpha(\nu) = \infty$. We cite the following lemma from [12]:

Lemma 2.3 (Lemma 2.13 in [12])**.** *Let μ and λ be Radon measures on $\mathbb{R}^n$, $0 < t < \infty$ and $A \subset \mathbb{R}^n$. If for all $x \in A$*

$$\underline{\lim}_{r\to 0} \frac{\lambda(B(x,r))}{\mu(B(x,r))} \leq t$$

then $\lambda(A) \leq t\mu(A)$.

Applying this with $\lambda = \mu_i$, we have that for any $\alpha' < \alpha'' < \alpha$ the μ_i measure of those points

$$A_i = \left\{ x : \underline{\lim}_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} < \alpha' \text{ but } \underline{\lim}_{r\to 0} \frac{\log \mu_i(B(x,r))}{\log r} > \alpha'' \right\}$$

is zero. However, since $\mathcal{E}_\alpha(\mu_i) < \infty$ we have already seen that μ_i almost surely

$$\underline{\lim}_{r\to 0} \frac{\log \mu_i(B(x,r))}{\log r} > \alpha''$$

so

$$\mu_i \left\{ x : \underline{\lim}_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} < \alpha' \right\} = 0. \tag{2.8}$$

The Lebesgue monotone convergence theorem now gives that taking the limit of (2.8) as $i \to \infty$ one has that

$$\mu \left\{ x : \underline{\lim}_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} < \alpha' \right\} = 0, \tag{2.9}$$

establishing the proposition. □

As for converse, we have the following, which is completely standard, and follows easily from (2.6):

Proposition 2.4. *Let μ be a probability measure on the metric space (X, d). Then for any $\alpha < \underline{\dim}\mu$, there is an increasing sequence of subsets A_n with $\mu(A_n) \to 1$, so that*

$$\mathcal{E}_\alpha(\mu|_{A_n}) < \infty$$

Proof. Fix a $\alpha < \alpha' < \underline{\dim}\mu$ and take A_n to be

$$A_n = \{x : \mu(B(x,r)) < r^{\alpha'} \text{ for all } r < n^{-1}\}.$$

□

We now return from the general theory to a specific case at hand. Let $\tilde{U}$ be as in (2.5). Since the original measure μ is invariant under the geodesic flow, $\Psi^{-1}(\tilde{\mu})$ is a product measure of some measure on $[-1, 1] \times [1, 1]$ with Lebesgue measure on $[0, 1]$. We will call any measure supported on U with this property **locally invariant**, even if it is not a restriction of any measure on SM invariant under the geodesic flow.

As we will see momentarily, using these results on the connection between dimension and energy, Proposition 2.1 follows from the following:

Proposition 2.5. *Let $\tilde{\nu}$ be any locally invariant probability measure supported on $\tilde{U}$. If $\tilde{\nu}$ has finite α-energy for some $\alpha < 2$ then $\nu' = \Pi(\tilde{\nu})$ also has finite α-energy. If $\tilde{\nu}$ has finite α-energy for $\alpha \geq 2$ then ν' is regular with respect to Lebesgue measure and furthermore*

$$\frac{d\nu'}{d\,vol} \in L^2(\text{vol}).$$

Remark: As mentioned in the introduction, this proposition can be used also to show that a planar Kakeya set (i.e. a compact subset $K \subset \mathbb{R}^2$ containing a segment of length one in any direction) has full dimension. Since this result is well-known we do not give details, but the key is defining from any Kakeya set K a probability measure

μ_K on K which modulo some trivial modifications can be made to be locally invariant: we describe μ_K by explaining how to pick a μ_K-random element of K. Pick uniformly a direction $\theta \in S^1$, and from all unit segments in this direction contained in K choose the one whose end point is greatest in lexicographical order (since K is compact, this would also be an end point of such a segment). So far we have chosen a random unit segment: to choose a random element of K simply choose a point from this random segment with uniform distribution.

Proof of Proposition 2.1 assuming Proposition 2.5. Let $\alpha < \underline{\dim}\,\tilde{\mu}$. Find an increasing sequence of subsets A_n of $\tilde{U}$ as in Proposition 2.4 so that $\tilde{\mu}(A_n) \to 1$ and $\mathcal{E}_\alpha(\tilde{\mu}|_{A_n}) < \infty$. We set $\tilde{\nu}_n = \tilde{\mu}|_{A_n}$ and $\nu'_n = \Pi(\tilde{\nu}_n)$.

Assume first $\underline{\dim}\,\tilde{\mu} \le 2$. Then by Proposition 2.5, we have that for all n

$$\mathcal{E}_\alpha(\nu'_n) < \infty.$$

But notice that

$$f_n := \frac{d\nu'_n}{d\mu'} = \mathbf{E}_{\tilde{\mu}}(1_{A_n} \mid \text{ Borel sigma-algebra on } M),$$

so $f_n(x) \uparrow 1$ for μ'-almost every x. Applying Proposition 2.2 we get that $\underline{\dim}\,\mu' \ge \alpha$.

A similar argument works in the case of $1 < \alpha < \underline{\dim}\,\tilde{\mu}$, where instead of Proposition 2.2 one uses the Lebesgue monotone convergence theorem. Note that even though each ν'_n has L^2-density, in the limit we only get absolute continuity with respect to Lebesgue measure. □

The key to the proof of Proposition 2.5 is the following straightforward transversality type result:

Lemma 2.6. *Let $p_1, q_1 \in \mathcal{C}_1$ and $p_2, q_2 \in \mathcal{C}_2$. Set*

$$\mathbf{x}(t) = \Phi(\gamma_{p_1,p_2}(t)) \qquad \mathbf{y}(t) = \Phi(\gamma_{q_1,q_2}(t)).$$

and take

$$\mathbf{v}(t) = \frac{\frac{d}{dt}\Phi(\gamma_{p_1,p_2}(t))}{\left|\frac{d}{dt}\Phi(\gamma_{p_1,p_2}(t))\right|} \qquad \mathbf{w}(t) = \frac{\frac{d}{dt}\Phi(\gamma_{q_1,q_2}(t))}{\left|\frac{d}{dt}\Phi(\gamma_{q_1,q_2}(t))\right|}.$$

Then for any $t_p \in [0, d_M(p_1,p_2)], t_q \in [0, d_M(q_1,q_2)]$

$$|\mathbf{x}(t_p) - \mathbf{y}(t_q)| + |\mathbf{v}(t_p) - \mathbf{w}(t_q)| \ge c[d_M(p_1,q_1) + d_M(p_2,q_2)]. \quad (2.10)$$

Proof. Set $I_1 = [-1,1] \times \{-1\}$ and $I_2 = [-1,1] \times \{1\}$, so that $\Phi(I_i) = \mathcal{C}_i$ for $i = 1, 2$. $\mathbf{x}$ is a solution of the ODE

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{g}^{-1}\xi \\ \dot{\xi}^i &= -\xi^T \frac{d(\mathbf{g}^{-1})}{dx^i}\xi \end{aligned} \quad (2.11)$$

where $\mathbf{g} = (g_{ij})$, and ξ and $\mathbf{v}$ are related by

$$\mathbf{v}(t) = \frac{\mathbf{g}^{-1}\xi}{|\mathbf{g}^{-1}\xi|}.$$

It would be more convenient for us to parametrize $\mathbf{x}$ and $\mathbf{y}$ by their second coordinate a instead of by t, a permissible reparameterization by (2.2). We will still use the same notation $\mathbf{x}$ and $\mathbf{y}$ for these paths, which should not cause any confusion since for the rest of the proof this is a parameterization we use. $(\mathbf{x}(a), \mathbf{v}(a))$ satisfy a first order ODE of the same general form

$$\begin{aligned} \frac{d\mathbf{x}}{da} &= F_1(\mathbf{x}, \mathbf{v}) \\ \frac{d\mathbf{v}}{da} &= -F_2(\mathbf{x}, \mathbf{v}), \end{aligned} \tag{2.12}$$

and while F_1 and F_2 may have singularities, the conditions set on U above guarantee that $(\mathbf{x}, \mathbf{v})$ will remain away from these singularities (uniformly in p_1, p_2 and a).

Let now $a_p, a_q \in [-1, 1]$ be arbitrary. For simplicity, assume $a_p \geq a_q$; the other case is handled in precisely the same way. Setting $\tau(a) = |\mathbf{x}(a + a_p) - \mathbf{y}(a + a_q)| + |\mathbf{v}(a + a_p) - \mathbf{w}(a + a_q)|$, equation (2.12) implies that

$$\tau(a) \leq \tau(0) \exp(c_0|a|). \tag{2.13}$$

Which in particular means that

$$|\mathbf{x}(-1 + a_p - a_q) + \mathbf{y}(-1)| < C\tau(0).$$

Clearly,

$$\begin{aligned} \tau(0) &\geq \left|x^2(a_p) - y^2(a_q)\right| = |a_p - a_q|\,, \\ |\mathbf{x}(-1) - \mathbf{x}(-1 + a_p - a_q)| &< 2\,|a_p - a_q| \end{aligned} \tag{2.14}$$

where $\mathbf{x} = (x^1, x^2)$ etc. Note that (2.14) follows from (2.1) and (2.2). Thus

$$|\Phi(p_1) - \Phi(q_1)| = |\mathbf{x}(-1) + \mathbf{y}(-1)| < C'\tau(0). \tag{2.15}$$

and similarly for $|\Phi(p_2) - \Phi(q_2)|$. □

The following result will help us use transversality to prove an energy estimate:

Lemma 2.7. *Let $\gamma : [0, \ell]^M$ be a shortest geodesic with $\gamma(0) \in \mathcal{C}_1$ and $\gamma(\ell) \in \mathcal{C}_2$. Let $p \in \Pi(\tilde{U})$, which does not lie on the geodesic segment γ.*

Let $\delta = d_M(p, \gamma[0, \ell])$, i.e. the distance between p and the closest points in the geodesic segment γ. Then for any $\alpha > 0$

$$\int_0^\ell d_M(p, \gamma(t))^{-\alpha} dt < C\delta^{-\alpha+1}. \tag{2.16}$$

Proof. Indeed, this follows quite ready from (2.1) and (2.2): set $\mathbf{x} = (x^1, x^2) = \Phi(p)$, and $\mathbf{y}(t) = \Phi \circ \gamma(t)$. Let t_0 be such that $\mathbf{y}(t_0) \in \mathbb{R} \times \{x^2\}$. Then for any t, by the inequalities (2.1) and (2.2),

$$\begin{aligned} \left|x^2 - y^2(t)\right| &\geq 0.9\lambda \left|t - t_0\right| \\ \left|x^1 - y^1(t)\right| &\geq \min(0, \delta - 0.1\lambda \left|t - t_0\right|). \end{aligned}$$

Set $\tau = \frac{\delta}{\lambda}$; note that if $|t - t_0| > \tau$ then

$$\left|x^2 - y^2(t)\right| > 0.9\delta$$

and if $|t - t_0| < \tau$ then

$$\left|x^1 - y^1(t)\right| \geq \delta/2$$

Using this, we see that

$$\begin{aligned} \int_0^\ell d_M(p, \gamma(t))^{-\alpha} dt &= \int_{|t-t_0|<\tau} + \int_{|t-t_0|>\tau} d_M(p, \gamma(t))^{-\alpha} dt \\ &\leq (0.9\lambda)^{-1} \int_{0.9\delta}^\infty \rho^{-\alpha} d\rho + \tau\delta^{-\alpha} \\ &\leq C\delta^{-\alpha+1}. \end{aligned}$$

□

We can now prove the following:

Lemma 2.8. *Let $p_1, q_1 \in \mathcal{C}_1$ and $p_2, q_2 \in \mathcal{C}_2$. Let $\ell_p = d_M(p_1, p_2)$, i.e. the length of the geodesic connecting p_1 with p_2 and $\ell_q = d_M(q_1, q_2)$. Let $\delta = d_M(p_1, q_1) + d_M(p_2, q_2)$. Then*

$$\int_0^{\ell_p} \int_0^{\ell_q} d_M \left(\gamma_{p_1,p_2}(t), \gamma_{q_1,q_2}(s)\right)^{-\alpha} \, dt \, ds < C\delta^{-\alpha+1} \tag{2.17}$$

$$\mathcal{L} \times \mathcal{L} \left\{(s, t) : d_M((\gamma_{p_1,p_2}(t), \gamma_{q_1,q_2}(s)) < \rho\right\} < C\frac{\rho^2}{\delta}. \tag{2.18}$$

where γ_{p_1,p_2} and γ_{q_1,q_2} are as in (2.4)*.*

Proof. As in Lemma 2.6, we set $\mathbf{x}(a)$ to be the unique point on the Φ image of the geodesics γ_{p_1,p_2} with its second coordinate equal to a, and in a similar way we define $\mathbf{y}(a)$ for the geodesics γ_{q_1,q_2}. Take

$$\mathbf{v}(a) = \frac{\frac{d\mathbf{x}}{da}}{\left|\frac{d\mathbf{x}}{da}\right|}$$
$$\mathbf{w}(a) = \frac{\frac{d\mathbf{y}}{da}}{\left|\frac{d\mathbf{y}}{da}\right|}.$$

We start by proving (2.18), which by (2.1) would follow from

$$\mathcal{L} \times \mathcal{L}\left\{(a,b) \in [-1,1]^2 : |\mathbf{x}(a) - \mathbf{y}(b)| < c_1\rho\right\} < C_1\frac{\rho^2}{\delta} \tag{2.19}$$

for any sufficiently small c_1. An immediate observation is that since $|\mathbf{x}(a) - \mathbf{y}(b)| \geq |a - b|$, and since the direction of the curves $\mathbf{x}(a)$ and $\mathbf{y}(a)$ is constrained by (2.1) and (2.2) we have that

$$\begin{aligned}|\mathbf{x}(a) - \mathbf{y}(a)| &\leq |\mathbf{x}(a) - \mathbf{y}(b)| + |\mathbf{y}(a) - \mathbf{y}(b)| \\ &\leq |\mathbf{x}(a) - \mathbf{y}(b)| + 2\,|a - b| \\ &\leq 3\,|\mathbf{x}(a) - \mathbf{y}(b)|\,.\end{aligned}$$

In particular,

$$\begin{aligned}\mathcal{L} \times \mathcal{L}\left\{(a,b) \in [-1,1]^2 : |\mathbf{x}(a) - \mathbf{y}(b)| < c_1\rho\right\} \leq \\ 2c_1\rho\mathcal{L}\left\{a \in [-1,1] : |\mathbf{x}(a) - \mathbf{y}(a)| < 3c_1\rho\right\}.\end{aligned} \tag{2.20}$$

Consider the sets

$$Q_1 = \{a \in [-1,1] : |\mathbf{x}(a) - \mathbf{y}(a)| < 3c_1\rho\}$$
$$Q_2 = \{a \in [-1,1] : |\mathbf{x}(a) - \mathbf{y}(a)| < c_2\delta\}$$

where $c_2 = c/100$, with c the constant appearing in (2.10) in the statement of Lemma 2.6.

Sublemma 2.9. *Let A be a connected component of Q_2. Then*

(1) *there is a unique $a_0 \in A$ so that $\mathbf{x}(a_0) = \mathbf{y}(a_0)$,*
(2) *$Q_1 \cap A$ is connected,*
(3) *$|\mathbf{x}(a) - \mathbf{y}(a)|$ is monotone increasing for $a > a_0$ in A, and monotone decreasing for $a < a_0$ in A. Furthermore, for any $a \in A$,*
$$\left|\tfrac{d}{da}(x^1(a) - y^1(a))\right| \geq c_2\delta. \tag{2.21}$$

In order to prove the sublemma, one only needs to show that there is some $a_0 \in A$ so that $\mathbf{x}(a_0) = \mathbf{y}(a_0)$, and that (2.21) holds, as the rest of the assertions in this sublemma follow.

We start with (2.21). Clearly $a \mapsto (x^1(a) - y^1(a))$ is differentiable, and

$$|\mathbf{v}(a) - \mathbf{w}(a)| \leq 2\left|\tfrac{d}{da}x^1(a) - \tfrac{d}{da}y^1(a)\right| \tag{2.22}$$

so if $a \in A$ and (2.21) does not hold

$$|\mathbf{x}(a) - \mathbf{y}(a)| + |\mathbf{v}(a) - \mathbf{w}(a)| < 3c_2\delta \tag{2.23}$$

in contradiction to (2.10).

Take a_0 to be an element of $\overline{A}$ where $|\mathbf{x}(a) - \mathbf{y}(a)|$ is minimal. Since certainly $|\mathbf{x}(a_0) - \mathbf{y}(a_0)| < c_2\delta$, we see that in fact $a_0 \in A$. We also know that $\frac{d}{da}(x^1(a) - y^1(a))$ is nonzero at a_0; the only way to reconcile this with a_0 minimizing this difference is if $\mathbf{x}(a_0) - \mathbf{y}(a_0) = 0$. This establishes the sublemma.

Let $A' = A \cap Q_1$, and set

$$M = \min_{a \in \overline{A'}} \left| \tfrac{d}{da}(x^1(a) - y^1(a)) \right| \geq c_2\delta,$$

with $a_1 \in \overline{A'}$ the point where this minimum is achieved. Clearly the length of A' satisfies

$$\mathcal{L}(A') \leq \frac{3c_1\rho}{M}. \tag{2.24}$$

On the other hand, by (2.1) and (2.2)

$$|\mathbf{v}(a) - \mathbf{w}(a)| \leq 2 \left| \tfrac{d}{da}(x^1(a) - y^1(a)) \right| \leq 4 \left| \mathbf{v}(a) - \mathbf{w}(a) \right|;$$

by (2.13), we have that for any $a \in [-1, 1]$,

$$\begin{aligned} \left| \tfrac{d}{da}(x^1(a) - y^1(a)) \right| &\leq 2 \exp(c_0 \left| a - a_1 \right|) \left[|\mathbf{x}(a_1) - \mathbf{y}(a_1)| + |\mathbf{v}(a_1) - \mathbf{w}(a_1)| \right] \\ &\leq 2 \exp(c_0 \left| a - a_1 \right|)[3c_1\rho + 2M] \leq c_3 M, \end{aligned}$$

so that

$$|\mathbf{x}(a) - \mathbf{y}(a)| \leq c_3 M \left| a - a_1 \right| + 3c_1\rho$$

and so (assuming ρ is much smaller than δ)

$$\mathcal{L}(A) \geq \frac{c_2\delta - 3c_1\rho}{c_3 M} \geq \frac{c_2\delta}{2c_3 M} \tag{2.25}$$

Equations (2.25) and (2.24) imply that

$$\frac{\mathcal{L}(A')}{\mathcal{L}(A)} \leq \frac{6c_1c_3\rho}{c_2\delta}.$$

Summing over all connected components of Q_2, we get that $\mathcal{L}(Q_1) \leq \frac{6c_1c_3\rho}{c_2\delta}\mathcal{L}(Q_2)$, and since $\mathcal{L}(Q_2) \leq 2$, this completes the proof of (2.18).

We still have to prove (2.17). In terms of $\mathbf{x}, \mathbf{y}$ this amounts to showing that there is some constant, say C_2, so that

$$\int_{-1}^{1} da \int_{-1}^{1} db \, \left| \mathbf{x}(a) - \mathbf{y}(b) \right|^{-\alpha} < C_2 \delta^{-\alpha+1}. \tag{2.26}$$

Let Q_1, Q_2 be as above; since we will vary ρ, we will explicitly write this parameter, so in other words

$$Q_1(\rho) = \{a \in [-1,1] : |\mathbf{x}(a) - \mathbf{y}(a)| < 3c_1\rho\}.$$

Since $|\mathbf{x}(a) - \mathbf{y}(a)| \leq 2\min_{b\in[-1,1]} |\mathbf{x}(a) - \mathbf{y}(b)|$, for any $a \in [-1,1]\backslash Q_2$, by Lemma 2.7 we have that for any ρ

$$\int_{Q_1(2\rho)\backslash Q_1(\rho)} da \int_{-1}^{1} db\ |\mathbf{x}(a) - \mathbf{y}(b)|^{-\alpha} \leq C_3\rho^{-\alpha+1}. \tag{2.27}$$

We have shown that for $\rho < \frac{c_2\delta}{6c_1}$,

$$\mathcal{L}(Q_2(\rho)) \leq \frac{12c_1c_3\rho}{c_2\delta}, \tag{2.28}$$

so setting $c_4 = \frac{c_2}{6c_1}, \quad c_5 = \frac{12c_1c_3}{c_2}$

$$\begin{aligned}
&\int_{-1}^{1} da \int_{-1}^{1} db\ |\mathbf{x}(a) - \mathbf{y}(b)|^{-\alpha} \leq \\
&\quad \leq \int_{[-1,1]\backslash Q_2(c_4\delta)} da \int_{-1}^{1} db\ |\mathbf{x}(a) - \mathbf{y}(b)|^{-\alpha} + \\
&\qquad + \sum_{n>0} \int_{Q_2(c_4 2^{-n+1}\delta)\backslash Q_2(c_4 2^{-n}\delta)} da \int_{-1}^{1} db\ |\mathbf{x}(a) - \mathbf{y}(b)|^{-\alpha} \\
&\quad \leq C'\delta^{-\alpha+1}\left[1 + \sum_{n>0} \mathcal{L}\left(Q_2(c_4 2^{-n+1}\delta)\right) 2^{(n-1)(\alpha-1)}\right] \qquad (2.29) \\
&\quad \leq C'\delta^{-\alpha+1}\left[1 + c_4c_5 \sum_{n>0} 2^{-(n-1)} 2^{(n-1)(\alpha-1)}\right] \\
&\quad \leq C''\delta^{-\alpha+1}
\end{aligned}$$

with the sum converging since $1 \leq \alpha < 2$. □

Finally, we are in position to prove:

Proof of Proposition 2.5. Let $\tilde{\nu}$ be a locally invariant probability measure on $\tilde{U}$ with finite α-energy. Initially, assume $\alpha < 2$.

By definition of local invariance, there is a probability measure $\overline{\nu}$ on $\mathcal{C}_1 \times \mathcal{C}_2$, so that in order to choose randomly a point $\xi \in \tilde{U}$ according to $\tilde{\nu}$ one can choose a pair (p_1, p_2) according to the probability measure $\overline{\nu}$, and then choose a point p on the geodesics γ_{p_1,p_2} connecting p_1, p_2 uniformly according to the appropriately normalized arc length measure. ξ would be the tangent vector to γ_{p_1,p_2} at p. This is equivalent to the condition given earlier that $\Psi^{-1}(\tilde{\nu})$ is a product measure (Ψ as in

(2.4)). Since Ψ is a diffeomorphism, the fact that $\Psi^{-1}(\tilde{\nu})$ is a product measure of Lebesgue measure with the image under a diffeomorphism of $\overline{\nu}$ implies that

$$\mathcal{E}_\alpha(\tilde{\nu}) \asymp \mathcal{E}_{\alpha-1}(\overline{\nu}). \tag{2.30}$$

However, it is also true that

$$\mathcal{E}_{\alpha-1}(\overline{\nu}) \asymp \mathcal{E}_\alpha(\nu'),$$

since Lemma 2.8 gives us that

$$\begin{aligned}\mathcal{E}_\alpha(\nu') = &\iint_{(\mathcal{C}_1\times\mathcal{C}_2)^2} d\nu'(p_1,p_2)d\nu'(q_1,q_2)\int_0^{d_M(p_1,p_2)} ds \\ &\qquad\qquad\int_0^{d_M(q_1,q_2)} dt d_M(\gamma_{p_1,p_2}(s),\gamma_{q_1,q_2}(t))^{-\alpha} \\ &\le \iint_{(\mathcal{C}_1\times\mathcal{C}_2)^2} d\nu'(p_1,p_2)d\nu'(q_1,q_2)[d_M(p_1,q_1)+d_M(p_2,q_2)]^{-\alpha+1} \\ &\asymp \mathcal{E}_{\alpha-1}(\overline{\nu}).\end{aligned}$$

This concludes the proof of the proposition for $\alpha < 2$.

Suppose now that $\alpha \ge 2$. We need the following sublemma, which in a slightly different formulation was used in [14]:

Sublemma 2.10. *Let μ be a measure on a d-dimensional compact Riemannian manifold with boundary X, satisfying that*

$$\underline{\lim}\, r^{-d}\iint d\mu(x)d\mu(y)\mathbf{1}(d_X(x,y)<r)<\infty. \tag{2.31}$$

Then μ is absolutely continuous with respect to the Riemannian volume measure vol on X, and furthermore the Radon-Nikodym derivative $\frac{d\mu}{d\,vol}$ is in $L^2(vol)$.

For the convenience of the reader, we repeat the arguments from [14]. Let $B_X(x,r)$ denote the open ball of radius r around x in X. We can rewrite (2.31) as

$$\underline{\lim}\, r^{-d}\int \mu(B_X(x,r))d\mu(x)<\infty \tag{2.32}$$

By Fatou's lemma, we have that (2.32) implies that μ-almost everywhere, $\underline{\lim}\,\mu(B_X(x,r))/r^d<\infty$, hence by the standard theorems on the differentiability of measures (see [12]) ν is regular with respect to the Riemannian volume. The same theorems then give that almost surely

$$\frac{d\mu}{d\,vol} = \lim \mu(B_X(x,r))/r^d$$

so using (2.32) once again we get that $\frac{d\mu}{d\,vol} \in L^2$. This concludes the proof of the sublemma.

Applying the sublemma, it suffice to show that

$$r^{-2} \iint d\nu'(x) d\nu'(y) \mathbf{1}(d_M(x,y) < r) \tag{2.33}$$

remains bounded as $r \to 0$. We rewrite this integral as

$$r^{-2} \iint d\overline{\nu}(p_1, p_2) d\overline{\nu}(q_1, q_2) \iint ds\, dt\, \mathbf{1}(d_M(\gamma_{p_1,p_2}(s), \gamma_{q_1,q_2}(t)) < r). \tag{2.34}$$

By Lemma 2.8, equation (2.18),

$$\begin{aligned}(2.34) &\leq r^{-2} \iint d\overline{\nu}(p_1, p_2) d\overline{\nu}(q_1, q_2) \frac{r^2}{d_M(p_1, q_1) + d_M(p_2, q_2)} \\ &\asymp \mathcal{E}_1(\overline{\nu}) \asymp \mathcal{E}_2(\tilde{\nu}).\end{aligned}$$

□

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