CORRECTIONS AND ADDITIONS TO "ENTROPY OF QUANTUM LIMITS"

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1. Introduction and some general comments

In our paper [1], we have proved the following for Γ a congruence sublattice of a cocompact lattice in $\mathrm{SL}(2,\mathbb{R})$ coming from $\mathbb{R}$-split quaternion algebras over $\mathbb{Q}$. For every $\epsilon, \tau > 0$ let

$$B(\epsilon,\tau) = a((-\tau,\tau))u^-((-\epsilon,\epsilon)u^+((-\epsilon,\epsilon)) \qquad B(\epsilon) = B(\epsilon,\epsilon)$$

for $a(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, $u^-(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, $u^+(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Throughout this note we fix $\tau_0 = 1/50$ (say).

Theorem 1 ([1, Thm. 2.1])**.** *Let μ be an arithmetic quantum limit*[1] *for Γ. Then for every compact $\Omega \subset X = \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ for any $x \in \Omega$*

$$\mu(xB(\epsilon,\tau_0)) \ll_\Omega \epsilon^{\kappa'} \tag{1}$$

for $\kappa' = 2/9$.[2]

This theorem implies that every ergodic component of a quantum limit μ under the flow $a(t)$ has entropy[3] $\geq \kappa'$. In a later paper [2], the second named author has been able to use Theorem 1 in conjunction with a partial classification of measures on X invariant under the geodesic flow that satisfy a recurrence property under the Hecke correspondence to prove that the only arithmetic quantum limit is the Haar Lebesgue measure. For that application one only needs to know that the entropy of every ergodic component of a quantum limit μ under $a(t)$ is positive (or equivalently, that μ is $u^+(s)$ recurrent, see [2, Thm 7.6]).

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[1]See [1, p. 155] for definition; in particular μ is a finite measure on $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ invariant under the action of the diagonal group.

[2]We quote the theorem in a form which is also suitable for Γ nonuniform; for Γ cocompact one can simply take $\Omega = \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$.

[3]The speed of the geodesic flow is normalized so that the entropy of the Haar Lebesgue measure is 2

It should be noted, however, that the proof of Theorem 1 is completely effective whereas the ergodic theoretic arguments used to obtain arithmetic quantum unique ergodicity are ineffective. In addition, Theorem 1 holds not only for quantum limits but for a much wider class of measures, which need not even be $a(t)$-invariant, for example, all measures m of the form

$$m(A) = \int_A |\Phi(x)|^2 \, d\,vol \qquad \begin{array}{r}\text{for } \Phi \in L^2(X) \text{ eigenfunction} \\ \text{of all Hecke operators,}\end{array} \tag{2}$$

so even though for the main application the value of κ' is immaterial it may conceivably be of some interest in other applications.

In [1] Theorem 1 has also been claimed for congruence sublattices of $\mathrm{SL}(2, \mathbb{Z})$, which our of course not cocompact. In [1, Sect. 4] we explain how to modify the proof in the compact case to the finite area one. That argument, however, contains a gap. The difficulty is with the very last sentence of that section: the ring $\mathbb{Q}(\alpha)$ can be isomorphic either to a real quadratic field $\mathbb{Q}(\sqrt{D})$ or (and this possibility was overlooked in [1]) to $\mathbb{Q} \oplus \mathbb{Q}$ depending on whether the characteristic polynomial of a certain matrix $\alpha \in M_2(\mathbb{Z}) \cap \mathrm{GL}(2, \mathbb{R})$ is irreducible over $\mathbb{Q}$ or not.

As we shall explain presently, the second possibility arises when one tries to estimate a measure of a small ball around a point x very close to a point whose trajectory under $a(t)$ (which is essentially the geodesic flow on the unit cotangent bundle of $\Gamma\backslash\mathbb{H}$) goes to a cusp of X.

We recall the (classical) maximal inequality for flows: for every $f \in L^1(\mu)$ defined the maximal function $M(f, x)$ by

$$M(f, x) = \sup_{T>0} \frac{1}{T} \int_0^T f(xa(t))dt.$$

if μ is $a(t)$-invariant,

$$\mu \left\{ x \in X : M(f, x) \geq R \|f\|_{1,\mu} \right\} \leq \frac{1}{R}.$$

In the nonuniform case, the method of [1] actually gives the following slightly weaker form of Theorem 1 (which again holds more generally for any m as in (2)):

Theorem 2. *Let μ be an arithmetic quantum limit for Γ. Then for every compact $\Omega \subset X = \Gamma \backslash \mathrm{SL}(2, \mathbb{R})$ for any $x \in \Omega$, either*

$$\mu(xB(\epsilon, \tau_0)) \ll_\Omega \epsilon^{\kappa'} \tag{3}$$

or $M(1_{\Omega^\complement}, x) > 0.199$ (with the same $\kappa' = 2/9$ as above).

In particular, choosing Ω so that $\mu(\Omega^{\complement}) < \delta$ one sees that (3) holds outside of a set of measure 5.1δ (that does not depend on ϵ). It follows that for μ almost every x,

$$\varliminf_{\epsilon\to 0} \frac{\log \mu(xB(\epsilon,\tau))}{\log \epsilon} \geq \kappa'$$

from which it follows that the entropy of every ergodic component of μ is at least κ' even in the nonuniform case. This is, of course, more than enough for the main application [2], which in fact shows that μ is a constant multiple of the Haar-Lebesgue measure.

There is a variant of the method we have used in [1] which avoids the sieving argument of [1, Sect. 5] as well as the difficulty we discuss here regarding the non-compact case. This method actually allows one to prove Theorem 1 above (for quantum limits and for measures as in (2), possibly with a somewhat lower κ'). This approach has been found independently by us and by Lior Silberman and Akshay Venkatesh, who have developed it substantially further. However, we do not believe this approach supersedes the original method, and have opted to publish the argument presented below for several reasons:

First, it is the most direct way to overcome the gap in [1, Sect. 4], and has the advantage that the value of $\kappa' = 2/9$ need not be changed.

Second, as is hinted in [2] and will be more leisurely explained in [3], our assumptions regarding μ in [1] and this current note are quite weak and are applicable in other, completely different scenarios, such as the following: let $\tilde{\mu}$ be a probability measure on $\mathrm{SL}(2,\mathbb{Q})\backslash \mathrm{SL}(2,\mathbb{A}_{\mathbb{Q}})$ invariant under the Adelic diagonal group. Let μ be its push forward under the natural projection of $\mathrm{SL}(2,\mathbb{Q})\backslash \mathrm{SL}(2,\mathbb{A}_{\mathbb{Q}})$ to $\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})$. Then it can be shown that μ satisfies Theorem 2 (hence, in conjunction with [2], μ is the Haar-Lebesgue measure).

Third, Silberman and Venkatesh have extended our results, along the lines of [1] but with several new ideas, to more general $\Gamma\backslash G$. It is our hope that they will present the alternative method while explaining their more general results in the forthcoming [4].

Finally, which is not unrelated to the third reason, we believe the method presented in this note is of independent interest and potentially could resolve difficulties related to compactness in the more general situations considered by Silberman and Venkatesh.

We thank Lior Silberman and Akshay Venkatesh for pointing out to us the gap in [1, Sect. 4]. We also wish to take this opportunity to note a few typos in [1]: on p. 159, Thm. 3.5 part (4), $|W| \gg \epsilon^{-\kappa'}$ and not as shown, and in p. 160, line 6, p is a prime $\leq n_2^{1/2}$.

2. Proof of Theorem 2

Fix $\Omega \subset \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ compact, as well as a relatively compact subset $\tilde{\Omega} \subset \mathrm{SL}(2,\mathbb{R})$ projecting bijectively to Ω. We shall use the same letter x to denote a point in Ω as well as the corresponding point in $\tilde{\Omega}$. The proof in [1, Sect. 4] gives for any $x \in \Omega$ that either

$$\mu(xB(\epsilon,\tau_0)) \ll_\Omega \epsilon^{\kappa'} \tag{4}$$

or that there exists an irreducible[4] $\alpha \in M_2(\mathbb{Z})$ with the following properties:

(A-1) $N = \det\alpha$ is a positive integer $\ll_\Omega \epsilon^{-4\kappa'}$
(A-2) the eigenvalues of α are two distinct integers, say v_1, v_2.
(A-3) $N^{-\frac{1}{2}}\alpha x \in xB(4\epsilon, 3\tau_0)$.

it follows from (A-3) that $\|\alpha\| \ll N^{\frac{1}{2}}$ (so in particular $v_1, v_2 \ll N^{\frac{1}{2}}$)[5].

We now make the following three elementary observations:

Lemma 3. *Let $\alpha \in M_2(\mathbb{Z})$ have two distinct integer eigenvalues v_1, v_2. Then α has integer row eigenvectors $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{Z}^2$ with $\|\mathbf{z}_1\| \cdot \|\mathbf{z}_2\| < 2\|\alpha\|$.*

Proof. Consider $\alpha' = \alpha - v_1 I$. This is a degenerate 2×2 integer matrix, $\|\alpha'\| < 2\|\alpha\|$, and necessarily has the form $\begin{pmatrix} rp & rq \\ sp & sq \end{pmatrix}$. Direct calculation gives that the eigenvectors of this matrix are $(s, -r)$ and (p, q). □

Lemma 4. *There is some constant $c > 0$ so that for any n if $\epsilon_1, \dots, \epsilon_n$ are positive real numbers satisfying $\sum \epsilon_i < c$ then $B(\epsilon_1)B(\epsilon_2)\dots B(\epsilon_n) \subset B(1) \cap [B(1)^{-1}]$.*

Proof. For sufficiently small ϵ, $B(\epsilon)$ is comparable to a ϵ ball around the identity according to a right invariant metric on $\mathrm{SL}(2,\mathbb{R})$, where one can use the triangle inequality. □

Lemma 5. *Suppose $g \in B(\epsilon,\tau)$ and $\mathrm{tr}(g) = \cosh t_0$. Then either $g \in a(t_0)B(O_\tau(\epsilon))$ or $g \in a(-t_0)B(O_\tau(\epsilon))$.*

Proof. Suppose $g = a(t)u^-(s_-)u^+(s_+)$ with $|t| < \tau$ and $|s_\pm| < \epsilon$. A direct calculation shows $\mathrm{tr}(g) = e^t(1 + s_-s_+) + e^{-t}$, hence

$$|\cosh t_0 - \cosh t| < O_\tau(\epsilon^2).$$

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[4]I.e. is not a nontrivial integer multiple of another integer matrix

[5]for definiteness, we use the ℓ^∞-norm for vectors, and the corresponding operator norm for matrices

Without loss of generality assume $t_0 t > 0$. Integrating the inequality $|x| < |\sinh x|$, it follows that $|t_0^2 - t^2| < 2\,|\cosh t_0 - \cosh t|$, hence $|t_0 - t| < O_\tau(\epsilon)$, and

$$g = a(t_0)a(t-t_0)u^-(s_-)u^+(s_+) \in a(t_0)B(O_\tau(\epsilon)).$$

□

From (A-2), (A-3) together with Lemma 5, we have

$$\begin{aligned} N^{-\frac{1}{2}}\alpha x &\in xa(t_0)B(C\epsilon) \\ N^{-\frac{1}{2}}\bar{\alpha} x &\in xa(-t_0)B(C\epsilon) \end{aligned} \tag{5}$$

with $\bar{\alpha} = N\alpha^{-1} \in M_2(\mathbb{Z})$ and $2t_0 = \pm\log(v_1/v_2)$. From (A-3) we see that $|t_0| \ll 1$, and since v_1, v_2 are integers $\ll N^{\frac{1}{2}}$ we have that $N^{-\frac{1}{2}} \ll t_0 \ll 1$.[6]

Suppose (without loss of generality) $\|\mathbf{z}_1\| \geq \|\mathbf{z}_2\|$; we assume $t_0 > 0$ otherwise replace α by $\bar{\alpha}$ in what follows.

From (5) it follows that

$$\begin{aligned} N^{-m/2}\alpha^m x &\in xa(t_0)B(C\epsilon)a(t_0)B(C\epsilon)\dots a(t_0)B(C\epsilon) \\ &\subset xa(mt_0)B(C\epsilon)B(Ce^{2t_0}\epsilon)\dots B(Ce^{2(m-1)t_0}\epsilon).. \end{aligned}$$

Note that $C(\epsilon + \dots + e^{2(m-1)t_0}\epsilon \leq 2C\epsilon t_0^{-1}e^{2mt_0}$, so as long as

$$2C\frac{e^{2mt_0}}{t_0}\epsilon < c \tag{6}$$

we can conclude from Lemma 4 that

$$xa(mt_0) = N^{-m/2}\alpha^m xb \qquad \text{for some } b \in B(1).$$

Apply both sides of the above equation to the (row) integer eigenvector $\mathbf{z}_2$ of α with eigenvalue v_2. We get that

$$\|\mathbf{z}_2 xa(mt_0)\| = N^{-m/2}v_2^m\,\|\mathbf{z}_2 b\| \ll e^{-mt_0}N^{1/4} \tag{7}$$

where we estimated $\|\mathbf{z}_2\|$ by Lemma 3 and used the fact that $\|\mathbf{z}_2\| \leq \|\mathbf{z}_1\|$.

For any compact $\Omega \subset \Gamma\backslash\operatorname{SL}(2,\mathbb{R})$ ($\Gamma < \operatorname{SL}(2,\mathbb{Z})$ congruence) there is a constant $c_\Omega > 0$ so that for every $x' \in \tilde{\Omega}$, for every integer vector $\mathbf{z}$, we have that $\|\mathbf{z}x'\| > c_\Omega$ (this is easily seen directly and follows from

[6] Actually, it can be shown that if (4) fails, α can be chosen so that $v_1 = p$ or p^2 and $v_2 = q$ or q^2 with p, q distinct primes $\leq \epsilon^{-\kappa'}$, hence $t_0 \gg \epsilon^{-\kappa'}$. This improves the constant $1/5 - \delta$ of the theorem to $3/7 - \delta$.

Mahler's criterion). Thus it follows from $t_0 \gg N^{-\frac{1}{2}}$, (6) and (7) that for

$$N^{\frac{1}{2}} \ll_\Omega e^{2mt_0} \ll N^{-\frac{1}{2}}\epsilon^{-1} \tag{8}$$

we have that $xa(mt_0) \notin \Omega$. Since $t_0 \ll 1$, by slightly changing the implicit constants in (8) we have that $xa(t) \notin \Omega$ for every t in the range

$$N^{\frac{1}{2}} \ll_\Omega e^t \ll N^{-\frac{1}{2}}\epsilon^{-1}.$$

We conclude that for ϵ sufficiently small depending on Ω (and suitable arbitrarily small $\delta > 0$) if x is such that (4) fails

$$M(1_{\Omega^\complement}, x) > \frac{\log(N^{-\frac{1}{2}}\epsilon^{-1}) - \log(N^{\frac{1}{2}})}{\log(N^{-\frac{1}{2}}\epsilon^{-1})} - \delta \geq \frac{1-4\kappa'}{1-2\kappa'} - \delta \geq 0.199.$$

References

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[4] Lior Silberman and Akshay Venkatesh. On quantum unique ergodicity for locally symmetric spaces II. in preparation, 2004.