

EXISTENCE AND WEYL'S LAW FOR SPHERICAL CUSP FORMS

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ABSTRACT. Let \mathbf{G} be a split adjoint semisimple group over \mathbb{Q} and $K_\infty \subset \mathbf{G}(\mathbb{R})$ a maximal compact subgroup. We shall give a uniform, short and essentially elementary proof of the Weyl law for cusp forms on congruence quotients of $\mathbf{G}(\mathbb{R})/K_\infty$. This proves a conjecture of Sarnak for \mathbb{Q} -split groups, previously known only for the case $\mathbf{G} = \mathrm{PGL}(n)$. The key idea amounts to a new type of simple trace formula.

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1. INTRODUCTION

Let M be a Riemannian manifold. It was proved by Weyl that if M is compact, the number of Laplacian eigenvalues less than T is asymptotic to $c(M)T^{\dim(M)/2}$. Here $c(M)$ is the product of the volume of M , the volume of the (Euclidean) unit ball in $\mathbb{R}^{\dim(M)}$, and $(2\pi)^{-\dim(M)}$. In general, if M is noncompact but of finite volume, there is no reason to expect that the Laplacian has any nontrivial discrete spectrum. In particular, the work of Phillips-Sarnak [19] and Wolpert [25] indicates that for a generic nonuniform lattice $\Gamma \subset \mathrm{PGL}(2, \mathbb{R})$, the Laplacian of the quotient $\Gamma \backslash \mathbb{H}^2$ should have only finite discrete spectrum.

On the other hand, Selberg [22] has shown that for $M = \Gamma \backslash \mathbb{H}^2$, where Γ is a congruence subgroup of $\mathrm{PGL}(2, \mathbb{Z})$, the Weyl asymptotic holds for the discrete spectrum of the Laplacian, i.e. the asymptotics of the eigenvalues of the Laplacian behave as if the surface M were compact. In order to prove this, Selberg developed his celebrated trace formula.

It has been conjectured by Sarnak [21] that the same should hold in the general setting of congruence quotients of noncompact symmetric spaces, that is to say, M of the form $\Gamma \backslash \mathbf{G}(\mathbb{R})/K$ where G is a semisimple algebraic group defined over \mathbb{Q} , $K \subset \mathbf{G}(\mathbb{R})$ a maximal compact subgroup, and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ a congruence subgroup.

In his thesis, S. D. Miller [16] established this conjecture for $\mathbf{G} = \mathrm{PGL}(3)$. More recently, W. Müller [18] has established this conjecture for $\mathbf{G} = \mathrm{PGL}(n)$. The upper bound, i.e. that the number of cusp forms with Laplacian eigenvalue at most T is asymptotically no more than that given by Weyl’s bound, was proved for general \mathbf{G} by Donnelly [3].

The work of Selberg, Miller, and Müller rely essentially on the theory of Eisenstein series, in particular, estimates on their behavior near the unitary axis. These estimates – which are quite delicate in the higher rank case – are used to explicitly control the contribution of the noncuspidal spectrum to an appropriate trace formula (in the work of Miller, to a pretrace formula).

In the present paper we give a simple proof of the Weyl law for cusp forms, valid for any split adjoint group \mathbf{G} over \mathbb{Q} , that is to say:

Theorem 1. *Let \mathbf{G} be a split adjoint semisimple group over \mathbb{Q} , $G_\infty = \mathbf{G}(\mathbb{R})$ and $K_\infty \subset G_\infty$ a maximal compact subgroup, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ a congruence subgroup. Let $M = \Gamma \backslash G_\infty / K_\infty$ be the associated locally symmetric space, endowed with the Riemannian metric that corresponds to the Killing form on the Lie algebra of G_∞ (see page 8). Let $N(T)$ be the number of cuspidal (see (4.1)) eigenfunctions of the Laplacian with eigenvalue $\leq T$. Then*

$$N(T) \sim c(M)T^{\dim(M)/2},$$

where $c(M)$ is as above.

Our approach is based on the observation that there are strong relations between the spectrum of the Eisenstein series at different places. This allows us, after passing to an S -arithmetic setting, to construct convolution operators whose image can be seen directly (even without knowing anything about Eisenstein series) to be purely cuspidal. In this way, the delicate estimates on Eisenstein series which were the hard part in the works of Selberg, Miller and Müller are entirely avoided. Once these operators are constructed (and using Donnelly’s upper bound), the Weyl law can be proved for general split groups by applying a pretrace formula (5.2) in a way that is very similar to that used by Miller in his proof of Weyl’s law for $\mathrm{PGL}(3, \mathbb{Z})$.

Already in the classical case of Maass forms on $\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathrm{PGL}(2, \mathbb{R})$ – i.e., even Maass forms on the modular surface – the proof seems to be new. Indeed, we are not aware of any proof that there are any even cusp forms on $\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}$ that does not require explicit bounds on the constant term of the Eisenstein series. In order to illustrate several of the main ideas of the paper in a relatively simple context we give an elementary and self-contained proof for the existence of infinitely many such forms in §2.

The idea of using convolution operators with purely cuspidal image is not completely new, and is for example the basis of what is known as the “simple trace formula”: see, for example, [6]. However, previous constructions used convolution operators which factor into the composition of independent convolution operators at each place. Such operators have not only the Eisenstein series but any everywhere unramified automorphic form in their kernel (all Hecke-Maass forms on $\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}$ are by definition everywhere unramified). This type of trace formula was used by Labesse and Müller [13] to prove a “weak Weyl law”, but cannot be used to prove the full Weyl law for precisely the reason just observed. In a sense, the main observation of the present paper is that there is a much larger class of

convolution operators with cuspidal image than those which have been previously utilized.

We have not aimed for the greatest generality. The approach described will probably apply much more generally than split adjoint groups over \mathbb{Q} ; we have restricted ourselves to this case mainly for simplicity of exposition. In order both to illustrate some of the techniques and to normalize the constants involved with minimal amount of computation we also prove Weyl's law for compact quotients — i.e. for groups which have \mathbb{Q} -rank zero. For these compact quotients Weyl's law is well known — indeed, the original Weyl's law is applicable, and moreover an explicit analysis of the error term using the tools of harmonic analysis on Lie groups was given by Duistermaat, Kolk and Varadarajan [5].

We do not believe that the present approach supersedes the approach via the trace formula. For instance, while we have not attempted to extract explicit estimates on the error term, we expect the trace formula approach, when it can be made to work, should at least in theory yield a substantially better error term than what can be extracted from the proof we give here. Also, our approach gives no information about cusp forms whose spectral parameters at the different places happen to satisfy the same relations as those satisfied by Eisenstein series.

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2. EXISTENCE OF CUSP FORMS FOR $\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}$

We first briefly give a proof of the infinitude of cusp forms for the special case of $\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}$. In this case, we can interpret the method quite concretely in terms of a wave equation on hyperbolic space; in the higher rank case, we will use convolution operators instead. As we will explain at the end of this section, the operator we construct explicitly using the wave equation can also be interpreted as a convolution operator.

Let $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ be the hyperbolic Laplacian on \mathbb{H} . Then $L^2(\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}) = \langle 1 \rangle \oplus L_{\mathrm{cts}}^2 \oplus L_{\mathrm{cusp}}^2$, where L_{cts}^2 is spanned by the continuous spectrum of Δ on $\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}$, L_{cusp}^2 is the cuspidal part, on which Δ acts discretely. Explicitly, L_{cts}^2 is spanned by the Eisenstein series $E_{1/2+ir}$ (see, e.g. [11]). We shall write L_0^2 for the orthogonal complement of the constant function in L^2 .

We wish to show that $L_{\mathrm{cusp}}^2 \neq \{0\}$.

For any prime p we also have the Hecke operator T_p , which acts on functions on $\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}$ via the rule:

$$(2.1) \quad T_p f(z) = \frac{1}{\sqrt{p}} \left(f(pz) + \sum_{k=0}^{p-1} f\left(\frac{z+k}{p}\right) \right).$$

Then T_p commutes with Δ . The Eisenstein series $E_{1/2+ir}(z)$ are joint eigenfunctions of Δ and T_p :

$$(2.2) \quad \Delta E_{1/2+ir} = \left(\frac{1}{4} + r^2\right) E_{1/2+ir}$$

$$(2.3) \quad T_p E_{1/2+ir} = (p^{ir} + p^{-ir}) E_{1/2+ir}.$$

Let us proceed formally for a moment to indicate the main idea of the method. From (2.2) and (2.3), the operator $\aleph := T_p - p^{\sqrt{1/4-\Delta}} - p^{-\sqrt{1/4-\Delta}}$ annihilates $E_{1/2+ir}$. The operator \aleph may be given a rigorous interpretation either in terms of the wave equation or using convolution operators. In the present section we shall use the wave equation. In any case, to show that $L_{\text{cusp}}^2 \neq \{0\}$ it suffices to find a *single nonconstant function* not annihilated by \aleph ; this we do by choosing an appropriate test function supported high in the cusp. A key ingredient will be the fact that solutions to the wave equation propagate at finite speed.

We now detail how \aleph may be understood in terms of the wave equation. Since we have given a detailed treatment of the (more general) convolution approach in the subsequent sections, we will only sketch this approach, omitting careful justification of issues concerning existence and uniqueness of solutions to the wave equation.

Equations (2.2) and (2.3) admit a nice interpretation in terms of the automorphic wave equation

$$(2.4) \quad u_{tt} = -\Delta u + \frac{u}{4}.$$

A solution $u = u(x + iy, t)$ to (2.4) may be regarded as describing the amplitude of a wave propagating in the hyperbolic plane. The low order term of $u/4$ is natural for the hyperbolic Laplacian; see [14, pp. 7-11].

Fix $r \in \mathbb{R}$. Consider the unique solution $u[r](x + iy, t)$ of (2.4) with initial conditions $u[r]|_{t=0} = E_{1/2+ir}$ and $(u[r])_t|_{t=0} = 0$, i.e. the only solution of (2.4) invariant under the time reversal with the prescribed value $E_{1/2+ir}$ at time zero. Then, by (2.2), $u[r](x + iy, t) = \frac{1}{2} E_{1/2+ir}(x + iy)(e^{itr} + e^{-itr})$. From (2.3) it follows that:

$$(2.5) \quad T_p E_{1/2+ir}(x + iy) = 2u[r](x + iy, \log p).$$

The important property of the above equation is that it gives $T_p E_{1/2+ir}$ in terms of propagation by the wave equation for fixed time $\log p$ which does not depend on r .

For every $t \in \mathbb{R}$ we can define a linear endomorphism U_t of $L^2(\text{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}) \cap C^\infty(\text{PGL}(2, \mathbb{Z}) \backslash \mathbb{H})$ to itself, taking a function $f(x + iy)$ to $2u(x + iy, t)$, where u is the solution to (2.4) with $u|_{t=0} = f$, $u_t|_{t=0} = 0$. One may show that this operator is well-defined in a standard way; moreover, it is self-adjoint (“time reversal symmetry.”) Formally speaking, one may write $U_t = e^{t\sqrt{1/4-\Delta}} + e^{-t\sqrt{1/4-\Delta}}$; in fact, U_t gives a rigorous meaning to the right-hand side.

Equation (2.5), and a simple computation for the constant function, allow us to conclude the following basic fact:

Proposition 1. *For every $f \in L_{\text{cts}}^2 \oplus \langle 1 \rangle$,*

$$(2.6) \quad T_p f = U_{\log p} f.$$

Both T_p and $U_{\log p}$ are self-adjoint. We therefore deduce from Prop. 1 that:

Corollary 1. *For every smooth $f \in L^2(\text{PGL}(2, \mathbb{Z}) \backslash \mathbb{H})$, $[T_p - U_{\log p}]f \in L_{\text{cusp}}^2$.*

The operator $T_p - U_{\log(p)}$ thus gives a rigorous interpretation to \aleph .

In order to show that L_{cusp}^2 is nonzero, we only need to find one function in $L_0^2(\text{PGL}(2, \mathbb{Z}) \backslash \mathbb{H})$ that does not satisfy (2.6). We will show that there are many, by constructing them “high in the cusp.”

For every $R > 0$ let Ω_R be the “Siegel domain” $\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \backslash \{x + iy : y > R\}$. One easily sees that for $R > 1$, the natural projection $\Omega_R \rightarrow \text{SL}(2, \mathbb{Z}) \backslash \mathbb{H}$ is injective, so the image of Ω_R in $\text{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}$ is the quotient of Ω_R by the reflection $x + iy \mapsto -x + iy$. Let $L_{0, \text{even}}^2$ be the subspace of $L^2(\Omega_R)$ consisting of functions with integral 0 and which are invariant under $-x + iy \mapsto x + iy$. Then we may regard the Hilbert space $L_{0, \text{even}}^2(\Omega_R)$ as being embedded isometrically in $L_0^2(\text{PGL}(2, \mathbb{Z}) \backslash \mathbb{H})$. Similarly, we define $C_{0, \text{even}}^\infty(\Omega_R)$.

For $0 \neq n \in \mathbb{Z}, R > 1$ we set $V_{n, R}$ to be the subspace of $C_{0, \text{even}}^\infty(\Omega_R)$ consisting of functions of the form $f(x + iy) = h(y) \cos(2\pi nx)$.

Suppose $R > e^t$. The solutions to (2.4) propagate at speed at most 1 on the hyperbolic plane. It follows that for $f \in V_{n, R}$ the function $U_t f$ is supported in a t -neighbourhood of Ω_R , i.e. in $\Omega_{Re^{-t}}$. Moreover, the domain $\Omega_{Re^{-t}}$ admits an action of \mathbb{R}/\mathbb{Z} , namely $sf(x + iy) = f(x + s + iy)$ for $s \in \mathbb{R}/\mathbb{Z}$. This action commutes with the Laplacian and so also with U_t . An element $f \in C_{0, \text{even}}^\infty(\Omega_R)$ belongs to $V_{n, R}$ exactly when it transforms under some linear combination of the characters of $s \mapsto e^{2\pi ins}$ and $s \mapsto e^{-2\pi ins}$ under this action.

It follows that $U_t f \in V_{n, R}$ if $f \in V_{n, R}$. We deduce:

$$(2.7) \quad U_t V_{n, R} \subset V_{n, Re^{-t}} \quad R > e^t.$$

On the other hand, it follows from (2.1) that, for $R > p$, we have:

$$(2.8) \quad T_p V_{n, R} \subset \begin{cases} V_{pn, R/p} & \text{if } p \nmid n \\ V_{pn, R/p} \oplus V_{n/p, pR} & \text{if } p \mid n \end{cases}.$$

Since $V_{n, R}$ and $V_{n', R'}$ are orthogonal for every $n \neq n'$, we conclude from (2.7) and (2.8) that $T_p V_{n, R} \perp U_{\log(p)} V_{n, R}$ if $n \neq 0$ and $R > p$. Consequently, if $f \in V_{n, R}$ and $T_p f \neq 0$, then $[T_p - U_{\log(p)}]f \neq 0$.

On the other hand, one easily verifies, by looking at the Fourier expansion, that T_p is injective on $V_{n, R}$.

We have proved:

Proposition 2. *For any $n \neq 0, R > p$, the map $(U_{\log(p)} - T_p)$ is an injective map of $V_{n, R}$ into $L_{\text{cusp}}^2(\text{PGL}(2, \mathbb{Z}) \backslash \mathbb{H})$.*

Evidently this implies the infinitude of cusp forms in a quantifiable fashion. Indeed, one may deduce by a standard variational argument a weak form of the Weyl law, i.e. that there is a constant $c > 0$ such that there are at least cT cusp forms of eigenvalue less than T . Since we prove the full Weyl law later, we omit the easy proof. We refer to the beginning of Sec. 6 for an outline of the idea of the proof of the Weyl law. For now we just comment that one problem is that, because \aleph kills the Eisenstein series, it will also act by a very small scalar on any cusp form that suitably “mimics” an Eisenstein series. Thus any approach which uses \aleph will lose information about these forms; but it will turn out that “fake Eisenstein series” are spectrally very sparse.

The operator \aleph can be interpreted as a convolution operator, but this requires a slightly different viewpoint. Since this will be used later in the proof of the Weyl law, we explain this in some detail.

We first explain what we mean by a convolution operator on $L^2(\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H})$. We identify $L^2(\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H})$ with the subspace $L^2(\mathrm{PGL}(2, \mathbb{Z}) \backslash G/K)$ of $K = \mathrm{PO}(2, \mathbb{R})$ -invariant functions on $\mathrm{PGL}(2, \mathbb{Z}) \backslash G$ with $G = \mathrm{PGL}(2, \mathbb{R})$. If $k \in C_c^\infty(K \backslash G/K)$, i.e. is a bi- K -invariant compactly supported smooth function¹, it gives rise to an operator $f \rightarrow f \star k$ on $L^2(\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}) = L^2(\mathrm{PGL}(2, \mathbb{Z}) \backslash G/K)$ given by

$$f \star k(g) = \int_G f(gh^{-1})k(h)dh.$$

More generally, one can consider convolution with compactly supported distributions instead of functions (in this case the convolution operator is well-defined only on suitably smooth functions f , for example on $C^\infty(\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H})$.)

The operator \aleph is a sum of two operators: the Hecke operator T_p , and the operator $U_{\log p} = p^{\sqrt{1/4-\Delta}} + p^{-\sqrt{1/4-\Delta}}$. The second of these operators, $U_{\log p}$, is already a convolution operator, though not with a function but with a distribution.

Explicitly, let $\Xi_s(z)$ denote the spherical function with parameter s , i.e. the unique function on \mathbb{H} which is

- (1) spherically symmetric, i.e. depending only on the hyperbolic distance of z from the point $i \in \mathbb{H}$
- (2) satisfies $\Delta \Xi_s = (\frac{1}{4} + s^2)\Xi_s$
- (3) $\Xi_s(i) = 1$.

We can identify it with a K -bi-invariant function on G .

For any bi- K -invariant compactly supported function (or distribution) k and s , the function $\Xi_s \star k$ also satisfies (1) and (2) above, and so is equal to $\hat{k}(s)\Xi_s$; the map $k \rightarrow \hat{k}(s)$ is called the spectral transform. Its inverse is given by (see [9, Theorem 4.8]²)

$$k(z) = \frac{1}{4\pi^2} \int_{\mathbb{R}} \hat{k}(s)\Xi_s(z)|c(s)|^{-2}ds \quad c(s) = \pi^{-\frac{1}{2}} \frac{\Gamma(is)}{\Gamma(is + 1/2)}$$

Again, we may identify the right K -invariant function k with a function on the upper half plane \mathbb{H} .

By definition of $U_{\log p}$, its spectral transform is $p^{is} + p^{-is}$, so formally the corresponding spherical distribution is given by

$$k_{U_{\log p}}(z) = \frac{1}{4\pi^2} \int (p^{is} + p^{-is})\Xi_s(z)|c(s)|^2 ds.$$

It follows from the finite propagation speed of the wave equation that $k_{U_{\log p}}(z)$ is supported in the hyperbolic disc of radius $\log p$ around i ; this can also be deduced from a distributional variant of Paley-Wiener type theorem [9, Theorem 4.7].

One may avoid the use of distributions in the following fashion: If $H \in C_c^\infty(K \backslash G/K)$, then for any $f \in C^\infty(\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H})$,

$$U_{\log p}(f) \star H = f \star k_1$$

¹Thus k is a function on G , not an element of K ; we hope this notation does not cause confusion.

²The hyperbolic metric used by Helgason and the one we use here differ by a factor of 2, and all other quantities defined by the metric, such as the Laplacian, the spherical transform etc. need to be scaled accordingly.

with k_1 now a smooth compactly supported function given explicitly by

$$k_1(z) = \frac{1}{4\pi^2} \int \hat{H}(s)(p^{is} + p^{-is})\Xi_s(z)|c(s)|^2 ds$$

(it is easy to see this defines a smooth function; compact support requires use of the properties of the wave equation and/or a Paley-Wiener type theorem).

The operator T_p cannot be directly interpreted as a convolution operator on $\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}$. However, one has the following fundamental isomorphism: in the notations of the next section, we consider the set of places $S = \{\infty, p\}$, and take $G_S = \mathrm{PGL}(2, \mathbb{R}) \times \mathrm{PGL}(2, \mathbb{Q}_p)$, $\Gamma = \mathrm{PGL}(2, \mathbb{Z}[1/p])$ and $K_S = K \times \mathrm{PGL}(2, \mathbb{Z}_p)$, with Γ considered as a discrete subgroup of G_S using the diagonal embedding. Then

$$\Gamma \backslash G_S / K_S \cong \mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}.$$

This isomorphism between spaces gives us an isomorphism ι^* of $L^2(\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H})$ with the space of right K_S invariant functions in $L^2(\Gamma \backslash G_S)$. A straightforward calculation now verifies that for $f \in L^2(\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H})$

$$\iota^*(T_p f) = \iota^*(f) \star k_{T_p}$$

for the K_S invariant distribution k_{T_p} on G_S defined by

$$k_{T_p}(g_\infty, g_p) = p^{-\frac{1}{2}} \delta_1(g_\infty) 1_{K_p} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}_{K_p} (g_p) \quad K_p = \mathrm{PGL}(2, \mathbb{Z}_p),$$

with δ_1 being Dirac's delta measure. Again we see that if further convolved by $H_\infty \in C_c^\infty(K \backslash G / K)$, or more precisely by $H(g_\infty, g_p) = H_\infty(g_\infty) 1_{K_p}(g_p)$ (which is now a bi- K_S -invariant, compactly supported function on G_S) we have that

$$\iota(T_p(f) \star H_\infty) = \iota(f) \star k'_1$$

with k'_1 the function

$$k'_1(g_\infty, g_p) = p^{-\frac{1}{2}} H_\infty(g_\infty) 1_{K_p} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}_{K_p} (g_p).$$

In this way we see that \aleph on $L^2(\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H})$ can be viewed as a convolution operator on the space $L^2(\Gamma \backslash G_S / K_S)$ by the distribution $k_\aleph = k_{U_{\log p}} - k_{T_p}$, and for any $H = H_\infty 1_{K_p}$, the composition $f \mapsto (\aleph f) \star H$ under the isomorphism ι^* becomes the convolution operator $f \mapsto f \star (k_1 - k'_1)$, with k_1, k'_1 continuous compactly supported bi- K_S -invariant functions as above. Convolution operators by smooth functions with a purely cuspidal image will be very useful for us later, when we prove the Weyl law, since they can be given a simple spectral expansion that converges pointwise (5.2).

After introducing the necessary formalism in the next section, in §4 we show how to find such convolution operators with cuspidal image for a general group.

3. THE SATAKE MAP, PLANCHEREL MEASURE, AND THE SPHERICAL TRANSFORM

In this section we review some facts from the harmonic analysis of S -algebraic groups, which we will need to generalize the constructions of the previous section and to prove Weyl's law.

Let S be a set of places of \mathbb{Q} containing ∞ . Set $\mathbb{Q}_S = \prod_{v \in S} \mathbb{Q}_v$ and let $\mathbb{Z}[S^{-1}]$ be the ring of S -integers. For $x = (x_v)_{v \in S} \in \mathbb{Q}_S^*$ we set $|x|_S = \prod_{v \in S} |x_v|_v$.

Let \mathbf{G} be a split adjoint semisimple group over \mathbb{Q} ; we shall in fact choose a model defined over $\mathbb{Z}[1/B]$ for some positive integer B . We shall use bold-face letters to designate algebraic groups (always over \mathbb{Q}) and usual letters to denote their points over \mathbb{R} or \mathbb{Q}_S . By $Z(\star)$ we mean the center of the group \star .

We fix (once and for all) a pair $\mathbf{T} \subset \mathbf{B}$ of a \mathbb{Q} -split torus \mathbf{T} contained in a Borel subgroup \mathbf{B} . Let \mathbf{N} be the unipotent radical of \mathbf{B} , so $\mathbf{N} = R_u(\mathbf{B})$ and $\mathbf{B} = \mathbf{T} \cdot \mathbf{N}$. If \mathbf{P} is a parabolic subgroup containing \mathbf{B} , then we have a Levi decomposition $\mathbf{P} = \mathbf{M}_P \mathbf{N}_P$. Let $\mathbf{T}_P = Z(\mathbf{M}_P)$.

Set $G_S = \mathbf{G}(\mathbb{Q}_S)$, $A_S = \mathbf{T}(\mathbb{Q}_S)$, $A_{P,S} = \mathbf{T}_P(\mathbb{Q}_S)$, $N_S = \mathbf{N}(\mathbb{Q}_S)$, $B_S = \mathbf{B}(\mathbb{Q}_S)$, and similarly define $P_S, M_{P,S}, N_{P,S}$. Let K_∞ be a maximal compact subgroup of $G_\infty := \mathbf{G}(\mathbb{R})$ with the property that $G_\infty = N_\infty A_\infty^\circ K_\infty$ is an Iwasawa decomposition, where A_∞° is the identity component of A_∞ ; equivalently, the Cartan involution of G_∞ which fixes K_∞ should act by inversion on A_∞° .

We assume now that all finite places of S are prime to the integer B . Set now $K_S = K_\infty \cdot \prod_{v \in S, v \neq \infty} \mathbf{G}(\mathbb{Z}_v)$, where \mathbb{Z}_v is the maximal compact subring of \mathbb{Q}_v . Then K_S is a maximal compact subgroup of $G(\mathbb{Q}_S)$. We will moreover assume that S has the property that, for each finite $v \in S$ and for each parabolic \mathbf{P} containing \mathbf{B} , $K_v \cap \mathbf{M}_P(\mathbb{Q}_v)$ is the stabilizer in $\mathbf{M}_P(\mathbb{Q}_v)$ of a special vertex in the building of $\mathbf{M}_P(\mathbb{Q}_v)$; and moreover this vertex belongs to the apartment associated to the maximal torus \mathbf{T} . This condition is satisfied for almost all finite v , as follows from [23, 3.9.1]. Moreover, $K_\infty \cap \mathbf{M}_P(\mathbb{R})$ is a maximal compact subgroup of $\mathbf{M}_P(\mathbb{R})$, being the fixed points of a Cartan involution; so $K_S \cap M_{P,S}$ is a maximal compact subgroup of $M_{P,S}$.

We now describe some normalizations of metric and measure. Fix the Riemannian metric on the symmetric space G_∞/K_∞ so that it corresponds to the Killing form on the Lie algebra of G_∞ . (That is, one identifies the tangent space to G_∞/K_∞ at the identity coset with the orthogonal complement of $\text{Lie}(K_\infty)$ in $\text{Lie}(G_\infty)$; now use the Killing form to endow it with an inner product.) This fixes a normalization of the Laplacian operator, and also gives a measure on G_∞/K_∞ .

The map $N_{P,S} \times M_{P,S} \times K_S \rightarrow G_S$ is surjective (Iwasawa). We equip each $G(\mathbb{Q}_v)$, for v finite, with the Haar measure which assigns $\mathbf{G}(\mathbb{Z}_v)$ mass 1. We equip K_∞ with the Haar measure of mass 1, and then choose the Haar measure on G_∞ so that it is compatible with the measure on G_∞/K_∞ arising from the Riemannian metric. We arbitrarily choose Haar measures on the other mentioned groups.

Let W_S be the Weyl group of A_S in G_S , i.e. $\text{Norm}_{G_S}(A_S)/Z_{G_S}(A_S)$. Let Δ be a simple system of roots (relative to the system of positive roots for \mathbf{T} defined by \mathbf{N}); thus each $\alpha \in \Delta$ gives a map $\alpha : \mathbf{T} \rightarrow \mathbb{G}_m$.

Let $\delta : A_S \rightarrow \mathbb{R}^*$ be the square root of the modular character of A_S acting on N_S , i.e. $\delta(a) = |\prod_{\alpha \in \Phi^+} \alpha(a)|_S^{1/2}$, where Φ^+ is the set of all positive roots for \mathbf{T} .

Let Γ be a congruence subgroup of $G(\mathbb{Z}[S^{-1}])$. It is known that the number of Γ orbits on proper \mathbb{Q} -parabolic subgroups of \mathbf{G} is finite. These orbits (roughly speaking) index the ‘‘cusps’’ of $\Gamma \backslash G_S$. Fix a set of representatives $\mathcal{R} = \{\mathbf{Q}_1, \dots, \mathbf{Q}_r\}$. For each $1 \leq i \leq I$ choose $\delta_i \in \mathbf{G}(\mathbb{Q})$ such that $\delta_i^{-1} \mathbf{Q}_i \delta_i$ contains \mathbf{B} . Put $\mathbf{P}_i = \delta_i^{-1} \mathbf{Q}_i \delta_i$, with unipotent radical \mathbf{N}_i and Levi subgroup $\mathbf{M}_i \supset \mathbf{T}$.

We then put $N_{S,i} = N_{P_i,S}$, $A_{S,i} = A_{P_i,S}$, $M_{S,i} = M_{P_i,S}$. Further set $\Gamma_i = \delta_i^{-1} \Gamma \delta_i$, $\Gamma_{A,i} = A_{S,i} \cap \Gamma_i$, $\Gamma_{N,i} = N_{S,i} \cap \Gamma_i$. If $S \neq \{\infty\}$ then it is easy to verify that each $\Gamma_{A,i}$ is infinite.

Let $C_c^\infty(K_S \backslash G_S / K_S)$ be the space of compactly supported smooth functions on $K_S \backslash G_S / K_S$.

Set $\mathfrak{a}_S = A_S / (A_S \cap K_S)$. It is isomorphic to a sum of copies of \mathbb{R} and \mathbb{Z} . Set $\mathfrak{a}_S^* \stackrel{\text{def}}{=} \text{Hom}(\mathfrak{a}_S, \mathbb{C}^*)$. It is a complex manifold in a natural fashion. We denote by $\mathfrak{a}_{S,\text{temp}}^* \subset \mathfrak{a}_S^*$ the subset consisting of *unitary* characters. We define the operation of “conjugation” $\nu \mapsto \bar{\nu}$ on \mathfrak{a}_S^* via the rule $\bar{\nu}(a) = \overline{\nu(a)^{-1}}$. This is an involution that fixes exactly $\mathfrak{a}_{S,\text{temp}}^*$. The set $\mathfrak{a}_{S,\text{temp}}^*$ is an abelian group isomorphic to a product of copies of \mathbb{R} and \mathbb{R}/\mathbb{Z} . Fix a Haar measure μ_{Haar} on $\mathfrak{a}_{S,\text{temp}}^*$.

In the case $S = \{\infty\}$ we obtain spaces \mathfrak{a}_∞ , \mathfrak{a}_∞^* , $\mathfrak{a}_{\infty,\text{temp}}^*$. The exponential map gives an isomorphism of \mathfrak{a}_∞ with the Lie algebra $\text{Lie}(A_\infty)$; in particular, \mathfrak{a}_∞ and $\mathfrak{a}_{\infty,\text{temp}}^*$ have the structure of real vector spaces, and $\mathfrak{a}_\infty^* \cong \mathfrak{a}_{\infty,\text{temp}}^* \otimes_{\mathbb{R}} \mathbb{C}$ the structure of a complex vector space. We denote by $\langle \cdot, \cdot \rangle$ the (complex) symmetric bilinear form on \mathfrak{a}_∞^* deduced from the Killing form on $\text{Lie}(A_\infty)$. The form $\langle \cdot, \cdot \rangle$ is positive definite on $\mathfrak{a}_{\infty,\text{temp}}^*$; for $\nu \in \mathfrak{a}_{\infty,\text{temp}}^*$ we put $\|\nu\| = \sqrt{\langle \nu, \nu \rangle}$; we will also use $\|\cdot\|$ to denote the norm on \mathfrak{a}_∞ coming from the Killing form.

For $\nu \in \mathfrak{a}_S^*$, we denote by ν_∞ its image under the natural map $\mathfrak{a}_S^* \rightarrow \mathfrak{a}_\infty^*$.

For $\nu \in \mathfrak{a}_S^*$, regarding $\nu\delta$ as a character of A_S , extending trivially on N_S and inducing, we obtain a principal series representation of G_S which has a unique spherical subconstituent $\pi(\nu)$. The twist by δ guarantees that this is “normalized” i.e. $\pi(\nu)$ is unitary if $\nu \in \mathfrak{a}_{S,\text{temp}}^*$. For any ν the conjugate-linear dual of $\pi(\nu)$ is isomorphic to $\pi(\bar{\nu})$; in particular, if $\pi(\nu)$ is unitary, $\pi(\nu)$ must be isomorphic to $\pi(\bar{\nu})$ and $\nu = w\bar{\nu}$ for some $w \in W_S$.

Let $\nu \in \mathfrak{a}_S^*$, and fix a G_S -invariant sesquilinear pairing $[\cdot, \cdot]$ between $\pi(\nu)$ and $\pi(\bar{\nu})$. Let $v_\nu^0 \in \pi(\nu)$ and $v_{\bar{\nu}}^0 \in \pi(\bar{\nu})$ be K_S -invariant vectors such that $[v_\nu^0, v_{\bar{\nu}}^0] = 1$. Set $\Xi_\nu(g)$ to be the function on G_S defined by $[\pi(g)v_\nu^0, v_{\bar{\nu}}^0]$, i.e. Ξ_ν is the spherical function with parameter ν .

Note that if $\nu_\infty \in \mathfrak{a}_\infty^*$ and $\pi(\nu_\infty)$ is unitarizable, then $\langle \nu_\infty, \nu_\infty \rangle \in \mathbb{R}$. This follows from the fact that the Casimir operator must operate on the space of $\pi(\nu_\infty)$ by a real scalar. In fact, the Laplacian operator on G_∞ / K_∞ acts on the spherical vector of $\pi(\nu_\infty)$ by a scalar that differs from $\langle \nu_\infty, \nu_\infty \rangle$ by a fixed (additive) constant.

If $k \in C_c^\infty(K_S \backslash G_S / K_S)$ then for every $\nu \in \mathfrak{a}_S^*$ we have that Ξ_ν is an eigenfunction of the corresponding convolution operator. The **spherical transform** $\hat{k}(\nu)$ is defined as the corresponding eigenvalue, i.e.

$$k \star \Xi_\nu = \hat{k}(\nu) \Xi_\nu.$$

The spherical transform $k \mapsto \hat{k}$ extends to an L^2 -isometry between $L^2(K_S \backslash G_S / K_S)$ and $L^2(\mathfrak{a}_S^*, \mu_{\text{Planch}})$ for an appropriate W_S -invariant measure μ_{Planch} on \mathfrak{a}_S^* , the **Plancherel measure**, which is absolutely continuous with respect to μ_{Haar} (See [10] for the archimedean case and [15] for the p -adic case). In particular, for any $k \in C_c^\infty(K_S \backslash G_S / K_S)$:

$$(3.1) \quad \int_{g \in G_S} |k(g)|^2 dg = \int_{\nu \in \mathfrak{a}_{S,\text{temp}}^*} |\hat{k}(\nu)|^2 d\mu_{\text{Planch}}(\nu).$$

The inverse of the spherical transform is given explicitly in terms of the Plancherel measure by

$$(3.2) \quad k(g) = \int_{\mathfrak{a}_{S,\text{temp}}^*} \hat{k}(\nu) \Xi_\nu(g) d\mu_{\text{Planch}}(\nu);$$

in particular, we have that

$$(3.3) \quad k(e) = \int_{\nu \in \mathfrak{a}_{S,\text{temp}}^*} \hat{k}(\nu) d\mu_{\text{Planch}}(\nu).$$

Let $d = \dim(G_\infty/K_\infty)$ and $r = \dim(A_\infty)$. We recall the following estimates for the density of the Plancherel measure which follow from the explicit form of the Plancherel measure [10, 15]:

- (1) There is a constant c_1 such that, for any positive function g on $\mathfrak{a}_{S,\text{temp}}^*$, we have

$$(3.4) \quad \int_{\mathfrak{a}_{S,\text{temp}}^*} g(\nu) d\mu_{\text{Planch}}(\nu) \leq c_1 \int_{\mathfrak{a}_{S,\text{temp}}^*} (1 + \|\nu_\infty\|)^{d-r} g(\nu) d\mu_{\text{Haar}}(\nu).$$

- (2) Moreover, there exists $\alpha(\mathbf{G}) > 0$ such that

$$(3.5) \quad \int_{\nu \in \mathfrak{a}_{S,\text{temp}}^* : \|\nu_\infty\|^2 \leq T} d\mu_{\text{Planch}}(\nu) \sim \alpha(\mathbf{G}) T^{d/2},$$

as $T \rightarrow \infty$.

Our normalizations imply that $\alpha(\mathbf{G})$ actually does not depend on the choice of S (subject to S containing ∞).

If $f \in C_c^\infty(K_S \backslash G_S / K_S)$, we define the **Abel-Satake transform** (also referred to as the Harish transform)

$$\mathcal{S}f(a) = \delta(a)^{-1} \int_{n \in N_S} f(na) dn$$

for $a \in A_S$. $\mathcal{S}f$ is $A_S \cap K_S$ -invariant; as such, it may be regarded as a function on \mathfrak{a}_S . The Satake transform is an isomorphism of $C_c^\infty(K_S \backslash G_S / K_S)$ with $C_c^\infty(\mathfrak{a}_S)^{W_S}$ (this follows by combining [24] in the real case and [2, Thm 4.1] for the p -adic case) and the following diagram commutes:

$$(3.6) \quad \begin{array}{ccc} C_c^\infty(K_S \backslash G_S / K_S) & \xrightarrow{k \mapsto \hat{k}} & \mathcal{H}(\mathfrak{a}_S^*) \\ \mathcal{S} \downarrow & & \text{id} \downarrow \\ C_c^\infty(\mathfrak{a}_S)^{W_S} & \xrightarrow{FT} & \mathcal{H}(\mathfrak{a}_S^*) \end{array}$$

Here $\mathcal{H}(\mathfrak{a}_S^*)$ is the space of holomorphic functions on \mathfrak{a}_S^* , the right-hand arrow is the identity map, and the bottom arrow the (appropriately normalized) Fourier transform on the abelian group \mathfrak{a}_S , i.e., if $f \in C_c^\infty(\mathfrak{a}_S)$, then $FT(f)$, evaluated at $\nu \in \mathfrak{a}_S^*$, equals $\int_{\mathfrak{a}_S} f(a) \nu(a) da$ for a suitably normalized Haar measure da .

Finally, we will make (crucial!) use of the following Paley-Wiener type theorem for the spherical transform on real groups (which is essentially equivalent to the above description of the spherical transform and the properties of the Abel-Satake transform)

Theorem 2 (Gangolli [7], Helgason [8]). *Suppose $k \in C_c^\infty(K_\infty \backslash G_\infty / K_\infty)$, and suppose that $k(\exp a) = 0$ for every $a \in \mathfrak{a}_\infty$ with $\|a\| > R$. Then the spherical transform $\hat{k}(\nu)$ is an entire holomorphic function of ν , invariant under the Weyl group W_∞ . Moreover, given any integer N we can find $C_N > 0$ such that*

$$(3.7) \quad \left| \hat{k}(\xi + i\eta) \right| \leq C_N (1 + (\|\xi\|^2 + \|\eta\|^2)^{1/2})^{-N} e^{R\|\xi\|}, \quad \xi, \eta \in \mathfrak{a}_{\infty,\text{temp}}^*.$$

Conversely, suppose F is a W_∞ -invariant entire holomorphic function of $\nu \in \mathfrak{a}_\infty^*$, and $R > 0$ is such that there exists a constant C_N (for any integer N) so that (3.7) holds. Then there exists a unique function $k \in C_c^\infty(K_\infty \backslash G_\infty / K_\infty)$ such that $\hat{k} = F$. Moreover, $k(\exp a) = 0$ for every $a \in \mathfrak{a}_\infty$ with $\|a\| > R$.

4. A CONVOLUTION OPERATOR WITH SPHERICAL, CUSPIDAL IMAGE

Notations being as in the previous section, we recall that $\varphi : \Gamma \backslash G_S \rightarrow \mathbb{C}$ is *cuspidal* if

$$(4.1) \quad \int_{\Gamma_{N,i} \backslash N_{S,i}} \varphi(\delta_i n g) dn = 0$$

for almost all $g \in G_S$ and for each $1 \leq i \leq I$.

Proposition 3. *Suppose $k \in C_c^\infty(K_S \backslash G_S / K_S)$ satisfies one of the (equivalent) conditions:*

$$(4.2) \quad \sum_{\tau \in \Gamma_{A,i}} \mathcal{S}k(\tau a) = 0 \text{ whenever } 1 \leq i \leq I, a \in \mathfrak{a}_S$$

or

$$(4.3) \quad \hat{k}(\nu) = 0 \text{ whenever } \nu|_{\Gamma_{A,i}} = 0, 1 \leq i \leq I.$$

Then the convolution $f \mapsto f \star k$ maps $L_{\text{loc}}^1(\Gamma \backslash G_S)$ into the space of cuspidal functions.

Note that in (4.2) we regard $\Gamma_{A,i}$ as a subgroup of \mathfrak{a}_S via the natural maps $\Gamma_{A,i} \subset A_{S,i} \subset A_S \rightarrow \mathfrak{a}_S$. The equivalence of the two conditions of the Proposition is verified via (3.6). Remark moreover that the series in (4.2) is easily verified to be absolutely and uniformly convergent (indeed it is, locally, a finite sum).

We shall give two proofs. The first is shorter, and uses some (entirely elementary) facts about pseudo-Eisenstein series. In the second we give an explicit and direct proof from the definition of ‘‘cuspidal.’’ The main idea is that Eisenstein series associated to $\mathbf{Q}_i(\mathbb{Q}_S)$ arise from automorphic data on $\bar{\Gamma} \backslash \mathbf{M}_{\mathbf{Q}_i}(\mathbb{Q}_S)$, where $\mathbf{M}_{\mathbf{Q}_i}$ is the Levi subgroup of \mathbf{Q}_i , and $\bar{\Gamma}$ the projection of $\Gamma \cap \mathbf{Q}_i(\mathbb{Q}_S)$ to $\mathbf{M}_{\mathbf{Q}_i}(\mathbb{Q}_S)$. If $\bar{\Gamma}$ has a large center, this constrains the automorphic data and forces relations between the parameters of the Eisenstein series at different places.

First proof: The assumption that $\hat{k}(\nu) = 0$ whenever ν annihilates $\Gamma_{A,i}$ is easily seen to imply that $E \star k = 0$, whenever E is a pseudo-Eisenstein series attached to \mathbf{Q}_i : that is to say, $E = \sum_{\Gamma \cap \mathbf{Q}_i(\mathbb{Q}_S) \backslash \Gamma} f(\gamma g)$ for $f \in C_c^\infty(\mathbf{N}_{\mathbf{Q}_i}(\mathbb{Q}_S)(\Gamma \cap \mathbf{Q}_i(\mathbb{Q}_S)) \backslash G_S)$, where $\mathbf{N}_{\mathbf{Q}_i}$ is the unipotent radical of \mathbf{Q}_i .³

Let $\check{k}(g) = \overline{k(g^{-1})}$. Then, for any E as above and $f \in L^2(\Gamma \backslash G_S)$, we have $\langle f, E \star k \rangle = \langle f \star \check{k}, E \rangle$. On the other hand, the pseudo-Eisenstein series attached to proper parabolics span the orthogonal complement of the cuspidal spectrum (see [17, II.1.2] for a proof in the adelic context, which easily implies the asserted statement). Therefore $f \star \check{k}$ is cuspidal for any $f \in L^2(\Gamma \backslash G_S)$.

On the other hand (4.2) is true for k if and only if it is true for \check{k} . Thus $f \star k$ is cuspidal for $f \in L^2(\Gamma \backslash G_S)$. Since k is compactly supported, one may replace $f \in L^2$ by $f \in L_{\text{loc}}^1$. \square

³See [17] for an adelic treatment.

Second proof: Fix $1 \leq i \leq I$ and take $f \in L^1_{\text{loc}}(\Gamma \backslash G_S)$. Put $f_i(g) = f(\delta_i g)$; then $f_i \in L^1_{\text{loc}}(\Gamma_i \backslash G_S)$; moreover $f_i \star k = (f \star k)(\delta_i g)$. We wish to show that, under the condition (4.2), we have

$$(4.4) \quad \int_{\Gamma_{N,i} \backslash N_{S,i}} (f_i \star k)(ng) = 0$$

for all $1 \leq i \leq I$ and almost all $g \in G_S$.

Set $K(x, y) = \sum_{\gamma \in \Gamma_i} k(x^{-1}\gamma y)$ (for x, y lying in any fixed compact set, the γ -sum is finite). Then $(f_i \star k)(x) = \int_{y \in \Gamma_i \backslash G_S} K(x, y) f_i(y) dy$. It follows that (4.4) will be satisfied if for any $x, y \in G$ we have $\int_{\Gamma_{N,i} \backslash N_{S,i}} K(nx, y) dn = 0$. Equivalently:

$$(4.5) \quad \sum_{\gamma \in \Gamma_i} \int_{n \in \Gamma_{N,i} \backslash N_{S,i}} k(x^{-1}n^{-1}\gamma y) dn = \sum_{\gamma' \in \Gamma_{N,i} \backslash \Gamma_i} \int_{n \in N_{S,i}} k(x^{-1}n^{-1}\gamma' y) dn = 0$$

Writing the γ' -sum as a union over $\Gamma_{A,i}$ -cosets we see it suffices to check that (for every $1 \leq i \leq I, x, y$)

$$\sum_{\tau \in \Gamma_{A,i}} \int_{n \in N_{S,i}} k(x^{-1}n\tau y) dn = 0$$

Using the Iwasawa decomposition, the bi- K_S -invariance of k , and the fact that any $\tau \in \Gamma_{A,i}$ centralizes $M_{S,i}$ and normalizes (in a measure-preserving fashion) $N_{S,i}$, we see that it suffices to check for $m \in M_{S,i}$:

$$(4.6) \quad \sum_{\tau \in \Gamma_{A,i}} \int_{n \in N_{S,i}} k(n\tau m) dn = 0$$

It is not hard to verify that the sum and integral (4.6) converge absolutely; in particular, we can freely interchange order of summation and integration.

Consider the function $k_i : m \mapsto \int_{n \in N_{S,i}} k(nm) dn$ on $M_{S,i}$. It is a compactly supported smooth spherical function on $M_{S,i}$, i.e. $k_i \in C_c^\infty(K_S \cap M_{S,i} \backslash M_{S,i} / K_S \cap M_{S,i})$. (4.6) amounts to the assumption that $k'_i \stackrel{\text{def}}{=} \sum_{\tau \in \Gamma_{A,i}} k_i(\tau m)$ identically vanishes. Note that k'_i is a spherical function on $M_{S,i}$, but no longer compactly supported.

Let $N_{M,i} = N_S \cap M_{S,i}$, i.e. the \mathbb{Q}_S -points of the unipotent radical of the Borel subgroup $\mathbf{B} \cap \mathbf{M}_i$ of \mathbf{M}_i . To check that $k'_i \equiv 0$, we use the injectivity of the Satake map for $M_{S,i}$ rather than G_S .

We proceed formally: no convergence problems arise from the fact that k'_i is not compactly supported, since it fails to be compactly supported “only in central directions.” In particular the integrals we write down below are all absolutely and uniformly convergent.

It suffices to check that for all $a \in A_S$ that $\int_{n_2 \in N_{M,i}} k'_i(n_2 a) = 0$, i.e.

$$\int_{n_2 \in N_{M,i}} \sum_{\tau \in \Gamma_{A,i}} \int_{n \in N_{S,i}} k(n\tau n_2 a) = 0.$$

Since τ commutes with each n_2 , we may write this

$$\sum_{\tau \in \Gamma_{A,i}} \int_{n \in N_{S,i}} \int_{n_2 \in N_{M,i}} k(nn_2\tau a) dndn_2 = 0.$$

Since $N_S = N_{S,i} \cdot N_{M,i}$ (with a corresponding measure decomposition), this is equivalent to: $\sum_{\tau \in \Gamma_{A,i}} \mathcal{S}k(\tau a) = 0$. \square

Lemma 1. *Let A be an abelian group isomorphic to a sum of copies of \mathbb{R} and \mathbb{Z} , let W be a finite group of automorphisms of A , and let $T = \{A_1, \dots, A_n\}$ be a finite collection of nontrivial subgroups (i.e. $A_j \neq \{0\}$). Then there exists a nonzero distribution on A which is a finite linear combination of point masses, is W -invariant, and vanishes when evaluated on any A_i -invariant function.*

Proof. Enlarge T to be W -invariant. For h any distribution on A set $\check{h}(x) = \overline{h(-x)}$. For $1 \leq j \leq n$, choose an element $0 \neq a_j \in A_j$ and let f_j be the compactly supported distribution on A given by $f_j(g) = g(a_j) - g(0)$, for $g \in C_c^\infty(A)$. Set $f_A = f_1 \star f_2 \star \dots \star f_n$, $f_B = f_A \star \check{f}_A$, $f = \sum_{w \in W} f_B^w$. It is clear that f is a W -invariant compactly supported distribution on A whose A_j -averages vanish (more correctly, f vanishes when tested on any A_j -invariant function). To see that $f \neq 0$, remark that the ‘‘Fourier transform’’ of f_B is non-negative and not identically vanishing. Finally, it is clear by construction that f is a finite linear combination of point masses. \square

We remark that this Lemma shows in fact that there is a smooth nonzero W -invariant function $f \in C_c^\infty(A)^W$ such that each of the averages $\int_{x \in A_i} f(y+x)dx = 0$ (for any $y \in A$). To see this, simply convolve with a smooth W -invariant function on A of small support.

Corollary 2. *There exist nonzero $k \in C_c^\infty(K_S \backslash G_S / K_S)$ that satisfy (4.2).*

Proof. Apply the previous remark with $A = \mathfrak{a}_S$ and $T = \{\Gamma_{A,i}\}_{1 \leq i \leq I}$, and use the surjectivity of the Satake transform. \square

5. THE PRE-TRACE FORMULA AND A PROOF OF THE WEYL LAW IN THE COMPACT CASE

We put $r = \text{rank}(\mathbf{G}) = \dim(A_\infty)$ and $d = \dim(G_\infty / K_\infty)$. Suppose that the spherical transform of $k \in C_c^\infty(K_S \backslash G_S / K_S)$ is nonnegative and that k is such that the corresponding convolution operator has a purely cuspidal image.

There is a basis of cusp forms on $L^2(\Gamma \backslash G_S / K_S)$ that consists of eigenfunctions for the convolution algebra $C_c^\infty(K_S \backslash G_S / K_S)$. Let $\mathcal{E} \subset \mathfrak{a}_S^*$ be the multiset of eigenvalues of cusp forms on $\Gamma \backslash G_S / K_S$, counted with multiplicity. For each $\nu \in \mathcal{E}$, let φ_ν be the corresponding eigenfunction.

Put $K(x, y) = \sum_{\gamma \in \Gamma} k(x^{-1}\gamma y)$. By our condition on cuspidality of k we get the spectral expansion

$$(5.1) \quad K(x, y) = \sum_{\nu \in \mathcal{E}} \hat{k}(\nu) \varphi_\nu(x) \overline{\varphi_\nu(y)}, \quad (x, y \in G_S).$$

(5.1) is, *a priori*, an equality in L^2 , but it is easy to see that one can specialize pointwise to obtain

$$(5.2) \quad \sum_{\gamma \in \Gamma} k(g^{-1}\gamma g) = \sum_{\nu \in \mathcal{E}} \hat{k}(\nu) |\varphi_\nu(g)|^2, \quad (g \in G_S).$$

Let $\Omega \subset \Gamma \backslash G_S$ be any compact set (which we will take to have almost full measure). Let $\tilde{\Omega}$ be a compact subset of G_S whose projection contains Ω so that

the projection is almost everywhere one to one. Integrating (5.2) over $\tilde{\Omega}$ we have

$$(5.3) \quad \begin{aligned} k(1)\text{vol}(\Omega) + \sum_{\gamma \in Z} \int_{\tilde{\Omega}} k(g^{-1}\gamma g) dg &= \int_{\tilde{\Omega}} \sum_{\gamma \in \Gamma} k(g^{-1}\gamma g) dg \\ &= \sum_{\nu \in \mathcal{E}} \hat{k}(\nu) \int_{\tilde{\Omega}} |\varphi_{\nu}(g)|^2 dg \leq \sum_{\nu \in \mathcal{E}} \hat{k}(\nu) \end{aligned}$$

where

$$(5.4) \quad Z = (\Gamma \setminus \{e\}) \cap \left\{ gxg^{-1} : g \in \tilde{\Omega}, \text{ and } x \text{ in the support of } k \right\};$$

in particular, Z is finite and depends only on $\tilde{\Omega}$ and the support of k .

The idea of using integrating over a compact Ω and obtaining an inequality as in (5.3), rather than integrating over all of $\Gamma \backslash G$ and obtaining the usual trace formula, was used in Miller's proof [16] of the Weyl law for cusp forms in $\text{PGL}(3)$. Our proof will follow the same pattern, but we will be able to avoid entirely the Eisenstein series.

5.1. Weyl's law for compact quotients. We now briefly describe how to prove Weyl's law for compact quotients, starting with (5.3). Of course, this was proven by Weyl, and it is also a trivial consequence of the usual trace formula. Nevertheless we take the opportunity to explain it in the notation of the present paper, as a precursor to the more involved case of noncompact quotient.

So far in our discussion in this section we have not assumed anything on \mathbf{G} , and in particular everything we have said remains valid for compact quotient $\Gamma \backslash G_S$. Of course in this case \mathbf{G} is not \mathbb{Q} -split, but this was only used for constructing convolution operators with cuspidal image. If $\Gamma \backslash G_S$ is compact all of the spectrum is cuspidal, and so any convolution kernel in $C_c^\infty(K_S \backslash G_S / K_S)$ with positive spherical transform will satisfy (5.3) (indeed, in this case we may as well take $\Omega = \Gamma \backslash G_S$ in which case (5.3) becomes an identity).

For this subsection, we shall vary the notation slightly and assume that \mathbf{G} is an \mathbb{Q} -anisotropic semisimple \mathbb{Q} -group, S an arbitrary set of places containing ∞ , $G = \mathbf{G}(\mathbb{Q}_S)$, and Γ is a congruence subgroup of $\mathbf{G}(\mathbb{Z}[S^{-1}])$, so that $\Gamma \backslash G_S$ is compact. Since we are interested in this case mostly as an introduction to the split case, we will further assume that Γ is torsion free, which avoids some minor difficulties.

To obtain Weyl's law for compact quotients, we apply (5.3), with $\Omega = \Gamma \backslash G_S$, for a family of test functions which will be defined using a function $H^\infty \in C_c^\infty(K_\infty \backslash G_\infty / K_\infty)$. This function will be chosen to have the following properties:

Lemma 2. *Let $0 < \varepsilon < 1$. There exists $H^\infty \in C_c^\infty(K_\infty \backslash G_\infty / K_\infty)$ so that its spherical transform $h(\nu) := \widehat{H^\infty}(\nu)$ satisfies:*

- (1) $h(\nu)$ depends only on $\langle \nu, \nu \rangle$, for $\nu \in \mathfrak{a}_\infty^*$.
- (2) $h(\nu)$ is real and non-negative whenever ν and $\bar{\nu}$ are W_∞ -conjugate to each other (in particular, when $\pi(\nu)$ unitary).
- (3) $h(\nu) \leq 1$ for every $\nu \in \mathfrak{a}_\infty^*$ with $\langle \nu, \nu \rangle \in \mathbb{R}^+$.
- (4) $h(\nu) \geq 1 - \varepsilon$ whenever $\nu \in \mathfrak{a}_\infty^*$ and $\langle \nu, \nu \rangle \in \mathbb{R}$, $0 \leq \langle \nu, \nu \rangle \leq 1 - \varepsilon$.
- (5) As $t \rightarrow 0$, we have

$$t^d \int_{\mathfrak{a}_{S, \text{temp}}^*} h(t\nu_\infty) d\mu_{\text{Planch}}(\nu) = \alpha(\mathbf{G}) + O(\varepsilon),$$

$$(6) \quad \sup_{\|\nu\| \geq 1, \nu \in \mathfrak{a}_{\infty}^*, \text{temp}} (1 + \|\nu\|)^{d+1} |h(\nu)| < \epsilon$$

We defer the somewhat technical and not very illuminating proof of this lemma to the end of this section, showing first how, using this H^∞ , Weyl's law can be proved.

For $0 < t \leq 1$, let $H_t^\infty \in C_c^\infty(K_\infty \backslash G_\infty / K_\infty)$ be so that $\widehat{H_t^\infty}(\nu) = h(t\nu)$. That this is well defined is a consequence of the Paley-Wiener type theorem (Thm. 2) for symmetric spaces. Indeed, Theorem 2 implies that given any open neighborhood of the identity in $K_\infty \backslash G_\infty / K_\infty$, if t is small enough, the support of H_t^∞ is contained in this neighborhood. We construct a function $H_t \in C_c^\infty(K_S \backslash G_S / K_S)$ from H_t^∞ as follows: for each finite place $v \in S$ and any t , let H_t^v be the characteristic function of K_v , and define $H_t \in C^\infty(K_S \backslash G_S / K_S)$ via $H_t := \prod_{v \in S} H_t^v$. The spectral transform of H_t is the function $\nu \mapsto h(t\nu_\infty)$, where $\nu \mapsto \nu_\infty$ is the natural projection $\mathfrak{a}_S^* \rightarrow \mathfrak{a}_\infty^*$.

Since $\tilde{\Omega}$ is fixed, and since the support of H_t^∞ shrinks to K_∞ , for t small enough the set Z defined in (5.4) consists only of nontrivial $\gamma \in \Gamma$ which belongs to some conjugate of K_S . The assumption that Γ is torsion free shows this set to be empty. Now take t to be sufficiently small so that Z is empty. Then (5.3), applied with $k = H_t$ and $\Omega = \Gamma \backslash G_S$, implies that

$$\sum_{\nu \in \mathcal{E}} h(t\nu_\infty) = H_t(1) \text{vol}(\Gamma \backslash G_S)$$

By (3.3) and (5) of Lemma 2,

$$H_t(1) = \int_{\nu \in \mathfrak{a}_{S, \text{temp}}^*} h(t\nu_\infty) d\mu_{\text{Planch}}(\nu) = t^{-d}(\alpha(\mathbf{G}) + O(\epsilon))$$

so

$$(5.5) \quad \sum_{\nu \in \mathcal{E}} h(t\nu_\infty) = t^{-d}(\alpha(\mathbf{G}) + O(\epsilon)) \text{vol}(\Gamma \backslash G_S).$$

Since for any $\nu \in \mathcal{E}$, the inner product $\langle \nu_\infty, \nu_\infty \rangle = \|\nu_\infty\|^2$ differs from the Laplacian eigenvalue of the corresponding eigenfunction φ_ν by a fixed real constant, we have by (3) of Lemma 2 that

$$(5.6) \quad h(t\nu_\infty) \leq 1 \quad \text{for all but finitely many } \nu \in \mathcal{E};$$

and for all $\nu \in \mathcal{E}$ we have that $h(t\nu_\infty)$ is nonnegative, real, and uniformly bounded for $t \in (0, 1]$.

In view of (4) of Lemma 2, equation (5.5) implies immediately that for all t sufficiently small

$$(5.7) \quad |\{\nu \in \mathcal{E} : \|\nu\|_\infty < t^{-1}\}| \leq t^{-d}(\alpha(\mathbf{G}) + O(\epsilon)) \text{vol}(\Gamma \backslash G_S).$$

On the other hand, applying (5.7) together with Lem. 2, (6) and (5.6) gives that

$$(5.8) \quad \left| \sum_{\nu \in \mathcal{E}: \|\nu_\infty\| > t^{-1}} h(t\nu_\infty) \right| \leq \sum_{n=0}^{\infty} \left| \sum_{\nu \in \mathcal{E}: \|\nu_\infty\| \in [2^n, 2^{n+1}]} h(t\nu_\infty) \right| \\ \leq \sum_n \frac{2\epsilon t^{-d} \alpha(\mathbf{G}) \text{vol}(\Gamma \backslash G_S) 2^{(n+1)d}}{(1 + 2^n)^{d+1}} = O(\epsilon t^{-d})$$

so by (5.5) and (5.6)

$$(5.9) \quad \begin{aligned} |\{\nu \in \mathcal{E} : \|\nu\|_\infty < t^{-1}\}| &\geq O(1) + \sum_{\nu \in \mathcal{E} : \|\nu\|_\infty < t^{-1}} h(t\nu_\infty) \\ &\geq \sum_{\nu \in \mathcal{E}} h(t\nu_\infty) - O(\epsilon t^{-d}) = t^{-d}(\alpha(\mathbf{G}) - O(\epsilon)) \text{vol}(\Gamma \backslash G_S), \end{aligned}$$

concluding the proof of Weyl's law in the compact case.

Proof of Lemma 2. Let \mathbb{O} be the orthogonal group of $(\mathfrak{a}_{\infty, \text{temp}}^*, \langle \cdot, \cdot \rangle)$. Then \mathbb{O} is a maximal compact subgroup of its complexification $\mathbb{O}_{\mathbb{C}}$, which is identified with the orthogonal group of $(\mathfrak{a}_{\infty}^*, \langle \cdot, \cdot \rangle)$. Moreover, $W_\infty \subset \mathbb{O}$.

Let χ be the characteristic function of the ball of radius 1 in $\mathfrak{a}_{\infty, \text{temp}}^*$. There exists a nonempty open set of Schwarz functions ψ on the real Euclidean space $\mathfrak{a}_{\infty, \text{temp}}^*$ satisfying:

$$(5.10) \quad 0 \leq \psi(\nu) < 1, \quad (\|\nu\| \leq 1, \nu \in \mathfrak{a}_{\infty, \text{temp}}^*)$$

$$(5.11) \quad \psi(\nu) > \sqrt{1 - \varepsilon}, \quad (\|\nu\| \leq \sqrt{1 - \varepsilon}, \nu \in \mathfrak{a}_{\infty, \text{temp}}^*)$$

$$(5.12) \quad \sup_{\|\nu\| \geq 1, \nu \in \mathfrak{a}_{\infty, \text{temp}}^*} (1 + \|\nu\|)^{d+1} |\psi(\nu)| < \varepsilon/2$$

$$(5.13) \quad \int_{\mathfrak{a}_{\infty, \text{temp}}^*} |\psi(\nu) - \chi(\nu)| (1 + \|\nu\|)^{d-r} d\nu < \varepsilon/2.$$

Since the set of functions whose Fourier transform has compact support is dense in the Schwarz space, we can find a ψ satisfying (5.10)–(5.13) whose Fourier transform has compact support. Averaging over \mathbb{O} does not affect the validity of (5.10)–(5.13); we may thereby assume that:

$$(5.14) \quad \psi(k\nu) = \psi(\nu), \quad \nu \in \mathfrak{a}_{\infty, \text{temp}}^*, k \in \mathbb{O}.$$

Since $W_\infty \subset \mathbb{O}$, ψ is W_∞ -invariant. Moreover, since the Fourier transform of ψ is compactly supported, it follows that ψ extends to a holomorphic function on \mathfrak{a}_{∞}^* . It follows that, considered as a function on \mathfrak{a}_{∞}^* , ψ is actually $\mathbb{O}_{\mathbb{C}}$ -invariant. Since two nonzero vectors $\nu_1, \nu_2 \in \mathfrak{a}_{\infty}^*$ are $\mathbb{O}_{\mathbb{C}}$ -conjugate precisely when $\langle \nu_1, \nu_1 \rangle = \langle \nu_2, \nu_2 \rangle$, we deduce that $\psi(\nu)$ depends only on $\langle \nu, \nu \rangle$ for $\nu \in \mathfrak{a}_{\infty}^*$. Note that, for ε sufficiently small, the conditions (5.10) and (5.12) guarantee that $\sup_{\nu \in \mathfrak{a}_{\infty, \text{temp}}^*} |\psi(\nu)| \leq 1$.

By Theorem 2, there is a function $F \in C_c^\infty(K_\infty \backslash G_\infty / K_\infty)$ whose spherical transform $\hat{F} = \psi$. Put $H^\infty = F \star \check{F}$, where $\check{F}(g) := \overline{F(g^{-1})}$.

Then $h(\nu) = \psi(\nu) \overline{\psi(\bar{\nu})}$, whence claim (2) follows at once. Since, for $\nu \in \mathfrak{a}_{\infty, \text{temp}}^*$, we have in fact $h(\nu) = |\psi(\nu)|^2$, we obtain also claim (1), (3), (4) and (6).

Let c_1 be as in (3.4). Then:

$$\begin{aligned}
(5.15) \quad & \limsup_{t \rightarrow 0} \left| t^d \int_{\mathfrak{a}_{S,\text{temp}}^*} h(t\nu_\infty) d\mu_{\text{Planch}}(\nu) - \alpha(\mathbf{G}) \right| \\
& \leq c_1 \limsup_{t \rightarrow 0} \int_{\mathfrak{a}_{S,\text{temp}}^*} |t^d h(t\nu_\infty) - t^d \chi(t\nu_\infty)| (1 + \|\nu_\infty\|)^{d-r} d\mu_{\text{Haar}}(\nu) \\
& \leq c_1 \limsup_{t \rightarrow 0} \int_{\mathfrak{a}_{S,\text{temp}}^*} |h(\nu_\infty) - \chi(\nu_\infty)| (t + \|\nu_\infty\|)^{d-r} d\mu_{\text{Haar}}(\nu) \\
& \leq c_1 \int_{\mathfrak{a}_{S,\text{temp}}^*} |h(\nu_\infty) - \chi(\nu_\infty)| (1 + \|\nu_\infty\|)^{d-r} d\mu_{\text{Haar}}(\nu).
\end{aligned}$$

Recalling that $\sup_{\nu \in \mathfrak{a}_{\infty,\text{temp}}^*} |\psi(\nu)| \leq 1$ we see that $|h(\nu) - \chi(\nu)| \leq 2|\psi(\nu) - \chi(\nu)|$ for $\nu \in \mathfrak{a}_{\infty,\text{temp}}^*$. Applying (5.15) and (5.13), we see that there is $c_2 > 0$ such that

$$\limsup_{t \rightarrow 0} \left| t^d \int_{\nu \in \mathfrak{a}_{S,\text{temp}}^*} h(t\nu_\infty) d\mu_{\text{Planch}}(\nu) - \alpha(\mathbf{G}) \right| \leq c_2 \varepsilon.$$

Reducing ε as necessary, we conclude that h satisfies also claim (5) of the Lemma. \square

6. THE PROOF OF WEYL LAW FOR \mathbb{Q} -SPLIT GROUPS

We now return to the more interesting case of \mathbf{G} a \mathbb{Q} -split group, in which case $\Gamma \backslash G_S$ is not compact, and the existence and the abundance of cusp forms is substantially more delicate because of the presence of continuous spectrum. As we have remarked in the introduction, the other approaches to the Weyl law in this context have required explicit estimates on the Eisenstein series to show that they are spectrally “negligible”; we will avoid this issue by taking, from the outset, convolution operators that satisfy (4.2) and therefore kill the continuous spectrum.

Our approach will be similar to the one presented in the previous section. We proceed by expanding spectrally a family of convolution operator with kernels $k_t \in C_c^\infty(K_S \backslash G_S / K_S)$ for $t \rightarrow 0$. Actually, to derive the full Weyl law we will need to use more than one such family. We will avoid the complications of having continuous spectrum in the spectral expansion of $f \mapsto k_t \star f$ by taking kernels k_t satisfying (4.2), and hence the corresponding convolution operators will have a purely cuspidal image.

It is clear from (4.2) that the support of the kernels k_t cannot possibly shrink to K_S as in the compact case. However, by using fairly crude estimates we will show that for t small the terms $\gamma \neq e$ in the finite volume analog to (5.3) are still negligible.

It also follows from the construction of §4 that even if one is interested only in studying cusp forms on $\Gamma_\infty \backslash G_\infty / K_\infty$ with $\Gamma_\infty < \mathbf{G}(\mathbb{Q})$ a congruence subgroup, one needs take S to contain at least one finite place. The passage from Weyl law on the S -algebraic $\Gamma \backslash G_S / K_S$ to that on $\Gamma_\infty \backslash G_\infty / K_\infty$ is fairly straightforward and is explained in §6.3. In the case of $\text{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}$ this is particularly straightforward: we take for example $S = \{\infty, p\}$, $\mathbf{G} = \text{PGL}(2)$ and note that $\text{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}$ can be identified with $\mathbf{G}(\mathbb{Z}[1/p]) \backslash G_S / K_S$, and so Weyl's law on $\mathbf{G}(\mathbb{Z}[1/p]) \backslash G_S / K_S$ is equivalent to that on $\text{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}$.

Our kernels k_t will be of the form $k \star H_t$ with H_t as in the previous section, and k a compactly supported spherical *distribution* satisfying an appropriate version of (4.2). Again returning to the special case of $\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}$, at the end of section 2 we have shown how the operator \aleph defined on functions on $\mathrm{PGL}(2, \mathbb{Z}) \backslash \mathbb{H}$ which has purely cuspidal image corresponds to such a spherical distribution on $K_S \backslash G_S / K_S$.

In order to get Weyl's law we will actually need k to be a distribution which approximates in an appropriate way the projection of $L^2(\Gamma \backslash G_S / K_S)$ on its cuspidal subspace $L^2_{\mathrm{cusp}}(\Gamma \backslash G_S / K_S)$.

For example, for $\mathrm{PGL}(2)$, we can use the following procedure: first note that the spectrum of \aleph is contained in $[-4\sqrt{p}, 4\sqrt{p}]$. By Weierstrass approximation theorem we can find a polynomial p with real coefficients and no constant term, so that $p(t) \approx 1$ in the range $\epsilon < |t| \leq 4\sqrt{p}$, and take k to be the distribution corresponding to the operator $p(\aleph)^2$.

6.1. General case: proof of lower bound in Weyl's law. We'll now prove the lower bound of the Weyl law (the full Weyl law follows by combining this with the work of Donnelly; see §6.3 for details).

Throughout this section we shall assume that, in addition to those conditions prescribed in Sec. 3, S contains at least one finite place in addition to ∞ . We denote by $\mathcal{D}(K_S \backslash G_S / K_S)$ the space of K_S -bi-invariant, compactly supported *distributions* on G_S . One defines the spherical transform $\hat{k}(\nu)$ for $k \in \mathcal{D}(K_S \backslash G_S / K_S)$ and $\nu \in \mathfrak{a}_S^*$ to be the scalar by which k acts on the spherical vector in $\pi(\nu)$. For any $k \in \mathcal{D}(K_S \backslash G_S / K_S)$ we put (the "spectral norm") $\|k\|_{\mathrm{spec}, \infty} = \sup_{\nu \in \mathfrak{a}_S^* : \pi(\nu) \text{ unitary}} |\hat{k}(\nu)|$, if finite.

We remark that our use of distributions is entirely for *notational convenience*. Indeed, the distributions we will use always be convolved with a smooth function when they are used in the eventual applications. Thus, although we will at a certain point use an analogue of the Paley-Wiener theorem Thm. 2 for distributions, this is cosmetic and can be replaced with the use of Thm. 2 as stated.

Let $k \in \mathcal{D}(K_S \backslash G_S / K_S)$ satisfy (4.3) and be so that $\|k\|_{\mathrm{spec}, \infty} < \infty$. Let $H^\infty \in C_c^\infty(K_S \backslash G_S / K_S)$; defining, for $t > 0$, H_t as in Sec. 5.1, we put $k_t = k \star H_t$. Then $k_t \in C_c^\infty(K_S \backslash G_S / K_S)$ and also satisfies (4.3).

The space $\Gamma \backslash G_S / K_S$ may be regarded as a finite union of locally symmetric spaces, each with universal covering G_∞ / K_∞ . Define \mathcal{E} as prior to (5.1). It follows from the result of Donnelly [3] that

$$(6.1) \quad \#\{\nu \in \mathcal{E} : \langle \nu_\infty, \nu_\infty \rangle \leq T\} \leq CT^{d/2},$$

for some constant C . (We will have need of the precise constant, but only at the end of the argument: see (6.11) where we specify the constant. For most of the argument, any constant C would do).

Lemma 3. *Let $F \subset G_S$ be a compact subset so that the projection of F to G_∞ does not intersect K_∞ . Let $k \in \mathcal{D}(K_S \backslash G_S / K_S)$ be such that $\|k\|_{\mathrm{spec}, \infty} < \infty$. Then there is a constant c , depending possibly on k and H^∞ , such that*

$$(6.2) \quad \sup_{g \in F} |k_t(g)| \leq ct^{-d+1/2}, \quad (0 < t \leq 1).$$

Moreover, there exists a sequence $k_n \in \mathcal{D}(K_S \backslash G_S / K_S)$ such that $\|k_n\|_{\mathrm{spec}, \infty} < \infty$ and moreover:

- (1) Each k_n satisfies (4.3),

- (2) $\widehat{k}_n(\nu) \geq 0$ whenever $\pi(\nu)$ is unitary, and:
(3) Suppose h is a non-negative Schwarz function on $\mathfrak{a}_{\infty, \text{temp}}^*$. Then

$$\lim_{n \rightarrow \infty} \left(\liminf_{t \rightarrow 0} \frac{\int \widehat{k}_n(\nu) h(t\nu_{\infty}) d\mu_{\text{Planch}}(\nu)}{\|k_n\|_{\text{spec}, \infty} \int h(t\nu_{\infty}) d\mu_{\text{Planch}}(\nu)} \right) = 1.$$

We defer the (tedious but straightforward) proof of this Lemma until the next section.

Proof. (of the lower bound in the Weyl law)

We shall take $k \in \mathcal{D}(K_S \backslash G_S / K_S)$ to be one of the sequence k_n constructed in Lem. 3, and $H^{\infty} \in C_c^{\infty}(K_{\infty} \backslash G_{\infty} / K_{\infty})$ will be as provided by Lem. 2. In particular, with these choices, $\widehat{k}(\nu) \geq 0$ and $h(t\nu_{\infty}) \geq 0$ for all $\nu \in \mathcal{E}$ and all $0 \leq t \leq 1$. Moreover, k_t satisfies (4.3).

Let $\Omega \subset \Gamma \backslash G_S$ be any compact set. Let $\tilde{\Omega}$ be any compact subset of G_S whose projection to Ω is almost everywhere one to one. Define Z as in (5.4), except replacing ‘‘support of k ’’ by any compact set that contains the support of k_t by $0 < t < 1$.

For each $\gamma \in Z$, let F_{γ} be the set of $g_{\infty} \in G_{\infty}$ such that $g_{\infty}^{-1} \gamma g_{\infty}$ belongs to K_{∞} . Then F_{γ} has zero measure (note that this requires that γ is not central, so uses the adjointness of \mathbf{G}). Let $\tilde{\Omega}' \subset \tilde{\Omega}$ be a compact subset of $\tilde{\Omega}$ whose projection to G_{∞} does not contain $\cup_{\gamma \in Z} F_{\gamma}$; since each F_{γ} is a null set, we may choose $\tilde{\Omega}'$ to have measure arbitrarily close to that of $\tilde{\Omega}$.

From (5.3), with $k = k_t$, we deduce:

$$(6.3) \quad \sum_{\nu \in \mathcal{E}} \widehat{k}(\nu) h(t\nu_{\infty}) \geq k_t(1) \text{vol}(\tilde{\Omega}') + \sum_{\gamma \in Z} \int_{\tilde{\Omega}'} k(g^{-1} \gamma g) dg$$

The set $\{g^{-1} \gamma g : g \in \tilde{\Omega}', \gamma \in Z\}$ is a compact subset of G_S , whose projection to G_{∞} does not intersect K_{∞} . So, for all $\gamma \in Z$, we have by (6.2) that $|\int_{g \in \tilde{\Omega}'} k_t(g^{-1} \gamma g)| \leq ct^{-d+1/2}$, for $0 \leq t \leq 1$, and for some constant c depending on $k, H^{\infty}, \tilde{\Omega}$ and $\tilde{\Omega}'$. So, applying Plancherel inversion (3.3), we see that:

$$\sum_{\nu \in \mathcal{E}} \widehat{k}(\nu) h(t\nu_{\infty}) \geq \text{vol}(\tilde{\Omega}') \int \widehat{k}(\nu) h(t\nu_{\infty}) d\mu_{\text{Planch}}(\nu) - c|Z|t^{-d+1/2}, \quad (0 < t \leq 1).$$

It follows:

$$\liminf_{t \rightarrow 0} t^d \sum_{\nu \in \mathcal{E}} h(t\nu_{\infty}) \geq \text{vol}(\tilde{\Omega}') \liminf_{t \rightarrow 0} t^d \frac{\int \widehat{k}(\nu) h(t\nu_{\infty}) d\mu_{\text{Planch}}(\nu)}{\|k\|_{\text{spec}, \infty}}.$$

We apply Lem. 3 to choose k , taking $k = k_n$ for some sufficiently large n and then take $n \rightarrow \infty$. Note moreover that we can choose $\text{vol}(\tilde{\Omega}')$ to be arbitrarily close to $\text{vol}(\tilde{\Omega}) = \text{vol}(\Omega)$, which in turn (by choosing Ω sufficiently large) can be made arbitrarily close to $\text{vol}(\Gamma \backslash G_S)$. We see that:

$$(6.4) \quad \liminf_{t \rightarrow 0} t^d \sum_{\nu \in \mathcal{E}} h(t\nu_{\infty}) \geq \text{vol}(\Gamma \backslash G_S) \liminf_{t \rightarrow 0} \left(t^d \int h(t\nu_{\infty}) d\mu_{\text{Planch}}(\nu) \right).$$

We choose H^{∞}, h as in Lem. 2; this choice depends on a parameter ϵ , which we then let approach 0. The same reasoning as used in Sec. 5.1 (see esp. (5.8)

and (5.9)) allows one to deduce from (6.4) a lower bound in the Weyl law; in the present context, we must use as an *a priori* input Donnelly's upper bound (6.1). In any case, we obtain:

$$(6.5) \quad \liminf_{T \rightarrow \infty} \frac{\#\{\nu \in \mathcal{E} : \langle \nu_\infty, \nu_\infty \rangle \leq T\}}{T^{d/2}} \geq \text{vol}(\Gamma \backslash G_S) \alpha(\mathbf{G}).$$

This is the lower bound in the Weyl law. (It can be verified by computation that the lower bound is "correct", i.e., the constant coincides with that described at the start of the Introduction; we sketch a different proof of this correctness in §6.3.) \square

6.2. Proof of the main lemma. We now turn to the proof of Lem. 2. We shall first prove a preparatory result which asserts, in essence, that linear combinations of trigonometric functions do not take small values too often.

Lemma 4. *Let $A = \mathbb{R}^d \times (\mathbb{R}/\mathbb{Z})^e$, and let $B(T)$ be the subset of A consisting of elements whose projection to \mathbb{R}^d lies in a Euclidean ball of radius T and centered at 0. Let χ_1, \dots, χ_n be distinct unitary characters of A , let $0 \neq a_i \in \mathbb{C}$, and put $F = \sum_{i=1}^n a_i \chi_i$. Fix a Haar measure $d\mu$ on A .*

Then:

$$(6.6) \quad \lim_{\epsilon \rightarrow 0} \left(\lim_{T \rightarrow \infty} \frac{\mu\{x \in B(T) : |F(x)| \leq \epsilon\}}{T^d} \right) = 0.$$

Proof. The map $\underline{\chi} = (\chi_1, \dots, \chi_n)$ maps A to the torus $\mathbb{T} := \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| = 1 \text{ for all } i\}$. Let L be the function $(z_1, \dots, z_n) \rightarrow \sum_i a_i z_i$ on \mathbb{T} , and let \mathbb{T}' be the closure of the image $\underline{\chi}(A)$. Then \mathbb{T}' is a subtorus of \mathbb{T} . L does not vanish identically on \mathbb{T}' , since otherwise $F = 0$, contradicting the linear independence of characters.

Let $d\mu_T$ be the measure $\frac{d\mu}{\mu(B(T))}$ restricted to $B(T)$, and let $\nu_T = \underline{\chi}_* \mu_T$. Any weak limit (as $T \rightarrow \infty$) of the measures ν_T is supported on \mathbb{T}' and invariant by \mathbb{T}' ; consequently, the ν_T converge to the invariant probability measure ν on \mathbb{T}' . (6.6) amounts to the assertion that the zero-locus of L on \mathbb{T}' has zero measure w.r.t ν ; but this is obvious, as L is real-analytic and nonvanishing. \square

Proof. (of Lem. 3) For fixed g s.t. $g_\infty \notin K_\infty$, the spherical functions $\Xi_\nu(g)$ decay as $\nu_\infty \rightarrow \infty$. An elegant quantification of this has been given by Duistermaat, Kolk and Varadarajan in [4, Cor. 11.2]; it shows in particular that for g_∞ in a fixed compact subset excluding K_∞ , we have a bound of the form $|\Xi_\nu(g)| \leq c'(1 + \|\nu_\infty\|)^{-1/2}$ for some c' . Combining with Plancherel inversion (3.2) and (3.4), it follows that there is c'' such that

$$(6.7) \quad \sup_{g \in F} t^{d-1/2} |k_t(g)| \\ \leq c'' t^{d-1/2} \|k\|_{\text{spec}, \infty} \int_{\nu_\infty \in \mathfrak{a}_\infty^*} h(t\nu_\infty) (1 + \|\nu_\infty\|)^{d-r-1/2} d\mu_{\text{Haar}}(\nu_\infty) \\ \leq c'' \|k\|_{\text{spec}, \infty} \int_{\mathfrak{a}_\infty^*} h(\nu_\infty) (t + \|\nu_\infty\|)^{d-r-1/2} d\mu_{\text{Haar}}(\nu_\infty),$$

whence (6.2).

Next, we turn to the construction of the k_n s. Apply Lem. 1 with $A = \mathfrak{a}_S$, $W = W_S$, $T = \{\Gamma_{A,i} : 1 \leq i \leq I\}$. Lem. 1 produces a nonzero, compactly supported, W_S -invariant distribution f on \mathfrak{a}_S which vanishes on any $\Gamma_{A,i}$ -invariant

function, for $1 \leq i \leq I$; moreover, this f is a finite linear combination of point masses. Let $\check{f}(x) = \overline{f(-x)}$, and replace f by $f \star \check{f}$; then f is still W_S -invariant and a linear combination of point masses, and $\hat{f}(\nu)$ is a non-negative real number whenever $\nu, \bar{\nu}$ are W_S -conjugate.

The function $\nu \mapsto \hat{f}(\nu)$ is a finite linear combination of characters. Put $K = \sup_{\nu \in \mathfrak{a}_S^* : \pi(\nu) \text{ unitary}} \hat{f}(\nu)$. The supremum is easily verified to be finite. Now set $P_n(x) = 1 - (1 - \frac{x^2}{K^2})^n$. Then P_n satisfies:

- (1) For each n , $P_n(0) = 0$.
- (2) For each n , $0 \leq P_n(x) \leq 1$ for $x \in [-K, K]$.
- (3) The P_n converge uniformly on any compact subset of $[-K, K] - \{0\}$ to the constant function with value 1.

Equip the space of compactly supported distributions on \mathfrak{a}_S with the algebra structure in which multiplication corresponds to convolution. With this in mind, set $f_n = P_n(f)$; in other words, $\widehat{f_n}(\nu) = P_n(\widehat{f}(\nu))$. Thus f_n is a W_S -invariant distribution on \mathfrak{a}_S , which is indeed a finite linear combination of point masses.

We now choose $k_n \in \mathcal{D}(K_S \backslash G_S / K_S)$ such that $\mathcal{S}k_n = f_n$; or equivalently (by (3.6)) $\widehat{k_n}(\nu) = \widehat{f_n}(\nu)$. (Here we have used a distributional extension of the Satake isomorphism; however, as we have remarked, one can avoid this entirely and use Thm. 2 as stated, because k_n enters only through its convolution with an appropriate H_t). It is clear that k_n satisfy (1) and (2) of Lem. 3. Moreover, by choice, $\|k_n\|_{\text{spec}, \infty} \leq 1$.

It remains to verify (3) of Lem. 3. For this it suffices to check that, for h a non-negative Schwarz function on $\mathfrak{a}_{\infty, \text{temp}}^*$, we have:

$$(6.8) \quad \lim_{n \rightarrow \infty} \left(\liminf_{t \rightarrow 0} \frac{\int h(t\nu_{\infty}) \widehat{f_n}(\nu) d\mu_{\text{Planch}}(\nu)}{\int h(t\nu_{\infty}) d\mu_{\text{Planch}}(\nu)} \right) = 1.$$

It is easy to see that $\int h(t\nu_{\infty}) d\mu_{\text{Planch}}(\nu) \sim ct^{-d}$, for some $c > 0$. To check (6.8), it suffices to check that the limit

$$L := \lim_{\epsilon \rightarrow 0} \left(\limsup_{t \rightarrow 0} t^d \int_{\nu \in \mathfrak{a}_{S, \text{temp}}^* : |\hat{f}(\nu)| \leq \epsilon} h(t\nu_{\infty}) d\mu_{\text{Planch}}(\nu) \right)$$

equals 0. From (3.4), we deduce that

$$(6.9) \quad L \leq c_1 \lim_{\epsilon \rightarrow 0} \left(\limsup_{t \rightarrow 0} t^d \int_{\nu \in \mathfrak{a}_{S, \text{temp}}^* : |\hat{f}(\nu)| \leq \epsilon} (1 + \|\nu_{\infty}\|)^{d-r} |h(t\nu_{\infty})| d\mu_{\text{Haar}}(\nu) \right).$$

Let c'_1 be so that $\sup_{\nu \in \mathfrak{a}_{\infty, \text{temp}}^*} |h(\nu)|(1 + \|\nu\|)^{d+1} \leq c'_1$. Then:

$$(6.10) \quad L \leq c_1 c'_1 \lim_{\epsilon \rightarrow 0} \limsup_{t \rightarrow 0} t^r \int_{\nu \in \mathfrak{a}_{S, \text{temp}}^* : |\hat{f}(\nu)| \leq \epsilon} \frac{(t + \|\nu_{\infty}\|)^{d-r}}{(1 + \|\nu_{\infty}\|)^{d+1}} d\mu_{\text{Haar}}(\nu) \\ \leq c_1 c'_1 \lim_{\epsilon \rightarrow 0} \limsup_{t \rightarrow 0} t^r \int_{\nu \in \mathfrak{a}_{S, \text{temp}}^* : |\hat{f}(\nu)| \leq \epsilon} (1 + \|\nu_{\infty}\|)^{-r-1} d\mu_{\text{Haar}}(\nu)$$

From Lem. 4, we conclude that the quantity on the right of (6.10) is 0, i.e. $L = 0$ as claimed. This completes the verification of condition (3) of the Lemma. \square

6.3. Deduction of Thm. 1 from the S -arithmetic Weyl law. We now discuss the final reductions to complete the proof of Thm. 1. The argument, in words, is as follows: (6.5) asserts, in effect, the lower bound in Weyl law for the cuspidal spectrum on a certain finite union of locally symmetric spaces. Since one knows, by Donnelly’s work, the upper bound in the Weyl law for each of these spaces, the Weyl law for each individual locally symmetric space follows.

The number of $(\Gamma, G_\infty K_S)$ double cosets in G_S is finite; in particular, there is a finite collection (g_i) such that G_S is the disjoint union of the double cosets $\Gamma g_i G_\infty K_S$. Without loss of generality we may take $g_1 = 1$.

Put $\Gamma_{\infty, i} = g_i^{-1} \Gamma g_i \cap G_\infty K_S$, which we regard (by projection) as a lattice in G_∞ . Then the space $\Gamma \backslash G_S / K_S$ is the disjoint union of the locally symmetric spaces $\Gamma_{\infty, i} \backslash G_\infty / K_\infty$. Let $N(T)$ (resp. $N_i(T)$) be the number of cuspidal eigenfunctions of the Laplacian on $\Gamma \backslash G_S / K_S$ (resp. $\Gamma_{\infty, i} \backslash G_\infty / K_\infty$) with eigenvalue $\leq T$. Then $\sum_i N_i(T) = N(T)$.

Now (6.5) shows that $\liminf_{T \rightarrow \infty} \frac{N(T)}{T^{d/2}} \geq \alpha(\mathbf{G}) \text{vol}(\Gamma \backslash G_S)$. On the other hand, the main result of [3] asserts that

$$(6.11) \quad \limsup_{T \rightarrow \infty} \frac{N_i(T)}{T^{d/2}} \leq c(\Gamma_{\infty, i} \backslash G_\infty / K_\infty),$$

where $c(M)$ is defined in the first paragraph of the Introduction.

We now claim that

$$(6.12) \quad \sum_i c(\Gamma_{\infty, i} \backslash G_\infty / K_\infty) = \alpha(\mathbf{G}) \text{vol}(\Gamma \backslash G_S).$$

Actually, this can be done using the explicit forms of the Plancherel measure, etc.; however, let us sketch a slightly more “conceptual” approach, which has the disadvantage of invoking slightly more. It is a theorem of A. Borel [1] that $G_\infty = \mathbf{G}(\mathbb{R})$ admits a cocompact torsion-free arithmetic lattice Λ . By the discussion of Sec. 5.1 – applied in the case $S = \{\infty\}$ – the number of Laplacian eigenfunctions on $\Lambda \backslash G_\infty / K_\infty$ with eigenvalue $\leq T$ is $\sim \alpha(\mathbf{G}) \text{vol}(\Lambda \backslash G_\infty) T^{d/2}$. On the other hand, Weyl’s original proof applies to the compact Riemannian manifold $\Lambda \backslash G_\infty / K_\infty$ and shows that this number is also $\sim c(\Lambda \backslash G_\infty / K_\infty) T^{d/2}$; consequently we have the equality $\alpha(\mathbf{G}) \text{vol}(\Lambda \backslash G_\infty) = c(\Lambda \backslash G_\infty / K_\infty)$. Now it is easy to see that both sides of this equality change by the same multiplicative factor when Λ is replaced by any other (not necessarily cocompact) lattice Λ' ; in particular, $\alpha(\mathbf{G}) \text{vol}(\Gamma_{\infty, i} \backslash G_\infty) = c(\Gamma_{\infty, i} \backslash G_\infty / K_\infty)$; summing over i , and noting that $\text{vol}(\Gamma \backslash G_S) = \sum_i \text{vol}(\Gamma_{\infty, i} \backslash G_\infty)$, we have proven (6.12).

From (6.12), (6.5) and (6.11), it follows that for each i we have $\lim_{T \rightarrow \infty} \frac{N_i(T)}{T^{d/2}} = c(\Gamma_{\infty, i} \backslash G_\infty / K_\infty)$, i.e. Weyl’s law holds for the symmetric space $\Gamma_{\infty, i} \backslash G_\infty / K_\infty$. On the other hand, any congruence subgroup of $\mathbf{G}(\mathbb{Z})$ arises as $\Gamma_{\infty, i}$ for some S -arithmetic congruence subgroup $\Gamma \subset \mathbf{G}(\mathbb{Z}[S^{-1}])$ and some i . Theorem 1 follows. \square

APPENDIX A. EXISTENCE OF CUSP FORMS FOR GENERAL \mathbf{G} — A PROOF USING WHITTAKER FUNCTIONS

The considerations of the previous sections give the full Weyl law. However, we would also like to explain a short proof of the *existence* of cusp forms that uses Whittaker functions; this does not give the Weyl law but, unlike the proof of the Weyl law, gives a very explicit method for constructing cuspidal functions. It is

easy to modify the present proof to show (e.g.) the existence of infinitely many cusp forms which do not lie in the image of a fixed set of functorial lifts.

For $T \geq 0$, set $A_{S, > T} = \{a \in A_S : |\alpha(a)|_S > T, \forall \alpha \in \Delta\}$. Define $\mathfrak{S}_T \subset G_S$ via $\mathfrak{S}_T = N_S \cdot A_{S, > T} \cdot K_S$ and set $\Gamma_B = B_S \cap \Gamma$, $\Gamma_A = A_S \cap \Gamma$, $\Gamma_N = N_S \cap \Gamma$. Then $\Gamma_B \cdot \mathfrak{S}_T \subset \mathfrak{S}_T$. The image of \mathfrak{S}_T in $\Gamma \backslash G_S$ is called a ‘‘Siegel set.’’

We shall need the following properties:

- (1) For every T , (the image of) \mathfrak{S}_T intersects every connected component of G_S/K_S .

(Let $\mathfrak{t} = \text{Lie}(\mathbf{T})$, so we have an exponential map $\exp : \mathfrak{t}(\mathbb{R}) \rightarrow \mathbf{T}(\mathbb{R}) \subset \mathbf{G}(\mathbb{R})$. Let $a \in \mathfrak{t}$ belong to the interior of the positive Weyl chamber. Then for any $x \in G_S/K_S$ and sufficiently large $t > 0$, $\exp(t.a)x$ belongs to \mathfrak{S}_T .)

- (2) If T is chosen sufficiently large, the natural map

$$\iota_T : \Gamma_B \backslash \mathfrak{S}_T \rightarrow \Gamma \backslash G_S$$

is a homeomorphism onto an open subset \mathcal{S}_T of $\Gamma \backslash G_S$.

(This is S -arithmetic reduction theory.)

- (3) If $Z \subset G_S$ is any compact subset there exists $T' > T$ so that $\mathfrak{S}_{T'} \cdot Z \subset \mathfrak{S}_T$.

(Set $Z' = K_S \cdot Z$. Since Z' is compact, there exists $L \geq 1$ such that $Z' \subset N_S \cdot \{a \in A_S, L^{-1} < |\alpha(a)|_S < L, \forall \alpha \in \Delta\} \cdot K_S$. Then $T' = LT$ suffices.)

Proposition 4. *Suppose $k \in C_c^\infty(K_S \backslash G_S / K_S)$ is nonzero. Then the convolution operator $f \mapsto f \star k$ is nontrivial on $L_{\text{loc}}^1(\Gamma \backslash G_S)$.*

Proof. We construct a function f on $\Gamma_B \backslash G_S$ such that $f \star k = \lambda f$ for some $\lambda \neq 0$, and so that $f|_{\mathfrak{S}_T} \neq 0$ for all T .

This immediately implies the assertion of Proposition 4 by ‘‘transporting’’ f to $\Gamma \backslash G_S$. More formally: let T be sufficiently large. Then $f \circ \iota^{-1}$ defines a function on \mathcal{S}_T . Extending this function by 0 off \mathcal{S}_T , we obtain a function $f' \in L_{\text{loc}}^1(\Gamma_S \backslash G_S)$. We claim $f' \star k \neq 0$. Indeed, let $T' > T$ be so that $\mathfrak{S}_{T'} \cdot \text{supp}(k) \subset \mathfrak{S}_T$. Then for $x \in \mathfrak{S}_{T'}$ we have $f' \star k(\iota(x)) = f \star k(x) = \lambda f(x)$. Since $\lambda \neq 0$ and $f|_{\mathfrak{S}_{T'}} \neq 0$, we are done.

Let ψ be a nondegenerate character of N_S trivial on Γ_N . (Nondegenerate means its stabilizer in A_S is central in G_S , i.e. trivial by the adjointness assumption.) Let π_S be a generic spherical representation of G_S upon which k acts nontrivially. (It exists, since the spherical constituent of $\pi(\nu)$ is generic for an open dense set of $\nu \in \mathfrak{a}_S^*$ ([12], [20]) and the transform $\hat{k}(\nu)$, being holomorphic, cannot vanish identically on such a set.) Let W be a spherical Whittaker function corresponding to π_S , i.e. W is a function on G_S of moderate growth whose right translates realize the representation π_S , and satisfying $W(n.g) = \psi(n)W(g)$. Then $W \star k = \lambda W$ for some $\lambda \neq 0$.

Set

$$(A.1) \quad f(g) = \sum_{\tau \in \Gamma_N \backslash \Gamma_B} W(\tau g).$$

The series converges absolutely and uniformly in compacta; this may be deduced from the ‘‘rapid decay’’ of W .

f defines a function on G_S that is left-invariant by Γ_B and right-invariant by K_S . To check that $f|_{\mathfrak{S}_T}$ is nonvanishing it suffices to check that the function $g \mapsto \int_{n \in \Gamma_N \backslash N_S} f(n.g)\psi(n)dn$ is not identically zero for $g \in \mathfrak{S}_T$. Note that each

term $W(\tau g)$ in the definition (A.1) of f transforms (on the left) by the character $n \mapsto \psi(\tau n \tau^{-1})$ of N_S , and these characters are all distinct since ψ is nondegenerate. Thus $\int_{n \in \Gamma_N \setminus N_S} f(n g) \psi(n) dn = W(g)$ and it suffices to check that $W(g)|_{\mathfrak{S}_T} \neq 0$ for any T .

However, note that G_S/K_S is a countable union of real symmetric spaces, and $W(g)$ defines an eigenfunction of the Laplacian on each component; in particular, it is real-analytic on each component. It follows that there is at least one component such that $W(g)$ does not vanish identically on any open set.

Since the image of \mathfrak{S}_T intersects every connected component of G_S/K_S it follows that $f|_{\mathfrak{S}_T}$ is nonzero, and we are done. \square

That there exists at least one cusp form now follows from Prop. 3 and Prop. 4 and Cor. 2. It is simple to modify this argument to show, e.g., that there are infinitely many cusp forms, or infinitely many cusp forms which are not self-dual.

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