Diagonalizable flows on locally homogeneous spaces and number theory

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Abstract. We discuss dynamical properties of actions of diagonalizable groups on locally homogeneous spaces, particularly their invariant measures, and present some number theoretic and spectral applications. Entropy plays a key role in the study of theses invariant measures and in the applications.

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1. Introduction

Flows on locally homogeneous spaces are a special kind of dynamical systems. The ergodic theory and dynamics of these flows are very rich and interesting, and their study has a long and distinguished history. What is more, this study has found numerous applications throughout mathematics.

The spaces we consider are of the form $\Gamma\backslash G$ where G is a locally compact group and Γ a discrete subgroup of G. Typically one takes G to be either a Lie group, a linear algebraic group over a local field, or a product of such. Any subgroup $H < G$ acts on $\Gamma\backslash G$ and this action is precisely the type of action we will consider here. One of the most important examples which features in numerous number theoretical applications is the space $\mathrm{PGL}(n,\mathbb{Z})\backslash \mathrm{PGL}(n,\mathbb{R})$ which can be identified with the space of lattices in $\mathbb{R}^n$ up to homothety.

Part of the beauty of the subject is that the study of very concrete actions can have meaningful implications. For example, in the late 1980s G. A. Margulis proved the long-standing Oppenheim conjecture by classifying the closed orbits of the group of matrices preserving an indefinite quadratic form in three variables in $\mathrm{PGL}(3,\mathbb{Z})\backslash \mathrm{PGL}(3,\mathbb{R})$ — a concrete action of a three-dimensional group on an eight-dimensional space.

An element h of a linear group G (considered as a group of $n \times n$ matrices over some field K) is said to be *unipotent* if $h - e$ is a nilpotent matrix, e being

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the identity. Using the adjoint representation one can similarly define unipotent elements for Lie groups. Thanks to the work of M. Ratner, actions of groups H generated by unipotent elements are well understood, and this has numerous applications to many subjects. We refer to [37, Chapter 3], [51] and [59] for more information on this important topic.

In this paper, we focus on the action of diagonalizable groups which of course contain no nontrivial unipotent elements. A prototypical example is the action of the group A of diagonal matrices on $\mathrm{PGL}(n,\mathbb{Z})\backslash \mathrm{PGL}(n,\mathbb{R})$. There is a stark difference between the properties of such actions when $\dim A = 1$ and when $\dim A \geq 2$. In the first case the dynamics is very flexible, and there is a wealth of irregular invariant probability measures and irregular closed invariant sets (though we present some results for one-dimensional actions in §2.2 under an additional recurrence condition). If $\dim A \geq 2$, the dynamics changes drastically. In particular, it is believed that in this case the invariant probability measures (and similarly the closed invariant sets) are much less abundant and lend themselves to a meaningful classification. Another dynamical property which is less often considered in this context but which we believe is important is the distribution of periodic orbits, i.e. closed orbits of the acting group with finite volume. The purpose of this paper is to present some results in these directions, particularly with regards to the classification of invariant measures, and their applications.

A basic invariant in ergodic theory is the ergodic theoretic entropy introduced by A. Kolmogorov and Ya. Sinai. This invariant plays a surprisingly big role in the study of actions of diagonalizable groups on locally homogeneous spaces as well as in the applications. We discuss entropy and how it naturally arises in several applications in some detail.

In an attempt to whet the reader's appetite, we list below three questions on which the ergodic theoretic properties of diagonalizable flows give at least a partial answer:

- Let $F(x_1,\dots,x_n)$ be a product of n linear forms in n variables over $\mathbb{R}$. Assume that F is not proportional to such a form with integral coefficients. What can be said about the values F attains on $\mathbb{Z}^n$? In particular, is $\inf_{0\neq x\in\mathbb{Z}^n} |F(x)| = 0$?

- Let ϕ_i be a sequence of Hecke-Maass cusp forms[1] on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$. What can be said about weak* limits of the measure $|\phi_i|^2\, dm$ (m being the uniform measure on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$)?

- Suppose $n \geq 3$ is fixed. Is it true that any ideal class in a totally real[2] number field K of degree n has a representative of norm $o(\sqrt{\mathrm{disc}(K)})$?

E.L. is scheduled to give a presentation based on this work in the ordinary differential equations and dynamical systems section of the 2006 International Congress

[1] See §5 for a definition.

[2] A number field K is said to be totally real if any embedding of K to $\mathbb{C}$ is in fact an embedding to $\mathbb{R}$.

of Mathematicians. Since much of this is based on our joint work, we have decided to write this paper jointly. Some of the results we present here are joint with P. Michel and A. Venkatesh and will also be discussed in their contribution to these proceedings [50].

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2. Entropy and classification of invariant measures

2.1. Measures invariant under actions of big diagonalizable groups.

2.1.1. We begin by considering a special case where it is widely expected that there should be a complete measure classification theorem for the action of a multidimensional diagonal group. The space we will be considering is $X_n = \mathrm{PGL}(n,\mathbb{Z})\backslash\mathrm{PGL}(n,\mathbb{R})$, which can be identified with the space of lattices in $\mathbb{R}^n$ up to homothety.

Conjecture 2.1. *Let A be the group of diagonal matrices in $\mathrm{PGL}(n,\mathbb{R})$, $n \geq 3$. Then any A-invariant and ergodic probability measure μ on the space X_n is homogeneous*[3]*, i.e. is the L-invariant measure on a closed orbit of some group $L \geq A$.*

While at present this conjecture remains open, the following partial result is known:

Theorem 2.2 (E., Katok, L. [14]). *Let A be the group of diagonal matrices as above and $n \geq 3$. Let μ be an A-invariant and ergodic probability measure on $\mathrm{PGL}(n,\mathbb{Z})\backslash\mathrm{PGL}(n,\mathbb{R})$. If for some $a \in A$ the entropy $h_\mu(a) > 0$ then μ is homogeneous.*

It is possible to explicitly classify the homogeneous measures in this case (see e.g. [42]), and except for measures supported on a single A-orbit none of them is compactly supported. It follows that:

Corollary 2.3. *Let $n \geq 3$. Any compactly supported A-invariant and ergodic probability measure on $\mathrm{PGL}(n,\mathbb{Z})\backslash\mathrm{PGL}(n,\mathbb{R})$ has $h_\mu(a) = 0$ for all $a \in A$.*

For an application of this corollary to simultaneous Diophantine approximation and values of products of linear forms see §4.

[3] The adjective “algebraic” is also commonly used for this purpose. We follow in this the terminology of [51].

2.1.2. We now give a general conjecture, which is an adaptation of conjectures of A. Katok and R. Spatzier [33, Main conjecture] and G. A. Margulis [49, Conjecture 2]. Similar conjectures were made by H. Furstenberg (unpublished).

Let S be a finite set of places for $\mathbb{Q}$ (i.e. a subset of the set of finite primes and ∞). By an S-algebraic group we mean a product $G_S = \prod_{v\in S} G_v$ with each G_v an algebraic group over $\mathbb{Q}_v$. A $\mathbb{Q}_v$-algebraic group G_v is *reductive* if its unipotent radical is trivial. An S-algebraic group G_S is reductive (semisimple) if each of the G_v is reductive (respectively, semisimple). For any group G and $\Gamma \subset G$ a discrete subgroup, we will denote the image of $g \in G$ under the projection $G \to \Gamma\backslash G$ by $(\!(g)\!)_\Gamma$ or simply $(\!(g)\!)$ if Γ is understood. We shall say that two elements a_1, a_2 of an Abelian topological group A are *independent* if they generate a discrete free Abelian subgroup.

Conjecture 2.4. *Let S be a finite set of places for $\mathbb{Q}$, let $G_S = \prod_{v\in S} G_v$ be an S-algebraic group, $G \leq G_S$ closed, and $\Gamma < G$ discrete. For each $v \in S$ let $A_v < G_v$ be a maximal $\mathbb{Q}_v$-split torus, and let $A_S = \prod_{v\in S} A_v$. Let A be a closed subgroup of $A_S \cap G$ with at least two independent elements. Let μ be an A-invariant and ergodic probability measure on $\Gamma\backslash G$. Then at least one of the following two possibilities holds:*

1. *μ is homogeneous, i.e. is the L-invariant measure on a single, finite volume, L-orbit for some closed subgroup $A \leq L \leq G$.*

2. *There is some S-algebraic subgroup L_S with $A \leq L_S \leq G_S$, an element $g \in G$, an algebraic homeomorphism $\phi : L_S \to \tilde{L}_S$ onto some S-algebraic group $\tilde{L}_S$, and a closed subgroup $H < \tilde{L}_S$ with $H \geq \phi(\Gamma)$ so that (i) $\mu((L_S \cap G).(\!(g)\!)_\Gamma) = 1$, (ii) $\phi(A)$ does not contain two independent elements and (iii) the image of μ to $H\backslash\tilde{L}_S$ is not supported on a single point.*

Examples due to M. Rees [60] show that μ need not be algebraic, even if $G = \mathrm{SL}(3,\mathbb{R})$ and Γ a uniform lattice; see [12, Section 9] for more details. Such μ arise from algebraic rank one factors of locally homogeneous subspaces as in case 2. of Conjecture 2.4.

2.1.3. We note that the following conjecture is a special case of Conjecture 2.4:

Conjecture 2.5 (Furstenberg). *Let μ be a probability measure on $\mathbb{R}/\mathbb{Z}$ invariant and ergodic under the natural action of the multiplicative semigroup $\{p^n q^l\}_{k,l\in\mathbb{Z}^+}$ with p, q multiplicatively independent integers*[4]*. Then either μ is Lebesgue or it is supported on finitely many rational points.*

For simplicity, assume p and q are distinct prime numbers. Then Conjecture 2.5

[4] I.e. integers which are not both powers of the same integer.

is equivalent to Conjecture 2.4 applied to the special case of

$$\begin{aligned} A_S &= \{(t_\infty, t_p, t_q) : t_v \in \mathbb{Q}_v^* \text{ for } v \in S\}, \qquad S = \{\infty, p, q\}, \\ A &= \left\{(t_\infty, t_p, t_q) \in A_S : |t_\infty| \cdot |t_p|_p \cdot |t_q|_q = 1\right\}, \\ G &= A \ltimes (\mathbb{R} \times \mathbb{Q}_p \times \mathbb{Q}_q), \\ \Gamma &= \Lambda \ltimes \mathbb{Z}[1/pq], \end{aligned}$$

with Λ the group $\{p^k q^l\}_{k,l \in \mathbb{Z}}$ embedded diagonally in A, and $\mathbb{Z}[1/pq]$ embedded diagonally in $\mathbb{R} \times \mathbb{Q}_p \times \mathbb{Q}_q$.

2.1.4. Following is a theorem towards Conjecture 2.4 generalizing Theorem 2.2. We note that the proof of this more general theorem is substantially more involved.

Theorem 2.6 (E., L. [19]). *Let S be a finite set of places for $\mathbb{Q}$ as above, $G_S = \prod_{v \in S} G_v$ a reductive S-algebraic group, and $\Gamma < G_S$ discrete. Let $S' \subset S$ and for any $v \in S'$, let A_v be a maximal $\mathbb{Q}_v$-split torus in G_v. Set $A_{S'} = \prod_{v \in S'} A_v$, and assume $A_{S'}$ has at least two independent elements*[5]*. Let μ be an $A_{S'}$-invariant and ergodic probability measure on $\Gamma \backslash G_S$. Then at least one of the following two possibilities holds:*

1. *There is some nontrivial semisimple $H < G_S$ normalized by $A_{S'}$ so that (i) μ is H invariant, (ii) there is some $g \in G_S$ so that $\mu(N_{G_S}(H).(\!(g)\!)_\Gamma) = 1$, and (iii) for any $a \in C_{A_{S'}}(H)$, the entropy $h_\mu(a) = 0$.*
2. *there is some $v \in S$ and a reductive $L_v \subset G_v$ of $\mathbb{Q}_v$-rank one satisfying the following: setting $N = C_{G_S}(L_v)$ and $L = N L_v$ (so that $N \lhd L$), there is some $g \in G_S$ so that $\mu(L.(\!(g)\!)_\Gamma) = 1$ and the image of $g^{-1}\Gamma g \cap L$ under the projection $L \to L/N$ is closed.*

Note that option 2. above precisely corresponds to the existence of an algebraic rank one factor of the action as in Conjecture 2.4.

2.2. Recurrence as a substitute for bigger invariance.

It is well known that invariance under a one-parameter diagonalizable group is not sufficient to obtain a useful measure classification theorem. On the other hand, it seems that in many situations one can replace additional invariance with a weaker requirement: that of recurrence under some further action.

2.2.1. Let X be a measurable space, equipped with a measure μ, and L a locally compact second countable group acting on X. We first give a definition of recurrence.

Definition 2.7. We say that μ is *recurrent* under L (or L-recurrent) if for every set $B \subset X$ with $\mu(B) > 0$ for a.e. $x \in B$ the set $\{\ell \in L : \ell.x\}$ is unbounded, i.e. has non-compact closure.

[5] Equivalently, that $\operatorname{rank}(A_{S'}) \geq 2$.

This condition is also called conservativity of μ; we find recurrence a more natural term when dealing with the action of general group actions. It can be defined alternatively in terms of *cross-sections*: a set $Y \subset X$ is said to be a cross-section for L if $\mu(L.Y) > 0$ and for every $y \in Y$ there is a neighborhood U of the identity in L so that $\ell.y \notin Y$ for every $\ell \in U \setminus \{e\}$. A cross-section is said to be *complete* if $\mu(X \setminus L.Y) = 0$. We can define recurrence using cross-sections as follows: a measure μ is recurrent under L if there is no cross-section intersecting each L-orbit in at most a single point. This definition is equivalent to the one given in Definition 2.7. An advantage of this viewpoint is that it allows us to consider more refined properties of the action:

Definition 2.8. We say that the L-recurrence of a measure μ is *dominated* by H if there is a complete cross-section $Y \subset X$ for L so that for every $y \in Y$

$$\{\ell \in L : \ell.y \in Y\} \subset H.$$

We say that the L-recurrence of a measure μ is *weakly dominated* by H if there is a (not necessarily complete) cross-section $Y \subset X$ satisfying the same.

2.2.2. We now give a specific rigidity theorem employing recurrence as a substitute for invariance under a multidimensional group. For an application of this theorem to arithmetic quantum unique ergodicity, see §5.

Theorem 2.9 (E., L. [20]). *Let v be either ∞ or a finite prime, and let G_v be a semisimple algebraic group over $\mathbb{Q}_v$ with $\mathbb{Q}_v$-rank one. Let A_v be a $\mathbb{Q}_v$-split torus in G_v an let L be an S-algebraic group (S a finite set of places for $\mathbb{Q}$ as above). Let $\Gamma < G_v \times L$ be a discrete subgroup so that $|\Gamma \cap \{e\} \times L| < \infty$. Suppose μ is an A_v-invariant, L-recurrent probability measure on $\Gamma\backslash G_v \times L$, and that for a.e. A_v-ergodic component μ_ξ the entropy $h_{\mu_\xi}(A_v) > 0$. Then a.e. A_v-ergodic component is homogeneous.*

The case $G_v = \mathrm{SL}(2, \mathbb{R})$ was proved in [39] and was used to prove arithmetic quantum unique ergodicity (see §5).

It would be interesting to prove a version of Theorem 2.9 where G_v is replaced by a higher rank algebraic group, e.g. $\mathrm{SL}(3, \mathbb{Q}_v)$, and A_v any algebraic embedding of $\mathbb{Q}_v^* \to G_v$. This seems like a feasible undertaking, but would require new ideas.

2.2.3. It seems desirable to have a general conjecture similar to Conjecture 2.4 where additional invariance is replaced by recurrence. The following seems not completely implausible:

Conjecture 2.10. *Let S be a finite set of places for $\mathbb{Q}$, and $G_S = \prod_{v \in S} G_v$ an S-algebraic group and $\Gamma < G_S$ discrete as above. Fix $v \in S$ and let A_v be a rank one $\mathbb{Q}_v$-split torus. Let $L < G_S$ be a closed subgroup commuting with A_v such that $|A_v \cap L| < \infty$. Suppose μ is an A_v-invariant, L-recurrent probability measure. Then there is a $A_v L$-invariant Borel set X' of positive μ-measure so that at least one of the following holds:*

1. *There are closed subgroups $\tilde{A} \geq A_v$ and $\tilde{L}$ so that (i) $\tilde{A}$ and $\tilde{L}$ commute and have compact intersection*[6] *(ii) for every $x \in X'$ both $\tilde{L}.x$ and $\tilde{A}.x$ are closed and furthermore $\tilde{A}.x$ has finite $\tilde{A}$-invariant measure (iii) $\mu|_{X'}$ is $\tilde{A}$-invariant (iv) the L-recurrence of $\mu|_{X'}$ is dominated by $\tilde{A} \cdot \tilde{L}$.*

2. *There is a closed subgroup $\tilde{L} < G_S$ commuting with A_v such that for a set of positive μ-measure of $x \in X'$ we have that $\tilde{L}.x$ is closed, and the L-recurrence of $\mu|_{X'}$ is dominated by $\tilde{L}$.*

If true, this conjecture implies many (if not all) cases of Conjecture 2.4, in particular Conjecture 2.1.

2.2.4. In the notations of the above conjecture, let a be an element in A_v which does not generate a bounded subgroup[7] of A_v. Let

$$\begin{aligned} G_a^+ &= \left\{ g \in G_v : a^{-n} g a^n \to e \quad \text{as } n \to \infty \right\} \\ G_a^- &= \left\{ g \in G_v : a^{n} g a^{-n} \to e \quad \text{as } n \to \infty \right\} \\ G_a^0 &= C_{G_v}(a). \end{aligned}$$

In the paper [20] we give some nontrivial information on an A_v-invariant, L-recurrent probability measure μ for A_v, L as in the conjecture, under the additional conditions that μ is U_v-recurrent for some $\mathbb{Q}_v$-algebraic subgroup $U_v \leq G_v^-$ such that

1. U_v commutes with L,

2. the U_v recurrence of μ is not weakly dominated by any proper A_v-normalized algebraic subgroup of U_v

3. for any $g \in G_v^+$, there is a $u \in U_v$ so that $ugu^{-1} \notin G_v^0 G_v^+$.

The exact conclusions we derive about μ in this case is somewhat technical but in particular they imply Theorem 2.9. Note that one consequence of these conditions is that $h_\mu(A_v) > 0$ (see §3).

2.3. Joinings.

2.3.1. Let (X, μ) and (Y, ν) be two measure spaces, and suppose that A is some locally compact group that acts on both (X, μ) and (Y, ν) in a measure preserving way. A *joining* between (X, μ) and (Y, ν) is a measure ρ on $X \times Y$ whose push forward under the obvious projection to X and Y are μ and ν respectively, and which is invariant under the diagonal action of A on $X \times Y$.

One example of a joining which always exists is taking $\rho = \mu \times \nu$ (the *trivial joining*). If $\phi : X \to Y$ is a measure preserving map which is A equivariant (i.e. $a.\phi(x) = \phi(a.x)$ for all $a \in A$ and a.e. $x \in X$) then $\rho = (\mathrm{Id} \times \phi)_* \mu$ is a nontrivial

[6] Note that $\tilde{L}$ may be trivial.

[7] I.e. a subgroup with noncompact closure.

joining between (X, μ) and (Y, ν). Note that this joining is supported on the graph of ϕ. Let (Z, η) be another measure space on which A acts preserving the measure. (Z, η) is a *factor*[8] of (X, μ) if there is an A equivariant measurable map $\psi : X \to Z$ so that $\eta = \psi_* \mu$. Any common factor of (X, μ) and (Y, ν) can also be used to give a nontrivial joining called the *relatively independent joining*.

Typically the most interesting case is studying the joinings of a space (X, μ) with itself (called *self joinings*).

2.3.2. We now consider joinings between locally homogeneous spaces $\Gamma \backslash G_S$ on which we have an action of a higher rank diagonalizable group. Even though we currently do not have a complete understanding of invariant measures in this context, we are able to give a complete classification of joining between two such actions in many cases.

Theorem 2.11 (E., L. [18, 17]). *Let S be a finite set of places for $\mathbb{Q}$, and $G_i = \prod_{v \in S} G_{i,v}$ for $i = 1, 2$ two S-algebraic semisimple groups, $\Gamma_i < G_i$ be lattices*[9]*, and m_i Haar measure on $\Gamma_i \backslash G_i$ normalized to have total mass one. Let $A = \prod_{v \in S'} A_v$ with each A_v a $\mathbb{Q}_v$-split torus and $S' \subseteq S$ satisfying* $\operatorname{rank} A \geq 2$. *Let τ_i $(i = 1, 2)$ be embeddings of A into G_i with the property that*

1. *$\tau_i(A)$ is generated by the subgroups $\tau_i(A) \cap H$ where H runs through the $\mathbb{Q}_v$ simple normal subgroups of $G_{i,v}$ $(v \in S)$.*

2. *For both $i = 1, 2$, there is no S-algebraic group L and an S-algebraic homomorphism $\phi : G_i \to L$ so that $\phi(\Gamma_i)$ is discrete and* $\operatorname{rank}(\phi \circ \tau_i(A)) \leq 1$.

Then any ergodic joining between $(\Gamma_1 \backslash G_1, m_1)$ and $(\Gamma_2 \backslash G_2, m_2)$ is homogeneous[10].

The assumptions of the theorem imply that the action of A on $(\Gamma_i \backslash G_i, m_i)$ $(i = 1, 2)$ is ergodic, and so any joining can be written as the integral of ergodic joinings. The second assumption in Theorem 2.11 regarding the non-existence of rank one factors is clearly necessary. There is no reason to believe the same is true regarding the first assumption.

A special case of the theorem is when G_i are simple algebraic groups over $\mathbb{Q}_v$, v either a finite prime or ∞, and τ_i any algebraic embeddings of $(\mathbb{Q}_v^*)^k$ to G_i, $k \geq 2$. In this case the two assumptions regarding τ_i are automatically satisfied. This case has been treated in [17] using the methods developed by M. E. and A. Katok in [13] (to be precise, only $v = \infty$ is considered in [17] , but there are no difficulties in extending that treatment to $\mathbb{Q}_v$ for any v). The proof of the more general Theorem 2.11 requires also the results in [20].

[8] It may be more consistent with standard mathematical terminology to call (Z, η) a quotient, but factor is the standard term in ergodic theory.

[9] I.e. discrete subgroups of finite covolume.

[10] A joining is in particular a measure on $\Gamma_1 \backslash G_1 \times \Gamma_2 \backslash G_2$ invariant under the diagonal action of the group A. Properties of invariant measures such as ergodicity, homogeneity etc. are in particular equally applicable to joinings.

2.3.3. A different approach to studying joinings was carried out by B. Kalinin and R. Spatzier in [32] using the methods developed by A. Katok and R. Spatzier in [33, 34]. A basic limitation of this technique is that e.g. for actions of split algebraic tori on semisimple or reductive algebraic groups they are able to analyze joinings only if the joining is ergodic not only under the action of the full acting group A but also under the action of certain one parameter subgroups of A. Typically, this is a fairly restrictive assumption, but for joinings which arise from isomorphisms as discussed in §2.3.1 this assumption is indeed satisfied[11]. This has been used by B. Kalinin and R. Spatzier to classify all measurable isomorphisms between actions of $\mathbb{R}^k$ on $\Gamma_i \backslash G_i$ $(i = 1, 2)$ with G_i a Lie group, $\Gamma_i < G_i$ a lattice and the action of $t \in \mathbb{R}^k$ on $\Gamma_i \backslash G_i$ is given by right translation by $\rho_i(t)$, ρ_i a proper embedding[12] of the group $\mathbb{R}^k$ in G_i whose image is Ad-diagonalizable over $\mathbb{C}$ under some mild conditions on the action of $\rho_i(\mathbb{R}^k)$ on $\Gamma_i \backslash G_i$.

2.3.4. We end our discussion of joinings by noting that extending these joining results to the general context considered in Conjecture 2.4 is likely to be difficult; at the very least it would directly imply Conjecture 2.5.

To see this, let p, q be two multiplicative independent integers, m Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ and μ any other continuous probability measure on $\mathbb{R}/\mathbb{Z}$ invariant and ergodic under the action of the multiplicative semigroup S generated by p and q. Let ρ denote the map $(x, y) \to (x, x + y)$ from $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ to itself. Then since m is weakly mixing for S, the measure $m \times \mu$ is ergodic for S and so $\rho_*(m \times \mu)$ is an ergodic self joining of the action of S on $(\mathbb{R}/\mathbb{Z}, m)$. What we have seen is that a counterexample to Furstenberg's Conjecture 2.5 would give a nonhomogeneous ergodic self joining of $(\mathbb{R}/\mathbb{Z}, m)$, which can be translated to a nonhomogeneous ergodic self joining of the group action considered in §2.1.3.

One can, however, classify joinings of the actions of commuting endomorphisms of tori with no algebraic projections on which the action degenerates to the action of a virtually cyclic group up to this problem of zero entropy factors. This has been carried out for actions by a group of commuting toral automorphisms satisfying a condition called total nonsymplecticity by B. Kalinin and A. Katok [31] using an adaptation of the methods of A. Katok and R. Spatzier. In [16] the authors deal with general actions of commuting toral automorphisms (without the total nonsymplecticity condition).

2.4. Historical discussion.

2.4.1. In 1967 Furstenberg proved that any orbit of the multiplicative semigroup $\left\{p^k q^l\right\}_{k,l\in\mathbb{Z}^+}$ on $\mathbb{R}/\mathbb{Z}$ for p, q multiplicatively independent integers is either finite or dense and conjectured Conjecture 2.5 apparently at around the same time. This conjecture seems to have appeared in print only much later (and by other authors quoting Furstenberg). Furstenberg's work was extended to the case of

[11] The same observation in the context of toral automorphisms was used by A. Katok, S. Katok and K. Schmidt in [36].

[12] I.e. the preimage of compact sets is compact.

automorphisms of tori and other compact abelian groups by D. Berend (see e.g. [5]).

The first substantial result towards Conjecture 2.5 was published in 1988 by R. Lyons [46], who proved it under the assumption that μ has completely positive entropy for the action generated by the single element p (in particular, μ is ergodic for the single transformation $x \mapsto px \bmod 1$). D. Rudolph [62] showed for p, q relatively prime that it is sufficient to assume that $h_\mu(p) > 0$; this is still the best result known in this case. The restriction that p, q were relatively prime was lifted by A. Johnson [29]. As Rudolph explicitly pointed out in his paper his proof significantly simplifies if one assumes that μ is ergodic under $x \mapsto px \bmod 1$. Other proofs of this result were given by J. Feldman [24] and B. Host [26] (B. Host actually proves a stronger result that implies Rudolph's). We also note that Host's proof employed recurrence for a certain action which does not preserve the measure, and was one of the motivations for Theorem 2.9.

2.4.2. The first results towards measure classification for actions of diagonalizable groups on quotients of Lie groups and automorphisms of tori were given by A. Katok and R. Spatzier [33, 34]; certain aspects of their work were clarified in [30]. Their proof replaces Rudolph's symbolic description by more geometric concepts, and in particular highlighted the role of conditional measures on invariant foliations on which a subaction acts isometrically. In most cases Katok and Spatzier needed to assume both a condition about entropy and an assumption regarding ergodicity of these subactions. Removing the extra ergodicity assumptions proved to be critical for arithmetic and other applications. M.E. and Katok [12, 13] and E.L. [39] developed two completely different and complementary approaches to proving measure rigidity results in the locally homogeneous context without additional ergodicity assumptions. Both of these techniques were used in [14]. We note that in [39] essential use was made of techniques introduced by M. Ratner to study unipotent flows, particularly her work on rigidity of horocycle flows [56, 54, 55]. Ratner's measure classification theorem [57] and its extensions [58, 48] are used in all these approaches.

In the context of action by automorphisms on tori no ergodicity assumption was needed by Katok and Spatzier under an assumption they term total nonsymplecticity. A uniform treatment for the general case, using entropy inequalities which should be of independent interest, was given by the authors in [16]. Host [27] has a treatment of some special cases (and even some non commutative actions) by other methods.

2.4.3. We have restricted our attention in this section solely to the measure classification question, but it is interesting to note that already in 1957, J. W. S. Cassels and H. P. F. Swinnerton-Dyer [8] stated a conjecture regarding values of products of three linear forms in three variables (case $n = 3$ of Conjecture 4.1) which is equivalent to Conjecture 4.4 regarding behavior of orbits of the full diagonal group on $\mathrm{SL}(3,\mathbb{Z})\backslash\mathrm{SL}(3,\mathbb{R})$, and which can be derived from Conjecture 2.1. It seems that the first to observe the connection between Furstenberg's work and that of

Cassels and Swinnerton-Dyer was G. A. Margulis [47].

3. Brief review of some elements of entropy theory

In this section we give several equivalent definitions of entropy in the context of actions of diagonalizable elements on locally homogeneous spaces, and explain the relations between them.

3.1. General definition of entropy.

3.1.1. Let (X, μ) be a probability space. The entropy $H_\mu(\mathcal{P})$ of a finite or countable partition of X into measurable sets measures the average information of $\mathcal{P}$ in the following sense. The partition can be thought of as an experiment or observation whose outcome is the partition element $P \in \mathcal{P}$ the point $x \in X$ belongs to. The information obtained about x from this experiment is naturally measured on a logarithmic scale, i.e. equals $-\log \mu(P)$ for $x \in P \in \mathcal{P}$. Therefore, the average information or *entropy* of $\mathcal{P}$ (with respect to μ) is

$$H_\mu(\mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P).$$

One basic property of entropy is sub-additivity; the entropy of the refinement $\mathcal{P} \vee \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$ satisfies

$$H_\mu(\mathcal{P} \vee \mathcal{Q}) \le H_\mu(\mathcal{P}) + H_\mu(\mathcal{Q}). \tag{3.1}$$

However, this is just a starting point for many more natural identities and properties of entropy, e.g. equality holds in (3.1) if and only if $\mathcal{P}$ and $\mathcal{Q}$ are independent.

The ergodic theoretic entropy $h_\mu(a)$ associated to a measure preserving map $a : X \to X$ measures the average amount of information one needs to keep track of iterates of a. To be more precise we need to start with a fixed partition $\mathcal{P}$ (either finite or countable with $H_\mu(\mathcal{P}) < \infty$) and then take the limit

$$h_\mu(a, \mathcal{P}) = \lim_{N \to \infty} \frac{1}{N} H_\mu \left(\bigvee_{n=0}^{N-1} a^{-n} \mathcal{P} \right).$$

To get independence of the choice of $\mathcal{P}$ the ergodic theoretic entropy is defined by

$$h_\mu(a) = \sup_{\mathcal{P} : H_\mu(\mathcal{P}) < \infty} h_\mu(a, \mathcal{P}).$$

The ergodic theoretic entropy was introduced by A. Kolmogorov and Ya. Sinai and is often called the Kolmogorov-Sinai entropy; it is also somewhat confusingly called the metric entropy even though X often has the additional structure of a metric space and in that case there is a *different* (though related) notion of entropy, the topological entropy (see §4.2.2), which is defined using the metric on X.

3.1.2. Entropy has many nice properties and is manifest in many different ways. We mention a few which will be relevant in the sequel.

A partition $\mathcal{P}$ is said to be a *generating partition* for a and μ if the σ-algebra $\bigvee_{n=-\infty}^{\infty} a^{-n}\mathcal{P}$ (i.e. the σ-algebra generated by the sets $\{a^n.P : n \in \mathbb{Z}, P \in \mathcal{P}\}$) separates points, that is for μ-almost every x, its atom with respect to this σ-algebra is $\{x\}$.[13] The Kolmogorov-Sinai theorem asserts the nonobvious fact that $h_\mu(a) = h_\mu(a, \mathcal{P})$ whenever $\mathcal{P}$ is a generating partition.

Entropy is most meaningful when μ is ergodic. In this case, positive entropy $h_\mu(a) > 0$ means that the entropy of the repeated experiment grows linearly, i.e. every new iteration of it reveals some new information of the point. In fact, one can go to the limit here and say that the experiment reveals new information even when one already knows the outcome of the experiment in the infinite past. Similarly, zero entropy means that the observations in the past completely determine the present one. If μ is an a-invariant but not necessarily ergodic measure, with an ergodic decomposition $\mu = \int \mu_x^{\mathcal{E}} d\mu(x)$,[14] then

$$h_\mu(a) = \int h_{\mu_x^{\mathcal{E}}}(a) d\mu(x), \tag{3.2}$$

i.e. the entropy of a measure is the average of the entropy of its ergodic components.

3.2. Entropy on locally homogeneous spaces.

3.2.1. Let $G = \prod_{v \in S} G_v$ be an S-algebraic group, $\Gamma < G$ discrete, and set $X = \Gamma \backslash G$. The Lie algebra of G_S can be defined as the product of the Lie algebra of the G_v, and the group G acts on its Lie algebra of G by conjugation; this action is called the adjoint representation and for every $a \in G$ the corresponding Lie algebra endomorphisms is denoted by $\mathrm{Ad}\, a$. Fix an $a \in G$ for which $\mathrm{Ad}\, a$ restricted to the Lie algebra of each G_v is diagonalizable over $\mathbb{Q}_v$. We implicitly identify between $a \in G$ and the corresponding map $x \mapsto a.x$ from X to itself.

The purpose of this subsection is to explain how the entropy $h_\mu(a)$ of an a-invariant measure μ relates to more geometric properties of X. A good reference for more advanced results along this direction is [48, Section 9] which contains an adaptation of results of Y. Pesin, F. Ledrappier, L. S. Young and others to the locally homogeneous context.

3.2.2. Fundamental to the dynamics of a are the stable and unstable horospherical subgroups G_a^- and G_a^+ introduced in §2.2.4. Both G_a^- and G_a^+ are unipotent algebraic groups and the Lie algebras of G_a^- (resp. G_a^+) are precisely the sums of the eigenspaces of the adjoint Ad_a of eigenvalue with absolute value less than (resp. bigger than) one. For any $x \in X$ the orbits $G_a^-.x$ and $G_a^+.x$ are precisely the stable and unstable manifolds of x. We will also need the group

$$G_a^0 = \left\{ g \in G : \text{the set } \{a^n g a^{-n}, n \in \mathbb{Z}\} \text{ is bounded} \right\}.$$

[13] Recall that the atom of x with respect to a countably generated σ-algebra $\mathcal{A}$ is the intersection of all $B \in \mathcal{A}$ containing x and is denoted by $[x]_{\mathcal{A}}$.

[14] This decomposition has the property that $\mu_x^{\mathcal{E}}$ is ergodic and for a.e. x, the ergodic averages of a function f along the orbit of x converge to $\int f d\mu_x^{\mathcal{E}}$

Under our assumptions G_a^0 can be shown to be an algebraic subgroup of G, and if 1 is the only eigenvalue of $\operatorname{Ad} a$ of absolute value one, $G_a^0 = C_G(a)$. This subgroup G_a^0 together with G_a^- and G_a^+ give a local coordinate system of G, i.e. there are neighborhoods $V^- \subset G_a^-$, $V^+ \subset G_a^+$, and $V^0 \subset G_a^0$ of e for which $V^+V^-V^0$ is a neighborhood of e in G and the map from $V^+ \times V^- \times V^0$ to $V^+V^-V^0$ is a bijection.

3.2.3. Suppose X is compact. If $\mathcal{P}$ is a finite partition with elements of small enough diameter, then the atoms of x with respect to $\mathcal{A} = \bigvee_{n=1}^{\infty} a^{-n}\mathcal{P}$ is a subset of $V^-V^0.x$ for all $x \in X$ as x and y are in the same $\mathcal{A}$-atom if and only if $a^n.x$ and $a^n.y$ are in the same partition element of $\mathcal{P}$ for $n = 1, 2, \ldots$. In particular, they must be close-by throughout their future, which can only be if the V^+-component of their relative displacement is trivial. Similarly, the atom of x with respect to $\bigvee_{-\infty}^{\infty} a^{-n}\mathcal{P}$ is a subset of $V^0.x$ for all $x \in X$.

Thus even though $\mathcal{P}$ might not be a generator, the atoms of the σ-algebra generated by $a^{-n}\mathcal{P}$ for $n \in \mathbb{Z}$ are small, with each atom contained in a uniformly bounded subset of a single G_a^0-orbit. G_a^0 possesses a metric invariant under conjugation by a, and this implies that a acts isometrically on these pieces of G_a^0-orbits. Such isometric extensions cannot produce additional entropy, and indeed a modification of the proof of the Kolmogorov-Sinai theorem gives that

$$h_\mu(a, \mathcal{P}) = h_\mu(a) \qquad \text{for } \textit{all } a\text{-invariant measures } \mu. \tag{3.3}$$

In the non-compact case there is a somewhat weaker statement of this general form that is still sufficient for most applications.

3.2.4. Positive entropy can be characterized via geometric tubes as follows. Let B be a fixed open neighborhood of $e \in G$, and define $B_n = \bigcap_{k=-n}^{n} a^k B$. A tube around $x \in X = \Gamma\backslash G$ is a set of the form $B_n.x$ for some n. Then for a as in §3.2.1, it can be shown using the Shannon-McMillan-Breiman theorem that if B is sufficiently small, for any measure μ with ergodic decomposition $\int \mu_x^{\mathcal{E}} d\mu(x)$

$$h_{\mu_x^{\mathcal{E}}}(a) = \lim_{n\to\infty} \frac{-\log \mu(B_n.x)}{2n} \qquad \text{for } \mu\text{-a.e. } x. \tag{3.4}$$

In particular, positive entropy of μ is equivalent to the exponential decay of the measure of tubes around a set of points which has positive μ-measure, and positive entropy of all ergodic components of μ is equivalent to the same holding for a conull subset of X. We note that (3.4) is a variant of a more general result of Y. Brin and A. Katok [7].

3.2.5. Positive entropy can also be characterized via recurrence. For μ, a as above the following are equivalent: (i) $h_{\mu_x^{\mathcal{E}}}(a) > 0$ for a.e. a-ergodic component $\mu_x^{\mathcal{E}}$ (ii) μ is G_a^--recurrent (iii) μ is G_a^+-recurrent (see e.g. [39, Theorem 7.6]).

3.2.6. A quite general phenomenon is upper semi-continuity of entropy $h_\mu(a)$ as a function of μ in the weak* topology[15]. Deep results of Yomdin, Newhouse and

[15] A sequence of measures μ_i converges in the weak* topology to μ if for every compactly supported continuous function f one has that $\int f d\mu_i \to \int f d\mu$.

Buzzi establish this for general C^∞ diffeomorphisms of compact manifolds, but in our context establishing such semi-continuity is elementary, particularly in the compact case. For non-compact quotients $X = \Gamma\backslash G$ and a sequence of a-invariant probability measures we might have escape of mass in the sense that a weak* limit might not be a probability measure. In that case entropy might get lost (even if some mass remains). However, if we assume that the weak* limit is again a probability measure then upper semi-continuity still holds in this context. The case where the whole sequence is supported on a compact set is discussed in [14, Cor. 9.3]. The general case follows along similar lines, the key step is showing that there is a finite partition capturing almost all of the ergodic theoretic entropy uniformly for the sequence, cf. §3.2.3.

4. Entropy and the set of values obtained by products of linear forms

4.1. Statements of conjectures and results regarding products of linear forms.

4.1.1. In this section we consider the following conjecture:

Conjecture 4.1. *Let $F(x_1, x_2, \ldots, x_n)$ be a product of n linear forms in n variables over the real numbers with $n \geq 3$, and assume that F is not proportional to a homogeneous polynomial with integer coefficients. Then*

$$\inf_{0 \neq x \in \mathbb{Z}^d} |F(x)| = 0. \tag{4.1}$$

We are not sure what is the proper attribution of this conjecture, but the case $n = 3$ was stated by J. W. S. Cassels and H. P. F. Swinnerton-Dyer in 1955 [8]. In that same paper, Cassels and Swinnerton-Dyer show that Conjecture 4.1 implies the following conjecture of Littlewood:

Conjecture 4.2 (Littlewood (c. 1930)). *For any $\alpha, \beta \in \mathbb{R}$,*

$$\liminf_{n \to \infty} n \,\|n\alpha\|\, \|n\beta\| = 0,$$

where for any $x \in \mathbb{R}$ we denote $\|x\| = \min_{n \in \mathbb{Z}} |x - n|$.

We let $\mathcal{F}_n$ denote the set of products of n linear forms in n-variables, considered as a subvariety of the space of degree n homogeneous polynomials in n-variables, and $P\mathcal{F}_n$ the corresponding projective variety with proportional forms identified. For any $F \in \mathcal{F}_n$ we let $[F]$ denote the corresponding point in $P\mathcal{F}_n$.

4.1.2. The purpose of this section is to explain how measure classification results (specifically, Corollary 2.3) can be used to prove the following towards the above two conjectures:

Theorem 4.3 (E., Katok, L. [14]). *1. The set of products $[F] \in P\mathcal{F}_n$ for which $\inf_{0 \neq x \in \mathbb{Z}^n} |F(x)| > 0$ has Hausdorff dimension zero.*

2. The set of $(\alpha, \beta) \in \mathbb{R}^2$ for which $\liminf_{n\to\infty} n \|n\alpha\| \|n\beta\| > 0$ has Hausdorff dimension zero.

Even though Conjecture 4.1 implies Conjecture 4.2, part 2 of Theorem 4.3 does not seem to be a formal consequence of 1 of that theorem. The proofs, however, are very similar. Related results by M.E. and D. Kleinbock in the S-arithmetic context can be found in [15].

4.1.3. As noted by G.A. Margulis [47], Conjecture 4.1 is equivalent to the following conjecture regarding the orbits of the diagonal group on $X_n = \mathrm{PGL}(n, \mathbb{Z}) \backslash \mathrm{PGL}(n, \mathbb{R})$ (cf. Conjecture 2.1 above):

Conjecture 4.4. *Let A be the group of diagonal matrices in $\mathrm{PGL}(n, \mathbb{R})$, and X_n as above. Then for any $x \in X_n$ its orbit under A is either periodic (closed of finite volume) or unbounded*[16].

The equivalence between Conjecture 4.1 and Conjecture 4.4 is a consequence of the following simple proposition (the proof is omitted):

Proposition 4.5. *The product of n linear forms*

$$F(x_1, \dots, x_n) = \prod_{i=1}^{n} (\ell_{i1}x_1 + \dots + \ell_{in}x_n)$$

satisfies $\inf_{0 \neq x \in \mathbb{Z}^n} |F(x)| \geq \delta$ if and only if there is no $(\!(g)\!) \in A.(\!(\ell)\!)$ where $\ell = (\ell_{ij})_{i,j=1}^n$ and a nonzero $x \in \mathbb{Z}^n$ so that

$$\|xg\|_\infty^n < \det(g)\delta.$$

In particular, by Mahler's compactness criterion, (4.1) holds if and only if $A.(\!(\ell)\!)$ is unbounded.

The map $F \mapsto N_G(A).(\!(\ell)\!)$, $N_G(A)$ being the normalizer of A in $G = \mathrm{PGL}(n, \mathbb{R})$ gives a bijection between $\mathrm{PGL}(n, \mathbb{Z})$ orbits in $P\mathcal{F}_n$ and orbits of $N_G(A)$ in X_n. Note that $N_G(A)$ is equal to the semidirect product of A with the group of $n \times n$ permutation matrices.

4.1.4. In a somewhat different direction, G. Tomanov [71] proved that if F is a product of n linear forms in n variables, $n \geq 3$, and if the set of values $F(\mathbb{Z}^n)$ is discrete, then F is proportional to a polynomial with integer coefficients. Translated to dynamics, this statement reduces to a classification of all *closed* A-orbits (bounded or unbounded) which was carried out by Tomanov and B. Weiss in [72].

4.2. Topological entropy and A-invariant closed subsets of X_n.

[16] I.e. does not have compact closure.

4.2.1. Corollary 2.3 regarding A-invariant measures on X_n which have positive ergodic theoretic entropy with respect to some $a \in A$ implies the following purely topological result towards Conjecture 4.4 (this theorem is essentially [14, Theorem 11.2]):

Theorem 4.6. *Let Y be a compact A-invariant subset of X_n. Then for every $a \in A$ the topological entropy $h_{\mathrm{top}}(Y, a) = 0$.*

Theorem 4.3 can be derived from Theorem 4.6 by a relatively straightforward argument that in particular uses Proposition 4.5 to translate between the orbits of A and Diophantine properties of products of linear forms (and a similar variant to relate orbits of a semigroup of A with the behavior of $\liminf_{k\to\infty} k \|k\alpha\| \|k\beta\|$).

4.2.2. We recall the definition of topological entropy, which is the topological dynamical analog of the ergodic theoretic entropy discussed in §3.1: Let (Y, d) be a compact metric space and $a : Y \to Y$ a continuous map[17]. Two points $y, y' \in Y$ are said to be k, ϵ*-separated* if for some $0 \leq \ell < k$ we have that $d(a^\ell.y, a^\ell.y') \geq \epsilon$. Set $N(Y, a, k, \epsilon)$ to be the maximal cardinality of a k, ϵ-separated subset of Y. Then the topological entropy of (Y, a) is defined by

$$\begin{aligned} H(Y, a, \epsilon) &= \liminf_{k\to\infty} \frac{\log N(Y, a, k, \epsilon)}{k} \\ h_{\mathrm{top}}(Y, a) &= \lim_{\epsilon\to 0} H(Y, a, \epsilon). \end{aligned}$$

We note that in analogy to §3.2.3, for the systems we are considering, i.e. $a \in A$ acting on a compact $Y \subset X_n$ there is some $\epsilon(Y)$ so that

$$h_{\mathrm{top}}(Y, a) = H(Y, a, \epsilon) \qquad \text{for } \epsilon < \epsilon(Y).$$

4.2.3. Topological entropy and the ergodic theoretic entropy are related by the *variational principle* (see e.g. [25, Theorem 17.6] or [35, Theorem 4.5.3])

Proposition 4.7. *Let Y be a compact metric space and $a : Y \to Y$ continuous. Then*

$$h_{\mathrm{top}}(Y, a) = \sup_\mu h_\mu(a)$$

where the sup *runs over all a-invariant probability measures supported on Y.*

Note that when $\mu \mapsto h_\mu(a)$ is upper semicontinuous (see §3.2.6), in particular if $a \in A$ (identified with the corresponding translation on X_n) and $Y \subset X_n$ compact, this supremum is actually attained by some a-invariant measure on Y.

4.2.4. We can now explain how Theorem 4.6 can be deduced from the measure classification results quoted in §2

[17]For Y which is only locally compact, one can extend a to a map $\tilde{a}$ on its one-point compactification $\tilde{Y}$ fixing ∞ and define $h_{\mathrm{top}}(Y, a) = h_{\mathrm{top}}(\tilde{Y}, \tilde{a})$

Proof of Theorem 4.6 assuming Corollary 2.3. Let $Y \subset X_n$ be compact, and $a \in A$ be such that $h_{\mathrm{top}}(Y, a) > 0$. Then by Proposition 4.7 there is an a-invariant probability measure μ supported on Y with $h_\mu(a) > 0$. Let $S_r = \left\{a \in A : \|a\|, \left\|a^{-1}\right\| < e^r\right\}$. Since A is a commutative group

$$\mu_r = \int_{S_r} (a'.\mu) dm_A(a')$$

(where m_A is Haar measure on A) is also a-invariant, and in addition it follows directly from the definition of entropy that $h_{a'.\mu}(a) = h_\mu(a)$ for any $a' \in A$. Using (3.2) it follows that $h_{\mu_r}(a) = h_\mu(a)$.

Let ν be any weak* limit point of μ_r. Then by semicontinuity of entropy,

$$h_\nu(a) \geq \liminf_{r \to \infty} h_{\mu_r}(a) > 0$$

and since S_r is a Folner sequence in A we have that ν is A-invariant. Finally, since Y is A-invariant, and μ is supported on Y, so is ν. But by Corollary 2.3, ν cannot be compactly supported — a contradiction. □

5. Entropy and arithmetic quantum unique ergodicity

Entropy plays a crucial role also in a completely different problem: arithmetic quantum unique ergodicity. Arithmetic quantum unique ergodicity is an equidistribution question, but unlike most equidistribution questions it is not about equidistribution of points but about equidistribution of eigenfunction of the Laplacian. A more detailed discussion of this topic can be found in [40] and the surveys [63, 65] as well as the original research papers, e.g. [61, 39].

5.1. The quantum unique ergodicity conjecture.

5.1.1. Let M be a complete Riemannian manifold with finite volume which we initially assume to be compact. Then since M is compact, $L^2(M)$ is spanned by the eigenfunctions of the Laplacian Δ on M. Let ϕ_n be a complete orthonormal sequence of eigenfunctions of Δ ordered by eigenvalue. These can be interpreted for example as the steady states for Schrödinger's equation

$$-i \frac{\partial \psi}{\partial t} = \Delta \psi$$

describing the quantum mechanical motion of a free (spinless) particle of unit mass on M (with the units chosen so that $\hbar = 1$). According to Bohr's interpretation of quantum mechanics $\tilde{\mu}_n(A) := \int_A |\phi_n(x)|^2 \, dm_M(x)$ is the probability of finding a particle in the state ϕ_n inside the set A at any given time, m_M denoting the Riemannian measure on M, normalized so that $m_M(M) = 1$. A. I. Šnirel′man, Y.

Colin de Verdière and S. Zelditch [69, 9, 75] have shown that whenever the geodesic flow on M is ergodic, for example if M has negative curvature, there is a subsequence n_k of density one on which $\tilde{\mu}_{n_k}$ converge in the weak* topology to m, i.e. $\tilde{\mu}_n$ become equidistributed *on average*. Z. Rudnick and P. Sarnak conjectured that in fact if M has negative sectional curvature, $\tilde{\mu}_n$ become equidistributed *individually*, i.e. that $\tilde{\mu}_n$ converge in the weak* topology to the uniform measure m_M.

5.1.2. As shown in [69, 9, 75] any weak* limit $\tilde{\mu}$ of a subsequence of the $\tilde{\mu}_i$ is the projection of a measure μ on the unit tangent bundle SM of M invariant under the geodesic flow. This measure μ can be explicitly constructed directly from the ϕ_i. We shall call μ the *microlocal lift* of $\tilde{\mu}$. We shall call any measure μ on SM arising in this way a *quantum limit*. A stronger form of Rudnick and Sarnak's conjecture regarding $\tilde{\mu}_n$ is the following (also due to Rudnick and Sarnak)

Conjecture 5.1 (Quantum unique ergodicity conjecture [61]). *Let M be a compact Riemannian manifold with negative sectional curvature. Then the uniform measure m_{SM} on SM is the only quantum limit.*

There is numerical evidence towards this conjecture in the analogous case of 2D concave billiards by A. Barnett [3], and some theoretical evidence is given in the next two subsections, but whether this conjecture should hold for general negatively curved manifolds remains unclear.

5.2. Arithmetic quantum unique ergodicity.

5.2.1. Consider now the special case of $M = \Gamma\backslash\mathbb{H}$, for Γ one of the following:

1. Γ is a congruence sublattice of $\mathrm{PGL}(2,\mathbb{Z})$;
2. D is a quaternion division algebra over $\mathbb{Q}$, split over $\mathbb{R}$ (i.e. $D(\mathbb{R}) := D\otimes\mathbb{R} \cong M(2,\mathbb{R})$). Let $\mathcal{O}$ be an Eichler order in D.[18] Then the norm one elements in $\mathcal{O}$ are a co-compact lattice in $D(\mathbb{R})^*/\mathbb{R}^*$. Let Γ be the image of this lattice under the isomorphism $D(\mathbb{R})^*/\mathbb{R}^* \cong \mathrm{PGL}(2,\mathbb{R})$.

We shall call such lattices *lattices of congruence type* over $\mathbb{Q}$.

5.2.2. If Γ is as in case 1 of §5.2.1, then $M = \Gamma\backslash\mathbb{H}$ has finite volume, but is not compact. A generic hyperbolic surface of finite volume is expected to have only finitely many eigenfunctions of Laplacian in L^2; consequently Conjecture 5.1 needs some modification to remain meaningful in this case. However, a special property of congruence sublattices of $\mathrm{PGL}(2,\mathbb{Z})$ congruence is the abundance of cuspidal eigenfunctions of the Laplacian (in particular, the existence of many eigenfunctions in L^2) on the corresponding surface, and so Conjecture 5.1 as stated is both meaningful and interesting for these surfaces. The abundance of cuspidal eigenfunctions follows from Selberg's trace formula [66]; see [41] for an elementary treatment.

[18] A subring $\mathcal{O} < D$ is said to be an order in D if $1 \in \mathcal{O}$ and every for $\beta \in \mathcal{O}$ its trace and its norm are in $\mathbb{Z}$. An order $\mathcal{O}$ is an Eichler order if it is the intersection of two maximal orders.

We remark that for congruence sublattices of $\mathrm{PGL}(2,\mathbb{Z})$ the continuous spectrum of the Laplacian is given by Eisenstein series; equidistribution (appropriately interpreted) of these Eisenstein series has been established by W. Luo and P. Sarnak and by D. Jakobson[44, 28].

5.2.3. The lattices given in §5.2.1 have the property that for all but finitely many primes p, there is a lattice Λ_p in $\mathrm{PGL}(2,\mathbb{R}) \times \mathrm{PGL}(2,\mathbb{Q}_p)$ so that

$$\Gamma\backslash \mathrm{PGL}(2,\mathbb{R}) \cong \Lambda_p\backslash \mathrm{PGL}(2,\mathbb{R}) \times \mathrm{PGL}(2,\mathbb{Q}_p)/K_p \tag{5.1}$$

with $K_p = \mathrm{PGL}(2,\mathbb{Z}_p) < \mathrm{PGL}(2,\mathbb{Q}_p)$. For example, for $\Gamma = \mathrm{PGL}(2,\mathbb{Z})$ one can take $\Lambda_p = \mathrm{PGL}(2,\mathbb{Z}[1/p])$ (embedded diagonally in $\mathrm{PGL}(2,\mathbb{R}) \times \mathrm{PGL}(2,\mathbb{Q}_p)$).

5.2.4. The isomorphism (5.1) gives us a map $\pi_p : \Lambda_p\backslash\mathbb{H} \times \mathrm{PGL}(2,\mathbb{Q}_p) \to \Gamma\backslash\mathbb{H}$. The group $\mathrm{PGL}(2,\mathbb{Q}_p)$ acts on $\Lambda_p\backslash\mathbb{H} \times \mathrm{PGL}(2,\mathbb{Q}_p)$ by right translation, and using this action we set for every $x \in \Gamma\backslash\mathbb{H}$

$$T_p(x) = \pi_p\left(\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}.\pi_p^{-1}(x)\right);$$

this is a set of $p+1$ points called the *p-Hecke correspondence*. Using this correspondence we define the *p-Hecke operator* (also denoted T_p) on functions on $\Gamma\backslash\mathbb{H}$ by

$$[T_p(f)](x) = p^{-1/2} \sum_{y \in T_p(x)} f(y).$$

The Hecke operators (considered as operators on $L^2(M)$) are self adjoint operators that commute with each other and with the Laplacian, so one can always find an orthonormal basis of the subspace of $L^2(M)$ which corresponds to the discrete part of the spectrum consisting of such joint eigenfunctions. Furthermore, if the spectrum is simple (as is conjectured e.g. for $\mathrm{PGL}(2,\mathbb{Z})$), eigenfunctions of the Laplacian are automatically eigenfunctions of all Hecke operators. These joint eigenfunctions of the Laplacian and all Hecke operators are called *Hecke-Maass cusp forms*.

5.2.5. We define an *arithmetic quantum limit* to be a measure μ on SM which is a quantum limit constructed from a sequence of Hecke-Maass cusp forms (see §5.1.2). The *arithmetic quantum unique ergodicity* question, also raised by Rudnick and Sarnak in [61] is whether the uniform measure on SM is the only arithmetic quantum limit. Some partial results towards answering this question were given in [61, 38, 74, 64]. Assuming the Riemann hypothesis for suitable automorphic L-functions, T. Watson [73] has shown that the only arithmetic quantum limit for both types of lattices considered in §5.2.1 is the normalized volume measure. In fact, to obtain this conclusion one does not need the full force of the Riemann hypothesis but only subconvexity estimates on the value of these L-functions at $1/2$, which are known for some families of L-functions but not for the ones appearing in Watson's work. Assuming the full force of the Riemann hypothesis gives a rate of convergence of the $\tilde{\mu}_k$ to the uniform measure that is known to be best possible [45].

Using measure classification techniques one can unconditionally prove the following:

Theorem 5.2 (L. [39]). *Let M be $\Gamma\backslash\mathbb{H}$ for Γ one of the lattice as listed in §5.2.1. Then if M is compact the arithmetic quantum limit is the uniform measure m_{SM} on SM (normalized to be a probability measure). In the noncompact case, any arithmetic quantum limit is of the form cm_{SM} some $c \in [0,1]$.*

It is desirable to prove unconditionally that even in the non-compact case m_{SM} is the only arithmetic quantum limit. A weaker version would be to prove unconditionally that for any sequence of Hecke-Maass cusp forms ϕ_i and $f, g \in C_c(M)$ with $g \geq 0$

$$\frac{\int f(x)\left|\phi_i(x)\right|^2 dm(x)}{\int g(x)\left|\phi_i(x)\right|^2 dm(x)} \rightarrow \frac{\int f dm(x)}{\int g dm(x)}.$$

5.2.6. We briefly explain how measure rigidity results are used to prove Theorem 5.2. Let ϕ_i be a sequence of Hecke-Maass cusp forms on $\Gamma\backslash\mathbb{H}$, and let μ be the associated arithmetic quantum limit, which we recall is a measure on SM which is essentially $\Gamma\backslash\operatorname{PGL}(2,\mathbb{R})$. Using the isomorphism (5.1) and in the notations of §5.2.3, one can identify μ with a right K_p-invariant measure, say μ', on $\Lambda_p\backslash\operatorname{PGL}(2,\mathbb{R}) \times \operatorname{PGL}(2,\mathbb{Q}_p)$. Using the fact that ϕ_i is an eigenfunction of the Hecke operators and some elementary fact regarding the regular representations of $\operatorname{PGL}(2,\mathbb{Q}_p)$ one can show that μ' is recurrent under $\operatorname{PGL}(2,\mathbb{Q}_p)$ (see Definition 2.7).

Building upon an idea of Rudnick and Sarnak from [61], J. Bourgain and E. L. [6] have shown that any ergodic component $\mu_x^{\mathcal{E}}$ of μ with respect to the action of the group $a(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ (equivalently, any ergodic component of μ' with respect to the action of the group $a'(t) = (a(t), e))$ has positive entropy (in fact, $h_{\mu_x^{\mathcal{E}}}(a) \geq 2/9$). One can now use Theorem 2.9 for $\operatorname{PGL}(2,\mathbb{R})$ to deduce that μ' and hence the arithmetic quantum limit μ is proportional to the Haar measure.

The entropy bound of [6] is proved by giving a uniform bound for every x in a compact $K \subset \Gamma\backslash\operatorname{PGL}(2,\mathbb{R})$ on the measure of geometric tubes around x of the form $\mu(B_k.x) < c_{B,K}\exp(-4k/9)$, with B_k as in §3.2.4. This bound holds already for appropriate lifts of the measures $\tilde{\mu}_n$, and depends only on ϕ_n being eigenfunctions of all Hecke operators (but does not use that they are also eigenfunctions of the Laplacian).

5.2.7. Using the same general strategy, L. Silberman and A. Venkatesh have been able to prove arithmetic quantum unique ergodicity for other $\Gamma\backslash G/K$, specifically for $G = \operatorname{PGL}(p,\mathbb{R})$ ($p > 2$ prime) and Γ a lattice arising from an order in a division algebra of degree p over $\mathbb{Q}$ (for $p = 2$ this gives precisely the lattices considered in part 2 of §5.2.1). While the strategy remains the same, several new ideas are needed for this extension, in particular a new micro-local lift for higher rank groups [68, 67].

Silberman has informed us that these methods can be used to establish analogs of Theorem 5.2 in some three-dimensional hyperbolic (arithmetic) manifolds; to the best of our knowledge such extensions are not known for any higher dimensional hyperbolic manifolds.

5.3. A result of N. Anantharaman. We conclude this section by discussing some very recent results of N. Anantharaman which shed some light on quantum limits in full generality (and not just in the arithmetic context). It can be deduced from her paper [1] that if M is a compact manifold with negative sectional curvature then every quantum limit has positive ergodic theoretic entropy. In the case of surfaces of constant curvature -1 Anantharaman actually proves that for any $\delta > 0$ any quantum limit has a positive measure of ergodic components with ergodic theoretic entropy $\geq (d-1)/2-\delta$; in this normalization the ergodic theoretic entropy of the uniform measure m_{SM} is $d-1$.

An exposition of some of her ideas as well as a different but closely related approach, both applied to a simpler toy model, can be found in [2] by N. Anantharaman and S. Nonnenmacher.

Note that in contrast to [6], this method inherently can only prove that some ergodic components have positive entropy. In the nonarithmetic situation it seems very hard to show that all ergodic components have positive entropy; indeed, this is false in the toy model considered in [2] as well as for an appropriately quantized hyperbolic toral automorphism [23].

6. Entropy and distribution of periodic orbits

Let G be a semisimple $\mathbb{R}$-split algebraic group, $\Gamma < G$ a lattice, and A a $\mathbb{R}$-split Cartan subgroup of G. Then there are always infinitely many periodic A orbits in $\Gamma\backslash G$ [53]. Uniform measure on these orbits give examples of A-invariant measures with zero entropy. It is surprising therefore that entropy plays a key role in our understanding of distribution properties of such compact orbits. The results described in this section are joint work of the authors with P. Michel and A. Venkatesh [21] and are also described from a somewhat different viewpoint in [50] in these proceedings. Unless otherwise stated, proofs of all the statements below can be found in [21].

Periodic orbits of $\mathbb{R}$-split Cartan subgroups have also been studied elsewhere. We mention in particular the papers [52] by H. Oh where finiteness theorems regarding these orbits are proved and [4] where Y. Benoist and H. Oh prove equidistribution of Hecke orbits of a fixed A-periodic orbit.

6.1. Discriminant and regulators of periodic orbits. For concreteness, we restrict to the case $G = \mathrm{PGL}(n, \mathbb{R})$, $\Gamma = \mathrm{PGL}(n, \mathbb{Z})$ and $A < G$ the group of diagonal matrices. Later, we will also allow Γ to be a lattice associated to an order in a division algebra D over $\mathbb{Q}$ of degree n with $D \otimes \mathbb{R} \cong M(n, \mathbb{R})$ (e.g. for $n = 2$ a lattice as in part 2 in §5.2.1).

6.1.1. We wish to attach to every periodic A orbit in $X_n = \Gamma\backslash G$ two invariants: discriminant and regulator. Before doing this, we we recall the following classical construction of such orbits:

Let K be a totally real extension of $\mathbb{Q}$ with $[K : \mathbb{Q}] = n$. Let $\mathcal{O}_K$ be the integers of K, and let $I \lhd \mathcal{O}_K$ be an ideal. Chose an ordering $\tau_1, \dots, \tau_n$ of the n embeddings of K in $\mathbb{R}$, and let $\tau = (\tau_1, \dots, \tau_n) : K \to \mathbb{R}^n$. Then $\tau(I)$ is a lattice in $\mathbb{R}^n$, hence corresponds to a point $(\!(g_I)\!) \in X_n$. If $\alpha_1, \dots, \alpha_n$ generate I as an additive group we can take $g_I = (\tau_j(\alpha_i))_{i,j=1}^n$. For any $v \in \mathbb{R}^n$ we let $\operatorname{diag}(v)$ be the diagonal matrix with entries $v_1, v_2, \dots, v_n$. If $\alpha \in \mathcal{O}_K^*$ then $I = \alpha I$ and

$$(\!(g_I)\!) = \operatorname{diag}(\tau(\alpha)).(\!(g_I)\!).$$

$\operatorname{diag}(\tau(\cdot))$ embeds $\mathcal{O}_K^*$ discretely in A, and Dirichlet's unit theorem gives us that $\mathcal{O}_K^*/\{\pm 1\}$ is a free Abelian group with $n-1$ generators. It follows that $(\!(g_I)\!)$ has periodic orbit under A. Two ideals $I, J \lhd \mathcal{O}_K$ are equivalent if $I = \alpha J$ for some $\alpha \in K$; in this case $\operatorname{diag}(\tau(\alpha)).(\!(g_I)\!) = (\!(g_J)\!)$ hence $A.(\!(g_I)\!) = A.(\!(g_J)\!)$ iff $I \sim J$. The number of equivalence classes of ideals is denoted by C_K and is called the *class number* of K; given τ we see that there are C_K distinct periodic A-orbits associated with $\mathcal{O}_K$.

A small variation of this construction allowing a general order $\mathcal{O} < \mathcal{O}_K$ and I a proper ideal of $\mathcal{O}$ gives all periodic A-orbits.

6.1.2. Let D be the algebra of all $n \times n$ (not necessarily invertible) diagonal matrices over $\mathbb{R}$. Let $x = (\!(g)\!)$ be a point with periodic A-orbit. Define $\Gamma_{A,g} := \Gamma \cap gAg^{-1}$. Then since x is periodic, $g^{-1}\Gamma_{A,g}g$ is a lattice in A. To $\Gamma_{A,g}$ we can also attach a subring $\mathbb{Q}[\Gamma_{A,g}]$ in $M(n, \mathbb{R})$ in an obvious way. Let $\Delta_{A,g} = g^{-1}(\mathbb{Q}[\Gamma_{A,g}] \cap M(n, \mathbb{Z}))g$; this ring can be shown to be a lattice in D (considered as an additive group).

We define the *regulator* $\operatorname{reg}(x)$ of a periodic $x \in X_n$ for $x = (\!(g)\!)$ to be the volume of $A/(g^{-1}\Gamma_{A,g}g)$ and the *discriminant* $\operatorname{disc}(x)$ by

$$\operatorname{disc}(x) = m_D(D/\Delta_{A,g})^2.$$

Note that $\operatorname{reg}(x)$ is simply the volume of the periodic orbit $A.x$. This defines both of these notions up to a global multiplicative constant corresponding to fixing a Haar measure on A and D respectively. This normalization can be chosen in such a way that $\operatorname{disc}(x)$ is an integer for every A-periodic x and so that for $(\!(g_I)\!)$, g_I as in §6.1.1 for $I \lhd \mathcal{O}_K$, the discriminant and regulator of $(\!(g_I)\!)$ coincide with the discriminant and regulator of the number field K. The regulator and discriminant of a periodic orbit are related, but in a rather weak way: in general for any periodic orbit $A.x$,

$$\log \operatorname{disc}(x) \ll \operatorname{reg}(x) \ll_\epsilon \operatorname{disc}(x)^{1/2+\epsilon}.$$

If e.g. K has no subfields other than $\mathbb{Q}$ the lower bound on $\operatorname{reg}(x)$ can be improved to $c'_n[\log \operatorname{disc}(x)]^{n-1}$, which is tight. However, if K, I and $x = (\!(g_I)\!)$ are as in §6.1.1, $C_K \operatorname{reg}(x)$, i.e. the total volume of all periodic A orbits coming from ideals in $\mathcal{O}_K$, is closely related to the discriminant:

$$\operatorname{disc}(x)^{1/2-o(1)} \le C_K \operatorname{reg}(x) \le \operatorname{disc}(x)^{1/2+o(1)}.$$

Properly formulated (the easiest formulation is Adelic) a similar relation holds also for periodic A orbits coming from non-maximal orders $\mathcal{O} < \mathcal{O}_K$.

6.2. Some distribution properties of periodic orbits. We would like to prove statements regarding how sequences of periodic orbits are distributed. Care must be taken however as even for the simplest cases such as X_3 it is not true that for any sequence of A-periodic points $x_i \in X_3$ with $\operatorname{disc}(x_i) \to \infty$ the orbits $A.x_i$ become equidistributed. Let $\mu_{A.x_i}$ be the unique A-invariant probability measure on $A.x_i$. One problem, for $n = 3$ or more generally, is that it is possible to construct sequences x_i such that $\mu_{A.x_i}$ converge weak* to a measure μ with $\mu(X_n) < 1$. However, as remarked by Margulis the following is a consequence of Conjecture 4.4:

Conjecture 6.1. *For any fixed compact $K \subset X_n$, $n \geq 3$, there are only finitely many periodic A-orbits contained in K.*

Using Corollary 2.3 and what we call the Linnik principle (see §6.3) we prove the following towards this conjecture:

Theorem 6.2 ([21]). *For any fixed compact $K \subset X_n$, $n \geq 3$, for any $\epsilon > 0$, the total volume of all periodic A-orbits contained in K of discriminant $\leq D$ is at most $O_\epsilon(D^\epsilon)$.*

In contrast, for $n = 2$, for any ϵ one can find a compact $K_\epsilon \subset X_2$ so that the total volume of all periodic A-orbits contained in K of discriminant $\leq D$ is $\gg D^{1-\epsilon}$.

Theorem 6.2 directly implies that for any $\epsilon > 0$ and $n \geq 3$ the number of totally real numbers fields K of degree $[K : \mathbb{Q}] = n$ and discriminant $\operatorname{disc}(K) \leq D$ for which for some ideal class there is no representative of norm $\leq \epsilon \operatorname{disc}(K)^{1/2}$ is $\ll D^\epsilon$, giving a partial answer to the third question posed in the introduction.

The same method gives the following in the compact case:

Theorem 6.3 ([21]). *Let Γ be a lattice in $\mathrm{PGL}(n, \mathbb{R})$ associated with a division algebra over degree n over $\mathbb{Q}$ and $\eta > 0$ arbitrary. For any i let $(x_{i,j})_{j=1,\dots,N_i}$ be a finite collection of A-periodic points with distinct A-orbits such that*

$$\sum_{j=1}^{N_i} \operatorname{reg}(x_{i,j}) \geq \max_j (\operatorname{disc}(x_{i,j}))^\eta.$$

Suppose that there is no locally homogeneous proper subset of $\Gamma\backslash G$ containing infinitely many $x_{i,j}$. Then $\overline{\bigcup_{i,j} A.x_{i,j}} = \Gamma\backslash G$.

6.3. The Linnik principle. In the proofs of Theorems 6.2 and 6.3 a crucial point is establishing that a limiting measure has positive entropy under some $a \in A$, which allows one to apply the measure classification results described in §2. Positivity of the entropy is established using the following proposition, which links entropy with the size (regulator) of a periodic orbit (or a collection of periodic orbits) compared to its discriminant(s). We give it below for some lattices Γ in

$G = \mathrm{PGL}(n,\mathbb{R})$, but this phenomenon is much more general. We call this relation between orbit size and entropy the *Linnik principle* in honor of Yu. Linnik in whose book [43] a special case of this relation is implicit.

Proposition 6.4 ([21]). *For every ℓ, let $A.x_{\ell,j}$ ($j = 1 \dots N_\ell$) be a finite collection of (distinct) periodic A-orbits in $\Gamma\backslash \mathrm{PGL}(n,\mathbb{R})$ with Γ either $\mathrm{PGL}(n,\mathbb{Z})$ or a lattice corresponding to an order in a division algebra of degree n over $\mathbb{Q}$. Let $\mu_{(\ell)}$ be the average of the measures $\mu_{A.x_{\ell,1}}, \dots, \mu_{A.x_{\ell,N_\ell}}$ weighted by regulator. Suppose that*

$$\sum_{j=1}^{N_\ell} \mathrm{reg}(x_{\ell,j}) \geq \max_j (\mathrm{disc}(x_{\ell,j}))^\eta.$$

and that the $\mu_{(\ell)}$ converge weak to a* probability *measure μ. Then for any regular*[19] *$a \in A$, there is an $c_{a,n} > 0$ (which can be easily made explicit) so that*

$$h_\mu(a) \geq c_{a,n}\eta. \tag{6.1}$$

If $\mathrm{disc}(x_{\ell,j}) = \mathrm{disc}(x_{\ell,j'})$ for all ℓ, j, j' then (6.1) can be improved to $h_\mu(a) \geq 2c_{a,n}\eta$.

The key observation lies in the fact that for any compact $K \subset \Gamma\backslash G$ if x and $x' \in K$ are periodic with discriminant $\leq D$ then either $x = a.x'$ for some small $a \in A$ or $d(x,x') > CD^{-2}$ where $C = C(K)$ depends on K (if $\mathrm{disc}(x) = \mathrm{disc}(x')$ then $d(x,x') > CD^{-1}$ hence the improved estimate in this case). This fact is used together with the subadditivity of $H_{\mu_{(\ell)}}(\cdot)$ (see §3.1.1) to show that $h_\mu(a) \geq c_{a,n}\eta$.

6.4. Packets of periodic orbits and Duke's theorem.

Another completely different way to establish positive entropy for limiting measures is given by subconvex estimates on L-functions.

Theorem 6.5 ([21]). *Let $K_{(\ell)}$ be a sequence of totally real degree three extensions of $\mathbb{Q}$, $C_{(\ell)}$ the class number of $K_{(\ell)}$, and $\tau_{(\ell)} : K_{(\ell)} \to \mathbb{R}^3$ a 3-tuple of embeddings. Let $A.x_{1,\ell}, \dots, A.x_{C_{(\ell)},\ell}$ be the periodic A orbits corresponding to the ideal classes of $K_{(\ell)}$ as in §6.1.1. Let $\mu_{(\ell)} = \frac{1}{C_{(\ell)}} \sum_i \mu_{A.x_{i,\ell}}$. Then $\mu_{(\ell)}$ convege in the weak* topology to the $\mathrm{PGL}(3,\mathbb{R})$ invariant probability measure m_{X_3} on X_3.*

Subconvexity estimates of W. Duke, J. Friedlander and H. Iwaniec [11] imply that for certain test functions f, the integrals $\int_{X_3} f d\mu_{(\ell)}$ converge to the right value (i.e. $\int_{X_3} f dm_{X_3}$). The space of test functions on which this convergence can be established from the subconvex estimates of Duke, Friedlander and Iwaniec is far from dense, but is sufficiently rich to show that any limiting measure of $\mu_{(\ell)}$ is a probability measure (i.e. there is no escape of mass to the cusp) and that the entropy of *every* ergodic component in such a limiting measure is greater than an explicit lower bound. Once these two facts have been established, Theorem 2.2 can be used to bootstrap entropy to equidistribution.

Theorem 6.5 is a generalization to the case $n = 3$ of the following theorem of Duke, proved using earlier and related subconvexity estimates of Duke and Iwaniec:

[19] I.e. with no multiple eigenvalues.

Theorem 6.6 (Duke [10]). *Let $K_{(\ell)} = \mathbb{Q}(\sqrt{D_\ell})$ be a sequence of real quadratic fields and $\mu_{(\ell)}$ an average of the corresponding measures on A-periodic orbits in X_2 as above. Then $\mu_{(\ell)}$ converge weak* to m_{X_2}.*

We note that Duke also gives an explicit rate of equidistribution of the $\mu_{(\ell)}$.

There is an alternative, ergodic theoretic, approach to this theorem that dates back to Yu. Linnik and B. F. Skubenko. Skubenko [70], building on work of Linnik [43], used this approach, which is closely related to techinques discussed in §6.3, to prove Theorem 6.6 under a congruence condition on the sequence D_ℓ (see [43, Chapter VI]). In [22] we show that a variation of this method can actually be used to give a complete proof of Theorem 6.6 using only ergodic theory and some properties of quadratic forms.

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