NON-LOCALIZATION OF EIGENFUNCTIONS ON LARGE REGULAR GRAPHS

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1. Introduction

The extent to which eigenfunctions of the Laplacian may localize in small sets, in the presence of a chaotic geodesic flow, is a question that has attracted much recent attention. In the case of negative sectional curvature, it is conjectured (see, eg., [IS95]) that the sup-norms of Laplace eigenfunctions satisfy strong bounds (subexponential in the eigenvalue). The Quantum Unique Ergodicity property, proved for certain arithmetic manifolds (see [Lin06], [SV07], [Sou09]), and conjectured for all manifolds of negative sectional curvature [RS94], gives a weaker, but still dramatic, notion of delocalization. Other work (eg., [AN07]) proves entropy estimates for eigenfunctions, a weaker notion of the inability to localize in small sets.

In this paper, we investigate eigenfunctions of the discrete Laplacian on large $d+1$-regular graphs, satisfying a mild condition (1) below (essentially asking that there not be too many short cycles through the same point). Much stronger results— in particular, strong sup-norm bounds— were obtained for random graphs by Tao-Vu in [TV09], expanding on the work of [ESY09], using probabilistic arguments. We will use methods more akin to the quantum chaos tools of [AN07] and others, and so our results hold for *all* graphs satisfying (1) and any eigenfunction, though the statement is much weaker. An earlier version of our result was also written in [Bro09].

To state the condition on our graphs, we let $\mathcal{T}_{d+1}$ be the $d+1$-regular tree (the universal cover of our $d+1$-regular graphs). For $f \in L^2(\mathcal{T}_{d+1})$, set

$$\tilde{S}_n(f)(x) = d^{-n/2} \sum_{d(x,y)=n} f(y)$$

to be a normalized average over the sphere of radius n, and write S_n for the projection of this operator to $L^2(\mathcal{G})$. We assume that there exist C and $\alpha > 0$, such that the matrix coefficients of S_n satisfy

$$\sup_{x\in\mathcal{G}} ||S_n\delta_x||_\infty \le C d^{-\alpha n} \text{ for all } n \le N \qquad (1)$$

For example, if we set N to be the radius of injectivity of the graph, then we may take $C = 1$ and $\alpha = 1/2$. Ideally, we would like to have $N \gtrsim \log |\mathcal{G}|$ (see below); at the very least, we should ensure that N is large relative to the other parameters, tending to ∞ with $|\mathcal{G}|$.

For random graphs, it is shown in [MWW04] that a large $d+1$-regular graph $\mathcal{G}$ almost surely does not have 2 cycles of length at most $\left(\frac{1}{4} - \epsilon\right) \log_d |\mathcal{G}|$ that share an edge; in particular, this means that the condition (1) holds for almost all graphs.

More generally, we will assume control over the norm of S_n as an operator from $L^p(\mathcal{G})$ to $L^q(\mathcal{G})$, for *some* conjugate pair $1 \leq p < 2 < q \leq \infty$ (see (2) below). This is equivalent to condition (1), up to changing the parameters C and α, but can give a better value for the bound δ in Theorem 1.

Theorem 1. *Let $\epsilon > 0$, and $\mathcal{G}$ a $d+1$-regular graph satisfying*

$$||S_n||_{L^p(\mathcal{G}) \to L^q(\mathcal{G})} \leq C d^{-\alpha n} \text{ for all } n \leq N \tag{2}$$

as an operator from $L^p(\mathcal{G})$ to $L^q(\mathcal{G})$, for some conjugate $1 \leq p < 2$ and $2 < q \leq \infty$ (i.e., satisfying $\frac{1}{p} + \frac{1}{q} = 1$). Then for any L^2-normalized eigenfunction ϕ on $\mathcal{G}$, any subset $E \subset \mathcal{G}$ satisfying

$$\sum_{x \in E} |\phi(x)|^2 > \epsilon$$

must be of size

$$|E| \gtrsim d^{\delta N}$$

as $N \to \infty$, where $\delta = \delta(\epsilon, \alpha, p)$ can be taken to be $\delta = 2^{-7} \frac{\alpha p}{(2-p)} \epsilon^2$. The implied constant depends on all parameters except N; namely d, C, α, p, and ϵ.

Note that if $N \gtrsim \log_d |\mathcal{G}|$, then the conclusion of Theorem 1 states that $|E| \gtrsim |\mathcal{G}|^{\delta'}$.

2. Some Harmonic Analysis on the $d+1$-Regular Tree

Throughout, we set $\mathcal{G}$ to be a $d+1$-regular graph of girth $g(\mathcal{G})$. We have the symmetric operator

$$T_d f(x) = \frac{1}{\sqrt{d}} \sum_{d(x,y)=1} f(y)$$

and an orthonormal basis $\{\phi_j\}_{j=1}^{|\mathcal{G}|}$ of $L^2(\mathcal{G})$ consisting of T_d-eigenfunctions[1]. (The discrete Laplacian on $\mathcal{G}$ can be written as $\Delta f = \left(\frac{\sqrt{d}}{d+1}T_d - 1\right) f$, and so the eigenfunctions of T_d are exactly the eigenfunctions of Δ.)

The universal cover of $\mathcal{G}$ is the $d+1$-regular tree, denoted $\mathcal{T}_{d+1}$. Harmonic analysis on $\mathcal{T}_{d+1}$ has been well studied, see eg. [FTP83]. For every $\lambda \in [-\frac{d+1}{\sqrt{d}}, \frac{d+1}{\sqrt{d}}]$, there exists a unique **spherical function** ϕ_λ satisfying:

- $T_d\phi_\lambda = \lambda\phi_\lambda$.
- ϕ_λ is radial; i.e., $\phi_\lambda(x) = \phi_\lambda(|x|)$ for all $x \in \mathcal{T}_{d+1}$, where $|x|$ denotes the distance from x to the origin in $\mathcal{T}_{d+1}$.
- $\phi_\lambda(0) = 1$.

The last condition is simply a convenient normalization.

We distinguish two parts of this spectrum: the **tempered spectrum** is the interval $[-2, 2]$, and the **untempered spectrum** is the part lying outside this interval, i.e. $\pm(2, \frac{d+1}{\sqrt{d}}]$. We will find it convenient to parametrize the spectrum by $\lambda = 2\cos\theta_\lambda$, where:

- $\theta_\lambda \in [0, \pi]$ for λ tempered.
- $i\theta_\lambda = r_\lambda \in (0, \log\sqrt{d})$ for λ untempered and positive.
- $i\theta_\lambda + i\pi = r_{-\lambda}$ for λ untempered and negative.

In this parametrization, we can write the spherical functions explicitly as [Bro91]:

$$\phi_\lambda(x) = d^{-|x|/2}\left(\frac{2}{d+1}\cos|x|\theta_\lambda + \frac{d-1}{d+1}\frac{\sin(|x|+1)\theta_\lambda}{\sin\theta_\lambda}\right)$$

It will be convenient to use the Chebyshev polynomials

$$\begin{aligned} P_n(\cos\theta) &= \cos n\theta \\ Q_n(\cos\theta) &= \frac{\sin(n+1)\theta}{\sin\theta} \end{aligned}$$

of the first and second kinds, respectively. With this notation the spherical functions become

$$\phi_\lambda(x) = d^{-|x|/2}\left(\frac{2}{d+1}P_{|x|}(\lambda/2) + \frac{d-1}{d+1}Q_{|x|}(\lambda/2)\right) \tag{3}$$

[1]The operator T_d matches the operator defined above as S_1, though here we wish to emphasize the degree d rather than the radius of the sphere.

For any compactly supported radial function $k = k(|x|)$ on $\mathcal{T}_{d+1}$, the **spherical transform** of k, denoted h_k, is given by

$$h_k(\lambda) = \sum_{x\in\mathcal{T}_{d+1}} k(x)\phi_\lambda(x) = k(0) + (d+1)\sum_{n=1}^{\infty} d^{n-1}k(n)\phi_\lambda(n)$$

for all $\lambda \in [-\frac{d+1}{\sqrt{d}}, \frac{d+1}{\sqrt{d}}]$ (the sum is actually finite since k is compactly supported).

We have the **Plancherel measure** dm on $[-2, 2]$ inverting the spherical transform on the tempered spectrum, i.e.

$$\int_0^\pi \phi_\lambda(x)dm(\theta_\lambda) = \delta_0(x)$$

where δ_0 is the δ function at 0 on $\mathcal{T}_{d+1}$ given by

$$\delta_0(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

The Plancherel measure is absolutely continuous (with respect to Lebesgue measure on the semi-circle) and symmetric about $\pi/2$, and so its Fourier series is of the form

$$\frac{dm}{d\theta} = \sum_{j=0}^{\infty} c_j \cos 2j\theta$$

The Plancherel measure is then given explicitly by [FTP83, Theorem 4.1]

$$\int_0^\pi \cos(2n\theta)dm = \frac{1-d}{2d^n}$$

for $n > 0$ (it is clear directly from the definitions that $\int_0^\pi dm = 1$).

The spectrum of T_d on $L^2(\mathcal{G})$ is contained in $[-\frac{d+1}{\sqrt{d}}, \frac{d+1}{\sqrt{d}}]$, and again we distinguish between the tempered eigenvalues in $[-2, 2]$ and the untempered eigenvalues outside this interval. Eigenfunctions of T_d are also eigenfunctions of convolution with radial kernels; in fact, for a "point-pair invariant" $k(x, y) = k(d(x, y))$ on $\mathcal{G}\times\mathcal{G}$, the eigenvalue for ϕ_j under convolution with k depends only on λ_j, and is given by the spherical transform $h_k(\lambda_j)$ [TW03].

3. The Main Estimate

Our result centers on the following estimate for matrix coefficients of $P_n(\frac{1}{2}T_d)$; recall that $P_n(\cos\theta) = \cos n\theta$ are the Chebyshev polynomials (of the first kind).

Lemma 1. *Let δ_0 be the δ-function supported at $0 \in \mathcal{T}_{d+1}$, and n a positive even integer. Then*

$$P_n(T_d/2)\delta_0(x) \quad = \quad \begin{cases} 0 & |x| \text{ odd} \quad \text{or} \quad |x| > n \\ \frac{1-d}{2d^{n/2}} & |x| < n \quad \text{and} \quad |x| \text{ even} \\ \frac{1}{2d^{n/2}} & |x| = n \end{cases}$$

In particular, we have

$$P_n(T_d/2)\delta_0(x) \lesssim d^{-n/2}$$

Proof: Write $\delta_0 = \int_0^\pi \phi_\lambda dm(\theta_\lambda)$. Then since $\frac{1}{2}T_d\phi_\lambda = \cos\theta_\lambda\phi_\lambda$, we have

$$\begin{aligned} P_n(T_d/2)\delta_0(x) &= \int_0^\pi (\cos n\theta_\lambda)\phi_\lambda(x)dm(\theta_\lambda) \\ &= d^{-|x|/2}\int_0^\pi (\cos n\theta_\lambda)\left(\frac{2}{d+1}P_{|x|}(\lambda/2) + \frac{d-1}{d+1}Q_{|x|}(\lambda/2)\right)dm(\theta_\lambda) \end{aligned}$$

by substituting (3) for the spherical functions.

Now since n is even, both $\cos n\theta_\lambda$ and the Plancherel measure are symmetric about $\pi/2$. But if $|x|$ is odd, then both

$$\begin{aligned} P_{|x|}(\lambda/2) &= \cos|x|\theta_\lambda \\ Q_{|x|}(\lambda/2) &= \frac{\sin(|x|+1)\theta_\lambda}{\sin\theta_\lambda} = \cos|x|\theta_\lambda + \cos\theta_\lambda\frac{\sin|x|\theta_\lambda}{\sin\theta_\lambda} \\ &= \cos|x|\theta_\lambda + \cos\theta_\lambda\left(1 + 2\sum_{j=1}^{(|x|-1)/2}\cos 2j\theta_\lambda\right) \end{aligned}$$

are odd functions with respect to $\pi/2$. Therefore the integral from 0 to $\pi/2$ cancels with the integral from $\pi/2$ to π, and $P_n(T_d/2)\delta_0(x)$ vanishes for $|x|$ odd.

Now consider $|x|$ even, in which case we can write

$$Q_{|x|}(\lambda/2) = \frac{\sin(|x|+1)\theta_\lambda}{\sin\theta_\lambda} = 1 + 2\sum_{j=1}^{|x|/2}\cos 2j\theta_\lambda$$

We will also make repeated use of the identity

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

If $|x| > n$, then we have

$$\begin{aligned} &d^{|x|/2}P_n(T_d/2)\delta_0(x) \\ &\quad = \int_0^\pi (\cos n\theta_\lambda)\left(\frac{2}{d+1}P_{|x|}(\lambda/2) + \frac{d-1}{d+1}Q_{|x|}(\lambda/2)\right)dm(\theta_\lambda) \end{aligned}$$

The left part of the integral yields

$$\begin{aligned}
&\frac{2}{d+1}\int_0^{\pi}\cos n\theta_\lambda \cos|x|\theta_\lambda dm(\theta_\lambda)\\
&= \frac{1}{d+1}\left(\int_0^{\pi}\cos\left(|x|-n\right)\theta_\lambda dm(\theta_\lambda)+\int_0^{\pi}\cos\left(|x|+n\right)\theta_\lambda dm(\theta_\lambda)\right)\\
&= \frac{1}{d+1}\left(\frac{1-d}{2d^{(|x|-n)/2}}+\frac{1-d}{2d^{(|x|+n)/2}}\right)\\
&= \frac{1-d}{d+1}\left(\frac{1}{2d^{(|x|-n)/2}}+\frac{1}{2d^{(|x|+n)/2}}\right)
\end{aligned}$$

The right part, on the other hand, is

$$\begin{aligned}
&\frac{d-1}{d+1}\int_0^{\pi}(\cos n\theta_\lambda)\left(1+2\sum_{j=1}^{|x|/2}\cos 2j\theta_\lambda\right)dm(\theta_\lambda)\\
&= \frac{d-1}{d+1}\int_0^{\pi}\left(\cos n\theta_\lambda+\sum_{j=1}^{|x|/2}[\cos\left(n+2j\right)\theta_\lambda+\cos\left(n-2j\right)\theta_\lambda]\right)dm(\theta_\lambda)\\
&= \frac{d-1}{d+1}\sum_{j=-|x|/2}^{|x|/2}\int_0^{\pi}\cos\left(n+2j\right)\theta_\lambda dm(\theta_\lambda)\\
&= \frac{d-1}{d+1}\left(\sum_{j=-|x|/2}^{n/2-1}\frac{1-d}{2d^{n/2-j}}+1+\sum_{j=n/2+1}^{|x|/2}\frac{1-d}{2d^{j-n/2}}\right)\\
&= \frac{d-1}{d+1}\left(\frac{1}{2d^{(n+|x|)/2}}+\frac{1}{2d^{(|x|-n)/2}}\right)
\end{aligned}$$

since the sum telescopes. Putting the two halves together gives $P_n(T_d/2)\delta_0(x) = 0$ for all $|x| > n$.

If, however, $|x| \leq n$, then the sum does not quite telescope as before. For $|x| < n$ we have instead

$$\begin{aligned}
&d^{|x|/2} P_n(T_d/2)\delta_0(x) \\
&= \frac{1}{d+1}\left(\frac{1-d}{2d^{(n-|x|)/2}} + \frac{1-d}{2d^{(n+|x|)/2}}\right) + \frac{d-1}{d+1}\sum_{j=(n-|x|)/2}^{(n+|x|)/2} \frac{1-d}{2d^j} \\
&= \frac{1-d}{d+1}\left(\frac{1}{2d^{(n-|x|)/2}} + \frac{1}{2d^{(n+|x|)/2}}\right) + \frac{d-1}{d+1}\left(-\frac{d}{2d^{(n-|x|)/2}} + \frac{1}{2d^{(n+|x|)/2}}\right) \\
&= \frac{1-d}{d+1}\left(\frac{1}{2d^{(n-|x|)/2}} + \frac{d}{2d^{(n-|x|)/2}}\right) \\
&= \frac{1-d}{2d^{(n-|x|)/2}}
\end{aligned}$$

which finally gives

$$P_n(T_d/2)\delta_0(x) = d^{-|x|/2}\frac{1-d}{2d^{(n-|x|)/2}} = \frac{1-d}{2d^{n/2}}$$

for $|x| < n$ and even.

If $|x| = n$, then we replace $\frac{1-d}{2d^{(n-|x|)/2}}$ with 1, and the above calculation becomes

$$\begin{aligned}
&d^{|x|/2} P_n(T_d/2)\delta_0(x) \\
&= \frac{1}{d+1}\left(1 + \frac{1-d}{2d^{(n+|x|)/2}}\right) + \frac{d-1}{d+1}\left(1 + \sum_{j=1}^{(n+|x|)/2} \frac{1-d}{2d^j}\right) \\
&= \frac{1}{d+1}\left(1 + \frac{1-d}{2d^{(n+|x|)/2}}\right) + \frac{d-1}{d+1}\left(\frac{1}{2} + \frac{1}{2d^{(n+|x|)/2}}\right) \\
&= \frac{1}{d+1} + \frac{d-1}{2(d+1)} \\
&= \frac{d+1}{2(d+1)} = 1/2
\end{aligned}$$

and so $P_n(T_d/2)\delta_0(x) = \frac{1}{2d^{|x|/2}} = \frac{1}{2d^{n/2}}$ for $|x| = n$, as required. $\square$

Corollary 1. *If $\mathcal{G}$ satisfies the hypothesis (2) of Theorem 1, then*

$$\left\|P_n\left(\frac{1}{2}T_d\right)\right\|_{L^p(\mathcal{G})\to L^q(\mathcal{G})} \lesssim d^{-\alpha n}$$

for all even positive integers $n \leq N$.

Proof: Thanks to Lemma 1, we have that

$$P_n(T_p/2) = \sum_{j=0}^{n/2-1} \frac{1-d}{2d^{n/2}} d^j S_{2j} + \frac{1}{2} S_n$$

$$\begin{aligned}
\left|\left|P_n\left(\frac{1}{2}T_d\right)\right|\right|_{L^p\to L^q} &\leq \sum_{j=0}^{n/2-1} \frac{d-1}{2d^{n/2}} ||d^j S_{2j}||_{L^p\to L^q} + \frac{1}{2}||S_n||_{L^p\to L^q} \\
&\leq d^{-n/2+1} \sum_{j=0}^{n/2} d^j ||S_{2j}|| \\
&\leq d^{-n/2+1} \sum_{j=0}^{n/2} C d^{j(1-2\alpha)} \\
&\lesssim_{d,\alpha} d^{-n/2} \cdot C d^{n/2-\alpha n} \lesssim d^{-\alpha n}
\end{aligned}$$

as required. □

4. Estimating the Mass of Small Sets

Now we turn to the proof of Theorem 1. We first consider tempered eigenfunctions; it will become clear how the argument is applied to untempered eigenfunctions as well.

Lemma 2. *Let $\epsilon > 0$. For any $\theta_0 \in [0, \pi]$, there exists a kernel k_0 on $\mathcal{T}_{d+1}$ such that:*

- *k_0 is supported on a ball of radius N.*
- *The operator of convolution with k_0, call it $K_{\theta_0}(f) = f * k_0$, has matrix coefficients bounded by*

  $$||K_{\theta_0}||_{L^p(\mathcal{G})\to L^q(\mathcal{G})} \lesssim_{d,\alpha} C d^{-\frac{1}{128}\alpha N \epsilon^2}$$

  as an operator from $L^p(\mathcal{G})$ to $L^q(\mathcal{G})$, where C and α are the parameters of the hypothesis (2).
- *The spherical transform of k_0 satisfies $h_{k_0} \geq -1$ everywhere, and $h_{k_0}(\theta_0) > \epsilon^{-1}$.*

Proof: Set $M = \lfloor \epsilon^{-1} \rfloor$ and $R = \lceil \frac{1}{8} N\epsilon \rceil$ (one should think of N as being much larger than ϵ^{-1}, so that R is large). By Dirichlet's Theorem, we can find a positive integer $r \leq R$ such that $|r\theta_0 \mod 2\pi| < 2\pi R^{-1} \leq \frac{2\pi}{N\epsilon}$. There exists an even multiple of r, say $r' = 2lr$, such that $\frac{1}{16} R\epsilon \leq r' \leq 2R$ (if $r \geq \frac{1}{32} R\epsilon$, we can simply take $l = 1$; otherwise there is a multiple of r between $\frac{1}{32} R\epsilon$ and $\frac{1}{16} R\epsilon$, so take twice that multiple). Moreover, since we can choose $2l \le \frac{1}{16}R\epsilon$, we have $|r'\theta_0 \mod 2\pi| < \frac{1}{8}\pi\epsilon \le \frac{\pi}{8M}$.

We now set the spherical transform of k_0 to be $h_{k_0}(\theta) = F_{2M}(r'\theta) - 1$, where F_{2M} is the Fejèr kernel of order $2M$. Since $r'\theta_0 \mod 2\pi \in \left[-\frac{\pi}{8M}, \frac{\pi}{8M}\right]$ is close enough to 0, we have

$$F_{2M}(r'\theta_0) = \frac{1}{2M}\frac{\sin^2(2Mr'\theta_0)}{\sin^2(r'\theta_0)} > M + 2$$

as long as $M \ge 4$, and therefore the eigenvalue of $\phi_{2\cos\theta_0}$ under convolution with k_0 will be $> M + 1 \ge \epsilon^{-1}$. Moreover, since F_{2M} is positive, the spherical transform of k_0 is bounded below by -1. It remains to check the first two properties.

Now, by Corollary 1 of the main estimate, we see that the kernel whose spherical transform is $\cos 2j\theta$— i.e., the kernel of $P_{2j}(\frac{1}{2}T_d)$— has norm $\lesssim d^{-2\alpha j}$ as a convolution operator from $L^p(\mathcal{G})$ to $L^q(\mathcal{G})$. The spherical transform of k_0 is a sum of terms of the form $\frac{2M-j}{M}\cos jr'\theta$, where $j = 1, 2, \ldots, 2M$ (note that we eliminated the $j = 0$ term by subtracting off the constant contribution to F_{2M}) and $r' \in 2\mathbb{Z}$. Thus

$$||K_{\theta_0}||_{L^p(\mathcal{G})\to L^q(\mathcal{G})} \lesssim \sum_{j=1}^{M} d^{-\alpha j r'} \lesssim_{d,\alpha} d^{-\alpha r'}$$

Then, since

$$r' \ge \frac{1}{16}R\epsilon \ge \frac{1}{128}N\epsilon^2$$

this concludes the proof of Lemma 2. □

We now wish to apply this convolution operator to examine the localization of eigenfunctions in small sets.

Proof of Theorem 1: Pick an eigenfunction ϕ_j of eigenvalue λ_j, and a set E satisfying

$$||\phi_j||^2_{L^2(E)} = ||\phi_j 1_E||^2_{L^2(\mathcal{G})} \ge \epsilon \tag{4}$$

Define the operator $K_j = K_{\theta_{\lambda_j}}$ of Lemma 2 (corresponding to $\theta_0 = \theta_{\lambda_j}$) if λ_j is tempered, or $K_j = K_{\theta=0}$ if λ_j is untempered. Observe that in either case K_j satisfies

$$\begin{aligned} \left|\langle K_j(\phi_j 1_E), \phi_j 1_E\rangle\right| &\le ||K_j(\phi_j 1_E)||_q ||\phi_j 1_E||_p \\ &\le ||K_j||_{L^p\to L^q} ||\phi_j 1_E||^2_p \end{aligned}$$

By Hölder's Inequality, we have

$$||\phi_j 1_E||^2_p = ||\phi_j^p 1_E||_1^{2/p} \le ||\phi_j^p||^{2/p}_{2/p} ||1_E||^{2/p}_{2/(2-p)} = ||\phi_j||^2_2 \cdot |E|^{\frac{2-p}{p}}$$

so that

$$\begin{aligned}\left|\langle K_j(\phi_j 1_E), \phi_j 1_E\rangle\right| &\leq ||K_j||_{L^p\to L^q} ||\phi_j 1_E||_p^2 \\ &\leq ||K_j||_{L^p\to L^q} ||\phi_j||_2^2 \cdot |E|^{\frac{2-p}{p}} \\ &\leq ||K_j||_{L^p\to L^q} \cdot |E|^{\frac{2-p}{p}} \\ &\lesssim_{d,\alpha} Cd^{-2^{-7}\alpha N\epsilon^2} \cdot |E|^{\frac{2-p}{p}} \qquad (5)\end{aligned}$$

by Lemma 2.

On the other hand, decompose $\phi_j 1_E$ spectrally as

$$\phi_j 1_E = \langle \phi_j 1_E, \phi_j\rangle \phi_j + g_{\text{temp}} + g_{\text{untemp}}$$

where g_{temp} and g_{untemp} are the tempered and untempered components of $\phi_j 1_E$, respectively, excluding the ϕ_j component. Notice that since

$$|\langle \phi_j 1_E, \phi_j\rangle| = ||\phi_j 1_E||_2^2 \geq \epsilon$$

we have

$$\begin{aligned}||g_{\text{temp}}||_2^2 &\leq ||\phi_j 1_E||_2^2 - |\langle \phi_j 1_E, \phi_j\rangle|^2 \\ &= ||\phi_j 1_E||_2^2 (1 - ||\phi_j 1_E||_2^2) \\ &\leq ||\phi_j 1_E||_2^2 (1-\epsilon) \qquad (6)\end{aligned}$$

Now the K_j-eigenvalue of any tempered eigenfunction is at least -1, and K_j must be positive on the untempered eigenfunctions, since each term in the Fourier expansion of the Fejer kernel is of the form $\cos 2j\theta = \cos -i2jr = \cosh(-2jr) > 0$ for $\theta = -ir$, and similarly $\cos(2j\theta) = \cos(-i2jr - 2j\pi) = \cosh(-2jr)$ for $\theta = -ir - \pi$. Therefore

$$\begin{aligned}\langle K_j(\phi_j 1_E), \phi_j 1_E\rangle &\geq |\langle \phi_j 1_E, \phi_j\rangle|^2 \langle K_j\phi_j, \phi_j\rangle - ||g_{\text{temp}}||_2^2 \\ &\geq ||\phi_j 1_E||_2^2 \Big(||\phi_j 1_E||_2^2 \langle K_j \phi_j, \phi_j\rangle - (1-\epsilon)\Big) \qquad (7)\end{aligned}$$

If λ_j is tempered, then Lemma 2 implies that the ϕ_j-eigenvalue of K_j is at least ϵ^{-1}, whereby

$$\langle K_j \phi_j, \phi_j\rangle \geq \epsilon^{-1} ||\phi_j||_2^2 = \epsilon^{-1}$$

If λ_j is untempered, then because $\cosh(2j\theta_j) > 1 = \cos(2j(0))$ we chose to use the same kernel as $\theta = 0$ from the tempered case, and get that

$$\langle K_{\theta=0}\phi_j, \phi_j\rangle \geq (F_{2M}(0) - 1)||\phi_j||_2^2 = 2M - 1 > \epsilon^{-1}$$

in the untempered case as well. Applying (4), we get from (7) that

$$\begin{aligned}\langle K_j(\phi_j 1_E), \phi_j 1_E\rangle &\geq ||\phi_j 1_E||_2^2 (||\phi_j 1_E||_2^2 \cdot \epsilon^{-1} - 1 + \epsilon) \\ &\geq ||\phi_j 1_E||_2^2 (1 - 1 + \epsilon) \\ &\geq \epsilon(\epsilon) = \epsilon^2 \qquad (8)\end{aligned}$$

and combining (5) with (8) yields

$$|E|^{\frac{2-p}{p}} \gtrsim_{d,\alpha} C^{-1}\epsilon^2 d^{2^{-7}\alpha\epsilon^2 N} \gtrsim d^{2^{-7}\alpha\epsilon^2 N}$$

which gives the bound of Theorem 1. □

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