Riemannian Geometry of Diffeomorphism Groups — Lecture notes

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These lecture notes were made for a course I gave during the spring term of 2021 at the Hebrew University, as part of David Kazhdan's Sunday seminars. Some of the topics that appear I did no cover, or did cover but not in the order they appear here (in particular, I taught Section 3 between Sections 2.3 and 2.4). I would not have been able to give this course without the help of friends who are much more knowledgable in these topics, to whom I am greatly indebted: Martin Bauer, Philipp Harms, Boris Khesin, Peter Michor and Klas Modin. In particular, much of the content of Section 2 is taken from notes of Martin, Philipp and Peter. I am also indebted of all the attendees of the course for their invaluable questions and comments, and in particular to David. All mistakes and inaccuracies are, of course, my fault alone.

The bibliography is not intended to be a comprehensive bibliography of the subject; it mainly includes books and articles I used while making these notes.

Week-by-week, the material covered was:

- Week 1: Introduction and motivation (§ 1)
- Week 2: Functional analytic preliminaries and definition of manifolds (§ 2.1–2.2)
- Week 3: Manifolds of smooth mappings (§ 2.3)
- Week 4: Manifolds of mappings of finite regularity (§ 3.1–3.2)
- Week 5: Volumorphisms and non-compact base manifold (§ 3.3–3.5), weak and strong Riemannian metrics (beginning of § 2.4)
- Week 6: Right-invariant metrics on diffeomorphism groups (§ 4.1), geodesic equation the Christoffel symbols (§ 2.4.1)
- Week 7: Sprays and the Exponential map (§ 2.4.2)
- Week 8: Geodesic distance, curvature, Hopf–Rinow theorem (§ 2.4.3–2.4.5)
- Week 9: The exponential map of the right-invariant L^2 metric (§ 4.2)
- Week 10: The Ebin–Marsden spray method, Camassa–Holm equation (§ 4.3–4.4)
- Week 11: Incompressible Euler equation, regularity (§ 4.5–4.6)
- Week 12: Boundary regularity (§ 4.6.1), Hunter–Saxton equation (§ 4.7)
- Week 13: Geodesic distance (§ 5), without the diameter section (§ 5.3)
- Week 14: Completeness (§ 6), discussion of some open problems

Contents

Chapter 1

Motivation and outline

Let M, N be (finite dimensional, complete) smooth manifolds. Let $\mathrm{Diff}(M)$ be the set of all smooth diffeomorphisms on it; it is obviously a group, that acts on M. Similarly, one may consider the set of all immersions $\mathrm{Imm}(M, N)$ from M to N (or embeddings $\mathrm{Emb}(M, N)$). But they are more than sets — they are, in fact, infinite dimensional manifolds. On these manifolds one can prescribe a variety of Riemannian metrics, turning them into Riemannian manifolds; however, as they are infinite dimensional, many of the things that are standard textbook material in finite dimensional Riemannian geometry — like existence of geodesics, behavior of the distance function the Riemannian metric induce, etc. — become highly non-trivial. The study of these topics is the goal of this course.

We will now give a few motivations for studying these topics, followed by a more detailed outline of the course.

1.1 Motivation #1: comparing shapes

Say we have two immersions $f_0, f_1 \in \mathrm{Imm}(S^1, \mathbb{R}^2)$, which we want to compare:

- How different f_0 and f_1 are?
- What is an "optimal" way of matching the sets $f_0(S^1)$ and $f_1(S^1)$?
- Is there a natural flow $f_t : [0, 1] \to \mathrm{Imm}(S^1, \mathbb{R}^2)$ from f_0 to f_1 that is "optimal" or "short"?

These questions arise naturally in many imaging applications: The first one appears in "geometric statistics", where one wants to automatically group a large set of images into clusters ("cups" versus "knives"); the second one is natural appears, if, say, we want to understand which parts of the first image correspond to a part of the second; the third appears in applications where one wants to create motion from finitely many frames.

To be more accurate, if f_1 is only a re-parametrization of f_0, that is, if $f_1 = f_0 \circ \varphi$ for some $\varphi \in \mathrm{Diff}(S^1)$, we would like to identify them, that is, to look on the **space of unparametrized curves** $\mathrm{Imm}(S^1, \mathbb{R}^2)/\,\mathrm{Diff}(S^1)$.

A natural way of addressing these questions is by introducing a Riemannian metric on $\mathrm{Imm}(S^1, \mathbb{R}^2)$. The distance between f_0 and f_1 would be the geodesic distance, and the natural flow would be the geodesic flow.

The tangent space of $\mathrm{Imm}(S^1, \mathbb{R}^2)$ at f is

$$T_f \,\mathrm{Imm}(S^1, \mathbb{R}^2) \cong \Gamma(f^* T\mathbb{R}^2) \cong C^\infty(S^1; \mathbb{R}^2).$$

So we can simply put an inner-product on $C^\infty(S^1; \mathbb{R}^2)$. The simplest one is probably

$$\langle u, v\rangle = \int_{S^1} u(\theta) \cdot v(\theta)\, d\theta.$$

However, this one has a major drawback: This metric is not invariant under reparametrizations of θ; as a consequence, it does not induce a metric on the space of unparametrized curves. Instead, we can look on the metric

$$\langle u, v\rangle_{L^2} = \int_{S^1} u(\theta) \cdot v(\theta)\, ds,$$

where $ds = |f'(\theta)| d\theta$ which is reparametrization-invariant, or on higher order metrics, say

$$\langle u, v\rangle_{H^1} = \int_{S^1} (u \cdot v + \nabla_{\partial_s} u \cdot \nabla_{\partial_s} v)\, ds$$
$$\langle u, v\rangle_{H^2} = \int_{S^1} (u \cdot v + \nabla_{\partial_s} u \cdot \nabla_{\partial_s} v + \nabla^2_{\partial_s} u \cdot \nabla^2_{\partial_s} v)\, ds$$

and so on. Here $\partial_s = \frac{1}{|f'(\theta)|}\partial_\theta$.

Natural questions are then:

- Does these Riemannian metrics yield a true metric space structure on $\mathrm{Imm}(S^1, \mathbb{R}^2)$? This is obvious in finite dimensions, but may fail in infinite dimensions (indeed, the answer is negative for L^2 but positive for H^1 and H^2).

- Do geodesics exist? For all time (geodesic completeness)? (no for L^2 and H^1, yes for H^2)

- Can two immersions be connected by a minimizing geodesic? (unknown for L^2 and H^1, yes for H^2)

As we will see, the reparametrization-invariance of the metrics is also a feature that plays a role in studying the geodesic equations (in particular, their regularity theory). There are, of course, also more applied questions — can we compute these geodesics? Are there efficient algorithms for implementing these ideas to real data — but we will not discuss them here.

1.2 Motivation #2: geometric approach to hydrodynamics

A second motivation is what is known as "geometric hydrodynamics" or "topological hydrodynamics", that was initiated by the seminar 1966 paper of Arnold [Arn66]. His key idea was that many equations in hydrodynamics can be cast as geodesic equations of various diffeomorphism groups —the full diffeomorphism group, volume preserving diffeomorphisms, and quotients and extensions thereof — with respect to different Riemannian metrics.[1] This reveals the geometric nature of these equations, and provides new tools to study them.

In hydrodynamics one considers a fluid domain (a manifold M), and for each time t, $u(t,x)$ is the velocity of the fluid at the point x in the domain. That is, $u(t,\cdot) \in \mathfrak{X}(M)$ is a vector field. The evolution of the vector field is given by an equation. The simplest one (which is more of a toy model than a description of an actual fluid) is the (inviscid) *Burgers' equation* on $\mathbb{R}$ (or S^1):[2]

$$u_t + uu_x = 0.$$

[1] The paper [Arn66] refers to the incompressible-Euler equation, but this was later extended to many other equations.

[2] This equation is, among other things, probably the simplest example of equations that develops shockwaves.

A more realistic (and famous) equation is the *incompressible Euler* equation:

$$\begin{cases} u_t + \nabla_u u = -\nabla p \\ \operatorname{div} u = 0. \end{cases}$$

Here M is a Riemannian manifold, $\operatorname{div} u = 0$ is an incompressible constraint, and p is a scalar function representing the pressure.

From a physics perspective, these equations are given in **Eulerian coordinates**; that is, x is the coordinate of the ambient space, and $u(t,x)$ is the velocity at time t at a point x. The complement view is that of **Lagrangian coordinates** — we let x be the position of a fluid particle, and we want to describes how it evolves in time; that is, we are looking at $\varphi(t,x)$, the position in ambient space, at time t, of the particle that was at time 0 at x. The Lagrangian and Eulerian viewpoints are related: $\varphi(t,x)$ is simply the flow of the vector field u, that is,

$$\begin{cases} \varphi_t(t,x) = u(t,\varphi(t,x)) \\ \varphi(0,x) = x. \end{cases} \tag{1.1}$$

For each time t for which the flow exists, we have that $\varphi(t,\cdot) \in \operatorname{Diff}(M)$. So while the Eulerian viewpoint describes the system as an evolution of a vector field, the Lagrangian viewpoint describes an evolution of a diffeomorphism. What equation does φ satisfy? For the Burgers' equation the answer is simple: From (1.1) we have

$$\begin{aligned} \varphi_{tt}(t,x) &= u_t(t,\varphi(t,x)) + u_x(t,\varphi(t,x))\varphi_t(t,x) \\ &= u_t(t,\varphi(t,x)) + u_x(t,\varphi(t,x))u(t,\varphi(t,x)) = 0. \end{aligned}$$

That is, each particle x flows freely, along a geodesic. However, the flow $t \mapsto \varphi(t,\cdot)$ can also be viewed from a *global* point of view, as a curve on the space $\operatorname{Diff}(S^1)$ (for simplicity), and in fact, as a geodesic flow: Here we have that $T_\varphi \operatorname{Diff}(S^1) \cong \Gamma(\varphi^* TS^1)$ — we will prove that later, but intuitively, for $v \in T_\varphi \operatorname{Diff}(S^1)$, at each point $x \in S^1$, $v(x)$ is a vector at $T_{\varphi(x)} S^1$ — so $v \circ \varphi^{-1}$ is a vector field. The Riemannian metric that will yield the Burgers' equation is then

$$\langle u, v \rangle_{L^2} = \int_{S^1} u \circ \varphi^{-1} \cdot v \circ \varphi^{-1} \, d\theta,$$

or more generally, on a Riemannian manifold (M,g), we have for $u,v \in T_\varphi \operatorname{Diff}(M)$,

$$\langle u,v\rangle_{L^2} = \int_M g(u\circ\varphi^{-1}, v\circ\varphi^{-1})\operatorname{Vol}_g = \int_M g(u,v)\operatorname{Vol}_{\varphi^* g}.$$

Incompressible Euler, turns out to be the same Riemannian metric, restricted to the submanifold $\operatorname{Diff}_{\operatorname{Vol}_g}(M)$ of volume-preserving diffeomorphisms.

The full diffeomorphism group $\operatorname{Diff}(M)$, and the group of volume-preserving diffeomorphisms $\operatorname{Diff}_\mu(M)$ are Lie groups (the group actions are smooth). The construction of the metrics above can be generalized as follows: Given a Lie group G, with a Lie algebra $\mathcal{G} = T_e G$, we have that $T_g G = \mathcal{G}g$. Thus, given an inner-product $\langle\,,\,\rangle$ on $\mathcal{G}$ (in the above example this is the L^2-inner product on vector fields), we obtain a Riemannian metric by

$$\langle u,v\rangle_g := \left\langle ug^{-1}, vg^{-1}\right\rangle.$$

This exact setting, on different groups and different inner-products (similar to the Sobolev metrics mentioned above for immersions), yields many equations that arise in physical contexts, and the symmetry of the problem provides tools to study them.

1.3 Other motivations

There are more places where such metrics play a role; these are not far away from the topics of this course, but we will not focus on them:

- In *image registration*, one models a three-dimensional greyscale image by a map $I : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ of a certain regularity. Let us denote by X the space of all such mappings. Given two images $I_0, I_1 \in X$, the first apporach to the image registration problem would be to search for $\varphi \in \operatorname{Diff}(\Omega)$, such that $I_0 \circ \varphi = I_1$. Two things can go wrong. First, such a φ may not exist and second, if it exists, it may not be unique. To address these problems, we can introduce a distance $d(\varphi,\psi)$ on the set of transformations and a distance $\rho(I,J)$ on the set of images and search for the minimizer of

$$\rho(I_0\circ\varphi, I_1)^2 + \varepsilon d(\operatorname{Id},\varphi)^2$$

among $\varphi \in \mathrm{Diff}(\Omega)$. The first term addresses the case of not having an exact matching, and the second term chooses between possible matchings the simplest one. The parameter ε chooses between accuracy and simplicity. The *large deformation diffeomorphic metric mapping (LDDMM)* approach to image registration takes the metric d to be a right-invariant Riemannian-metric on $\mathrm{Diff}(\Omega)$. Here one wants a metric that will induce a distance function d with "good" properties, but also that d will be computable.[3]

- In *symplectic geometry*, one considers the Lie group of Hamiltonian symplectomorphisms on a manifold M, $\mathrm{Diff}_\omega(M)$. An important tool is the **Höfer metric**, which is a bi-invariant Finsler metric on this group, which induces a non-degenerate distance function (by Finsler metric it means that instead of an inner product on the Lie algebra there is a norm).

- The space $\mathrm{Met}(M)$ of all Riemannian metrics on a manifold M is a manifold, and has reparametrization-invariant L^2 metric that is related to Teichmüller theory.

- The 2-Wasserstein distance on the space of volume forms can be realized as a distance induced by a Riemannian metric (which is closely related to the metric of the incompressible Euler equation).

1.4 Outline

We plan to cover the following material (though not necessarily in this order):

1. A short functional-analytic background: mainly, what we mean by smoothness in various topological vector spaces.

2. Infinite dimensional geometry: basic definitions (what is a manifold), main examples (immersions, diffeomorphisms), Riemannian metrics (weak and strong), what goes through and what does not from finite dimensional Riemannian geometry.

[3] This introduction was taken from [BH15].

3. Focus on diffeomorphism groups of various kinds (different regularities, non-compact manifolds, etc.).

4. The Sobolev hierarchy of Riemannian metrics on $\mathrm{Diff}(M)$ and their geodesic equations, curvature, etc.

5. A few miracles: special cases that can be solved explicitly.

6. Metric theory: collapse and non-collapse of the geodesic distance. Diameter of the manifolds.

7. Local theory: local existence and well-posedness of the geodesic equations, regularity theory, blow-up of solutions.

8. Global theory: global existence of geodesics (geometric completeness), metric completeness.

Chapter 2

Infinite dimensional manifolds

In this section we intend to define smooth infinite dimensional manifolds. These are going to be modelled after infinite dimensional topological vector spaces, mainly on Hilbert spaces, Banach spaces and Fréchet spaces; recall the basic definitions:

Definition 2.1

1. *A **Hilbert space** is a complete inner-product space.*

2. *A **Banach space** is a complete normed space.*

3. *A **Fréchet space** is a locally convex t.v.s. whose topology is induced by a complete, translation invariant metric; equivalently, it is a t.v.s. whose topology is induced by a countable family of seminorms, which is point seperating and yields a complete space.*

The topological vector space that are important for us are spaces of functions, in particular:

Example 2.2

1. The space $C^k(M)$ of k-times continuously differentiable maps on a compact manifold M (possibly with boundary). This is a Banach space with the norm

$$\|f\|_{C^k} = \sum_{i=0}^{k} \sup_{x\in M} |\nabla^i f(x)|.$$

2. The space $W^{k,p}(M)$, which is the closure of the smooth functions with respect to the norm

$$\|f\|_{W^{k,p}} = \left(\sum_{j=0}^{k}\int_M |\nabla^j f|^p\right)^{1/p}.$$

Here (M,g) is a compact Riemannian manifold, $p \in [1,\infty)$ and $k \in \mathbb{N} \cup \{0\}$. For $p = 2$ this is a Hilbert space, which is also denoted by $H^k(M)$.

3. The space $C^\infty(M)$ of smooth functions over a compact manifold M, with the topology of uniform convergence on each derivative separately. This is a Fréchet space with the seminorms $\{\|\cdot\|_{C^k}\}_{k\geq 0}$ or with the seminorms $\{\|\cdot\|_{W^{k,p}}\}_{k\geq 0}$ for some p.

4. The space $C^\infty(\mathbb{R})$ of smooth functions over $\mathbb{R}$, with the topology of uniform convergence on each derivative separately on each compact subset. This is a Fréchet space with the seminorms $\{\|\cdot\|_{C^k([-n,n])}\}_{k\geq 0, n>0}$. The same is true for functions between finite dimensional spaces.

There are important spaces of functions that are not even Fréchet , for example the space $C_c^\infty(\mathbb{R})$ of compactly supported smooth functions, with the direct limit topology of $C^\infty([-n,n])$, meaning that a sequence converges if all of it is supported in a compact set, and the sequence and of all its derivatives uniformly converges over this set. For simplicity, we will not focus on these. Also, the dual space[1] of a Fréchet space, with the natural topology of uniform convergence on bounded sets,[2] is never a Fréchet unless the space was normable to begin with [Köt83, §29.1(7)], [Vog00, Theorem 2.3]. This tells us that, in some sence, once we consider the Fréchet category, we need to go beyond it, if, say, we want to consider the cotangent bundle of a Fréchet manifold.

[1] If E is a Fréchet space, then the space of bounded linear functionals and continuous linear functionals coincide [Vog00, Theorem 2.2].

[2] The structure of E^* are detailed in [Vog00, Corollary 2.6].

2.1 Infinite dimensional t.v.s. and differentiability

As we will require the transitions maps being smooth in the infinite dimensional manifold to be smooth, we should first address the issues of differentiability and smoothness in infinite dimensional t.v.s.

Definition 2.3 *(Fréchet derivative) Let E and F be normed spaces. A function $f : U \to F$ defined on an open subset $U \subseteq E$ is called **Fréchet differentiable** at $x \in U$ with derivative $\mathrm{d}f(x) \in L(E, F)$ if*

$$\lim_{E \setminus \{0\} \ni v \to 0} \frac{f(x+v) - f(x) - \mathrm{d}f(x)(v)}{\|v\|} = 0.$$

The function f is called continuously Fréchet differentiable if it is Fréchet differentiable at every $x \in U$ with continuous Fréchet derivative

$$\mathrm{d}f : U \to L(E, F).$$

Higher-order Fréchet derivatives, provided they exist, are defined iteratively as

$$\mathrm{d}^k f = \mathrm{d}(\mathrm{d}^{k-1} f) : U \to L(E, L(\underbrace{E, \dots, E}_{k-1 \ times}; F) \cdots)) \cong L(\underbrace{E, \dots, E}_{k \ times}; F),$$

where the isomorphism is the standard canonical map. The function f is called Fréchet smooth if it is Fréchet C^k for all $k \in \mathbb{N}$.

However, it is not clear how this definition generalizes to more general locally-convex t.v.s.; in fact, it is not clear that $L(E, F)$ is in the same category as E and F, what locally-convex topology to put on it, etc. We therefore consider "directional derivatives" instead of a "full derivative":

Definition 2.4 *(Gâteaux differentiability) Let E and F be Hausdorff locally convex topological vector spaces. A function $f : U \to F$ defined on an open subset $U \subseteq E$ is called **Gâteaux differentiable** at $x \in U$ if for every $hv \in E$ there exists $D_v f(x) \in F$ such that*

$$\lim_{\mathbb{R} \setminus \{0\} \ni t \to 0} \frac{f(x+tv) - f(x) - tD_v f(x)}{t} = 0.$$

The function f is called continuously Gâteaux differentiable if it is Gâteaux differentiable at every $x \in U$ with continuous Gâteaux derivative

$$Df : U \times E \ni (x, v) \mapsto D_v f(x) \in F.$$

Higher-order Gâteaux derivatives, provided they exist, are defined iteratively as

$$D^k f(x; v_1, \dots, v_k) \coloneqq (D_{v_k} D_{v_{k-1}} \cdots D_{v_1} f)(x) : U \times E^k \to F.$$

The function f is called Gâteaux smooth if it is Gâteaux C^k for all $k \in \mathbb{N}$.

Note that our continuity requirement is rather mild: we require that $Df : U \times E \to F$ is continuous, rather than $Df : U \to L(E, F)$ is. This is, because, as we mentioned earlier, there is an ambiguity on which topology to put on $L(E, F)$ (beyond the Banach category). Moreover, even in the Banach category, these notions of continuity are different:

Example 2.5 consider $L : \mathbb{R} \times C(S^1) \to C(S^1)$, defined by $L(t, f)(x) = f(x+t)$. Then L is continuous, however the map $\mathbb{R} \to L(C(S^1), C(S^1))$ is not, since $\sup_{x \in S^1} |f(x+t) - f(x)|$ can be as large as 2 when $\|f\|_\infty = 1$, regardless of t, hence $L(t, \cdot)$ does not converge to $L(0, \cdot)$. Note also that $L : \mathbb{R} \times C(S^1) \to C(S^1)$, is not Gâteaux differentiable at all points, since $\partial_t L(0, f)$ must be $f'(0)$, which is not defined over all $C(S^1)$.

Proposition 2.6

1. *The continuity of $D^k f$ implies that $D^k f(x, \cdot)$ is linear (this is not clear a priori).*
2. *Continuous linear functions are Gâteaux smooth.*
3. *The chain rule holds for Gâteaux C^k functions.*
4. *If E and F are normed spaces, then Gâteaux C^k functions are Fréchet C^{k-1}.*
5. *In particular, for normed spaces there is no difference between Fréchet smoothness and Gâteaux smoothness.*

We will not prove that here.

A t.v.s. that will be important for us is the following:

Definition 2.7 (The space of smooth curves) Let F be a locally-convex t.v.s.. The space of (Gâteaux) smooth curves $C^\infty(\mathbb{R}, F)$, is endowed with the locally convex topology of uniform convergence of the function and all its derivatives on all compact subsets separately, where uniform convergence is the uniform convergence with respect to all Minkowski functionals on all absolutely-convex, bounded subsets.

2.1.1 Smooth functions

The previous example shows that the correspondence

$$C(E \times F, G) \cong C(E, C(F, G))$$

fails, even in the category of Banach spaces. Here we will see that for smooth functions, the situation is much better, even for Fréchet spaces.

We have seen the for normed spaces, Gâteaux and Fréchet smoothness are the same. For Fréchet spaces, there is another notion of smoothness that is equivalent to Gâteaux smoothness, and is useful for discussing smoothness in more general spaces:

Proposition 2.8 Let E, F be Fréchet spaces, and let $U \subset E$ be open. A map $f : U \to F$ is Gâteaux smooth if and only if $f \circ c : \mathbb{R} \to F$ is smooth for every smooth $c : \mathbb{R} \to E$.

We will not prove this here.

From now on, by smoothness we will mean exactly that:

Definition 2.9 A map $f : U \subset E \to F$ between a Fréchet space E and a locally-convex t.v.s. F will be called smooth if $f \circ c : \mathbb{R} \to F$ is smooth for every smooth $c : \mathbb{R} \to E$.

The fact that smoothness can be tested by composition with smooth curves will expedite calculations. In particular, we have the following useful property:

Proposition 2.10 If E, F, G are finite dimensional normed space, and $\alpha \in C^\infty(F, G)$, then the map $C^\infty(E, F) \to C^\infty(E, G)$ defined by $f \mapsto \alpha \circ f$ is smooth, where $C^\infty(F, G)$ is with its standard Fréchet topology.

We also have the following useful property (see [KM97, Cor. 3.13] for a more general setup):

Proposition 2.11 *Let E, F, G be finite dimensional normed spaces, then a map $f : E \times F \to G$ is smooth, if and only if $f : E \to C^\infty(F, G)$ is smooth, where $C^\infty(F, G)$ is with its standard Fréchet topology.*

In fact, this also holds if E, F, G are Fréchet spaces, provided we have the correct topology on $C^\infty(E, F)$, as follows.[3]

We denote by $C^\infty(U, F)$ the space of smooth functions from $U \subset E$ to F. It is endowed by the topology induced by all the maps $c^* : C^\infty(U, F) \to C^\infty(\mathbb{R}, F)$, $f \mapsto f \circ c$ for $c \in C^\infty(\mathbb{R}, U)$. This is, in general, not a Fréchet space, however it is a locally-convex t.v.s. [KM97, §3.11].

2.2 Hilbert, Banach and Fréchet manifolds

Definition 2.12 *A smooth **Hilbert/Banach/Fréchet manifold** is a set $\mathcal{M}$ together with the following data:*

1. *A cover of $\mathcal{M}$ by subsets $(U_\alpha)_{\alpha\in A}$, and*
2. *For each $\alpha \in A$, an injective functions $u_\alpha : U_\alpha \to E_\alpha$ with values in a Hilbert/Banach/Fréchet space E_α, such that*
3. *For all $\alpha, \beta \in A$, the image $u_\alpha(U_{\alpha\beta})$ of the set $U_{\alpha\beta} := U_\alpha \cap U_\beta$ is open in E_α (in particular, $u_\alpha(U_\alpha)$ is open), and*
4. *The mapping $u_{\alpha\beta} := u_\alpha \circ u_\beta^{-1} : u_\beta(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$ is smooth.*

The spaces E_α are called modeling vector spaces of M,[4] the tuples (U_α, u_α) are called a charts, the mappings $u_{\alpha\beta}$ are called a chart changings, and the

[3] A remark to myself: There is no contradiction between Lemma II.3.2 and Section II.3.9 in the notes of Bauer, Harms and Michor for the case of E being a Fréchet space, because of a delicate issue with the product topology, which is finer in Section II.3.9 than in Lemma II.3.2. Note that the convenient topology induced on E^* from $C^\infty(E, \mathbb{R})$ is the same as standard functional analytic one (uniform convergence on bounded sets).

[4] Note that we do not assume that all E_α are the same space.

collection $(U_\alpha, u_\alpha)_{\alpha\in A}$ is called an atlas. Two atlases are called equivalent if their union is again an atlas. An equivalence class of atlases is sometimes called a manifold structure. The union of all atlases in an equivalence class is again an atlas, the maximal atlas for this manifold structure.

As in finite dimensional manifolds, the manifold structure induces a topology on $\mathcal{M}$ in the standard way — $A \subset \mathcal{M}$ is open if $u_\alpha(A \cap U_\alpha)$ is open in E_α for every α. Similarly, we can define continuous and smooth functions on $\mathcal{M}$. We can also define fiber-bundles over $\mathcal{M}$ and so on.

Definition 2.13 *Let $\mathcal{M}$ be a Fréchet manifold with an atlas $(u_\alpha : U_\alpha \to E_\alpha)_{\alpha\in A}$. The tangent bundle is a Fréchet manifold $T\mathcal{M}$ defined the set*

$$\coprod_{\alpha\in A} U_\alpha \times E_\alpha := \bigcup_{\alpha\in A} \{\alpha\} \times U_\alpha \times E_\alpha$$

modulo the relation

$$(\alpha, x, s) \sim (\beta, y, t) \quad \Leftrightarrow \quad x = y \quad \text{and} \quad du_{\alpha\beta}(u_\beta(x))t = s,$$

with the bundle maps

$$\psi_\alpha : \{\alpha\} \times U_\alpha \times E_\alpha \to U_\alpha \times E_\alpha, \qquad \psi_\alpha(\alpha, x, s) = (x, s).$$[5]

$T\mathcal{M}$ can be identified with $C^\infty(\mathbb{R}, \mathcal{M})$ modulo equivalence relations of curves, as usual, via the map $[c(t)] \mapsto [(\alpha, c(0), (u_\alpha \circ c)'(0))]$, for any α such that $c(0) \in U_\alpha$.

Proposition 2.14 *For every $x \in \mathcal{M}$, $\mathcal{M}$ is locally diffeomorphic to $T_x\mathcal{M}$. That is, we can consider $\mathcal{M}$ as modelled by $T_x\mathcal{M}$.*

Proof: Let $\alpha \in A$ such that $x \in U_\alpha$, then $T_x\mathcal{M}$ is diffeomorphic to E_α by construction. Therefore the coordinate chart $u_\alpha : U_\alpha \to E_\alpha$ defines the wanted diffeomorphism. ■

[5] This induces the manifold atlas on $T\mathcal{M}$ by composing with u_α.

Proposition 2.15 *A map $f : \mathcal{M} \to \mathcal{N}$ between two Fréchet manifolds is smooth if and only if $f \circ c : \mathbb{R} \to \mathcal{N}$ is smooth for any $c \in C^\infty(\mathbb{R}, \mathcal{M})$.*

This follows from the definitions of smoothness via charts, and the fact that for the Fréchet spaces that appear in the charts, smoothness can be tested via smooth curves (see Proposition 2.8). See III.3.8.

2.3 Manifolds of mappings

Let M be a compact, finite dimensional Riemannian manifold, and let $\pi : V \to M$ be a vector bundle over M. We denote by $\Gamma(M;V)$ the vector space of smooth sections. It is a Fréchet space with the topology of uniform convergence of all derivatives (separately).

Now let M be a compact finite dimensional Riemannian manifold, and let N be a finite dimensional Riemannian manifold. We start by showing that $C^\infty(M,N)$, with its natural topology (convergence of all derivatives, each one uniformly[6]), is a Fréchet manifold.[7]

There exists a neighborhood W_0 of the zero section of TN such that the map

$$(\pi_N, \exp) : W_0 \to N \times N, \qquad (\pi_N, \exp)(w_p) = (p, \exp_p(w))$$

is a diffeomorphism onto its image, which we denote by $W_{N\times N}$. Given $f \in C^\infty(M,N)$, define the inverse charts (V_f, v_f) by

$$\begin{aligned} &V_f := \{h \in \Gamma(f^*TN) \;:\; h(M) \subset W_0\} \\ &v_f : V_f \to C^\infty(M,N), \quad v_f(h)(x) := \exp_{f(x)} h(x). \end{aligned}$$

The chart (U_f, u_f) is then given by

$$\begin{aligned} &U_f := v_f(V_f) = \{g \in C^\infty(N,M) \;:\; (f,g)(M) \subset W_{N\times N}\} \\ &u_f : U_f \to \Gamma(f^*TN), \quad u_f(g)(x) = \exp^{-1}_{f(x)}(g(x)). \end{aligned}$$

[6] Here, uniform convergence can be viewed either in coordinate charts, or via jets.

[7] The fact that $C^\infty(M,N)$ is a manifold is true in much greater generality (N needs not to be finite dimensional, M needs not to be compact), however the manifold may not be Fréchet in this case, and there are delicate topological issues (the manifold topology on $C^\infty(M,N)$ in the non-compact case differs from the ∞-compact-open or ∞-whitney topologies, and we will not get into it here. See [KM97, Chapter IX] for more details.

Proposition 2.16 *The charts $(U_f, u_f)_{f\in C^\infty(M,N)}$ define a manifold structure on $C^\infty(M,N)$. The topology of the manifold is that of uniform convergence of functions and each of their derivatives.*

Corollary 2.17

1. *Following the discussion above on the tangent space (see Proposition 2.14) we can identify $T_f C^\infty(M,N) \cong \Gamma(f^*TN)$. Moreover, we can identify the whole tangent space as $TC^\infty(M,N) \cong C^\infty(M,TN)$; this identification is not only as sets but also as manifolds.*

2. *Following Proposition 2.11, we have that if M is a compact manifold, and S, N finite-dimensional manifolds, then a map $F : S \to C^\infty(M,N)$ if and only if the associated map $F^\wedge : S \times M \to N$ is smooth.*

Proof: Note that v_f is injective by the definition of V_f and W_0. For given $f, \bar{f} \in C^\infty(M,N)$, consider the set

$$U_f \cap U_{\bar{f}} = \{g \in C^\infty(N,M) \ : \ (f,g)(M), (\bar{f},g)(M) \subset W_{N\times N}\}.$$

If $g \in U_f \cap U_{\bar{f}}$ and g' is sufficiently close to g in C^0, then $g' \in U_f \cap U_{\bar{f}}$ as well, as $W_{N\times N}$ is open. It follows that $u_f(U_f \cap U_{\bar{f}})$ is open in $\Gamma(f^*TN)$ (since the topology on $\Gamma(f^*TN)$ is stronger than C^0).

We now show that the transition maps are smooth: To this end, define, for a given $f \in C^\infty(M,N)$, the sets

$$\begin{aligned} V_f^{f^*TN} &:= \{w \in f^*TN \ : \ (p^*f)(w) \in W_0\} \\ V_f^{M\times N} &:= \{(x,y) \in M \times N \ : \ (f(x), y) \in W_{N\times N}\}, \end{aligned}$$

where $p : f^*TN \to M$ is the projection (and thus $(p^*f)(x,w) = (f(x), w)$ is simply the canonical identification of $(f^*TN)_x \cong (TN)_{f(x)}$). Then the map

$$\Sigma_f : V_f^{f^*TN} \to V_f^{M\times N}, \quad \Sigma_f(w) = (\pi(w), \exp_{f(\pi(w))} w),$$

is obviously smooth and invertible. Now, we have that

$$u_{\bar{f}} \circ u_f^{-1} : u_f(U_f \cap U_{\bar{f}}) \to u_{\bar{f}}(U_f \cap U_{\bar{f}})$$

is simply $u_{\bar{f}} \circ u_f^{-1}(h) = \Sigma_{\bar{f}}^{-1} \circ \Sigma_f \circ h$, and since the composition operator with smooth maps is smooth (see Proposition 2.10), $u_{\bar{f}} \circ u_f^{-1}$ is smooth.

The fact that the topology is that of uniform convergence of the maps and all derivatives follows from the natural topology on $\Gamma(f^*TN)$ and the fact that the exponential map is smooth. ∎

One can generalize this result is for the case where N is not necessarily finite dimension: this will be useful when we consider the path space in an infinite dimensional manifold $\mathcal{M}$, when we discuss geodesics. For this, we need to assume that N has a **local addition**:

Definition 2.18 *(Local addition) Let N be a smooth manifold. A local addition on N is a smooth mapping $\Sigma : W_{TN} \to N$ fitting into the following commutative diagram:*

$$\begin{tikzcd}
TN & W_{TN} \arrow[l, hook', "\textit{open}"] \arrow[rr, "(\pi_N{,}\Sigma)", "\cong"'] & & W_{N\times N} \arrow[r, hook, "\textit{open}"'] & N \times N \\
& & N \arrow[ul, hook', "\textit{zero section}"] \arrow[ur, hook, "\textit{diagonal}"'] & &
\end{tikzcd}$$

For finite dimensional manifolds, Σ can be taken to be the exponential map of any Riemannian metric. Moreover, any Lie group G admits a local addition, since $TG \cong G \times T_eG$ via $(g, X) \mapsto TL_g(X)$, and a neighborhood of $0 \in T_eG$ is diffeomorphic to a neighborhood of $e \in G$. Once we have a local addition, the previous proof works verbatim.

This construction is, in fact, a special case of smooth fiber bundles sections, $\Gamma(M; B)$ of a smooth bundle $\pi : B \to M$ over a compact M (see [Ham82, §I.4]). The idea is similar: B takes the role of $M \times N$, and instead of f^*TN we have the bundle $f^*VB \to M$ consisting of all vectors of the vertical bundle $VB = \ker(T\pi) \subset TB$ that lie above f. In order to prove the manifold structure, all we need is to have, instead of the exponential map (or a local addition), is a map $\Sigma : W_{VB} \subset VB \to B$, such that the following commutative diagram holds:

$$\begin{tikzcd}
VB & W_{VB} \arrow[l, hook', "\text{open}"] \arrow[rr, "(\pi_N{,}\Sigma)", "\cong"'] & & W_{B\times B} \arrow[r, hook, "\text{open}"'] & B \times B \\
& & B \arrow[ul, hook', "\text{zero section}"] \arrow[ur, hook, "\text{diagonal}"'] & &
\end{tikzcd}$$

Example 2.19 We may consider also the space $\mathrm{met}(M)$ of all Riemannian metrics over a compact M, since the positively definite matrices is open in the symmetric matrices.

2.3.1 Immersions, embeddings, diffeomorphisms

Theorem 2.20 *Let M be a compact manifold, and let N be a finite-dimensional manifold with $\dim(M) \leq \dim(N)$.*

1. *The set $\mathrm{Imm}(M,N)$ of all smooth functions $f \in C^\infty(M,N)$ whose differential $T_x f : T_x M \to T_{f(x)} N$ is injective at every point $x \in M$ is an open subset of the Fréchet manifold $C^\infty(M,N)$.*

2. *The set $\mathrm{Emb}(M,N)$ of all immersions which are a homeomorphism onto their range is an open subset of the Fréchet manifold $\mathrm{Imm}(M,N)$.*

3. *The set $\mathrm{Diff}(M)$ of all smooth mappings $M \to M$ with smooth inverse coincides with the Fréchet manifold $\mathrm{Emb}(M,M)$ and is a Lie group, i.e., inversion and composition are smooth.*

Proof:

1. The set $\mathrm{Imm}(M,N)$ is open in $C^\infty(M,N)$ because the manifold topology of $C^\infty(M,N)$ is finer than the topology of uniform convergence of the first spatial derivative.

2. The manifold topology of $C^\infty(M,N)$ coincides with the final topology with respect to the set of all smooth curves. Thus, a set is open in $C^\infty(M,N)$ if and only if its pre-images under all smooth curves are open. Let $c : \mathbb{R} \to \mathrm{Imm}(M,N)$ be a smooth curve such that $c_0 := c(0)$ is an embedding. We have to show that then $c(t)$ remains an embedding for small t. Note that an injective immersion $f : M \to N$ is a homeomorphism as a map $M \to f(M)$, since M is compact: Indeed, let $U \subset M$ be open, then U^c is compact, hence $f(U^c)$ is compact and in particular closed, hence $f(U) = f(U^c)^c$ is open.

   Thus, it is sufficient to prove that the mapping $c(t)$ stays injective for t near 0: Otherwise, there are $t_n \to 0$ and $x_n \neq y_n$ in M with $c(t_n)(x_n) = c(t_n)(y_n)$. Passing to subsequences we may assume that $x_n \to x$ and $y_n \to y$ in M. Since c as a map $\mathbb{R} \times M \to N$ is also continuous, we get $c_0(x) = c_0(y)$, so $x = y$. The mapping $(t,z) \mapsto (t, c(t)(z))$

is a diffeomorphism near $(0, x)$, since it is an immersion. But then $c(t_n)(x_n) \neq c(t_n)(y_n)$ for large n, a contradiction.

3. The sets $\mathrm{Diff}(M)$ and $\mathrm{Emb}(M, M)$ coincide: M is the disjoint union of its connected components M_i, which are compact manifolds. For any $f \in \mathrm{Emb}(M, M)$, the set $f(M_i)$ is open since f is a local diffeomorphism and closed since M_i is compact. Thus, $f(M_i)$ equals all of M_j for some j. As f is injective, no two connected components are mapped into the same connected component. As M has only finitely many connected components, every connected component appears as the image of some connected component. Thus, f is surjective. The inverse of f is smooth by the inverse function theorem. Thus, f is a diffeomorphism.

   Composition $\mathrm{comp} : \mathrm{Diff}(M) \times \mathrm{Diff}(M) \to \mathrm{Diff}(M)$ is smooth: Consider two smooth curves $f, g : \mathbb{R} \to \mathrm{Diff}(M)$. Because of Proposition 2.15, it is sufficient to prove that $f \circ g : \mathbb{R} \to \mathrm{Diff}(M)$ is a smooth map. As discussed in Corollary 2.17, we can identify f and g with smooth maps $f^\wedge, g^\wedge : \mathbb{R}_+ \times M \to M$. But then, by standard composition rule for finite-dimensional manifolds, the map $\mathbb{R} \times M \ni (t, x) \mapsto f^\wedge(t, g^\wedge(t, x)) \in M$ is smooth. We can now do the same identification in the other direction, and identify this map with a smooth map $\mathbb{R} \ni t \mapsto f(t) \circ g(t) \in \mathrm{Diff}(M)$. Thus, the composition map is smooth along smooth curves. Therefore, it is smooth.

   Inversion $\mathrm{inv} : \mathrm{Diff}(M) \to \mathrm{Diff}(M)$ is smooth: Consider a smooth curve $f : \mathbb{R} \to \mathrm{Diff}(M)$. Equivalently, by the exponential law of convenient calculus, the map $f^\wedge : \mathbb{R} \times M \to M \ni (t, x) \mapsto f(t)(x)$ is smooth. Let $g := \mathrm{inv} \circ f : \mathbb{R} \to \mathrm{Diff}(M)$ be the inverse of f. Then $y := g^\wedge(t, x)$ satisfies the implicit equation $f^\wedge(t, y) - x = 0$. Note that the left-hand side has a non-degenerate ∂_y-derivative because $f^\wedge(t, \cdot)$ is a diffeomorphism. Therefore, the implicit function theorem (in finite dimensions!) implies that y depends smoothly on (t, x), i.e., $g^\wedge$ is smooth. Thus, by the exponential law of convenient calculus, g is smooth. We have shown that the inversion map is smooth along smooth curves. Therefore, it is smooth.

■

2.4 Weak and strong Riemannian metrics

Definition 2.21 (Weak and strong Riemannian metrics) A weak Riemannian metric *on a manifold $\mathcal{M}$ is a smooth map $g : T\mathcal{M} \times_{\mathcal{M}} T\mathcal{M} \to \mathbb{R}$ such that g_x is an inner product on $T_x\mathcal{M}$ for every $x \in \mathcal{M}$. The metric g is called* strong *if the inner product g_x induces the locally convex topology on $T_x\mathcal{M}$ for every $x \in \mathcal{M}$.*

The following result shows the scarcity of strong Riemannian metrics in infinite dimensions and thus motivates the study of weak Riemannian geometry.

Theorem 2.22 (Strong Riemannian metrics) Let g be a weak Riemannian metric on a manifold $\mathcal{M}$. Then the following are equivalent:

1. *g is a strong Riemannian metric on $\mathcal{M}$.*
2. *$\mathcal{M}$ is a Hilbert manifold and $g^\vee : T\mathcal{M} \to T^*\mathcal{M}$ is surjective.*
3. *$\mathcal{M}$ is a Hilbert manifold and $g^\vee : T\mathcal{M} \to T^*\mathcal{M}$ is a vector bundle isomorphism.*

Proof: (1) ⇒ (2). Fix a point $x \in \mathcal{M}$. By assumption, the locally convex topology of $T_x\mathcal{M}$ is normable by $\|\cdot\|_{g_x}$. This norm is complete since $T_x\mathcal{M}$ is a Fréchet space. Thus, $(T_x\mathcal{M}, g_x)$ is a Hilbert space. This implies that $\mathcal{M}$ is a Hilbert manifold because $\mathcal{M}$ is locally diffeomorphic to $T_x\mathcal{M}$, as is easily seen in a chart (Proposition 2.14). Moreover, $g^\vee : T\mathcal{M} \to T^*\mathcal{M}$ is surjective by the Riesz representation theorem.

(2) ⇒ (3) follows from the open mapping theorem.

(3) ⇒ (1). Fix a point $x \in \mathcal{M}$ and define $U := \{u \in T_x\mathcal{M} : \|u\|_{g_x} < 1\}$. Then U is absolutely convex (convex and balanced). As $g_x^\vee$ is surjective, every bounded linear functional $\varphi \in T_x^*\mathcal{M}$ is of the form $\varphi(u) = g_x(u, v)$ for some $v \in T_x\mathcal{M}$. Every such functional is bounded on U because $\sup_{u \in U} |\varphi(u)| = |g(v, u)| \le \|v\|_g$. A weakly-bounded set in a locally convex t.v.s. is bounded (see [KM97, 52.19]), hence U is bounded in $T_x\mathcal{M}$. Since g_x is smooth, it is in particular continuous; thus the norm $\|\cdot\|_{g_x}$ is continuous, and consequently U

is open in $T_x\mathcal{M}$. Now U is an absolutely convex and bounded 0-neighborhood in $T_x\mathcal{M}$, and is thus $T_x\mathcal{M}$ is normable, with $\mu_U(v) \coloneqq \inf\{t > 0 : v \in tU\}$ a norm on $T_x\mathcal{M}$. As $\mu_U = \|\cdot\|_{g_x}$, the topology induced by g coincides with the topology of $T_x\mathcal{M}$. This holds for all $x \in \mathcal{M}$, and therefore g is a strong Riemannian metric. ■

2.4.1 Geodesic equation and the Christoffel symbol

In this section we discuss the geodesic equation of a Riemannian manifold $(\mathcal{M}, g)$. Recall that we can define the geodesic in two different ways:

- As a curve $c : [0,1] \to \mathcal{M}$ which is of zero acceleration:

$$\nabla_{\dot{c}}\dot{c} = 0. \tag{2.1}$$

  For this we need to define the **Levi-Civita** connection of the metric, that is, an operator $\mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ that satisfies the connection axioms, as well as being symmetric and compatible with the metric g.

- As a critical point of the (kinetic) energy,

$$E(c) \coloneqq \frac{1}{2}\int_0^1 g_c(\dot{c}, \dot{c})\mathrm{d}t, \tag{2.2}$$

  with fixed endpoints at times $t = 0$ and $t = 1$.

Much of the theory in this case is similar to the theory in finite dimensions, with a big caveat: For weak Riemannian metrics, the **Levi-Civita connection may fail to exist**. The easiest way to see it is in coordinate charts — recall that if $\mathcal{M}$ is a finite dimensional manifold, then we have that the Levi-Civita connection can be defined by

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k, \qquad \Gamma^k_{ij} = \frac{1}{2}g^{km}(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}).$$

The problem lies in multiplication with g^{km}, which is basically using the isomorphism $T^{\mathcal{M}} \to T\mathcal{M}$ induced by the metric. However, this isomorphism does not exist in general for weak Riemannian metrics. Nevertheless, for a lot of Riemannian metrics the Christoffel symbols, and thus the Levi-Civita connection, do exist.

Let us begin by seeing how the Christoffel symbols are defined for infinite-dimensional manifolds. Christoffel symbols, are, as usual, not intrinsic objects; they "live" in a coordinate system, and as such, we will define them on charts, that is, on manifolds that are open subsets of Fréchet spaces.

Definition 2.23 (Christoffel symbol) *Let $\mathcal{M}$ be an open subset of a Fréchet space F, endowed with a weak Riemannian metric g. Then the Christoffel symbol, provided it exists, is defined as the unique map $\Gamma : \mathcal{M} \times F \times F \to F$ such that*

$$g_x(\Gamma_x(u,v),w) = \frac{1}{2}(D_{x,u}g)(v,w) + \frac{1}{2}(D_{x,v}g)(u,w) - \frac{1}{2}(D_{x,w}g)(u,v)$$

holds for all $x \in M$ and $u,v,w \in F$, where $D_{x,u}$ is the directional derivative at x in the direction u.

The Christoffel symbol is defined precisely to make the following theorem work.

Theorem 2.24 (Geodesic equation) *Let x,y be points in an open connected subset $\mathcal{M}$ of a Fréchet space. Then the kinetic energy E is a smooth function on the path space $C^\infty_{x,y}([0,1],\mathcal{M})$, and its derivative satisfies for any $m \in T_c C^\infty_{x,y}([0,1],\mathcal{M})$ that*

$$\mathrm{d}E(c).m = \int_0^1 \Big(-g_c(c_{tt},m) - (D_{c,c_t}g)(c_t,m) + \frac{1}{2}(D_{c,m}g)(c_t,c_t)\Big)\mathrm{d}t,$$

where $c_t = \partial_t c$ and $c_{tt} = \partial_t^2 c$. If the Christoffel symbol Γ exists, this can be rewritten as

$$\mathrm{d}E(c).m = \int_0^1 g_c(-c_{tt} - \Gamma_c(c_t,c_t),m)\mathrm{d}t.$$

Thus, in this case the first-order optimality condition $\mathrm{d}E(c) = 0$ is equivalent to the geodesic equation

$$c_{tt} + \Gamma_c(c_t,c_t) = 0. \tag{2.3}$$

Here $C^\infty_{x,y}([0,1],\mathcal{M})$ is the space of all smooth curves c with $c(0) = x$ and $c(1) = y$. It is a manifold with $T_c C^\infty_{x,y}([0,1],\mathcal{M})$ being the subspace of the vector fields $C^\infty(c^*T\mathcal{M})$ that vanish at $t = 0,1$.

In other words, the theorem states that if the Christoffel symbol does not exist, the geodesic equation exists only in a weak form.

Proof: By the "kinetic" interpretation of tangent vectors as equivalence relations of curves, the derivative of E can be computed along a variational family $c : \mathbb{R} \to C^\infty_{x,y}([0,1], M)$, where $c(0)$ corresponds to the given curve and $c'(0)$ to the tangent vector m in the statement of the theorem. As discussed Corollary 2.17(2), we can identify this variational family with a smooth function $c : \mathbb{R} \times [0,1] \to M$ subject to the boundary conditions $c(\cdot, 0) = x$ and $c(\cdot, 1) = y$. Then

$$\begin{aligned}\partial_s|_0 E(c(s,\ )) &= \frac{1}{2}\partial_s|_0 \int_0^1 g_c(c_t, c_t)\, dt = \int_0^1 \Big(\tfrac{1}{2}(D_{c,c_s} g)(c_t, c_t) + g_c(c_{st}, c_t)\Big)\, dt \\ &= \int_0^1 \Big(\tfrac{1}{2}(D_{c,c_s} g)(c_t, c_t) - (D_{c,c_t} g)(c_s, c_t) - g_c(c_s, c_{tt})\Big)\, dt \\ &= \int_0^1 g_c(c_s, -c_{tt} - \Gamma_c(c_t, c_t))\, dt,\end{aligned}$$

where the step from the first to the second line uses integration by parts with respect to t, and the step from the second to the third line uses the definition of the Christoffel symbol, provided it exists. ■

Lemma 2.25 (Existence of the Christoffel symbol) *Let M be an open subset of a Fréchet space, endowed with a Riemannian metric g.*

1. *If g is a weak Riemannian metric then the Christoffel symbol is unique but may fail to exist.*
2. *If g is a strong Riemannian metric, then the Christoffel symbol exists.*

Proof: Uniqueness of the Christoffel symbol is guaranteed by the injectivity of the metric $g_x^\vee : T_x M \to T_x^* M$. Existence of the Christoffel symbol may fail for weak Riemannian metrics due to the non-surjectivity of $g_x^\vee$; see Example 2.26. This cannot happen for strong Riemannian metrics, where the map $g_x^\vee$ is invertible. ■

Example 2.26 (Non-existence of the Christoffel symbol) For any sequence $m : \mathbb{N} \to (0, \infty)$ with $\lim_{i\to\infty} m_i = 0$, the Christoffel symbol of the following weak Riemannian metric on ℓ^2 does not exist:

$$g_x(u,v) := e^{-\frac{1}{2}\|x\|^2_{\ell^2}} \langle u,v\rangle_{\ell^2_m} := e^{-\frac{1}{2}\|x\|^2_{\ell^2}} \sum_i m_i u_i v_i, \qquad x \in \ell^2, \quad u,v \in T_x\ell^2 = \ell^2.$$

To see this, note that the directional derivative of the metric with respect to the foot point is given by

$$D_{x,w} g_x(u,v) = -\langle x,w\rangle_{\ell^2}\ g_x(u,v) = -\langle u,v\rangle_{\ell^2_m}\ g_x(m^{-1}x, w),$$

provided that the element-wise product $m^{-1}x$ belongs to ℓ^2. Thus, the Christoffel symbol, if it exists, satisfies

$$\begin{aligned} g_x(\Gamma_x(u,v),w) &= \frac{1}{2}(D_{x,u}g)(v,w)\frac{1}{2}(D_{x,v}g)(u,w) - \frac{1}{2}(D_{x,w}g)(u,v), \\ \Gamma_x(u,v) &= -\frac{1}{2}\langle x,u\rangle_{\ell^2}\ v - \frac{1}{2}\langle x,v\rangle_{\ell^2}\ u + \frac{1}{2}\langle u,v\rangle_{\ell^2_m}\ m^{-1}x. \end{aligned}$$

Thus, at any point $x \in \ell^2$ such that $m^{-1}x$ is not square integrable, the right-hand side above does not belong to ℓ^2, and consequently the Christoffel symbol does not exist. However, the Christoffel symbol does exist if the same metric is considered on the subspace of rapidly decreasing sequences, provided that m has polynomial decay. Similarly, as shall be seen later on, the Christoffel symbol of Sobolev metrics of order $< 1/2$ on groups of non-smooth diffeomorphisms fails to exist, whereas it exists on groups of smooth diffeomorphisms.

Example 2.27 Consider the right-invariant L^2 metric on $\mathrm{Diff}(S^1)$: For $u, v \in T_\varphi \mathrm{Diff}(S^1)$,

$$g_\varphi(u,v) = \int_{S^1} u \circ \varphi^{-1} v \circ \varphi^{-1}\, dx = \int_{S^1} uv\varphi_x\, dx,$$

where we identify S^1 with the interval $[0,1]$ so that $T_\varphi \mathrm{Diff}(S^1) \cong C^\infty(S^1)$. This is obviously a weak metric. This identification induced a coordinate system on $\mathrm{Diff}(S^1)$, in which we calculate the Christoffel symbol and show it exists:

$$(D_{\varphi,u}g)(v,w) = \lim_{s\to 0} \frac{1}{s}\left(\int_{S^1} vw(\varphi + su)_x\, dx - \int_{S^1} vw\varphi_x\, dx \right) = \int_{S^1} vwu_x\, dx$$

and thus

$$g_\varphi(\Gamma_\varphi(u,v),w) = \frac{1}{2}\int_{S^1}(vwu_x+uwv_x-uvw_x)\,dx = \int_{S^1}(vu_x+uv_x)w\,dx = g_\varphi\left(\frac{vu_x+uv_x}{\varphi_x},w\right)$$

hence

$$\Gamma_\varphi(u,v) = \frac{vu_x+uv_x}{\varphi_x}.$$

This tells us that the Christoffel symbol exists on $\mathrm{Diff}(S^1)$, but also that it does not exist in $\mathrm{Diff}^{H^k}(S^1)$ for any k, since if $u,v,\varphi \in H^k$, then $\Gamma_\varphi(u,v)$ is only in H^{k-1}. The geodesic equation is given by

$$\varphi_{tt} + 2\frac{\varphi_{tx}}{\varphi_x}\varphi_t = 0.$$

In Eulerian coordinates, using

$$\varphi_t = u\circ\varphi, \quad \varphi_{tx} = u_x\circ\varphi\varphi_x, \quad \varphi_{tt} = u_t\circ\varphi + u_x\circ\varphi\varphi_t,$$

we obtain

$$u_t + 3uu_x = 0,$$

the Burgers' equation.

Example 2.28 (Geodesic equations of right-invariant metrics on Lie groups) The previous example is a special case of right-invariant metrics on Lie groups:[8] Let G be a Lie group, with a Lie algebra $\mathfrak{g}$. We then have that $T_gG \sim \mathfrak{g}g$. Given an inner product $\langle\,,\,\rangle$ on $\mathfrak{g}$, we obtain a smooth **right-invariant metric** on G by

$$\langle\xi,\eta\rangle_g := \left\langle\xi\circ g^{-1},\eta\circ g^{-1}\right\rangle.$$

For $u\in G$, denote by $\mathrm{ad}_u : \mathfrak{g}\to\mathfrak{g}$ the adjoint operator $\mathrm{ad}_u(v) := [u,v]$. We then have that the geodesic equation, in a weak form, is

$$\int_0^1 \langle u_t,v\rangle + \langle u,\mathrm{ad}_u(v)\rangle\,dt = 0 \quad \forall v\in C_0^\infty((0,1);\mathfrak{g})$$

for every $u(t) = g_t(t)g(t)^{-1}$ is the velocity. If the adjoint operator has a well defined adjoint relative to $\langle\,,\,\rangle$ is well-defined, that is, if $\langle\mathrm{ad}_u v,w\rangle = \left\langle v,\mathrm{ad}_u^T w\right\rangle$, we obtain the strong form

$$u_t + \mathrm{ad}_u^T u = 0. \tag{2.4}$$

[8]The presentation below borrows from [Kol17, §1]. See also [KW09, §I.4.3]

This equation was first introduced by Arnold in [Arn66] (for left-invariant metrics, but this is the same up to a sign). This equation has an integral version, known as *conservation of momemtum*:

$$\mathrm{Ad}^T_{g(t)} u(t) = u_0,$$

where $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is defined by $\mathrm{Ad}_g := TL_g \circ TR_{g^{-1}}$.

Let us obtain (2.4), assuming that ad^T exists: Let $g(0,t)$ be a geodesic, and let $g(s,t)$ be a variation of it, with $g_s(0,t)g^{-1}(0,t) = v$, $g_t g^{-1} = u$. We then have

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\Big|_{s=0} \int_0^1 \left\langle g_t g^{-1}, g_t g^{-1} \right\rangle dt &= \int_0^1 \left\langle (g_t g^{-1})_s, g_t g^{-1} \right\rangle dt \\
&= \int_0^1 \left\langle (g_s g^{-1})_t + [g_s g^{-1}, g_t g^{-1}], g_t g^{-1} \right\rangle dt \\
&= \int_0^1 -\left\langle g_s g^{-1}, (g_t g^{-1})_t \right\rangle + \left\langle [g_s g^{-1}, g_t g^{-1}], g_t g^{-1} \right\rangle dt \\
&= \int_0^1 -\langle v, u_t \rangle + \langle [v,u], u \rangle \, dt = -\int_0^1 \langle v, u_t \rangle + \langle \mathrm{ad}_u v, u \rangle \, dt \\
&= -\int_0^1 \left\langle v, u_t + \mathrm{ad}^T_u u \right\rangle dt.
\end{aligned}$$

In the transition to the second line we used the fact that

$$(g_t g^{-1})_s - (g_s g^{-1})_t = [g_s g^{-1}, g_t g^{-1}].$$

For matrix Lie groups, this follows from the fact that $(g^{-1})_t = -g^{-1} g_t g^{-1}$ and similarly for the s derivative, and the fact that $g_{ts} = g_{st}$. For genera Lie groups this is more complicated, see [BKMR96, Proposition 5.1]

Note that the existence of ad^T_u is different, and is not implied by the existence of the Christoffel symbol (or the Levi-Civita connection, see below). In fact, in most of the cases that will interest us, it rarely exists. For example, in the case of $G = \mathrm{Diff}(S^1)$, we have $\mathfrak{g} = \mathfrak{X}(S^1)$ and thus $[u,v] = uv_x - vu_x$. For the L^2 metric, we have

$$\langle w, [u,v] \rangle = \int_{S^1} w(uv_x - vu_x)\, dx = \int_{S^1} (-(wu)_x - wu_x) v \, dx = \int_{S^1} (-w_x u - 2wu_x) v \, dx,$$

and thus $\mathrm{ad}^T_u(w) = -w_x u - 2wu_x$, and indeed we retrieve the Burgers equation (There is a sign error somewhere here). On the other hand, for the right-invariant H^1-metric

$$\langle u, v \rangle := \int_{S^1} uv + u_x v_x \, dx,$$

we have

$$\langle u, [u, v]\rangle = \int_{S^1} u(uv_x - vu_x) + u_x(uv_x - vu_x)_x \, dx$$

and it is not so trivial how to write this as $\int_{S^1} f(u)v + f(u)_x v_x \, dx$ for some function $f(u)$ (though it is possible, as we will later see, see (4.7) and the footnote that follows). Nevertheless, we will see later in § 4.3 that the Christoffel symbol exists and can be calculated in this case (also for finite smoothness, in which $\mathrm{ad}_u^T u$ does not exist).

We now define the Levi-Civita connection using the Christoffel symbol. Alternatively, it can be defined as a symmetric, metrically-compatible connection, showing that if it exists, than it is unique (using the Koszul formula).

Definition 2.29 Let $(\mathcal{M}, g)$ be a Riemannian manifold such that the Christoffel symbol Γ^α exists in every chart (U_α, u_α) of some smooth atlas. Then, the ***Levi-Civita covariant derivative*** *$\nabla_X Y \in \mathfrak{X}(M)$ of $Y \in \mathfrak{X}(M)$ in the direction of $X \in \mathfrak{X}(M)$ is defined as*[9]

$$\nabla_X Y(x) = (x, \mathrm{d}\bar{Y}(x).\bar{X}(x) + \Gamma_x(\bar{Y}(x), \bar{X}(x))),$$

where $X(x) = (x, \bar{X}(x))$ and $Y(x) = (x, \bar{Y}(x))$ for any $x \in M$.

✎ *Exercise 2.1* Verify that $\nabla_X Y$ is indeed a vector field (i.e., that it does not depend on the chart), and that the geodesic equation is indeed given by (2.1).

2.4.2 Exponential map

Recall that a vector field on a manifold $\mathcal{M}$ is a section of the tangent bundle $T\mathcal{M}$.

Definition 2.30 (Integral curves and local flows) Let X be a vector field on a manifold $\mathcal{M}$. An integral curve of X is a smooth curve $c : J \to \mathcal{M}$ defined on an interval $J \subseteq \mathbb{R}$ such that $c'(t) = X(c(t))$ holds for all $t \in \mathbb{R}$. A local flow of X is a smooth mapping $\mathrm{Fl}^X : \mathcal{M} \times \mathbb{R} \supseteq U \to \mathcal{M}$ defined on an open neighborhood U of $\mathcal{M} \times \{0\}$ such that

[9] $\nabla_X Y$ can also be defined via the **connector** $K : TTM \to TM$, defined locally by $K(x, h; k, \ell) := (x, \ell - \Gamma_x(k, h))$.

1. *$U \cap (\{x\} \times \mathbb{R})$ is a connected open interval.*
2. *If $\mathrm{Fl}^X_s(x)$ exists then $\mathrm{Fl}^X_{t+s}(x)$ exists if and only if $\mathrm{Fl}^X_t(\mathrm{Fl}^X_s(x))$ exists, and we have equality.*
3. *$\mathrm{Fl}^X_0(x) = x$ for all $x \in \mathcal{M}$.*
4. *$\frac{d}{dt}\mathrm{Fl}^X_t(x) = X(\mathrm{Fl}^X_t(x))$.*

In formulas similar to (4) we will often omit the point x for sake of brevity, without signalizing some differentiation in a space of mappings. The following lemma summarizes some existence and uniqueness properties of local flows: while existence may fail beyond Banach manifolds, uniqueness is a consequence of the flow property.

Lemma 2.31 (Existence and uniqueness) Let X be a vector field on a manifold $\mathcal{M}$. If $\mathcal{M}$ is a Banach manifold, then X has a local flow, and if X is smooth, so is the flow. If X has a local flow, then every integral curve c of X satisfies $c(t) = \mathrm{Fl}^X_t(c(0))$ as long as both sides are well-defined. Thus, in this case there exists a unique maximal flow. Furthermore, X is Fl^X_t-related to itself, i.e., $T(\mathrm{Fl}^X_t) \circ X = X \circ \mathrm{Fl}^X_t$.

Proof: Existence of a local flow on Banach manifolds, and its smoothness, can be verified in charts using the Picard–Lindelöf theorem, and the standard argument for continuous dependence (used iteratively on derivatives of the ODE) on initial conditions using Grönwall's inequality.[10] Uniqueness follows from the computation

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{Fl}^X(-t, c(t)) &= -\tfrac{\mathrm{d}}{\mathrm{d}s}|_{s=-t}\mathrm{Fl}^X(s, c(t)) + \tfrac{\mathrm{d}}{\mathrm{d}s}|_{s=t}\mathrm{Fl}^X(-t, c(s)) \\ &= -\tfrac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\mathrm{Fl}^X_{-t}\mathrm{Fl}^X(s, c(t)) + T(\mathrm{Fl}^X_{-t}).c'(t) \\ &= -T(\mathrm{Fl}^X_{-t}).X(c(t)) + T(\mathrm{Fl}^X_{-t}).X(c(t)) = 0,\end{aligned}$$

which implies that $\mathrm{Fl}^X_{-t}(c(t)) = c(0)$ is constant, and therefore $c(t) = \mathrm{Fl}^X_t(c(0))$. The last assertion follows from $X \circ \mathrm{Fl}^X_t = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Fl}^X_t = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\mathrm{Fl}^X_{t+s} = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}(\mathrm{Fl}^X_t \circ \mathrm{Fl}^X_s) = T(\mathrm{Fl}^X_t) \circ \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\mathrm{Fl}^X_s = T(\mathrm{Fl}^X_t) \circ X$, where we omit the point $x \in \mathcal{M}$ for the sake of brevity. ■

[10] For more details see [Lan99, IV], in particular theorems 1.11 and 1.16.

The **geodesic spray** is an object, related to the connection and the Christoffel symbols, that enables us to define the geodesic equations as a flow. Generally, **spray** on $\mathcal{M}$ is a vector field on $T\mathcal{M}$ satisfying

$$T\pi_{\mathcal{M}} \circ S = \mathrm{Id}_{T\mathcal{M}}$$

and

$$S \circ m_t^{\mathcal{M}} = T(m_t^{\mathcal{M}}).m_t^{T\mathcal{M}}.S,$$

where $m_t^{\mathcal{N}} : T\mathcal{N} \to T\mathcal{N}$ is the scalar multiplication by t. In local coordinates, a spray is of the form

$$S(x,h) \coloneqq S(x,h) \coloneqq (x,h;h,\bar{\Gamma}_x(h)),$$

where $\bar{\Gamma}$ is quadratic in h (see [Mic20, Section 5.9]).

Definition 2.32 *Let $(\mathcal{M},g)$ be a Riemannian manifold such that the Christoffel symbol Γ^α exists in every chart (U_α,u_α) of some smooth atlas. Then, the* ***geodesic spray****, the vector field S on TM defined by*

$$S(x,h) \coloneqq (x,h;h,-\Gamma_x(h,h)).$$

✎ *Exercise 2.2* Show that S is indeed a spray (in particular, that it is invariant under coordinate changes and thus a vector field.

Assume that $(\mathcal{M},g)$ admits a geodesic spray (equiv., Christoffel symbols/Levi-Civita connection), and that the flow of the spray exists. Let $x \in \mathcal{M}$ and $v \in T_x\mathcal{M}$. Then, $\mathrm{Fl}_t^S(x,v) = (c(t),c_t(t))$, where c is a solution to the geodesic equation with initial data (x,v). Indeed, denote $\gamma(t) = \mathrm{Fl}_t^S(x,v)$, where $\gamma(t) = (x(t),h(t))$, then

$$(x,h,x_t,h_t) = \gamma_t = S(\gamma) = (x,h,h,-\Gamma_x(h,h)),$$

hence $x_{tt} + \Gamma_x(x_t,x_t) = 0$.

Definition 2.33 (Geodesic exponential map) *The exponential map of a spray S is defined as*

$$\exp(X) \coloneqq \pi_{\mathcal{M}}(\mathrm{Fl}_1^S(X)),$$

for all $X \in T\mathcal{M}$ such that the flow $\mathrm{Fl}_1^S(X)$ of the spray exists. The geodesic exponential map is the exponential map of the geodesic spray on a Riemannian manifold.

Theorem 2.34 (Geodesic exponential map) *Let $(\mathcal{M}, g)$ be a Riemannian manifold. If $\mathcal{M}$ is a Banach manifold and the spray exists and is smooth, then the exponential map is well-defined on a neighborhood U of the zero section in $T\mathcal{M}$, which may be chosen such that $(\pi_{\mathcal{M}}, \exp) : U \to \mathcal{M} \times \mathcal{M}$ is a diffeomorphism onto its range. In particular, this is always true for strong metrics on Hilbert manifolds.*

Proof: Assume the geodesic spray is a smooth vector field on the Banach manifold $T\mathcal{M}$. Thus, it has a local flow defined on some open set $V \subseteq T\mathcal{M} \times \mathbb{R}$ by Lemma 2.31. This local flow satisfies

$$\varepsilon \operatorname{Fl}^S_{\varepsilon t}(X) = \operatorname{Fl}^S_t(\varepsilon X), \qquad \varepsilon, t \in \mathbb{R}, \quad X \in T\mathcal{M},$$

whenever either side exists, thanks to the uniqueness of local flows and the following computation:

$$\begin{aligned} \tfrac{\partial}{\partial t} \varepsilon \operatorname{Fl}^S_{\varepsilon t}(X) &= \tfrac{\partial}{\partial t} m^{\mathcal{M}}_\varepsilon \operatorname{Fl}^S_{\varepsilon t}(X) = T(m^{\mathcal{M}}_\varepsilon) \tfrac{\partial}{\partial t} \operatorname{Fl}^S_{\varepsilon t}(X) \\ &= T(m^{\mathcal{M}}_\varepsilon) m^{T\mathcal{M}}_\varepsilon S(\operatorname{Fl}^S_{\varepsilon t}(X)) = S(m^{\mathcal{M}}_\varepsilon \operatorname{Fl}^S_{\varepsilon t}(X)) = S(\varepsilon \operatorname{Fl}^S_{\varepsilon t}(X)), \end{aligned}$$

where the second to last equality follows from the quadratic structure of sprays (recall that $m^{\mathcal{N}}_\varepsilon : T\mathcal{N} \to T\mathcal{N}$ is the scalar multiplication by ε). In particular, the open set $U = \pi_1(V \cap T\mathcal{M} \times \{1\})$, on which exp is defined, is a non-empty open set that contains the zero section of $T\mathcal{M}$.

Note that for $\exp_x := \exp|_{\pi^{-1}_{\mathcal{M}}\{x\} \cap U}$, the differential at 0_x is the identity: indeed, for $X \in T_x\mathcal{M}$, we have

$$\begin{aligned} \tfrac{d}{dt}|_{t=0} \pi_{\mathcal{M}}(\operatorname{Fl}^S_1(tX)) &= \tfrac{d}{dt}|_{t=0} \pi_{\mathcal{M}}(t \operatorname{Fl}^S_t(X)) = \tfrac{d}{dt}|_{t=0} \pi_{\mathcal{M}}(\operatorname{Fl}^S_t(X)) \\ &= T(\pi_{\mathcal{M}}).S(\operatorname{Fl}^S_0(X)) = T(\pi_{\mathcal{M}}).S(X) = X. \end{aligned}$$

where we used the properties of the spray described above. It follows that $(T\exp_x)_0 = \mathrm{Id}_{T_x\mathcal{M}}$. Thus, since $(\pi_{\mathcal{M}}, \exp)(x, X) = (x, \exp_x(X))$, it follows that $(T(\pi_{\mathcal{M}}, \exp))_{0_x}$ is invertible, hence $(\pi_{\mathcal{M}}, \exp)$ is locally a diffeomorphism near 0_x for every x, by the inverse function theorem (for Banach spaces), and thus a diffeomorphism onto its image for a sufficiently small neighborhood of the zero section. [Lan99, VIII Proposition 5.1]

For strong metrics on Hilbert manifolds, the geodesic spray is automatically a smooth vector field by Lemma 2.25. ■

We will later see that for the right-invariant L^2 metric on $\mathrm{Diff}(S^1)$, the spray is smooth, however the exponential map fails to be even a C^1 diffeomorphism on any neighborhood of the zero section. [CK02, Theorem 3]

Example 2.35 Let M be a compact manifold, with a volume form μ, and let (N,g) be a finite dimension Riemannian manifold. Define the (non-invariant) L^2 metric on $C^\infty(M,N)$ by

$$G_f(u,v) = \int_M g_{f(x)}(u(x),v(x))\,\mu(x), \qquad f \in C^\infty(M,N),\ u,v \in \Gamma(f^*TN). \tag{2.5}$$

For simplicity, let us assume that N can be covered by a single chart, that is, $N \subset \mathbb{R}^n$, and let $\gamma : N \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be the Christoffel symbol of g in this chart. We can identify $f \in C^\infty(M,N)$ with $f : M \to \mathbb{R}^n$, and similarly, a tangent vector $u \in T_f C^\infty(M,N)$ is identified with a function $u : M \to \mathbb{R}^n$ via $u = [f + tu]$. We then have

$$\begin{aligned}(D_{f,w}G)(u,v) &= \frac{d}{dt}\Big|_{t=0} \int_M g_{ij}(f(x)+tw(x))u^i(x)v^j(x)\,\mu(x)\\ &= \int_M \partial_k g_{ij}(f(x))u^i(x)v^j(x)\,\mu(x),\end{aligned}$$

from which a direct calculation shows that the Christoffel symbol of G, which we denote by $\Gamma : C^\infty(M,N) \times C^\infty(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n) \to C^\infty(M,\mathbb{R}^n)$, is given by

$$\begin{aligned}&\Gamma_f(u,v)^i(x)\\ &\quad= \frac{1}{2} g^{im}(f(x))\left[\partial_k g_{jm}(f(x)) + \partial_j g_{km}(f(x)) - \partial_k g_{jm}(f(x))\right] u^j(x)v^k(x)\\ &\quad= \gamma^i_{jk}(f(x))u^j(x)v^k(x) = \gamma^i_{f(x)}(u(x),v(x)),\end{aligned}$$

which we can simply write as $\Gamma(u,v) = \gamma \circ (u,v)$. In particular, since γ is smooth, we obtain that Γ is smooth. Therefore, we have that the geodesic equation is

$$0 = c_{tt}(x) + \Gamma_c(c_t,c_t)(x) = c_{tt}(x) + \gamma_{c(x)}(c_t(x),c_t(x)).$$

In other words, for every x, we obtain that $c(\cdot,x) : \mathbb{R} \to N$ is a geodesic in (N,g) — each particle evolves as a free particle along a geodesic in N. These results continue to hold even if N is not a subset of $\mathbb{R}^n$ (see [Bru18], where the derivation is somewhat different; see also [EM70, Theorem 9.1, Corollary 9.3]); namely, the geodesic spray is given by

$$S(u) = \sigma \circ u \qquad u \in \Gamma(f^*TN),$$

where $\sigma \in \mathfrak{X}(TN)$ is the geodesic spray of N, and the exponential map EXP of $C^\infty(M,N)$ is given by

$$\mathrm{EXP}(u) = \exp \circ u, \qquad u \in \Gamma(f^*TN),$$

that is, $\mathrm{EXP}_f(u)(x) = \exp_{f(x)}(u(x))$ for every $x \in M$. In particular, we obtain that if N is geodesically complete, so is $C^\infty(M,N)$.

Finally, note that $\mathrm{EXP}_f : T_f C^\infty(M,N) \to C^\infty(M,N)$ is a local diffeomorphism onto its image; indeed, it is the inverse of the coordinate charts that we defined! Therefore $(\pi, \mathrm{EXP}) : U \subset TC^\infty(M,N) \to C^\infty(M,N) \times C^\infty(M,N)$ is a diffeomorphism onto its range.

✎ *Exercise 2.3* In the above example, calculate the geodesic equation using the energy.

✎ *Exercise 2.4* Show that for the case $N = M = S^1$, if we restrict the metric to the diffeomorphism group $\mathrm{Diff}(S^1) \subset C^\infty(S^1, S^1)$, then the geodesic equation can be written in Eulerian coordinates $u \in \mathfrak{X}(S^1) \cong C^\infty(S^1)$, where $\varphi_t(x,t) = u(t, \varphi(t,x))$, as Burgers' equation

$$u_t + uu_x = 0,$$

Show that the $\mathrm{Diff}(S^1)$ with this metric is not geodesically complete.

2.4.3 Geodesic distance

Definition 2.36 (Geodesic distance) Let $(\mathcal{M}, g)$ be a Fréchet Riemannian manifold. The Riemannian metric induces a pseudo distance on the manifold, called the geodesic distance:

$$\mathrm{dist}^g(x,y) = \inf_{c \in C^\infty_{x,y}([0,1],\mathcal{M})} \mathrm{Len}(c),$$

where the Riemannian length of a path c is defined as

$$\mathrm{Len}(c) = \int_0^1 \sqrt{g_{c(t)}(\dot{c}(t), \dot{c}(t))}\mathrm{d}t.$$

The length and kinetic energy of a curve c are related by the Cauchy–Schwarz inequality $\mathrm{Len}(c) \le 2E(c)^{1/2}$ with equality for constant-speed curves. As any curve can be reparameterized to constant speed, it follows that length-minimization is equivalent to energy-minimization.

On finite-dimensional manifolds, the geodesic distance separates points and induces the manifold topology. The situation is similar for strong Riemannian metrics on infinite-dimensional manifolds. However, for weak Riemannian metrics, a surprising degeneracy can appear:

Theorem 2.37 (Geodesic distance) *Let* $(\mathcal{M}, g)$ *be a Riemannian manifold.*

1. *If g is a strong Riemannian metric, then the geodesic distance separates points and induces the original manifold topology.*
2. *If g is merely a weak Riemannian metric, then the geodesic distance may fail to separate points or even vanish completely. In particular, the manifold topology may differ from the topology induced by the geodesic distance.*

Proof:

1. We follow the presentation in [Lan99, Proposition VII.6.1]. Let $x_0 \neq y_0 \in \mathcal{M}$, and U be a chart at x_0. We can thus identify U with an open set in a Hilbert space $\mathcal{H}$. let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ be the Hilbert norm and inner product on $\mathcal{H}$. For sufficiently small $r > 0$, the closed ball $\bar{B}_r(x_0, r)$ with respect to $\|\cdot\|$ does not contain the point y. As the Riemannian metric g is strong, the inner product g_x depends smoothly and thus continuously on the foot point x. Thus, one may choose $r > 0$ even smaller to ensure that there exists a constant $C > 0$ such that

$$g_x(u,u) \ge C^2\|u\|^2, \qquad x \in \bar{B}_r(x_0, r), \quad u \in T_x\mathcal{M}.$$

Indeed, since g is strong, we have that

$$g_x(u,v) = \langle(u, A(x)v\rangle,$$

where $A(x) : \mathcal{H} \to \mathcal{H}$ is a positive definite, invertible symmetric operator. In particular, we have that $g_{x_0}(u,u) \geq C_0^2 \|u\|^2$. Since the dependence $x \mapsto A(x)$ is continuous, we have that for r small enough, $\|A(x) - A(x_0)\| < \frac{1}{2}C_0^2$. The result then follows with $C^2 = \frac{1}{2}C_0^2$.

We will show that this implies that $\operatorname{dist}^g(x_0, y_0) \geq Cr$. As the point y lies outside of the ball $\bar{B}_r(x_0, r)$, any path c that connects x to y has to cross the boundary $\partial \bar{B}_r(x_0, r)$. Let t_0 be the first time where the curve c crosses the boundary, i.e., $c(t) \in \bar{B}_r(x_0, r)$ for $t \leq t_0$. In the chart centered at x_0 one has $c(t_0) = ru$ for some unit vector u with respect to $\|\cdot\|$. Now we decompose the path c as $c(t) = s(t)u + w(t)$, where $\langle w(t), u\rangle = 0$. Then $s(0) = 0$, $s(t_0) = r$, and

$$\begin{aligned}\operatorname{Len}(c) &= \int_0^1 \sqrt{g_c(\dot{c},\dot{c})}\mathrm{d}t \geq \int_0^{t_0} \sqrt{g_c(\dot{c},\dot{c})}\mathrm{d}t \geq C \int_0^{t_0} \sqrt{\langle \dot{c},\dot{c}\rangle}\mathrm{d}t \\ &\geq C \int_0^{t_0} \sqrt{\langle \dot{s}u, \dot{s}u\rangle}\mathrm{d}t = C \int_0^{t_0} |\dot{s}|\mathrm{d}t \geq Cr.\end{aligned}$$

Thus, $\operatorname{dist}^g(x_0, y_0) \geq Cr$. This calculation shows that dist^g separates points and that every set which is open in a manifold chart contains a metric ball. Conversely, one easily sees that every metric ball contains some set which is open in a manifold chart. Thus, the manifold and metric topologies (induced by dist^g) coincide.

2. A weak Riemannian metric with vanishing geodesic distance on the space ℓ^2 is constructed below in Example 2.38. Further examples on spaces of diffeomorphisms, symplectomorphisms, and immersions are discussed later on.

■

Example 2.38 (Vanishing geodesic distance) The following example of a Riemannian manifold with vanishing geodesic distance is adapted from [MT20]. We consider the same weak Riemannian manifold as in Example 2.26. Namely, for a sequence $m : \mathbb{N} \to (0,\infty)$ with $\lim_{i\to\infty} m_i = 0$, we consider the weak Riemannian metric on ℓ^2 given by:

$$g_x(u,v) := e^{-\frac{1}{2}\|x\|^2_{\ell^2}} \langle u,v\rangle_{\ell^2_m} := e^{-\frac{1}{2}\|x\|^2_{\ell^2}} \sum_i m_i u_i v_i, \qquad x \in \ell^2, \quad u,v \in T_x\ell^2 = \ell^2.$$

Let $\{e_n\}_{n=1}^{\infty}$ denote the basis of canonical unit vectors in ℓ^2, and let x and y be two arbitrary elements of ℓ^2. We aim to construct paths of arbitrarily short length which connect x to y. Therefore, let for $t \in [0,1]$

$$c_1(t) = x + t m_n^{-1/4} e_n, \quad c_2(t) = x + m_n^{-1/4} e_n + t(y-x), \quad c_3(t) = y + (1-t) m_n^{-1/4} e_n.$$

Note that the concatenation of these three paths connects x to y. The velocities of these linear paths are

$$\dot{c}_1(t) = m_n^{-1/4} e_n, \qquad \dot{c}_2(t) = (y-x), \qquad \dot{c}_3(t) = -m_n^{-1/4} e_n.$$

The curve segments c_1 and c_3 are arbitrarily short for large n because

$$\operatorname{Len}(c_1) = \int_0^1 e^{-\frac{1}{4}\|c_1(t)\|_{\ell^2}^2} \|m_n^{-1/4} e_n\|_{\ell^2_m} \mathrm{d}t \le \int_0^1 \|m_n^{1/4} e_n\|_{\ell^2} \mathrm{d}t = \|m_n^{1/4} e_n\|_{\ell^2} = m_n^{1/4},$$

with a similar estimate for c_3, where we used in the last step that e_n has unit length in ℓ^2. The curve segment c_2 is also arbitrarily short for large n because the ℓ^2 norm is large along c_2,

$$\|c_2(t)\|_{\ell^2}^2 \ge m_n^{-1/2} - 2 m_n^{-1/4}(\|x\|_{\ell^2} + \|y-x\|_{\ell^2}) = m_n^{-1/4}(m_n^{-1/4} - 2(\|x\|_{\ell^2} + \|y-x\|_{\ell^2}))$$

leading to the following upper bound on the length of c_2:

$$\operatorname{Len}(c_2) = \int_0^1 e^{-\frac{1}{4}\|c_2(t)\|_{\ell^2}^2} \|y-x\|_{\ell^2_m} \mathrm{d}t \le e^{-\frac{1}{4} m_n^{-1/4}(m_n^{-1/4} - 2(\|x\|_{\ell^2} + \|y-x\|_{\ell^2}))} \|y-x\|_{\ell^2_m}.$$

Thus, the geodesic distance between any two points x and y is zero.

Note that in this example we do not have a geodesic spray (see Example 2.26). However, we can restrict ourselves to rapidly decreasing sequences, and assume that m has a polynomial decay, and then we have a smooth spray and thus a smooth exponential map, and still, the geodesic distance still collapses, with the same construction.

The above example shows for weak Riemannian manifolds that the manifold topology may differ from the (trivial) metric topology. The following lemma is a partial converse to this statement.

Proposition 2.39 (Manifold versus metric topology) *Let $(\mathcal{M}, g)$ be a Riemannian manifold. If the exponential map at every point $x \in \mathcal{M}$ is a local diffeomorphism and the geodesic distance induces the manifold topology on $\mathcal{M}$, then g is already a strong Riemannian metric.*

Proof: Assume for contradiction that g is merely a weak Riemannian metric. Then there exists a point $x \in \mathcal{M}$ and a sequence $h_n \in T_x\mathcal{M}$ such that $\|h_n\|_{g_x} \to 0$, but h_n does not converge to zero in $T_x\mathcal{M}$ with respect to the original topology on $T_x\mathcal{M}$. Let $x_n \coloneqq \exp_x(h_n)$. Since $\exp_x$ is a local diffeomorphism and $h_n \nrightarrow 0$ with respect to the original topology, then $x_n \nrightarrow x$ in the manifold topology on $\mathcal{M}$. On the other hand, $\mathrm{dist}^g(x_n, x) = \|h_n\|_{g_x} \to 0$. ■

Finally, we note that for strong Riemannian metrics, as for finite dimensional Riemannian manifolds, there exist normal neighborhoods, around points, in which geodesics are length minimizing. In particular, we obtain **local existence of minimal geodesic for strong metrics.** See [Lan99, VIII, §6] for details.

2.4.4 Curvature

Lemma 2.40 (Lie bracket) For any vector fields X and Y on a manifold $\mathcal{M}$, there exists a unique vector field $[X,Y]$ on M, which is called the Lie bracket of X and Y, such that

$$[X,Y]f = XYf - YXf, \qquad f \in C^\infty(\mathcal{M}),$$

where $Xf \in C^\infty(\mathcal{M})$ is defined by $Xf(x) = \mathrm{d}f(x).X(x)$ for all $x \in \mathcal{M}$.

Recall that we can think of vector fields as equivalence classes of curves in the manifolds, and thus as derivations. The lemma above and its proof below use this identification in both directions. For more details, see [Lan99, V, §1].

Proof: In finite dimensions the derivation $f \mapsto XYf(x) - YXf(x)$ automatically corresponds to a tangent vector at x. In infinite dimensions this can be verified in local coordinates as follows. Temporarily assume $\mathcal{M}$ to be an open subset of a Fréchet space F, and write $X = (\mathrm{Id}_{\mathcal{M}}, \bar{X})$ and $Y = (\mathrm{Id}_{\mathcal{M}}, \bar{Y})$ using the identification $T\mathcal{M} \cong \mathcal{M} \times F$. Then the vector field given by

$$[X,Y](x) \coloneqq (x, \mathrm{d}\bar{Y}(x).\bar{X}(x) - \mathrm{d}\bar{X}(x).\bar{Y}(x)) \in T_x\mathcal{M}, \qquad x \in \mathcal{M},$$

has the desired property that $[X,Y]f = XYf - YXf$. Moreover, it is invariant under coordinate changes, i.e., if $\varphi : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism to an open subset $\mathcal{N}$ of a convenient space and $y := \varphi(x)$, then some cancellations of $\mathrm{d}\varphi$ and $\mathrm{d}\varphi^{-1}$ imply that

$$\begin{aligned}[\varphi_* X, \varphi_* Y](y) &= [T\varphi \circ X \circ \varphi^{-1}, T\varphi \circ Y \circ \varphi^{-1}](y) \\ &= (y, \mathrm{d}\varphi(x).\mathrm{d}\bar{Y}(x).\bar{X}(x) - \mathrm{d}\varphi(x).\mathrm{d}\bar{X}(x).\bar{Y}(x)) = (\varphi_*[X,Y])(y).\end{aligned}$$

Therefore, the above formula for $[X,Y]$ is the coordinate expression of a unique vector field on the given manifold $\mathcal{M}$. ■

Lemma 2.41 (Riemann curvature) On any manifold $\mathcal{M}$ with covariant derivative ∇, there exists a unique $\binom{1}{3}$-tensor field R, which is called the curvature of ∇, such that for any vector fields X, Y, and Z,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Proof: We need to show that R is tensorial, i.e., $R(X,Y)Z|_x$ depends on the vector fields X, Y, and Z only through their values at the point x. This is essentially the same proof as in finite dimensions: Temporarily assume $\mathcal{M}$ to be an open subset of a Fréchet space F, identify $T\mathcal{M}$ with $\mathcal{M} \times F$, and choose arbitrary "constant" vector fields $X(x) = (x, \bar{X})$, $Y(x) = (x, \bar{Y})$, and $Z(x) = (x, \bar{Z})$ with $\bar{X}$, $\bar{Y}$, and $\bar{Z}$ in F (in finite dimensions we would have taken the coordinate vector fields $X(x) = \partial_i$ etc.). Then $[X,Y] = 0$, and the coordinate expression of the covariant derivative in Lemma **??** shows that $R(X,Y)Z|_x = (x, \bar{R}_x(\bar{X},\bar{Y})\bar{Z})$, where

$$\begin{aligned}\bar{R}_x(\bar{X},\bar{Y})(\bar{Z}) &= \nabla_X \nabla_Y Z|_x - \nabla_Y \nabla_X Z|_x - 0 \\ &= \nabla_X(-\Gamma(\bar{Y},\bar{Z}))|_x - \nabla_Y(-\Gamma(\bar{X},\bar{Z}))|_x \\ &= -d\Gamma_x(\bar{X})(\bar{Y},\bar{Z}) + \Gamma_x(\Gamma_x(\bar{Y},\bar{Z}),\bar{X}) + d\Gamma_x(\bar{Y})(\bar{X},\bar{Z}) - \Gamma_x(\Gamma_x(\bar{X},\bar{Z}),\bar{Y}).\end{aligned}$$

Clearly, the right-hand side is tensorial in X, Y, and Z. ■

Theorem 2.42 (Boundedness of the curvature tensor) Let $(\mathcal{M}, g)$ be a Riemannian manifold.

1. *If g is a strong Riemannian metric, then the curvature is locally bounded in g-norm.*

2. *If g is a weak Riemannian metric admitting a covariant derivative, then the curvature may be unbounded in g-norm not only locally but also as a multilinear operator on the tangent space at a single point.*

Proof:

1. As g is strong, $\mathcal{M}$ is a Hilbert manifold, and the curvature is a smooth section of the bundle of $\binom{1}{3}$-tensors over $\mathcal{M}$. In particular, it is continuous and locally bounded, in the sense that for every $x \in \mathcal{M}$, there exists a constant $C > 0$ and a neighborhood U of x such that $\|R(X,Y)Z\|_g \le C$ whenever the g-norm of $X, Y, Z \in TU$ is bounded by 1.

2. See Example 2.43 below.

■

Example 2.43 (Unbounded curvature) We restrict the Riemannian metric of Example 2.26 to the space $\mathfrak{s}$ of rapidly decreasing sequences. Thus, for some fixed sequence $m : \mathbb{N} \to (0, \infty)$ with a polynomial decay, we consider the weak Riemannian metric g on $\mathfrak{s}$ given by

$$g_x(u,v) := e^{-\frac{1}{2}\|x\|^2_{\ell^2}} \langle u, v\rangle_{\ell^2_m} := e^{-\frac{1}{2}\|x\|^2_{\ell^2}} \sum_i m_i u_i v_i, \qquad x \in \mathfrak{s}, \quad u, v \in T_x\mathfrak{s} = \mathfrak{s}.$$

By Example 2.26 the Christoffel symbol of this metric exists and is given by

$$\Gamma_x(u,v) = \frac{1}{2}\langle x, u\rangle_{\ell^2}\ v + \frac{1}{2}\langle x, v\rangle_{\ell^2}\ u - \frac{1}{2}\langle u, v\rangle_{\ell^2_m}\ m^{-1}x.$$

Accordingly, the derivative of the Christoffel symbol satisfies

$$\mathrm{d}\Gamma_x(u)(v,w) = \frac{1}{2}\langle u, v\rangle_{\ell^2} w + \frac{1}{2}\langle u, w\rangle_{\ell^2} v - \frac{1}{2}\langle v, w\rangle_{\ell^2_m} m^{-1}u.$$

By the coordinate expression for the curvature in the proof of Lemma 2.41, the curvature at $x = 0$ satisfies

$$\begin{aligned} g_0(R_0(u,v)v,u) &= -g_0(d\Gamma_0(u)(v,v),u) + g_0(d\Gamma_0(v)(u,v),u) \\ &= -\langle u, v\rangle_{\ell^2}\langle v, u\rangle_{\ell^2_m} + \frac{1}{2}\langle v, v\rangle_{\ell^2_m}\langle u, u\rangle_{\ell^2} + \frac{1}{2}\langle v, v\rangle_{\ell^2}\langle u, u\rangle_{\ell^2_m}. \end{aligned}$$

In particular, for $i \neq j$, the multiples $u := m_i^{-1/2} e_i$ and $v := m_j^{-1/2} e_j$ of the canonical basis vectors e_i and e_j are orthonormal with respect to g_0 and satisfy

$$g_0(R_0(u,v)v,u) = m_i^{-1} + m_j^{-1}.$$

Letting $i, j \to \infty$ shows that the sectional curvature at $x = 0$ is unbounded.

2.4.5 Completeness and the Hopf–Rinow theorem

Definition 2.44 (Completeness properties) Let $(\mathcal{M}, g)$ be a convenient Riemannian manifold.

1. ***Geodesic completeness*** *of $(\mathcal{M}, g)$ means that the exponential map is defined on all of $T\mathcal{M}$. Equivalently, geodesics exist for all time.*

2. ***Metric completeness*** *of $(\mathcal{M}, g)$ means that $\mathcal{M}$ equipped with the geodesic distance dist^g is a complete metric space. Note that this presupposes the geodesic distance to separate points.*

3. ***Existence of minimizing geodesics*** *on $(\mathcal{M}, g)$ means that any two points $x, y \in \mathcal{M}$ can be connected by a smooth curve c with $\operatorname{Len}(c) = \operatorname{dist}^g(x, y)$.*

Recall the theorem of Hopf–Rinow in finite dimensions [dC92, §7 Theorem 2.8].

Theorem 2.45 (Hopf–Rinow in finite dimensions) Let $(\mathcal{M}, g)$ be a finite dimensional Riemannian manifold. Then geodesic and metric completeness are equivalent, and either of them implies existence of minimizing geodesics.

In infinite dimensions most of these statements do not hold, as summarized in the following theorem:

Theorem 2.46 (Hopf–Rinow in infinite dimensions) Let $(\mathcal{M}, g)$ be a strong Riemannian manifold. Then metric completeness implies geodesic completeness, but all other statements of Hopf–Rinow may fail.

Proof: To prove that metric completeness implies geodesic completeness, let $c : J \to \mathcal{M}$ be a geodesic parameterized by arc length on its maximal interval of existence $J \subseteq \mathbb{R}$. By the existence and uniqueness theorem for differential equations, J is open in $\mathbb{R}$. Assume for contradiction that J is bounded above and let t_n be a sequence in J converging to the supremum of J. For any $n, m \in \mathbb{N}$ we have

$$\operatorname{dist}^g(c(t_n), c(t_m)) \le |t_n - t_m|, \tag{2.6}$$

and thus $c(t_n)$ is a Cauchy sequence in $(\mathcal{M}, \operatorname{dist}^g)$. As $(\mathcal{M}, \operatorname{dist}^g)$ is metrically complete, $c(t_n)$ converges to a point x in $(\mathcal{M}, \operatorname{dist}^g)$. Thus, $c(t_n)$ converges to x in the manifold topology of $\mathcal{M}$ by Theorem 2.37. By Theorem 2.34 there is $\varepsilon > 0$ such that the exponential map exp is defined on the ball of radius 2ε around $0_x \in T\mathcal{M}$, hence, for all sufficiently large $n \in \mathbb{N}$, $\exp_{c(t_n)}$ is defined in a ball of radius ε around $0_{c(t_n)} \in T_{c(t_n)}\mathcal{M}$. Consequently, the geodesic can be extended to an interval of length at least ε beyond t_n, thereby contradicting the maximality of J. It follows that J is unbounded above, and by the same argument J is unbounded below, which implies that all geodesics exist for all time.

An example that neither metric nor geodesic completeness implies existence of minimizing geodesics is Grossman's ellipsoid [Gro65] in Example 2.48 below. Atkin [Atk75] further extended this example to find a metrically and geodesically complete Riemannian manifold with two points that cannot be joined by any geodesic (not just a minimizing one).

We will see later several examples that geodesic completeness does not imply metric completeness in the context of weak Riemannian metrics on spaces of mappings; see also Atkin [Atk97] for an example of a metrically incomplete, geodesically complete metric on a Hilbert manifold in which any two points can be joined by a minimizing geodesic. ■

While Theorem 2.46 shows that metrically complete Riemannian manifolds may not have minimizing geodesics between all of their points, the following theorem by Ekeland [Eke78] states that this cannot be happen too often (the proof uses *Ekeland variational principle*):

Theorem 2.47 *Let x be a point in a metrically complete strong Riemannian manifold $(\mathcal{M}, g)$. Then the set of all points $y \in \mathcal{M}$ such that there exists a unique geodesic connecting x to y is dense in $\mathcal{M}$.*

Example 2.48 (Metric completeness does not imply existence of minimizing geodesics) Let $m : \mathbb{N} \to (0,1]$ satisfy $m_0 = 1$, $m_i < 1$ for all $i \geq 1$, and $\lim_{i\to\infty} m_i = 1$. Then the ellipsoid

$$\mathcal{M} = \left\{ x \in \ell^2 : \|x\|^2_{\ell^2_m} := \sum_i m_i x_i^2 = 1 \right\} \subset \ell^2$$

with the Riemannian metric inherited from ℓ^2 is metrically and consequently geodesically complete, but there is no minimizing geodesic between the north pole $(1,0,0,\dots)$ and the south pole $(-1,0,0,\dots)$ in $\mathcal{M}$. To see this, note that $\mathcal{M} = F(S)$ is the diffeomorphic image of the unit sphere $S = \{x \in \ell^2 : \|x\|_{\ell^2} = 1\}$ under the bounded linear map

$$F : \ell^2 \to \ell^2, \qquad x \mapsto m^{-1/2}x := (m_i^{-1/2}x_i)_{i\in\mathbb{N}}.$$

Let c be a smooth path in S from the north to the south pole. Then the lengths of c in S and of $F \circ c$ in $\mathcal{M}$ satisfy

$$\pi \leq \mathrm{Len}(c) = \int_0^1 \|\dot{c}(t)\|_{\ell^2} \mathrm{d}t \leq \int_0^1 \|m^{-1/2}\dot{c}(t)\|_{\ell^2} \mathrm{d}t = \mathrm{Len}(F \circ c).$$

This implies $\mathrm{dist}_{\mathcal{M}}(e_0, -e_0) \geq \pi$ because every smooth curve in $\mathcal{M}$ is of the form $F \circ c$ for some c. One actually has $\mathrm{dist}_{\mathcal{M}}(e_0, -e_0) = \pi$ because $\mathrm{Len}(F \circ c) = m^{-1/2}\pi$ if c is the half great circle joining the north and south poles in the (e_0, e_n)-plane, and this length tends to π as $n \to \infty$. However, there exists no minimizing geodesic between the north and south pole in S: this would imply $\mathrm{Len}(c) = \mathrm{Len}(F \circ c)$ and consequently $\dot{c}_i(t) = 0$ for all $i \geq 1$, which is possible only for the constant curve.

Chapter 3

Diffeomorphism groups

3.1 Manifolds of mappings revisited

Recall that we have seen that, given a compact manifold M and a finite dimensional manifold N, the space $C^\infty(M,N)$ is a Fréchet manifold, with

$$T_f C^\infty(M,N) \cong \Gamma(f^*TN),$$

which follows from the fact that $\Gamma(f^*TN)$ is the modelling space in the chart at f we constructed around f (Proposition 2.16). In particular, we have that

$$TC^\infty(M,N) \cong C^\infty(M,TN) = \bigcup_{f\in C^\infty(M,N)} \Gamma(f^*TN).$$

We have also seen (Theorem 2.20) that in the case $M = N$, the space $\mathrm{Diff}(M)$ is an open subspace of $C^\infty(M,N)$, hence also a Fréchet manifold, and is a Fréchet Lie group, whose Lie algebra is the space of vector fields

$$T_{\mathrm{Id}} \mathrm{Diff}(M) \cong \mathfrak{X}(M) := \Gamma(TM).$$

3.2 Spaces of non-smooth maps

We can also consider the space of non-smooth mappings. The upside of this is that in the Banach or Hilbert category, analysis is simpler, and that in the Hilbert category there are strong Riemannian metrics. The downside is that these spaces are somewhat "less clean" geometrically; in particular, in the non-smooth case, diffeomorphisms do not form a Lie-group.

We start with C^k maps, for $k \geq 1$.

Theorem 3.1 (Manifolds of C^k-mappings) *Let M be a compact manifold, let N be a finite-dimensional manifold, and let $k \in \mathbb{N} \cup \{0\}$. Then $C^k(M,N)$ is a smooth* ***Banach*** *manifold modeled on Banach spaces $\Gamma_{C^k}(f^*TN)$ with $f \in C^\infty(M,N)$.*

Here $\Gamma_{C^k}(f^*TN)$ stands for the sections of regularity C^k.

Proof: The construction is similar to Proposition 2.16: We define a neighborhood W_0 of the zero section of TN such that the map

$$(\pi_N, \exp) : W_0 \to N \times N, \qquad (\pi_N, \exp)(w_p) = (p, \exp_p(w))$$

is a diffeomorphism onto its image, which we denote by $W_{N\times N}$, and define the chart maps

$$\begin{aligned} V_f &:= \{h \in \Gamma_{C^k}(f^*TN) \ : \ h(M) \subset W_0\} \\ v_f &: V_f \to C^\infty(M,N), \quad v_f(h)(x) := \exp_{f(x)} h(x), \end{aligned}$$

and the chart maps by (U_f, u_f). The caveat is that the transition maps $u_{\bar f} \circ u_f^{-1}(h) = \Sigma_{\bar f}^{-1} \circ \Sigma_f \circ h$, where $\Sigma_f(w) = (\pi(w), \exp_{f(\pi(w))} w)$, are not necessarily smooth. However, if f and $\bar f$ are smooth, then Σ_f and $\Sigma_{\bar f}$ are (note that the argument of Σ_f is a point, and not a section), and consequently also $u_{\bar f} \circ u_f^{-1}$. Therefore, we take the charts only around $f \in C^\infty(M,N)$.

Since we do not take the charts around all functions in $C^k(M,N)$, it is left to check that the chart $\{U_f\}_{f\in C^\infty(M,N)}$ do cover $C^k(M,N)$. This follows from the density of $C^\infty(M,N)$ maps in $C^k(M,N)$ (with respect to C^k convergence): Say we have $g \in C^k(M,N)$, then from this density there exists $f \in U_g$, hence $(g,f) \in W_{N\times N}$. However, note that we could have chosen, to begin with, $W_{N\times N}$ to be symmetric; hence $(f,g) \in W_{N\times N}$, and so $g \in U_f$. Thus $g \in \bigcup\{U_f\}_{f\in C^\infty(M,N)}$. ■

Similarly, we can discuss mappings of Sobolev regularity H^k or $W^{k,p}$. To this end, we start by defining Sobolev spaces on sections of vector bundles [Weh04, Appendix B]:

Definition 3.2 (*Sobolev spaces*) *In the following, $p \in [1, \infty)$ and $k \in \mathbb{N} \cup \{0\}$.*

1. *Let $\pi : E \to M$ be a finite dimensional vector bundle over a compact manifold, endowed with a connection ∇. The* ***Sobolev space*** $\Gamma_{W^{k,p}}(E)$ *is the completion of $\Gamma(E)$ with respect to the norm*

$$\|h\|_{W^{k,p}} = \left(\sum_{j=0}^{k} \int_M |\nabla^j h|^p \right)^{1/p}.$$

   $\Gamma_{W^{k,p}}(E)$ *is a Banach space, and for $p = 2$, the space $\Gamma_{H^k}(E) = \Gamma_{W^{k,2}}(E)$ is a Hilbert space.*

2. *If N is a finite dimensional Riemannian manifold, then the Sobolev space $W^{k,p}(M, N)$ is given by*

$$\{f : M \to N \ : \ \iota \circ f \in \Gamma_{W^{k,p}}(M \times \mathbb{R}^D)\},$$

   where $\iota : N \to \mathbb{R}^D$ is a (fixed) isometric embedding.

Theorem 3.3 (Sobolev embedding)

1. *If $(k-m)p > \dim(M)$, then $\Gamma_{W^{k,p}}(E)$ is continuously embedded in $\Gamma_{C^m}(E)$.*

2. *If $(k-m)p > \dim(M)$, then $W^{k,p}(M, N) \subset C^m(M, N)$.*

3. *For $kp > \dim(M)$, the above definitions are equivalent to definitions via local charts. Moreover, in this case $C^\infty(M, N)$ is dense in $W^{k,p}(M, N)$ with respect to the $W^{k,p}$-topology induced by the embeddings (or equivalently, charts).*

Definitions via local charts also enable us to consider fractional Sobolev space $W^{s,p}$ (where $s \in [0, \infty)$ rather than an integer), but we will not focus on that here.

Theorem 3.4 (Manifolds of $W^{k,p}$-mappings) Let M be a compact manifold, let N be a finite-dimensional manifold, and let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty)$ such that $kp > \dim M$. Then $W^{k,p}(M, N)$ is a smooth ***Banach*** *manifold modeled on Banach spaces $\Gamma_{W^{k,p}}(f^* TN)$ with $f \in C^\infty(M, N)$. For $p = 2$, the space $H^k(M, N)$ is a* ***Hilbert*** *manifold.*

The proof is identical to the proof of Theorem 3.1.

Now, because we only have charts around smooth functions, we a-priori only have the identification $T_f W^{k,p}(M,N) \cong \Gamma_{W^{k,p}}(f^* TN)$ for smooth f. However, using these charts, one can construct the whole $TW^{k,p}(M,N)$, and it turns out that

$$TW^{k,p}(M,N) \cong W^{k,p}(M,TN)$$

(here, TN is endowed with the Sasaki metric that corresponds to the metric of N). In particular, we obtain that we can identify $T_f W^{k,p}(M,N)$ with $\{h \in W^{k,p}(M,TN) \,:\, \pi \circ h = f\}$ for any $f \in W^{k,p}(M,N)$ (see [EM70, Section 2]).

3.2.1 Lie groups and half-Lie groups

We now turn our focus to immersions, embedding and diffeomorphism groups. As long as we consider spaces of non-smooth maps, who are modeled on spaces in which the topology is at least as strong as C^1 convergence, then the results of Theorem 2.20(1)–(2) hold, with the same proof. That is, we have the following:

Theorem 3.5 *Let M be a compact manifold, and let N be a finite-dimensional manifold with $\dim(M) \le \dim(N)$.*

1. *For $k \ge 1$, the set $\mathrm{Imm}_{C^k}(M,N)$ of all smooth functions $f \in C^k(M,N)$ whose differential $T_x f : T_x M \to T_{f(x)} N$ is injective at every point $x \in M$ is an open subset of the Banach manifold $C^k(M,N)$.*
2. *The set $\mathrm{Emb}_{C^k}(M,N)$ of all immersions which are a homeomorphism onto their range is an open subset of the Banach manifold $\mathrm{Imm}_{C^k}(M,N)$.*

The same holds for $W^{s,p}$, instead of C^k, assuming that $(s-1)p > \dim M$ (so that $W^{s,p} \subset C^1$).

However, in the non-smooth case, the diffeomorphism groups do not form a Lie group [Ebi70, Section 3]:

Theorem 3.6 *Let M be a compact manifold, and let $k \geq 1$. The set* $\mathrm{Diff}_{C^k}(M)$ *of all C^k invertible maps $M \to M$ coincides with the Fréchet manifold* $\mathrm{Emb}_{C^k}(M, M)$, *and is a* ***half Lie-group****, i.e., inversion and composition are continuous, and right-multiplication is smooth.*

Proof: The proof that $\mathrm{Diff}_{C^k}(M)$ coincides with $\mathrm{Emb}_{C^k}(M, M)$ is the same as in Theorem 2.20(3).

Continuity and smoothness can be checked in charts of M; that is, we can assume that M is a (flat) closed unit ball.

Let $f_n, f, g_n, g \in \mathrm{Diff}_{C^k}(M)$ where $f_n \to f$ and $g_n \to g$. We aim to prove that $f_n \circ g_n \to f \circ g$ in C^k. For C^0, we have

$$\begin{aligned}\sup_{x \in M} d(f_n \circ g_n(x), f \circ g(x)) &\leq \sup_{x \in M} d(f_n \circ g_n(x), f \circ g_n(x)) + \sup_{x \in M} d(f \circ g_n(x), f \circ g(x)) \\ &= \sup_{y \in M} d(f_n(y), f(y)) + \sup_{x \in M} d(f \circ g_n(x), f \circ g(x))\end{aligned}$$

and the righthand-side tends to zero since $f_n \to f$ uniformly, and $g_n \to g$ uniformly and f is uniformly continuous. For the first derivative, we assume that $M \subset \mathbb{R}^{\dim M}$:

$$D(f_n \circ g_n) - D(f \circ g) = (Df_n - Df) \circ (g_n).Dg_n + Df \circ g_n.(Dg_n - Dg) + (Df \circ g_n - Df \circ g).Dg,$$

which converges to zero since $Df_n \to Df$ and $Dg_n \to Dg$ uniformly. Higher derivatives follow the same pattern.

Smoothness of right-translation: Let $f \in \mathrm{Diff}_{C^k}(M)$, and consider the action $R_f : \mathrm{Diff}_{C^k}(M) \to \mathrm{Diff}_{C^k}(M)$, $R_f(g) = g \circ f$. Consider now TR_f. We show that it is a continuous mapping $T\,\mathrm{Diff}_{C^k}(M) \to T\,\mathrm{Diff}_{C^k}(M)$. Let $v \in T_g\,\mathrm{Diff}_{C^k}(M) = \Gamma_{C^k}(g^*TM)$, and let $g(t)$ be a smooth curve such that $g(0) = g$ and $g'(0) = v$, then

$$TR_f v = \left.\frac{d}{dt}\right|_{t=0} R_f g(t) = \left.\frac{d}{dt}\right|_{t=0} g(t) \circ f = v \circ f \in \Gamma_{C^k}((g \circ f)^*TM).$$

Note that TR_f is thus indeed a map $T\,\mathrm{Diff}_{C^k}(M) \to T\,\mathrm{Diff}_{C^k}(M)$. It is continuous by the same argument that shows that the composition is continuous. Iterating this argument shows that this is a smooth function.

Note what goes wrong for left translation:

$$TL_f v = \frac{d}{dt}\bigg|_{t=0} L_f g(t) = (Tf \circ g)v.$$

Since Tf is only a C^{k-1} map, we have that TR_f maps $T\,\mathrm{Diff}_{C^k}(M)$ to sections of regularity C^{k-1}. [1]

Now consider the inversion operator $\mathrm{inv} : \mathrm{Diff}_{C^k}(M) \to \mathrm{Diff}_{C^k}(M)$, and let $f_n \to f$ in C^k. We then have

$$\begin{aligned}\sup_{x\in M} d(f_n^{-1}(x), f^{-1}(x)) &\le \sup_{y\in M} d(f_n^{-1}(f_n(y)), f^{-1}(f(y))) + \sup_{y\in M} d(f^{-1}(f(y)), f^{-1}(f_n(y))\\ &= \sup_{y\in M} d(f^{-1}(f(y)), f^{-1}(f_n(y))),\end{aligned}$$

which tends to zero by the same argument as for the composition. For the first derivative, we have

$$D(f^{-1}) = i \circ Df \circ f^{-1},$$

where $i : \mathrm{GL}_{\dim M} \to \mathrm{GL}_{\dim M}$ is the matrix inversion operator. Since i is smooth, the map $f \mapsto D(f^{-1})$ is a continuous function $C^1 \to C^0$. Higher derivative follow by induction.

Note that inv is not smooth: Let $g(t)$ be a smooth curve such that $g(0) = g$ and $g'(0) = v$, then we have that $\mathrm{inv}\, g(t, g(t, x)) = x$, from which it follows that

$$\partial_t \,\mathrm{inv}\, g(t, g(t, x)) + D(\mathrm{inv}\, g)(t, g(t, x)).\partial_t g(t, x) = 0,$$

or in other words

$$\partial_t \,\mathrm{inv}\, g(t, x) = -D(\mathrm{inv}\, g).\partial_t g \circ \mathrm{inv}\, g = (i \circ Dg.\partial_t g) \circ \mathrm{inv}\, g,$$

Therefore,

$$T\,\mathrm{inv}\, v = -(i \circ Dg.v) \circ \mathrm{inv}\, g,$$

but due to the derivative, this is not in $T_{\mathrm{inv}\, g}\,\mathrm{Diff}_{C^k}(M)$ since it is only a C^{k-1} vector field.[2] ■

[1] This loss of derivative does not happen, of course, if f is a smooth map, in which case one can show that L_f is indeed smooth. In more detail, we have that if $f \in \mathrm{Diff}_{C^{k+l}}(M)$, then $L_f : \mathrm{Diff}_{C^k}(M) \to \mathrm{Diff}_{C^k}(M)$ is a C^l map. In fact, as a map $\mathrm{Diff}_{C^{k+l}}(M) \times \mathrm{Diff}_{C^k}(M) \to \mathrm{Diff}_{C^k}(M)$, the composition map is C^l.

[2] This reasoning also show that if one considers $\mathrm{inv} : \mathrm{Diff}_{C^{k+l}}(M) \to \mathrm{Diff}_{C^k}(M)$, then it is a C^l map.

The same holds (including the footnotes in the previous page) in the Sobolev category:

Theorem 3.7 *Let M be a compact manifold, and let $k \in \mathbb{N}$, $p \in [1, \infty]$, such that $(k-1)p > \dim M$. The set $\mathrm{Diff}_{W^{k,p}}(M)$ of all invertible $W^{k,p}$-maps $M \to M$ is a* ***half Lie-group****, i.e., inversion and composition are continuous, and right-multiplication is smooth.*

The proof follows the same lines, but the calculations are more cumbersome so we will not repeat them here, and only mention the key estimates:

Proposition 3.8 (Sobolev algebra) *Let $k > n/p$, $l \le k$. Then the map $W^{k,p}(\mathbb{R}^n) \times W^{l,p}(\mathbb{R}^n) \to W^{l,p}(\mathbb{R}^n)$, $(f, g) \mapsto fg$ is a continuous bilinear map.*

See [Ebi70, Lemma 3.2], [IKT13, Lemma 2.3] for proofs.

Proposition 3.9 *Let D_n be the closed n-dimensional disc. Let $f_0 \in W^{k,p}(D_n; D_n)$, $g_0 \in W^{k,p}(D_n)$, where $(k-1)p > n$ and Df_0 invertible everywhere (note that f is C^1). Then the composition map $W^{k,p}(D_n; D_n) \times W^{k,p}(D_n) \to W^{k,p}(D_n)$ is jointly continuous near (f_0, g_0).*

See [Ebi70, Lemma 3.1]. The key issues in proving this are:

- The chain rule holds for composition $g \circ f$ of Sobolev maps of this regularity: $D(g \circ f) = (Dg \circ f)Df$. See [EG15, Theorem 4.4(ii)] and [IKT13, Lemma 2.4(ii)].
- The Jacobian $J(f)$ and its inverse are bounded uniformly in the vicinity of f_0 (because of the topology is stronger than C^1). This is repeatedly used when we change variables. This is also used when showing that the inverse map $\mathrm{inv} : \mathrm{Diff}_{W^{k,p}}(M) \to \mathrm{Diff}_{W^{k,p}}(M)$ is continuous, as the inverse operator for matrices involves a polynomial in the matrix entries (here Proposition 3.8 is useful), divided by the determinant.

More details can be found in [Ebi70, §3], [IKT13, §2] and [Kol17, §2].

Comment: We will often denote $\mathrm{Diff}_{H^k}(M)$ simply by $\mathrm{Diff}_k(M)$. For a compact manifold M, the space $\mathrm{Diff}(M)$ is the inverse limit of $(\mathrm{Diff}_k(M))_{k>\frac{\dim M}{2}+1}$. This makes $\mathrm{Diff}(M)$ into something with more structure than a general Fréchet manifold, namely an **inverse limit Hilbert (ILH) manifold** [Omo70]. In layman terms, this means that the topology of $\mathrm{Diff}(M)$ is the one induced by the inclusions $\mathrm{Diff}(M) \subset \mathrm{Diff}_k(M)$, and a map $\mathrm{Diff}(M) \to \mathrm{Diff}(M)$ is C^l if and only if for every k it has an extension to a C^l map $\mathrm{Diff}_{k'}(M) \to \mathrm{Diff}_k(M)$ for some k' [EM70, pp. 108–109].[3] This will turn out to be useful when we study existence of geodesics on $\mathrm{Diff}(M)$, since we can study them by studying the existence on Banach, and even Hilbert manifolds.

Finally, we note that the fact that we only get Lie group in the Fréchet setting and not in the Banach setting is actually related to a much more general phenomenon [Omo79, Omo96]:

Theorem 3.10 (Omori, 1978) If a connected Banach Lie group G acts effectively, transitively and smoothly on a compact manifold, then G is finite dimensional.

3.3 Volume preserving diffeomorphisms

*Definition 3.11 Let M be a finite dimensional compact manifold, endowed with a volume form μ. The space of all **volume preserving diffeomorphisms** is*

$$\mathrm{Diff}_\mu(M) = \{f \in \mathrm{Diff}(M) \ : \ f^*\mu = \mu\}.$$

Similarly, for k such that $k-1 > \dim(M)/2$, one can define $\mathrm{Diff}_\mu^{H^k}$.

We now show that this is a closed submanifold of the whole diffeomorphism group. This was originally shown in [EM70, Theorem 4.2], though we will follow a slightly different proof [Ebi15, §3].

[3] See also https://mathoverflow.net/questions/74064/fr%C3%A9chet-manifolds-vs-ilh-manifolds.

Theorem 3.12 $\mathrm{Diff}_\mu(M)$ *is a smooth closed Fréchet Lie subgroup of* $\mathrm{Diff}(M)$*, whose Lie algebra is*

$$T_{\mathrm{Id}}\,\mathrm{Diff}_\mu(M) = \mathfrak{X}_\mu(M) := \{v \in \mathfrak{X}(M) \;:\; \operatorname{div} v = 0\}\,.$$

A similar statement (changing Lie group to half-Lie group) holds for $\mathrm{Diff}_\mu^{H^k}$ *and* Diff_{H^k}*, and* $\mathrm{Diff}_\mu(M)$ *is the inverse limit Hilbert manifold of* $\mathrm{Diff}_\mu^{H^k}$*.*

Here, the divergence operator is defined by $\mathcal{L}_v\mu = \operatorname{div}(v)\mu$. In local coordinates, writing $\mu = \rho\, dx^1 \wedge \ldots \wedge dx^n$, it is given by $\operatorname{div}(v) = \rho^{-1}\partial_i(\rho v^i)$.

Before giving the proof, recall the implicit function theorem for finite dimensional manifolds: if $f : M \to N$ is smooth, and q is a regular value of f, then $f^{-1}(q) \subset M$ is a submanifold. The proof goes with local coordinates around q and $p \in f^{-1}(q)$, so $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$, and then we split $\mathbb{R}^m = \ker T_p f \oplus V$, and then we have that $T_p f|_V$ is invertible and use the standard implicit function theorem. The tangent space of $f^{-1}(q)$ at p is then $\ker T_p f$. Now, the implicit function theorem works in the Banach category, but the fact that $\ker T_p f$ splits T_pM needs to be assumed:

Definition 3.13 *Let* $f : M \to N$ *be a smooth map between Banach manifolds. We say that* p *is a* ***regular point*** *of* f *if* $T_p f$ *is onto and* $\ker T_p f$ *splits* T_pM *(that is, there exists a closed* $V \subset T_pM$ *such that* $T_pM = \ker T_p f \oplus V$*).* q *is a* ***regular value*** *of* f *if* $f^{-1}(q)$ *contains only regular points.*

In finite dimensional settings the splitting assumption is automatically satisfied, but this is also true in Hilbert settings, since one can take the orthogonal complement of $\ker T_p f$. This is the reason in the above theorem we only discuss $\mathrm{Diff}_\mu^{H^k}$ and not $\mathrm{Diff}_\mu^{W^{k,p}}$ (although the proof can probably be adapted to this case as well). More on splitting can be read in [Lan99, II §2] and [Ger75, §5] (the later also includes examples for non-splitting closed subspaces).

Proof: Denote $m = \dim M$, and denote by $H^k(\Lambda^m)$ the top-forms of M of regularity H^k, and let

$$H^k_\mu(\Lambda^m) = \left\{0 < \nu \in H^k(\Lambda^m) \;:\; \int_M \nu = \int_M \mu\right\},$$

where by $\nu > 0$ we mean that $\nu = f\mu$ for a strictly positive f. $H^k_\mu(\Lambda^m)$ is obviously an open subset of an affine subspace of codimension one of $H^k(\Lambda^m)$. Its tangent space (at every point!) is

$$H^k_0(\Lambda^m) = \left\{\lambda \in H^k(\Lambda^m) \;:\; \int_M \lambda = 0\right\}.$$

Define

$$\psi : \mathrm{Diff}^k(M) \to H^{k-1}_\mu(\Lambda^m), \qquad \psi(\eta) = \eta^*\mu.$$

It is immediate that ψ indeed maps $\mathrm{Diff}^k(M)$ to $H^{k-1}_\mu(\Lambda^m)$. We will show that ψ is a submersion (i.e., that all points are regular points of ψ), and thus it will follow that $\mathrm{Diff}^k_\mu(M) = \psi^{-1}(\mu)$ is a submanifold of $\mathrm{Diff}^k(M)$, whose tangent space at the identity is

$$T_{\mathrm{Id}}\,\mathrm{Diff}^k_\mu(M) = \ker T_{\mathrm{Id}}\psi = \{v \in \mathfrak{X}_{H^k}(M) \;:\; \operatorname{div} v = 0\}.$$

The closedness of $\mathrm{Diff}^k_\mu(M)$ in $\mathrm{Diff}^k(M)$ is obvious (since convergence in H^k is stronger than C^1), as well as the group properties; this will complete the proof for the Sobolev case. The smooth case will then follow by taking the intersection of the Sobolev maps as $k \to \infty$ (we will not detail that here).

Let us calculate the derivative of ψ at the identity: Let $\eta(t)$ be the flow of $v \in T_{\mathrm{Id}}\,\mathrm{Diff}^k(M)$, then

$$T_{\mathrm{Id}}\psi(v) = \left.\frac{d}{dt}\right|_{t=0} \psi(\eta(t)) = \left.\frac{d}{dt}\right|_{t=0} \eta(t)^*\mu = \mathcal{L}_v\mu = \operatorname{div}(v)\mu.$$

In order to show that ψ is a submersion at Id, we need to show that for every $\lambda \in H^{k-1}_0(\Lambda^m)$, there exists $v \in \mathfrak{X}_{H^k}(M)$ such that $\operatorname{div}(v) = \lambda/\mu$. Restricting ourselves to conservative vector fields, $v = \nabla f$ for some $f \in H^{k+1}(M)$, this yields the equation

$$\Delta f = \lambda/\mu.$$

Since M is closed and $\int_M \lambda = 0$, there exists a solution (existence follows for example by using Riesz representation theorem on the functional $f \mapsto \int_M f\lambda$ on the Hilbert space of $H^1(M)$ functions with zero mean with the $\dot{H}^1$ inner-product; regularity follows by standard elliptic regularity arguments). Thus, Id is a regular point.

The fact that all other points are regular follow by right-translation: let $\eta \in \mathrm{Diff}^k(M)$, then $T_\eta\,\mathrm{Diff}^k(M) = T_{\mathrm{Id}}\,\mathrm{Diff}^k(M) \circ \eta$, hence if $v \in T_{\mathrm{Id}}\,\mathrm{Diff}^k(M)$

with a flow $\varphi(t)$, then

$$T_\eta\psi(v\circ\eta) = \left.\frac{d}{dt}\right|_{t=0}(\varphi(t)\circ\eta)^*\mu = \eta^*\left(\left.\frac{d}{dt}\right|_{t=0}\varphi(t)^*\mu\right) = \operatorname{div}(v)\circ\eta\eta^*\mu.$$

Therefore we can similarly show that $\operatorname{div}(v) = (\lambda/\eta^*\mu)\circ\eta^{-1}$ has a solution for every $\lambda \in H_0^{k-1}(\Lambda^m)$. ■

In fact, one can prove a stronger result, namely that $\operatorname{Diff}(M)$ is diffeomorphic to $\operatorname{Diff}_\mu(M)\times C_\mu^\infty(\Lambda^m)$, and that since $C_\mu^\infty(\Lambda^m)$ is convex, $\operatorname{Diff}_\mu(M)$ is a deformation retract of $\operatorname{Diff}(M)$ [EM70, Theorem 5.1].

3.4 Other diffeomorphism groups

One can also consider other related spaces, in various regularities. We will not prove here that they are indeed manifolds.

1. Given symplectic 2-form ω, the diffeomorphism which preserve ω for the group of **symplectomorphisms**, whose Lie algebra consists of all symplectic vector fields

   $$\{X\in\mathfrak{X}(M)\ :\ \mathcal{L}_\omega X = 0\}.$$

   One can also consider all **Hamiltonian symplectomorphisms**.

2. Another related group that arises in hydrodynamics (in particular, to the KdV equation) is the **Virasoro group** (or Virasoro–Bott group).

3. Yet another group that arises in applications is the group of a diffeomorphisms that fix a given point (or a submanifold). For example, the Hunter–Saxton equation is a geodesic equation on the elements in $\operatorname{Diff}(S^1)$ that fix a point (equivalently, the quotient space of $\operatorname{Diff}(S^1)$ modulo rotations). [Len07]

3.5 Non-compact base manifold

In the above discussions we only considered compact base manifolds. For a non-compact base manifold, one needs to be more cautious, and specify conditions at infinity. To illustrate this, we will consider diffeomorphisms of the Euclidean space. Some of the spaces that we can consider include [Mic20, Section 6.9]:

1. The space $\mathcal{S}(\mathbb{R}^n)$ denotes the space of all Schwartz functions. It is a Fréchet space.

2. The space $W^{\infty,p}(\mathbb{R}^n) = \cap_{k=1}^{\infty} W^{k,p}$. It is a Fréchet space (and an inverse-limit Banach space).

3. The space $C_c^\infty(\mathbb{R}^n)$ of compactly supported smooth functions (where $f_n \to f$ if all of them are supported in the same compact set on which there is uniform convergence of all derivatives). This is not a Fréchet space, but an LF-space (a locally convex inductive limit of the Fréchet spaces $C^\infty(K_i)$ for and increasing sequence of compact sets $(K_i)_{i\in\mathbb{N}}$ that cover $\mathbb{R}^n$). In this case smoothness is trickier, and we will not get into details.

Theorem 3.14 *The diffeomorphism groups*

$$\begin{aligned}
\mathrm{Diff}_c(\mathbb{R}^n) &= \{f \in \mathrm{Diff}(\mathbb{R}^n) \ : \ f - \mathrm{Id} \in C_c^\infty(\mathbb{R}^n)^n\} \\
\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n) &= \{f \in \mathrm{Diff}(\mathbb{R}^n) \ : \ f - \mathrm{Id} \in \mathcal{S}(\mathbb{R}^n)(\mathbb{R}^n)^n\} \\
\mathrm{Diff}_{W^{\infty,p}}(\mathbb{R}^n) &= \{f \in \mathrm{Diff}(\mathbb{R}^n) \ : \ f - \mathrm{Id} \in W^{\infty,p}(\mathbb{R}^n)^n\}
\end{aligned}$$

are all smooth Lie groups, with Lie algebras being the vector fields with the appropriate decay. Moreover,

$$\mathrm{Diff}_c(\mathbb{R}^n) \subset \mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n) \subset \mathrm{Diff}_{W^{\infty,p}}(\mathbb{R}^n),$$

and the inclusions are smooth, and each one is normal in the ones containing it. The connected component of the identity in $\mathrm{Diff}_c(\mathbb{R}^n)$ *is simple.*

We will not prove it here, see [Mic20, Theorem 6.10]. For references on the simplicity of the connected components of the diffeomorphism groups, see [BBHM13, p. 15].

Chapter 4

Geodesic equations of right-invariant metrics

In this section we will focus on some natural/famous metrics on diffeomorphism group, calculate their geodesic equations and see some of their properties. In the next chapters we will discuss more some of their metric properties, existence of geodesics etc.

4.1 Right-invariant metrics

Our main focus will be right-invariant metrics, which will follow Example 2.28 for the case $G = \mathrm{Diff}(M)$, where M is a closed Riemannian manifold, or for $G = \mathrm{Diff}_c(\mathbb{R}^n)$ (or with other decay conditions on $\mathbb{R}^n$). Recall that in this case, an inner product $(\, , \,)$ on $\mathfrak{g}$, induces a **right-invariant metric** on G by

$$\langle u, v \rangle_g := (u \circ g^{-1}, v \circ g^{-1}).$$

The geodesic equation is then given by (2.4), although in most cases writing down (2.4) explicitly is equivalent to deriving the geodesic equation from the energy to begin with, which is the path we will usually take. Also, for proving existence of solutions, (2.4) typically not very useful.

Recall that for $G = \mathrm{Diff}(M)$, the Lie algebra $\mathfrak{g} = \mathfrak{X}(M)$ is the space of vector fields. We have the following natural metrics:

1. The L^2 metric:
$$(u,v)_{L^2} = \int_M g(u(x),v(x))\,\mathrm{dVol}_g(x)$$

2. The H^k metrics:
$$\begin{aligned}(u,v)_{H^k} &= \sum_{i=0}^{k}\int_M g(\nabla^i u(x),\nabla^i v(x))\,\mathrm{dVol}_g(x)\\ &= \sum_{i=0}^{k}\int_M g((-\Delta)^i u(x),v(x))\,\mathrm{dVol}_g(x)\end{aligned}$$
Here, in the first line, we consider g as an inner product on all tensors over M.

3. More generally, given a symmetric, positive definite, pseudo-differential operator $A:\mathfrak{X}(M)\to\mathfrak{X}(M)$, we obtain the A-metric
$$(u,v)_A = \int_M g(Au,v)\,\mathrm{dVol}_g$$
This enables us, for example, to define fractional Sobolev metrics on $\mathrm{Diff}(M)$.

4. The $H(\mathrm{div})$ metric:
$$(u,v)_{H(\mathrm{div})} = \int_M g(u,v) + \operatorname{div} u \operatorname{div} v\,\mathrm{dVol}_g = \int_M g(u,v) - g(\nabla \operatorname{div} u, v)\,\mathrm{dVol}_g$$
This space is simpler than the H^1 metric (if $\dim M = 1$ they coincide), but still richer than the L^1 metric.

5. The $\dot{H}^1$ (semi-)metric:
$$(u,v)_{\dot{H}^1} = \int_M g(\nabla u(x),\nabla v(x))\,\mathrm{dVol}_g(x).$$
This is not necessarily an inner product: It is on $\mathfrak{X}_c(\mathbb{R}^n)$, but not on any space of vector fields that contains a vector field u such that $\nabla u = 0$. For example, $\mathfrak{X}(S^1) = C^\infty(S^1)$, which allows for constant vector fields. However, the resulting Riemannian semi-metric is a true Riemannian metric on the right cosset $\mathrm{Rot}(S^1)/\mathrm{Diff}(S^1)$ of diffeomorphism groups modulo the rotations subgroup (this is not surprising since the flow of a constant vector field generates all the rotations).

Similarly, one can also consider the $\dot{H}^{\mathrm{div}}$ semi-metric.

Note that all these metrics are smooth on $\mathrm{Diff}(M)$, or $\mathrm{Diff}_\mu(M)$, since the group operations are smooth. However, for diffeomorphism groups of finite regularity, e.g., $\mathrm{Diff}^{H^k}(M)$, it is not obvious that this method produces a smooth metric, as it involves composition with the inverse map, and the inverse operator in non-smooth. We will ignore this subtlety for now, but later we will note how this is proven in many cases (in fact, in all of the cases above, except for the L-case, unless we make further assumptions).

4.2 Burgers Equation: L^2 metric on $\mathrm{Diff}(S^1)$

Let us calculate directly the geodesic equation associated with the right-invariant L^2 metric on $\mathrm{Diff}(S^1)$ (we already did it when we calculated its Christoffel symbols):[1] Given a path $\varphi : [0,1] \to \mathrm{Diff}(S^1)$, its energy is given by

$$E(\varphi) = \int_0^1 \int_{S^1} (\varphi_t \circ \varphi^{-1})^2 \, dx \, dt = \int_0^1 \int_{S^1} \varphi_t^2 \varphi_x \, dx \, dt,$$

and thus its variation, with respect to a family $\varphi : \mathbb{R} \times [0,1] \to \mathrm{Diff}(S^1)$ fixing the boundary conditions, is

$$\begin{aligned} \partial_s|_0 E(\varphi(s, \ )) &= \int_0^1 \int_{S^1} (2\varphi_{ts}\varphi_t\varphi_x + \varphi_t^2 \varphi_{tx}) \, dx \, dt \\ &= -\int_0^1 \int_{S^1} (2\varphi_s(\varphi_{tt}\varphi_x + 2\varphi_t\varphi_{tx}) \, dx \, dt, \end{aligned}$$

hence the geodesic equation is

$$\varphi_{tt} + 2\varphi_t \frac{\varphi_{tx}}{\varphi_x} = 0.$$

Moving to Eulerian coordinates, and using the identities:

$$\varphi_t = u \circ \varphi, \quad \varphi_{tx} = u_x \circ \varphi \varphi_x, \quad \varphi_{tt} = u_t \circ \varphi + u_x \circ \varphi \varphi_t,$$

we obtain **Burgers' equation**

$$u_t + 3uu_x = 0.$$

This equation can be solved by the method of characteristics: Consider the graph $(t, x, u(t,x))$ in $\mathbb{R}^3$; its normal is the vector field $(u_t, u_x, -1)$, so the

[1] In this subsection I follow some parts of [CK02].

equation tells us that the vector field $(1, 3u, 0)$ is tangent to the graph. In other words, the graph is an integral manifold of this vector field. We thus have to solve the system of ODEs

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = 3u, \quad \frac{du}{ds} = 0,$$
$$t|_{s=0} = 0, \quad x|_{s=0} = x_0, \quad u|_{s=0} = u_0(x_0),$$

hence $t = s$, $u(t(s), x(s)) = u_0(x_0)$ and $x(s) = x_0 + 3su_0(x_0)$, and so

$$u_0(x_0) = u(t(s), x(s)) = u(s, x_0 + 3su_0(x_0)), \tag{4.1}$$

or, after renaming,

$$u_0(x) = u(t, x + 3tu_0(x)) \qquad t \in [0, T), \quad x \in S^1. \tag{4.2}$$

Differentiating this equation with respect to x, we obtain

$$u_x(t, x + 3tu_0(x))(1 + 3tu_0'(x)) = u_0'(x),$$

hence, if $\min_{S^1} u_0'(x) < 0$, this equation breaks down when

$$T = \min_{x \in S^1} \frac{1}{-3u_0'(x)}.$$

Note that since $x \in S^1$ (periodic solutions) $\min_{S^1} u_0'(x)$ is always negative unless u_0 is constant. It can be shown that T is indeed precisely the blowup time of the equation. Note that from (4.2) we have that

$$\max_{x \in S^1} |u(t, x)| = \max_{x \in S^1} |u_0(x)|,$$

and that as $t \to T$,

$$\min_{x \in S^1} u_x(t, x) \to -\infty,$$

hence the solution looks like a shockwave. If we go to the Lagrangian coordinates, as $t \to T$, the map $\varphi(t, \cdot)$ stop being a diffeomorphism (it loses the immersion property). In geometric terms, we proved the following:

Proposition 4.1 *The exponential map* $\exp_{\mathrm{Id}} : U \subset \mathfrak{X}(S^1) \to \mathrm{Diff}(S^1)$ *is defined on the open set*

$$U = \left\{ u_0 \in C^\infty(S^1) \ : \ u_0'(x) > -\frac{1}{3} \right\}.$$

In other words, geodesics of the right-invariant L^2 metric on $\mathrm{Diff}(S^1)$ exist locally in time, but not globally in time (the above proposition discusses geodesics from the identity map, but all other base points behave in the same way). Note that U is indeed open in $C^\infty(S^1)$. We will later see that geodesics in this case are *never* length minimizing (Section 5.2)[2], and that while the exponential map is defined on an open set, it is not a local diffeomorphism on any open set.

We can write the geodesic equation in Lagrangian coordinates as

$$0 = \varphi_{tt}\varphi_x + 2\varphi_t\varphi_{tx} = \frac{\partial_t(\varphi_t(t,x)\varphi_x^2(t,x))}{\varphi_x(t,x)},$$

and thus we have

$$\varphi_t(t,x)\varphi_x^2(t,x) = \varphi_t(0,x)\varphi_x^2(0,x).$$

If we consider a geodesic that starts at the identity, i.e., $\varphi(0,x) = x$, we obtain

$$\varphi_t(t,x)\varphi_x^2(t,x) = u_0(x). \tag{4.3}$$

We now use this formula to prove the following result:

Theorem 4.2 ([CK02], Theorem 3) *The exponential map $\exp_{\mathrm{Id}} : U \to \mathrm{Diff}(S^1)$ is not a C^1-diffeomorphism on any open neighborhood of zero.*

Proof: We will show that if $\exp_{\mathrm{Id}}$ is a C^1 map, then $D\exp_{\mathrm{Id}}(v_n)$ is not invertible for a sequence v_n that converges to zero (note that $D\exp_{\mathrm{Id}}(0) = \mathrm{Id}_{\mathfrak{X}(S^1)}$ and thus invertible). In the following we will use the standard coordinate chart on $\mathrm{Diff}(S^1)$, that is, we will identify both diffeomorphisms and vector fields on S^1 with functions in $C^\infty(S^1)$.

Assume that $\exp_{\mathrm{Id}}$ is C^1, and let $v \in U$ and $w \in \mathfrak{X}(S^1)$. Let φ^ε the geodesic starting at Id in the direction $(v + \varepsilon w)$. From (4.3) we have that for ε small enough,

$$\varphi^\varepsilon(t,x) = x + \int_0^t \frac{v(x) + \varepsilon w(x)}{(\varphi_x^\varepsilon(s,x))^2}\, ds, \qquad |\varepsilon| < \varepsilon_0, \quad t \in [0,1], \quad x \in S^1.$$

[2] In fact, they are not even local minimizers, see [Bru13].

We therefore have,

$$\frac{\varphi^\varepsilon(t,x)-\varphi^0(t,x)}{\varepsilon} = \int_0^t \frac{w(x)}{(\varphi_x^\varepsilon(s,x))^2}\,ds - \int_0^t \frac{v(x)\left[\varphi_x^\varepsilon(s,x)+\varphi_x^0(s,x)\right]}{\left(\varphi_x^\varepsilon(s,x)\varphi_x^0(s,x)\right)^2}\,\frac{\varphi_x^\varepsilon(s,x)-\varphi_x^0(s,x)}{\varepsilon}\,ds. \tag{4.4}$$

We would like to take $\varepsilon \to 0$. Here the C^1 assumption plays a role.

Note that

$$\varphi^\varepsilon(t)-\varphi^0(t) = \exp_{\mathrm{Id}}(t(v+\varepsilon w)) - \exp_{\mathrm{Id}}(tv),$$

hence, for every t, we have that the following limits hold uniformly on $x \in S^1$,

$$\lim_{\varepsilon\to 0}\frac{\varphi^\varepsilon(t,x)-\varphi^0(t,x)}{\varepsilon} = d\exp_{\mathrm{Id}}(tv)(tw) =: \psi(t,x)$$
$$\lim_{\varepsilon\to 0}\frac{\varphi_x^\varepsilon(t,x)-\varphi_x^0(t,x)}{\varepsilon} = \psi_x(t,x),$$

since by assumption $\frac{\varphi^\varepsilon(t,x)-\varphi^0(t,x)}{\varepsilon} \to d\exp_{\mathrm{Id}}(tv)(tw)$ in $C^\infty(S^1)$. This is, however, still not enough in order to use the dominated convergence theorem to take $\varepsilon \to 0$ under the integral sign in (4.4).

For $t \in [0,1]$ and $\varepsilon > 0$ small enough, define

$$F:[0,1]\to C^\infty(S^1) \qquad F(s) = \frac{\exp_{\mathrm{Id}}(tv+\varepsilon sw)-\exp_{\mathrm{Id}}(tv)}{\varepsilon} - s\,d\exp_{\mathrm{Id}}(tv)w.$$

Note that since $\exp_{\mathrm{Id}}$ is C^1, so is $F(s)$, and so,

$$\begin{aligned}\|F(s_0)\|_{C^1(S^1)} = \|F(s_0)-F(0)\|_{C^1(S^1)} &\le \max_{s\in[0,1]}\|F'(s)\|_{C^1(S^1)} \\ &= \max_{s\in[0,1]}\|d\exp_{\mathrm{Id}}(tv+\varepsilon sw)w - d\exp_{\mathrm{Id}}(tv)w\|_{C^1(S^1)}.\end{aligned}$$

Now, by assumption $d\exp_{id}$ is continuous in the $C^\infty(S^1)$ topology, and in particular in the $C^1(S^1)$ topology; thus, it is norm-bounded on compact sets. In particular, there exists $M > 0$ such that for every $t \in [0,1]$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, we have

$$\max_{s\in[0,1]}\|d\exp_{\mathrm{Id}}(tv+\varepsilon sw)w - d\exp_{\mathrm{Id}}(tv)w\|_{C^1(S^1)} \le M,$$

hence

$$\max_{s,t\in[0,1],|\varepsilon|<\varepsilon_0} \|F(s)\|_{C^1}(S^1) = \max_{s,t\in[0,1],|\varepsilon|<\varepsilon_0} \left\| \frac{\exp_{\mathrm{Id}}(tv + \varepsilon t w) - \exp_{\mathrm{Id}}(tv)}{\varepsilon} - t\, d\exp_{\mathrm{Id}}(tv)w \right\|_{C^1(S^1)} \le M,$$

and thus, for $s = t$, we have

$$\max_{t\in[0,1],|\varepsilon|<\varepsilon_0} \left\| \frac{\varphi_x^\varepsilon(t,\cdot) - \varphi_x^0(t,\cdot)}{\varepsilon} - \psi_x(t,\cdot) \right\|_{C^0(S^1)} \le M.$$

Thus, by dominated convergence, we can take $\varepsilon \to 0$ in (4.4), and obtain

$$t\, d\exp_{\mathrm{Id}}(tv)(w) = \psi(t,x) = \int_0^t \frac{w(x)}{\varphi_x^2(s,x)}\, ds - \int_0^t \frac{2v(x)}{\varphi_x^3(s,x)} \psi_x(s,x)\, ds, \quad (4.5)$$

for every $t \in [0,1]$ and $x \in S^1$. Differentiating with respect to t, we obtain

$$\psi_t(t,x) = \frac{w(x)}{\varphi_x^2(t,x)} - \frac{2v(x)}{\varphi_x^3(t,x)} \psi_x(t,x).$$

Now, take $v(x) = c > 0$ be a constant vector field. In this case, it is easy to see (say from (4.3)) that $\varphi(t,x) = x + ct$, and thus

$$\psi_t(t,x) = w - 2c\psi_x.$$

Since $\psi(0,x) = d\exp_{\mathrm{Id}}(0)0 = 0$, the solution to this equation is

$$\psi(t,x) = \frac{1}{2c} \int_{x-2ct}^x w(y)\, dy,$$

and in particular, for $t = 1$, we obtain

$$(d\exp_{\mathrm{Id}}(v)w)(x) = \frac{1}{2c} \int_{x-2c}^x w(y)\, dy. \quad (4.6)$$

Now, for $v_n(x) = \frac{1}{n}$, take $w_n(x) = \sin(\pi n x)$, then $d\exp_{\mathrm{Id}}(v_n)w_n \equiv 0$, and in particular $d\exp_{\mathrm{Id}}(v_n)$ is not invertible. However $v_n \to 0$ in $C^\infty(S^1)$, and thus $d\exp_{\mathrm{Id}}$ is not invertible in any neighborhood of 0. ■

Comment: Note that, in principle, once we "guessed" v_n and w_n, we can go to (4.4), and try to estimate the limit $\varepsilon \to 0$ directly for these v_n and w_n (by estimating the specific solutions to these initial conditions for the geodesics), and obtain either the vanishing of $d\exp_{\mathrm{Id}}(v_n)w_n$, or its non-existence, directly.

Note that the proof works not only on smooth diffeomorphisms, but basically on diffeomorphisms on any regularity that controls C^1, in particular, for $\mathrm{Diff}_{C^k}(S^1)$ for $k \geq 1$ or $\mathrm{Diff}_{H^k}(S^1)$ for $k \geq 2$. In these Banach settings, if $\exp_{\mathrm{Id}}$ would have been a C^1 map, we would immediately obtain by the inverse function theorem that it is a local diffeomorphism in the vicinity of 0, since $d\exp_{\mathrm{Id}}(0) = \mathrm{Id}$ is invertible. Thus, the above proof, that shows that if $\exp_{\mathrm{Id}}$ is C^1 then there are $v_n \to 0$ in which $d\exp_{\mathrm{Id}}(v_n)$ is not invertible, actually shows that

Theorem 4.3 *For Banach diffeomorphisms of S^1 the exponential map* $\exp_{id}$ *of the right-invariant L^2 metric is not a C^1* map *in any neighborhood of zero.*

In particular, since in the Banach category, a smooth spray implies a smooth flow, we obtain

Corollary 4.4 *For Banach diffeomorphisms of S^1 with the right-invariant L^2 metric, the geodesic spray does not exist (equiv., the Christoffel symbol).*

Note we already seen this in Example 2.27 by calculating the Christoffel symbol explicitly for the smooth case, obtaining

$$\Gamma_\varphi(u, v) = \frac{vu_x + uv_x}{\varphi_x},$$

and showing that this formula only makes sense (as a Christoffel symbol) only for smooth diffeomorphisms.

Question: Is $\exp_{\mathrm{Id}}$ a C^1 map in some neighborhood of 0? Is the exponential map similarly badly behaved for the right-invariant L^2 metric on $\mathrm{Diff}_c(\mathbb{R})$ or $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R})$?

4.3 Camassa–Holm Equation: H^1 metric on $\mathrm{Diff}(S^1)$

The right-invariant H^1 metric on $\mathrm{Diff}(S^1)$ is much better behaved than the L^2 metric. In particular, we will prove that the exponential map exists, and is a smooth map on an open neighborhood of the zero section.[3] The proof of existence is of a type, known as the **Spray method of Ebin and Marsden** (or simply the Ebin-Marsden method), which was initiated for the study of the incompressible Euler equation in [EM70], which we will discuss later. The presentation in this section also follows later versions of the method, as presented in [Ebi15, Kol17, Bru17]. The main ingredients of the methods are:

1. **Working on Banach manifolds:** In general, a (geodesic) smooth spray on a Fréchet manifold does not guarantee existence of a (geodesic) flow. However, it does for Banach manifolds. Thus we will treat at first the larger space $\mathrm{Diff}_{H^k}(S^1)$ for some large enough k. This also gives us other analytic tools, like the inverse function theorem.

2. **Proving the spray is smooth:** The main goal is therefore to prove that the spray exists, and is smooth. Here we pay the price for working in the Banach category, as there are natural operations that are only continuous but not smooth in this case (e.g., the Lagrangian-to-Eulerian map $\varphi \mapsto u = \varphi_t \circ \varphi^{-1}$).

3. **Regularity:** Finally, we will show (quite generally) that for a right-invariant metric we have a **no-loss-no-gain** result — namely, that if the geodesic in $\mathrm{Diff}_{H^k}(M)$ has initial conditions of higher regularity, than this regularity remains throughout the existence of the geodesic. In particular, we will obtain short-time existence of geodesics also in the smooth category, as well as smooth dependence on the initial data. We will cover this in Section 4.6.

We will illustrate this method in detail for the right-invariant H^1-metric on

[3] $\exp_{\mathrm{Id}}$ is, in fact, a local diffeomorphism [CK02, Theorem 5], but we will not prove that here.

Diff(S^1).[4] Let us start by calculating its geodesic equation:

$$\begin{aligned}E(\varphi) &= \int_0^1 \int_{S^1} (u^2 + u_x^2)\, dx\, dt = \int_0^1 \int_{S^1} (\varphi_t^2 + (\varphi_{tx}/\varphi_x)^2) \circ \varphi^{-1}\, dx\, dt \\ &= \int_0^1 \int_{S^1} (\varphi_t^2 \varphi_x + \varphi_{tx}^2 \varphi_x^{-1})\, dx\, dt.\end{aligned}$$

Note that this calculation also shows that the metric itself is smooth. Thus we have

$$\begin{aligned}\delta E(\varphi)(h) &= \int_0^1 \int_{S^1} (2\varphi_t\varphi_x h_t + \varphi_t^2 h_x - \varphi_{tx}^2\varphi_x^{-2} h_x + 2\varphi_{tx}\varphi_x^{-1} h_{tx})\, dx\, dt \\ &= -\int_0^1 \int_{S^1} (2(\varphi_t\varphi_x)_t + (\varphi_t^2)_x - (\varphi_{tx}^2\varphi_x^{-2})_x - 2(\varphi_{tx}\varphi_x^{-1})_{tx}) h\, dx\, dt\end{aligned}$$

Hence we obtain

$$\begin{aligned}0 &= 2(\varphi_t\varphi_x)_t + (\varphi_t^2)_x - (\varphi_{tx}^2\varphi_x^{-2})_x - 2(\varphi_{tx}\varphi_x^{-1})_{tx} \\ &= 2(\varphi_{tt}\varphi_x + 2\varphi_t\varphi_{tx}) - (\varphi_{tx}^2\varphi_x^{-2})_x - 2(\varphi_{tx}\varphi_x^{-1})_{tx} \\ &= 2(u_t + 3uu_x) \circ \varphi\varphi_x - (\varphi_{tx}^2\varphi_x^{-2})_x - 2(\varphi_{tx}\varphi_x^{-1})_{tx} \\ &= 2(u_t + 3uu_x) \circ \varphi\varphi_x - (u_x^2 \circ \varphi)_x - 2(u_x \circ \varphi)_{tx} \\ &= 2(u_t + 3uu_x) \circ \varphi\varphi_x - 2u_x u_{xx} \circ \varphi\varphi_x - 2(u_{xx} \circ \varphi\varphi_x)_t \\ &= 2(u_t + 3uu_x) \circ \varphi\varphi_x - 2u_x u_{xx} \circ \varphi\varphi_x - 2(u_{xxt} \circ \varphi\varphi_x + uu_{xxx} \circ \varphi\varphi_x + u_{xx} \circ \varphi\varphi_{tx}) \\ &= 2(u_t + 3uu_x) \circ \varphi\varphi_x - 2u_x u_{xx} \circ \varphi\varphi_x - 2(u_{xxt} + uu_{xxx} + u_x u_{xx}) \circ \varphi\varphi_x\end{aligned}$$

where the transition to the third line is exactly as in the L^2 case, and the transition to the sixth line uses the fact that $(f(t, \varphi(t,x)))_t = f_t \circ \varphi + f_x \circ \varphi \varphi_t = (f_t + f_x u) \circ \varphi$. Therefore we obtain the **Camassa–Holm equation**

$$u_t + 3uu_x - 2u_x u_{xx} - uu_{xxx} - u_{xxt} = 0,$$

which we can also write as

$$Au_t + uAu_x + 2u_x Au = 0, \qquad A = 1 - \partial_x^2.$$

Now, note that $A : H^k(S^1) \to H^{k-2}(S^1)$ is a bounded linear operator (and in particular, smooth). It is also invertible (a solution of $u = A^{-1} f$ is immediately given by convolution using Fourier series); by the open mapping (Banach–Schauder) theorem A^{-1} is also bounded, hence smooth as well.

[4] This presentation is influenced by Klas Modin's lecture `https://slides.com/kmodin/diffeos`, and also by [CK02, §4].

The Camassa–Holm equation was introduced by Camassa and Holm in 1993 [CH93] as a model for shallow water waves, and was intensely studied since. It was observed to be a geodesic equation by Misiołek in 1998 [Mis98] (in fact, he showed that a slightly more general version of the equation is a geodesic equation of a right-invariant metric on an extension of the diffeomorphism group).

However, if we try to write this as an ODE, we run into problems:

$$u_t = -A^{-1}\left[uAu_x + 2u_xAu\right], \tag{4.7}$$

and, just counting derivatives, we get stuck — in order for it to be solved as an ODE on Banach spaces, then, assume that $u \in H^k(S^1)$, we would get a solution if we can write $u_t = F(u)$, where $F : H^k(S^1) \to H^k(S^1)$. However, if $u \in H^k(S^1)$, then $u_x \in H^{k-1}(S^1)$, so $uAu_x \in H^{k-3}(S^1)$, hence $A^{-1}(uAu_x) \in H^{s-1}(S^1)$ so we lose one derivative, hence $F : H^k(S^1) \to H^{k-1}(S^1)$ and we cannot use the contractive mapping theorem to conclude existence of solutions for short time.[5]

However, a miracle happens in Lagrangian coordinates:

$$\begin{aligned}
\varphi_{tt} &= (u\circ\varphi)_t = (u_t + uu_x)\circ\varphi \\
&= -A^{-1}\left(uAu_x + 2u_xAu - A(uu_x)\right)\circ\varphi \\
&= -A^{-1}\left(uAu_x + 2u_xAu - uAu_x + 3u_xu_{xx}\right)\circ\varphi \\
&= -A^{-1}\left(2uu_x + u_xu_{xx}\right)\circ\varphi
\end{aligned}$$

The operator $u \mapsto A^{-1}\left[2uu_x + u_xu_{xx}\right]$ is a smooth operator $H^k(S^1) \to H^k(S^1)$, so we are in good shape.[6]

However, we are not done: We are studying the system of ODEs

$$\begin{cases}\varphi_t = v \\ v_t = S_\varphi(v), \quad S_\varphi(v) = -A^{-1}\left(2(v\circ\varphi^{-1})(v\circ\varphi^{-1})_x + (v\circ\varphi^{-1})_x(v\circ\varphi^{-1})_{xx}\right)\circ\varphi,\end{cases}$$

[5] Note that, in the language of Example 2.28, this tells us that $\operatorname{ad}_u^T u = A^{-1}\left[uAu_x + 2u_xAu\right]$. This tells us, that the transpose to the adjoint operator exists for $\mathfrak{g} = C^\infty(S^1)$, but not for $\mathfrak{g} = H^k(S^1)$.

[6] A more conceptual version of this calculation is

$$\begin{aligned}
\varphi_{tt} &= -A^{-1}\left(uAu_x - A(uu_x) + 2u_xAu\right)\circ\varphi \\
&= -A^{-1}\left([\nabla_u, A]u + 2u_xAu\right)\circ\varphi.
\end{aligned}$$

but it is not obvious that that S is smooth since $\varphi \mapsto \varphi^{-1}$ is *not* smooth in $\mathrm{Diff}_{H^k}(S^1)$; this is the price we have to pay for working in the Banach category (in the Fréchet category S would have been smooth, but we would not have a "black-box" ODE methods).

The hope for proving that $(\varphi, v) \mapsto S_\varphi(v)$ is smooth even though the inversion is not comes from its conjugation structure:

$$S_\varphi(v) = (TR_\varphi \circ S \circ TR_{\varphi^{-1}})(v), \qquad S(u) = -A^{-1}\left((u^2 + \frac{1}{2}u_x^2)_x\right) = -A^{-1} \circ B \circ C(u),$$

where $B(u) = u^2 + \frac{1}{2}u_x^2$ and $C(u) = u_x$. The strategy is as follows: Define $(A^{-1})_\varphi = TR_\varphi \circ A^{-1} \circ TR_{\varphi^{-1}}$, and similarly B_φ and C_φ. We will prove that, for k large enough,

- $(\varphi, v) \mapsto (\varphi, C_\varphi(v))$ is smooth $\mathrm{Diff}_{H^k}(S^1) \times H^k(S^1) \to \mathrm{Diff}_{H^k}(S^1) \times H^{k-1}(S^1)$.
- $(\varphi, v) \mapsto (\varphi, B_\varphi(v))$ is smooth $\mathrm{Diff}_{H^k}(S^1) \times H^{k-1}(S^1) \to \mathrm{Diff}_{H^k}(S^1) \times H^{k-2}(S^1)$.
- $(\varphi, v) \mapsto (\varphi, (A^{-1})_\varphi(v))$ is smooth $\mathrm{Diff}_{H^k}(S^1) \times H^{k-2}(S^1) \to \mathrm{Diff}_{H^k}(S^1) \times H^k(S^1)$ in some neighborhood of $(\mathrm{Id}, 0)$

This will show that $\exp : T\,\mathrm{Diff}_{H^k}(S^1) \cong \mathrm{Diff}_{H^k}(S^1) \times H^k(S^1) \to \mathrm{Diff}_{H^k}(S^1)$ is smooth in a neighborhood of $(\mathrm{Id}, 0)$. However, this suffices for proving that exp is smooth in the neighborhood of the zero section: Indeed, fix $\varphi_0 \in \mathrm{Diff}_{H^k}(S^1)$, then, from the right-invariance, we have

$$\exp(\psi, v) = \exp(\psi \circ \varphi_0^{-1}, v \circ \varphi_0^{-1}) \circ \varphi_0 = R_{\varphi_0} \circ \exp \circ TR_{\varphi_0^{-1}}(\psi, v).$$

In a neighborhood of $(\varphi_0, 0)$, $TR_{\varphi_0^{-1}}(\psi, v)$ will be in the neighborhood of $(\mathrm{Id}, 0)$ in which we have shown that exp is smooth, and thus on this neighborhood exp will be smooth as a composition of smooth maps (since right translation is smooth, and φ_0 and φ_0^{-1} are fixed).

For simplicity, let us see first that $(\varphi, v) \mapsto (\varphi, B_\varphi(v))$ is smooth. Writing B_φ explicitly, we have

$$B_\varphi(v) = (v \circ \varphi^{-1})_x \circ \varphi = v_x \varphi_x^{-1},$$

which is a continuous map $\mathrm{Diff}_{H^k}(S^1) \times H^{k-1}(S^1) \to \mathrm{Diff}_{H^k}(S^1) \times H^{k-2}(S^1)$ as long as $k - 1 > 1/2$, since then φ_x is C^∞ and bounded away from zero, and by the module property of Sobolev spaces (see Proposition 3.8). Since B_φ is linear in v, in order to show it is C^∞ we only need to check the derivative with respect to φ. Let $t \mapsto \varphi(t)$ be some curve with $\varphi(0) = \varphi$ and $\varphi_t(0) = w \in H^k(S^1)$. Then, the derivative with respect to

$$\partial_\varphi B_\varphi(w)(v) = \frac{d}{dt}\bigg|_{t=0} B_\varphi(v) = \frac{d}{dt}\bigg|_{t=0} (v_x \varphi_x^{-1}) = -v_x \varphi_x^{-2} w_x,$$

which is again continuous as long as $k - 1 > 1/2$. Similarly,

$$\partial_\varphi^n B_\varphi(w_1, \ldots, w_n)(v) = (-1)^n \varphi_x^{-n-1} v_x (w_1)_x \ldots (w_n)_x.$$

Thus the map $(\varphi, v) \mapsto (\varphi, B_\varphi(v))$ is indeed smooth. The smoothness of $(\varphi, v) \mapsto (\varphi, C_\varphi(v))$ follows a similar path (with more cumbersome calculations).

We now turn to $(\varphi, v) \mapsto (\varphi, (A^{-1})_\varphi(v))$. First, we note that its inverse map is $(\varphi, v) \mapsto (\varphi, A_\varphi(v))$, and this map is a smooth map $\mathrm{Diff}_{H^k}(S^1) \times H^k(S^1) \to \mathrm{Diff}_{H^k}(S^1) \times H^{k-2}(S^1)$ by similar calculations as for B_φ (recall that $A = 1 - \partial_x^2$), provided that $k - 2 > 1/2$. In order to prove smoothness of the inverse, one can calculate the derivative of $(\varphi, v) \mapsto (\varphi, A_\varphi(v))$ at $(\mathrm{Id}, 0)$ and show that it is invertible. By the inverse function theorem, this will imply that $(\varphi, v) \mapsto (\varphi, (A^{-1})_\varphi(v))$ is smooth near $(\mathrm{Id}, 0)$. This is the approach we will take here. The more general approach (see [Kol17, §2, §4], [EK14]) is to look at the map $\varphi \mapsto A_\varphi$ as a map $\mathrm{Diff}_{H^k}(S^1) \to \mathrm{Aut}(H^{k-2}(S^1), H^k(S^1))$, and show that the map $P \mapsto P^{-1}$ on automorphisms of Banach spaces is smooth (in fact, real analytic). This will imply that $\varphi \mapsto (A^{-1})_\varphi$ is smooth, and consequently $(\varphi, v) \mapsto (\varphi, (A^{-1})_\varphi(v))$ is smooth, since the evaluation map of continuous linear maps between Banach spaces is a smooth operation.

Let us calculate the derivative of $(\varphi, v) \mapsto (\varphi, A_\varphi(v))$ at $(\mathrm{Id}, 0)$. Denote this map by $F = (F_1, F_2)$. Then

$$D_1 F_1 = \mathrm{Id} \qquad D_2 F_1 = 0,$$

Since $F_2(\varphi, v)$ is linear in v, we have

$$D_2 F_2(\varphi, v)(w) = F_2(\varphi, w) = w - w_{xx} \varphi_x^{-2} + w_x \varphi_x^{-3} \varphi_{xx}.$$

In particular, at $(\mathrm{Id}, 0)$, we have

$$D_2 F_2(\mathrm{Id}, 0)(w) = (1 - \partial_x^2)(w),$$

which is, as we have seen before, an isomorphism $H^k(S^1) \mapsto H^{k-2}(S^1)$. Thus we obtain that

$$DF_{(\mathrm{Id},0)} = \begin{pmatrix} \mathrm{Id} & 0 \\ * & 1-\partial_x^2 \end{pmatrix} \in L(H^k(S^1) \times H^k(S^1), H^k(S^1) \times H^{k-2}(S^1)).$$

which is invertible (in fact, $* = 0$ here). To conclude, we have completes the proof of the following result:

Theorem 4.5 *The exponential map of the right-invariant H^1 metric on $\mathrm{Diff}_{H^k}(S^1)$, $k \geq 3$, is a smooth map in a neighborhood of the zero section. Moreover, $\exp_{\mathrm{Id}} : H^k(S^1) \to \mathrm{Diff}_{H^k}(S^1)$ is a local diffeomorphism near zero.*

4.4 Local existence for other metrics on $\mathrm{Diff}_{H^k}(M)$

Here we follow [Kol17]. For a closed manifold (or $\mathbb{R}^n$), a general right-invariant metric on of the type

$$(u,v)_A = \int_M g(Au, v)\,\mathrm{dVol}_g$$

where $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a symmetric, positive definite, pseudo-differential operator, the geodesic equation for geodesic emanating from the identity (also know as *Euler-Poincaré equation (EPDiff)*) is given by

$$m_t + \nabla_u m + (\nabla u)^t m + (\operatorname{div} u) m = 0, \qquad m := Au,$$

where $(\nabla u)^t$ is the adjoint of ∇u with respect to the Riemannian metric g of M (adjoint in both the vector and co-vector parts, so it is still a $(1,1)$-tensor). Au is also known as the momentum. If A in invertible, can write the equation as

$$u_t = -A^{-1}\left(\nabla_u Au + (\nabla u)^t Au + (\operatorname{div} u) A_u\right),$$

which is known as the *Euler–Arnold equation* for $\mathrm{Diff}(M)$. Note that the red term shows us that this equation is not an ODE in the Eulerian variable u. Moving to the Lagrangian variable φ, we obtain that the equation is

$$\begin{cases} \varphi_t = v, \\ v_t = \{A^{-1}\left([A, \nabla_u]u - (\nabla u)^t Au - (\operatorname{div} u) A_u\right)\} \circ \varphi =: S(u) \circ \varphi \end{cases}$$

where $u = v \circ \varphi^{-1}$. Note the similarity to the H^1 case we discussed in detail. If A is a differential operator of order 1 at least (which corresponds to an H^l metric, $l \geq 1/2$), then the order of the red term is the same as A, so, at least from a perspective of counting derivative, there is a possibility of solving this system by ODE methods. This also shows, why in the right-invariant L^2-case, A is the identity map, this method does not work.

The geodesic spray in this case is given by

$$(\varphi, v) \mapsto (\varphi, v, v, S_\varphi(v)), \qquad S_\varphi = TR_\varphi \circ S \circ TR_{\varphi^{-1}}.$$

Proving local well-posedness of the geodesic equation in $\mathrm{Diff}_{H^k}(M)$ then boils down to prove that the map $(\varphi, v) \mapsto S_\varphi(v)$ is smooth (or at least Lipschitz). This can be done for a wide variety of metrics — basically, on $\mathrm{Diff}_{H^k}(M)$, for all H^l-metrics such that $l \geq 1/2$, and $k - 1 > \dim M/2$ and $k - 2l \geq 0$, [BEK15, Theorem 5.4] (fractional l on $\mathbb{R}^d$ and $\mathbb{T}^d$), [MP10] (integer l for general closed manifolds).

Note that if $k - 1 > \dim M/2$, then the H_k-metric on $\mathrm{Diff}_{H^k}(M)$ is a strong metric. In this case, the local well-posedness of the geodesic equation is immediate by general results on the exponential maps of strong metrics (Theorem 2.34). We will later show that in this case we also have long term existence, and, in fact, metric and geodesic completeness.

4.5 Incompressible Euler: L^2 metric on $\mathrm{Diff}_\mu(M)$

We now turn to the right-invariant L^2 metric on $\mathrm{Diff}_\mu(M)$. It is a restriction of the right-invariant L^2 metric on $\mathrm{Diff}(M)$, which seems to be bad news, since, as we have seen in Section 4.2, it is badly behaved. However, it is also the restriction of the *non-invariant* L^2 metric on $\mathrm{Diff}(M)$, which we defined in Example 2.35 on $C^\infty(M, M)$, and which has a good local behavior (smooth exponential map). We will denote the non-invariant L^2 metric on $\mathrm{Diff}(M)$ by $\bar{G}$, and by $G = \bar{G}|_{\mathrm{Diff}_\mu(M)}$ its restriction to $\mathrm{Diff}_\mu(M)$. Note, that indeed G is right-invariant: For $\varphi \in \mathrm{Diff}_\mu(M)$ we have

$$\begin{aligned} G_\varphi(u, v) &= \int_M g_{\varphi(x)}(u(x), v(x))\, \mu(x) = \int_M g_x(u \circ \varphi^{-1}(x), v \circ \varphi^{-1}(x))\, \varphi^* \mu(x) \\ &= \int_M g_x(u \circ \varphi^{-1}(x), v \circ \varphi^{-1}(x))\, \mu(x) = G_{\mathrm{Id}}(u \circ \varphi^{-1}, v \circ \varphi^{-1}), \end{aligned}$$

where the transition to the second line follows by the fact that φ is volume-preserving. We will therefore be able to use the simplicity of the non-invariant metric on $\mathrm{Diff}(M)$ to study the invariant metric on $\mathrm{Diff}_\mu(M)$. This is the approach used by Ebin and Marsden in their seminal paper [EM70]; the presentation below borrows also from [Ebi15].

First, let us calculate the geodesic equation:

Theorem 4.6 (Arnold 1966) Let (M,g) be a closed Riemannian manifold. The geodesic equation of the right-invariant L^2 metric on $\mathrm{Diff}_\mu(M)$,

$$G_\varphi(v_1,v_2) = \int_M g_{\varphi(x)}(v_1(x),v_2(x))\,\mu(x), \qquad v_i \in T_\varphi \mathrm{Diff}_\mu(M) \cong \mathfrak{X}_\mu(M)\circ\varphi$$

is, in Eulerian coordinates, in incompressible Euler equation

$$\begin{cases} u_t + \nabla_u u = -\nabla p \\ \mathrm{div}_\mu\, u = 0, \end{cases} \tag{4.8}$$

where u is a vector field and p a scalar funcion.

Before proving it, let us first recall the decomposition of a vector field into its solonoidal and gradient parts, that is, the Hodge-Helmholtz decomposition: Let u be a vector field; we can decompose it into

$$u = v + \nabla f, \qquad \mathrm{div}_\mu\, v = 0,$$

v is uniquely determined, and f is uniquely determined up to a constant. Moreover, v and ∇f are L^2-orthogonal (with respect to g), so we can write

$$v = Pu, \quad \nabla f = Qu,$$

where

$$P : \mathfrak{X}(M) \to \mathfrak{X}_\mu(M) \quad Q : \mathfrak{X}(M) \to \mathfrak{X}_\mu(M)^\perp$$

are the orthogonal projections. We can write Q explicitly via

$$Qu = \nabla\Delta^{-1}\,\mathrm{div}_\mu\, u. \tag{4.9}$$

Indeed:

$$\nabla\Delta^{-1}\,\mathrm{div}_\mu(v+\nabla f) = \nabla\Delta^{-1}\Delta f = \nabla f = Qu.$$

Note that we used the fact that Δ^{-1} is well defined in this case. This is true only functions of zero mean, since constant functions are harmonic functions on M; however, since they are the only harmonic functions (which follows from integration by parts), then $\nabla\Delta^{-1}$ is well-defined (one can also put the projection on zero mean function between div and Δ^{-1}). All the above arguments hold also for Sobolev vector fields $\mathfrak{X}^{H^k}(M)$.

We can now prove Theorem 4.6.

Proof: Let $\varphi : [0,1] \to \mathrm{Diff}_\mu(M)$, with $\varphi(0) = \mathrm{Id}$, whose energy is

$$E(\varphi) = \frac{1}{2}\int_0^1 \int_M g_{\varphi(x)}(\varphi_t(x), \varphi_t(x))\, d\mu(x)\, dt.$$

Let $\varphi : \mathbb{R} \times [0,1] \to \mathrm{Diff}_\mu(M)$ be a variation of φ, fixing the endpoint, then Then, repeatedly using the fact that $\varphi(s,t,\cdot)$ is volume preserving, we have

$$\begin{aligned}
\partial_s E(\varphi(s,\ )) &= \frac{1}{2}\int_0^1 \int_M \partial_s|_{s=0} g_\varphi(\varphi_t, \varphi_t)\, \mu\, dt = \int_0^1 \int_M g_\varphi(\nabla^g_{\partial_s}\varphi_t, \varphi_t)\, \mu\, dt \\
&= \int_0^1 \int_M g_\varphi(\nabla^g_{\partial_t}\varphi_s, \varphi_t)\, \mu\, dt \\
&= \int_0^1 \int_M \partial_t g_\varphi(\varphi_s, \varphi_t) - g_\varphi(\varphi_s, \nabla^g_{\partial_t}\varphi_t)\, \mu\, dt \\
&= \int_M g_\varphi(\varphi_s, \varphi_t)|_{t=1} - g_\varphi(\varphi_s, \varphi_t)|_{t=0}\, \mu - \int_0^1 \int_M g_\varphi(\varphi_s, \nabla^g_{\partial_t}\varphi_t)\, \mu\, dt \\
&= -\int_0^1 \int_M g_\varphi(\varphi_s, \nabla^g_{\partial_t}\varphi_t)\, \mu\, dt \\
&= -\int_0^1 \int_M g_{\mathrm{Id}}(\varphi_s \circ \varphi^{-1}, \nabla^g_{\partial_t}\varphi_t \circ \varphi^{-1})\, \mu\, dt,
\end{aligned}$$

Now denoting $u = \varphi_t \circ \varphi^{-1}|_{s=0}$ and $w = \varphi_s \circ \varphi^{-1}|_{s=0}$, both are maps $[0,1] \to \mathfrak{X}_\mu(M)$, we obtain

$$\delta E_\varphi(w) = -\int_0^1 \int_M g(w, u_t + \nabla_u u)\, \mu\, dt.$$

Therefore, if φ is a geodesic, this equals to zero for every $w : [0,1] \to \mathfrak{X}_\mu(M)$, and thus

$$u_t + \nabla_u u \in \mathfrak{X}_\mu(M)^\perp.$$

Since $\mathfrak{X}_\mu(M)^\perp$ consists of gradient of functions, we obtain the result. Note that we can also write the equation as

$$u_t + \nabla_u u = Q(u_t + \nabla_u u) = Q(\nabla_u u),$$

where the second equality holds since $\operatorname{div}_\mu(u_t) = \partial_t(\operatorname{div}_\mu(u)) = 0$. ■

We now want to prove local existence of the geodesic equation. The idea is similar — we will prove that for k large enough, the geodesic spray in $\mathrm{Diff}_\mu^{H^k}(M)$ is smooth, and as such, the exponential map exists as a smooth map on a neighborhood of the zero section. In the next section, we will see that from that, we get existence also in $\mathrm{Diff}_\mu(M)$.

Theorem 4.7 (Existence for incompressible Euler in H^k) *Let (M, g) be a closed Riemannian manifold, and let $k > \frac{\dim M}{2} + 1$. Then the exponential map of the right-invariant L^2-metric on $\mathrm{Diff}_\mu^{H^k}(M)$ is a smooth map on a neighborhood of the zero section. In particular, this shows that the incomporessible Euler equation (4.8) has a unique solution, at least for small times, for every initial condition u_0, and that the solution depends continuously on the u_0.*

Comment: Note that for the solution in Lagrangian coordinates we obtain smooth dependence on the initial condition, as the exponential map is smooth, that is $u_0 \mapsto \exp_{\mathrm{Id}}(u_0)$ is a smooth map. However, in Eulerian coordinates we only obtain only continuous dependence: Denote by $\varphi(t)$ the solution in Lagrangian coordinates, then $u_0 \mapsto u|t = 1$ is the map $u_0 \mapsto \varphi_t(1, \cdot) \circ \varphi^{-1}(1, \cdot)$, and the map $\varphi \mapsto \varphi^{-1}$ is merely continuous [EM70, Theorem 15.2(ii)–(iii)]. Note that this is not just a matter of our methods, but it has been shown that the solution map in H^k is not even uniformly continuous [HM10]. We will later see that the map $u_0 \mapsto \exp_{id}(u_0)$ is smooth also in the smooth category. In this case, since $\mathrm{Diff}_\mu(M)$ is a Lie group, the data-to-solution map in smooth also in the Eulerian variable.

Proof: Denote by $\bar{G}$ the non-invariant L^2 metric on $\mathrm{Diff}^k(M)$, and by $G = \bar{G}|_{\mathrm{Diff}_\mu^k(M)}$ the right-invariant L^2 metric on $\mathrm{Diff}_\mu^k(M)$. Recall that $\mathrm{Diff}_\mu^k(M)$ is a closed submanifold of $\mathrm{Diff}^k(M)$ (Theorem 3.12). In Example 2.35 we have shown the following, regarding the metric $\bar{G}$ (everything was discussed in smooth settings, but the same arguments hold also in Sobolev settings): We showed that the geodesic spray $\bar{S} \in \mathfrak{X}(T\,\mathrm{Diff}^k(M))$ exists and is given by

$$\bar{S}(u) = \sigma \circ u \qquad u \in \Gamma(\varphi^* TM),$$

where $\sigma \in \mathfrak{X}(TM)$ is the geodesic spray of M. In particular, the existence of $\bar{S}$ is equivalent to the existence of a covariant derivative $\bar{\nabla}$ (and both imply that the exponential map exists as a smooth map). In order to prove the the exponential map exists as a smooth map for G as well, it is sufficient to prove that its covariant derivative ∇ exists as a smooth map. However, there is a simple relation between $\bar{\nabla}$ and ∇, as in finite dimensional Riemannian geometry:

$$\nabla = \mathcal{P} \circ \bar{\nabla},$$

where $\mathcal{P} : T\,\mathrm{Diff}^k(M)|_{\mathrm{Diff}^k_\mu(M)} \to T\,\mathrm{Diff}^k_\mu(M)$ is the $\bar{G}$-orthogonal projection. Thus, it is sufficient to prove that $\mathcal{P}$ is a smooth map. Recall that we denoted by $P : \mathfrak{X}^k(M) \to \mathfrak{X}^k_\mu(M)$ the orthogonal projection on divergence-free vector fields. It is immediate that

$$\mathcal{P}_\varphi(v) = P(v \circ \varphi^{-1}) \circ \varphi.$$

Thus $\mathcal{P}$ is a twisted map based on P, of a similar structure as we have seen in the previous Sections. From (4.9) we have that

$$P = I - Q = I - \nabla \Delta^{-1} \operatorname{div}_\mu,$$

and thus $\mathcal{P}$ is a twisted pseudo-differential operator of order 0, whose smoothness follows similar lines as Theorem 4.5 and Section 4.4. ■

4.6 Regularity

In this section we will see how right-invariance can be used to obtain regularity. The idea dates back to Ebin–Marsden [EM70, §12], but we will follow a generalization due to Bruveris [Bru17]:

Theorem 4.8 (Exchanging regularities of a $\mathrm{Diff}(M)$-equivariant map) *Let M be a compact manifold, N and P smooth manifolds (without boundary). Let $k-1 > \dim M/2$, $0 \le l \le q$, and $F : U \subset H^k(M,N) \to H^k(M,P)$, where U is open, be a $\mathrm{Diff}_{H^k}(M)$-equivariant map, i.e., $F(u \circ \varphi) = F(u) \circ \varphi$. Then, if F is C^q, then it maps $H^{k+l}(M,N)$ to $H^{k+l}(M,P)$, and $F : U \cap H^{k+l}(M,N) \to H^{k+l}(M,P)$ is C^{q-l}.*

An immediate corollary is

Corollary 4.9 *Under the above hypotheses, if* $F : H^k(M,N) \to H^k(M,P)$ *is* C^∞*, then* $F : C^\infty(M,N) \to C^\infty(M,P)$ *is* C^∞*.*

In our case, let g be a smooth, right-invariant metric on a compact manifold without boundary M. Let $U \subset T\,\mathrm{Diff}_{H^k}(M) \cong H^k(M,TM)$ be a neighborhood of the zero section on which the exponential map is defined, and consider

$$F = \exp : U \to \mathrm{Diff}_{H^k}(M) \subset H^k(M,M)$$

We immediately obtain

Corollary 4.10 *Assume* G *is a smooth, right-invariant Riemannian metric on* $\mathrm{Diff}_{H^k}(M)$*, where* M *is a compact manifold without boundary* M*, and* $k-1 > \dim(M)/2$*. If the exponential map is smooth on an open neighborhood of the zero section of* $T\,\mathrm{Diff}_{H^k}(M)$*, then the exponential map exists and is smooth on* $\mathrm{Diff}_{H^{k+l}}(M)$ *for any* $l > 0$*, and on* $\mathrm{Diff}(M)$*.*

Thus we can lift all the results from the previous section to the smooth category. This result is known as a **no-loss-no-gain** result: namely, if we consider a right-invariant metric on $\mathrm{Diff}_{H^k}(M)$, then the geodesic will continue to be as regular as it initial condition. In particular, if the initial condition happens to be smooth, then the geodesic will continue being smooth as long as it exists.

We now prove Theorem 4.8. We will need the following generalization of Proposition 3.9, (see also Footnote 1):

Lemma 4.11 *Let* $k-1 > \dim M/2$ *and* $q \in \mathbb{N}$*. Then*

$$H^{k+q}(M,\mathbb{R}^n) \times \mathrm{Diff}_{H^k}(M) \to H^k(M,\mathbb{R}^n), \qquad (f,\varphi) \mapsto f \circ \varphi$$

is a C^q*-map.*

Proof: (of Theorem 4.8) To simplify notation, we will assume that F is define on the whole $H^q(M,N)$ (All our arguments will be local in nature so it will not matter).

Step I: reduction fo $N = \mathbb{R}^n$ and $P = \mathbb{R}^m$. Let us embed N and P into Euclidean space, and let $N_0 \subset \mathbb{R}^{n_0}$ and $P_0 \subset \mathbb{R}^{m_0}$ be tubular neighborhoods of N and P. Denote the inclusions and retraction maps by $\iota_N : N \to N_0$ and $r_N : N_0 \to N$ and similarly for p. Note that we can extend F to $F_0 : H^k(M, N_0) \to H^k(M, P_0)$ via

$$F_0(u_0) = \iota_p \circ F \circ r_N \circ u_0.$$

Note that if F is C^q, then F_0 is also C^q, since composition with C^∞ function is a C^∞ operator on Sobolev spaces. This extension is still $\mathrm{Diff}_{H^k}(M)$-equivariant, as

$$F_0(u_0 \circ \varphi) = \iota_p \circ F(r_N \circ u_0 \circ \varphi) = \iota_p \circ F(r_N \circ u_0) \circ \varphi = F_0(u_0) \circ \varphi.$$

Since $H^k(M, N_0)$ is an open subset of $H^k(M, \mathbb{R}^{n_0})$, and similarly for P_0. Thus, if the theorem is proven for Euclidean target spaces, the result holds for $F_0 : H^k(M, N_0) \to H^k(M, P_0)$ as well. Thus we obtain that $F_0 : H^{k+l}(M, N_0) \to H^{k+l}(M, P_0)$ is C^{q-l}; since

$$F(u) = r_p \circ \iota_p F(r_N \circ \iota_N \circ u) = r_p \circ F_0(\iota_N \circ u),$$

we obtain that $F : H^{k+l}(M, N) \to H^{k+l}(M, P)$ is C^{q-l} as well.

Step II: If $F : H^k \to H^k$ is C^1, then $F : H^{k+1} \to H^{k+1}$ is C^0. Let $X_1, \dots, X_A \in \mathfrak{X}(M)$ be smooth vector fields such that

$$\mathrm{span}\,\{X_1(x), \dots, X_A(x)\} = T_x M$$

for all $x \in M$. Then, an equivalent norm for $H^r(M, \mathbb{R}^n)$ is

$$\|u\|_{H^r} \sim \|u\|_{H^{r-1}} + \sum_{j=1}^{A} \|Tu.X_j\|_{H^{r-1}}. \tag{4.10}$$

Let $\varphi^j : (-\varepsilon, \varepsilon) \to \mathrm{Diff}(M)$ be any smooth map satisfying $\varphi^j(0) = \mathrm{Id}$, $\dot{\varphi}^j(0) = X_j$ (for example, take the flow of X_j). Fix $u \in H^{k+1}(M, \mathbb{R}^n)$. By Lemma 4.11, the map

$$\mathbb{R} \to H^k(M, \mathbb{R}^n), \qquad t \mapsto u \circ \varphi^j(t)$$

is C^1. Since $F : H^k \to H^k$ is C^1, we obtain that the map

$$\mathbb{R} \to H^k(M, \mathbb{R}^m), \qquad t \mapsto F(u \circ \varphi^j(t)) = F(u) \circ \varphi^j(t)$$

is C^1. Differentiating at $t = 0$, we obtain

$$DF_u(Tu.X_j) = T(F(u)).X_j,$$

and taking the H^k-norm on both sides we obtain

$$\|T(F(u)).X_j\|_{H^k} \le \|DF_u\|_{L(H^k,H^k)}\|Tu.X_j\|_{H^k}$$

hence by (4.10),

$$\|F(u)\|_{H^{k+1}} \lesssim \|F(u)\|_{H^k} + \|DF_u\|_{L(H^k,H^k)}\|u\|_{H^{k+1}},$$

hence $F(u) \in H^{k+1}(M, \mathbb{R}^m)$.

Continuity follows in a similar manner: By differentiating $t \mapsto F(u \circ \varphi^j(t)) - F(v \circ \varphi^j(t))$, we obtain

$$\|F(u)-F(v)\|_{H^{k+1}} \lesssim \|F(u)-F(v)\|_{H^k}+\|DF_u-DF_v\|_{L(H^k,H^k)}\|u\|_{H^{k+1}}+\|DF_v\|_{L(H^k,H^k)}\|u-v\|_{H^{k+1}}.$$

If u, v are close in H^{k+1}, then $\|u - v\|_{H^{k+1}}$ is small, and since they are close in H^k and $\|DF_u - DF_v\|_{L(H^k,H^k)}$ is small (since $F : H^k \to H^k$ is C^1). Thus $F : H^{k+1} \to H^{k+1}$ is continuous.

Step III: Induction. The rest of the proof does not involve the equivariance explicitly, but is rather a clever induction, and we will not repeat it here. See [Bru17, Steps 2–3, pp. 15–16]. ∎

Note that we can extend Theorem 4.8 to functions that are equivariant with respect to submanifolds of the whole diffeomorphism groups. Namely, the place where the equivariance was needed was in equivariance of the map with respect to the flow of the vector fields $X_1, \dots X_A$ (see Step II of the proof). Thus, whenever we have spanning vector fields such that F is invariant under their action, we obtain the same result. In particular, this is true for functions that are equivariant under volume-preserving diffeomorphisms:[7]

[7] This combines [Bru17] with the main idea of [EM70, Lemma 12.2].

Theorem 4.12 (Exchanging regularities of a $\mathrm{Diff}_\mu(M)$-equivariant map) *Let M be a compact manifold with a volume form μ, N and P smooth manifolds (without boundary). Let $k-1 > \dim M/2$, $0 \le l \le q$, and $F : U \subset H^k(M,N) \to H^k(M,P)$, where V is open, be a $\mathrm{Diff}_\mu^{H^k}(M)$-equivariant map, i.e., $F(u\circ\varphi) = F(u)\circ\varphi$. Then, if F is C^q, then it maps $H^{k+l}(M,N)$ to $H^{k+l}(M,P)$, and $F : U\cap H^{k+l}(M,N) \to H^{k+l}(M,P)$ is C^{q-l}.*

Proof: All we need to show is that we can construct divergence-free vector fields $X_1,\dots X_A \in \mathfrak{X}_\mu(M)$ such that, for every $x\in M$,

$$\operatorname{span}\{X_1(x),\dots,X_A(x)\} = T_xM.$$

Indeed, the flow of these vector fields generates volume-preserving diffeomorphisms, on which F is equivariant. By covering M with coordinate patches, we can assume that $M=\mathbb{R}^n$ and that we want need to prove this for x in the unit disc D^n. In these coordinates, we have that $\mu = \rho dx^1\wedge\dots\wedge dx^n$ for some positive smooth function ρ. Let $\lambda\in C_c^\infty(\mathbb{R}^n)$ be a bump function, such that $\lambda|_{D^n}\equiv 1$. Define $f(x) = \lambda(x)x_2$, and let

$$X_1 = \left(\frac{1}{\rho}\partial_2 f, -\frac{1}{\rho}\partial_1 f, 0,\dots,0\right).$$

X_1 is divergence-free: Indeed

$$\operatorname{div}_\mu(X_1) = \frac{1}{\rho}\partial_i(\rho X_1^i) = \frac{1}{\rho}(\partial_1\partial_2 f - \partial_2\partial_1 f) = 0.$$

Moreover,

$$X_1|_{D^n} = \frac{1}{\rho}e^1.$$

Similarly, we can find $X_i\in\mathfrak{X}_\mu(M)$ such that $X_i|_{D^n} = \rho^{-1}e^i$. This concludes the proof. ∎

Note that we cannot directly obtain the equivalent of Corollary 4.10 also holds for right-invariant metrics on $\mathrm{Diff}_\mu^{H^k}(M)$, since we do not have an identification of $T\,\mathrm{Diff}_\mu^{H^k}(M)$ with a Sobolev space. However, given the construction of X_i we can apply the same method of proof to obtain regularity of the exponential map (see [EM70, Theorem 12.1]).

4.6.1 Boundary value problem regularity

Note that what we proved so far is **initial value problem regularity** — that is, that if the initial data if smooth, the geodesic flow remains smooth as long as it exists. A related question is **boundary value problem regularity**:

Question:

1. Global question: Assume that $\varphi \in \mathrm{Diff}(M)$ such that there exists an H^k-geodesic from Id to φ.[8] Is the geodesic smooth for all time?
2. Local question: Corollary 4.10 implies that $\exp_{\mathrm{Id}} : U \subset \mathfrak{X}(M) \to \mathrm{Diff}(M)$ is a smooth map in many cases. Is it a local diffeomorphism? (Note that for the Banach case this is immediate by the inverse function theorem since $(D\exp_{\mathrm{Id}})_0 = \mathrm{Id}_{\mathfrak{X}(M)}$.)

Note that, in view of the initial value regularity, the first question can be rephrased as

If $\varphi = \exp_{id}(X) \in H^{k+l}$, is $X \in H^{k+l}$?

For this we can give a conditional result [Bru17, §5]:

Proposition 4.13 *Let g be a smooth, right-invariant metric on $\mathrm{Diff}^k(M)$ for some $k-1 > \dim M/2$, with a smooth exponential map $\exp : U \subset T\,\mathrm{Diff}^k(M) \to \mathrm{Diff}^k(M)$. Assume that $\varphi_1 = \exp(\varphi_0, X)$, where $\varphi_0, \varphi_1 \in \mathrm{Diff}^{k+l}(M)$ for some $1 \le l \le \infty$. Then, if $\pi \times \exp : U \to \mathrm{Diff}^k(M) \times \mathrm{Diff}^k(M)$ is locally invertible around X, then $X \in T\,\mathrm{Diff}^{k+l}(M)$.*

The proof is immediate by noting that the local inverse is still $\mathrm{Diff}^k(M)$-equivariant. This relates the boundary value problem to the question of **conjugate points**:

Definition 4.14 *Let $(\mathcal{M}, g)$ be a Riemannian manifold with a smooth exponential maps. Two points $p, q \in \mathcal{M}$, connected by a geodesic γ are called **conjugate points along** γ, if $d(\exp_p)_{\dot\gamma(0)} : T_p\mathcal{M} \to T_q\mathcal{M}$ is non invertible.*

[8] note that we showed that $\exp_{id} : V \subset \mathfrak{X}_{H^k}(M) \to \mathrm{Diff}_{H^k}(M)$ is a local diffeomorphism in many cases, so this assumption holds in an H^k neighborhoow of Id.

In our case, if φ_0 and φ_1 are nonconjugate in $(\mathrm{Diff}^k(M), g)$ along a geodesic $\varphi(t)$, then $\pi \times \exp : U \to \mathrm{Diff}^k(M) \times \mathrm{Diff}^k(M)$ is locally invertible around $\dot\varphi(0)$. We therefore obtain

Corollary 4.15 *Let g be a smooth, right-invariant metric on $\mathrm{Diff}^k(M)$ for some $k - 1 > \dim M/2$. Let $\varphi_0, \varphi_1 \in \mathrm{Diff}(M)$ that are nonconjugate along a geodesic $\varphi(t)$, then $\varphi(t)$ is smooth for all $t \in [0,1]$.*

The boundary value regularity problem is thus closely related to the study of conjugate points. Some results about them (basically that for metrics of order > 1/2 they are isolated along finite geodesic segments) can be found in [MP10].

Non-conditional boundary value results are often of a local nature:

- For the Camassa–Holm equation (H^1-metric on $\mathrm{Diff}(S^1)$), it is proven in [CK02, Lemma 4, Theorem 5] that $\exp_{\mathrm{Id}}$ is a local diffeo in the smooth category.
- In [KLT08], this was proven for H^k metrics, $k \geq 1$, on $\mathrm{Diff}(\mathbb{T}^2)$.
- A global, unconditional, boundary regularity result for the L^2 metric on $\mathrm{Diff}_\mu(M)$ (incompressible Euler equation), when $\dim M = 2$ was recently announced by Patrick Heslin.

4.7 The miracle of the Hunter–Saxton equation

In this last part of this chapter, we will another way in which the viewpoint of a PDE as a geodesic equation can give us tools to solve it. The PDE that we will study is the **Hunter–Saxton equation** that was introduced by Hunter and Saxton in 1991 in the study of nematic liquid crystals [HS91]. It was given a geometric interpretation by Khesin and Misiołek in 2003 [KM03]. The equation is

$$u_{txx} + 2u_x u_{xx} + u u_{xxx} = 0, \tag{4.11}$$

where u is a time-dependent vector field of a one-dimensional manifold. We will study two cases:

- The periodic case $u : [0,T) \times S^1 \to \mathbb{R}$.
- The non-periodic case $u : [0,T) \times \mathbb{R} \to \mathbb{R}$.

We will generally follow [Len07] for the periodic case and [BBM14] for the non-periodic case.

Following a calculation similar to § 4.3, it is easy to see that this is the geodesic equation of the $\dot{H}^1$ metric:[9]

$$G_\varphi(u,v) = \int_M (u \circ \varphi^{-1})_x (v \circ \varphi^{-1})_x \, dx = \int_M u_x v_x \varphi_x^{-1} \, dx \tag{4.12}$$

where M is either S^1 or $\mathbb{R}$. We need to be more precise now, since in the periodic case G is not a metric but just a semi-metric, and in the non-periodic case we need to specify decay conditions. We will therefore work on these spaces:

- In the periodic case we will consider G as a metric on $\mathrm{Rot}(\mathrm{S}^1) \setminus \mathrm{Diff}(S^1)$ of the right-cosets of diffeomorphism of S^1 modulo rotations, which we will identify with the set

  $$\mathcal{M}_{S^1} = \left\{ \varphi \in \mathrm{Diff}(S^1) \ : \ \varphi(0) = 0 \right\}.$$

  The Lie algebra in this case is

  $$\mathfrak{m}_{S^1} = \left\{ u \in C^\infty(S^1) \ : \ u(0) = 0 \right\}$$

- In the non-periodic case, we will consider G to be a metric on the space of H^∞ diffeomorphism that decay to the identity at minus infinity

  $$\mathcal{M}_{\mathbb{R}} = \left\{ \varphi = \mathrm{Id} + f : \mathbb{R} \to \mathbb{R} \ : \ f' \in W^{\infty,1}(\mathbb{R}),\ f' > -1,\ \lim_{x \to -\infty} f(x) = 0 \right\}.$$

  The Lie algebra in this case is

  $$\mathfrak{m}_{\mathbb{R}} = \left\{ f : \mathbb{R} \to \mathbb{R} \ : \ f' \in W^{\infty,1}(\mathbb{R}),\ \lim_{x \to -\infty} f(x) = 0 \right\}.$$

  We can also impose other decay conditions on f'.

[9] One can also observe that the Camassa–Holm equation (geodesic equation of H^1 metric) is essentially the Hunter–Saxton equation ($\dot{H}^1$ metric) plus the Burgers equation (L^2 metric).

Why in the non-periodic case we need to work with $\mathcal{M}_{\mathbb{R}}$ and not with the smaller space $\mathrm{Diff}^{W^{\infty,1}}(\mathbb{R})$? Let us look on the Christoffel symbol of G: Differentiating (4.12), we obtain that

$$DG_\varphi(w)(u,v) = -\int_{\mathbb{R}} u_x v_x w_x \varphi_x^{-2}\,dx,$$

and thus

$$G_\varphi(\Gamma_\varphi(u,v),w) = -\frac{1}{2}\int_{\mathbb{R}} \frac{u_x v_x}{\varphi_x} w_x \varphi_x^{-1}\,dx,$$

hence

$$\Gamma_\varphi(u,v)(x) = -\frac{1}{2}\int_{-\infty}^{x} \frac{u_x(y)v_x(y)}{\varphi_x(y)}\,dy.$$

Now, even if $\varphi = \mathrm{Id}$ and u,v are compactly supported, then $\Gamma_\varphi(u,v)$ does not decay as $x \to \infty$, but we only have that $\Gamma_\varphi(u,v) \in \mathfrak{m}_{\mathbb{R}}$. In other words, the Christoffel symbol does not exist on smaller spaces with decay on both boundaries.[10]

We now turn to explicitly solving the geodesic equation:

Theorem 4.16 (Periodic Hunter–Saxton is a flow on an ∞-dimensional sphere)
The map

$$\Psi : (\mathcal{M}_{S^1}, G) \to (C^\infty(S^1), L^2), \qquad \Psi(\varphi) = 2\sqrt{\varphi_x},$$

is an isometric embedding, whose image is an open subset of the sphere:

$$\Psi(\mathcal{M}_{S^1}) = \mathcal{U}_{S^1} := \left\{ f \in C^\infty(S^1) \ : \ f > 0,\ \|f\|_{L^2} = 2 \right\}.$$

Proof: Let $u,v \in T_\varphi\mathcal{M}_{S^1}$. Then we have

$$T\Psi_\varphi u = \frac{u_x}{\sqrt{\varphi_x}},$$

[10] In the periodic case the Christoffel symbol has a somewhat more complicated formula, since we need to maintain the constraint $\int_{S^1} u\,dx = 0$ also for the symbol. See [Len07, §2] for an explicit formula.

and thus, by (4.12),

$$\langle T\Psi_\varphi u, T\Psi_\varphi v\rangle_{L^2} = \int_{S^1} \frac{u_x}{\sqrt{\varphi_x}}\frac{v_x}{\sqrt{\varphi_x}}\,dx = G_\varphi(u,v),$$

hence Ψ is an isometric immersion. Its image is indeed $\mathcal{U}_{S^1}$, since $\int_{S^1}\varphi_x\,dx = 1$ (since we identify S^1 with $[0,1]$). It is obviously an embedding, as we can write the inverse map

$$\Psi^{-1} : \mathcal{U} \to \mathcal{M}_{S^1}, \qquad \Psi^{-1}(f)(x) = \frac{1}{4}\int_0^x f^2(t)\,dt.$$

■

Corollary 4.17 (Explicit solutions for periodic Hunter–Saxton) *Let $u : [0,T) \to \mathfrak{m}_{S^1}$ be a solution of (4.11) with initial data $u(0) = u_0$. Then the integral curve $\varphi : [0,T) \to \mathcal{M}_{S^1}$ of u, starting from the identity, is given by*

$$\varphi(t,x) = x - \frac{1-\cos(2t)}{8}\int_0^x (4 - u_0'(y)^2)\,dy + \frac{u_0(x)}{2}\sin(2t).$$

In particular, the maximal time of existence of the solution is given by

$$T^*(u_0) = \frac{\pi}{2} + \arctan\left(\frac{1}{2}\min_{x\in S^1} u_0'(x)\right) < \frac{\pi}{2}.$$

Proof: Since $\Psi : \mathcal{M}_{S^1} \to \mathcal{U}$ is an isometry, we can find the geodesics in $\mathcal{U}$ (with the induced metric from the L^2 metric on $C^\infty(S^1)$), and apply Ψ^{-1}. This is a standard calculus of variations with constraint problem: Let $f_0, f_1 \in \mathcal{U}$, and $f : [0,1] \to \mathcal{U}$ curve between them. We need to minimize the L^2 energy $\int_0^1\int_{S^1} f^2(t,x)\,dx\,dt$, subject to the constraint $\|f(t,\cdot)\|_{L^2} = 2$. This is equivalent to the minimization problem of the energy

$$E(f) = \int_0^1\int_{S^1} f_t^2(t,x)\,dx\,dt - \int_0^1 \lambda(t)\left(\int_{S^1} f^2(t,x) - 4\,dx\right)dt.$$

The variation of this energy is then

$$dE(f).\delta f = -2\int_0^1\int_{S^1} (f_{tt} + \lambda f)\delta f\,dx\,dt,$$

and thus we obtain the system

$$f_{tt}(t,x) + \lambda(t) f(t,x) = 0, \qquad \int_{S^1} f^2 \, dx = 4. \tag{4.13}$$

Multiplying the ODE by f and integrating, we obtain

$$\int_{S^1} f f_{tt} \, dx = -\lambda(t) \int_{S^1} f^2 \, dx = -4\lambda(t).$$

Differentiating the integral equality in (4.13) once and twice with respect to t, we obtain

$$2\int_{S^1} f f_t \, dx = 0, \qquad 2\int_{S^1} f f_{tt} + f_t^2 \, dx = 0. \tag{4.14}$$

Thus

$$\lambda(t) = \frac{1}{4}\int_{S^1} f_t^2 \, dx = \text{const},$$

since a geodesic is of constant speed. In order to simplify the calculations below, we will rescale time so that $\lambda(t) = 1$. Thus

$$f(t,x) = A(x)\cos(t) + B(x)\sin(t).$$

Now, let $f(t,\cdot) = \Psi(\varphi(t,\cdot))$. Since $\varphi(0,\cdot) = \mathrm{Id}$, we have that $A = f(0,\cdot) = 2$. Since $\varphi_t(0,\cdot) = u_0$, we have that $B = f_t(0,\cdot) = T\Psi_{id}(u_0) = u_0'$. Thus we obtain

$$\sqrt{\varphi_x(t,x)} = \frac{1}{2} f(t,x) = \cos(t) + \frac{u_0'(x)}{2}\sin(t)$$

from which the solution formula follows by integration.

The solution continues to exist as long as $f > 0$, that is, as long as t is such that $\cos(t) + \frac{u_0'(x)}{2}\sin(t) > 0$ for all $x \in S^1$, from which the formula for $T^*(u_0)$ follows immediately. ∎

Theorem 4.18 (Non-periodic Hunter–Saxton is a flow on a flat space) *The map*

$$\Psi : (\mathcal{M}_{\mathbb{R}}, G) \to (W^{\infty,1}(\mathbb{R}), L^2), \qquad \Psi(\varphi) = 2(\sqrt{\varphi_x} - 1),$$

is an isometric embedding, whose image is the open, convex subset:

$$\Psi(\mathcal{M}_{\mathbb{R}}) = \mathcal{U}_{\mathbb{R}} := \left\{ f \in C^\infty(S^1) \ : \ f > -2 \right\}.$$

The proof is essentially the same as of Theorem 4.16, noting that

$$\Psi^{-1}(f)(x) = x + \int_{-\infty}^{x} \left(\left(\frac{f(y)}{2} + 1 \right)^2 - 1 \right) dy.$$

Note that indeed $\Psi^{-1}(f) \in \mathcal{M}_{\mathbb{R}}$.

Corollary 4.19 (Explicit solutions for non-periodic Hunter–Saxton) *Let* $u : [0,T) \to \mathfrak{m}_{\mathbb{R}}$ *be a solution of* (4.11) *with initial data* $u(0) = u_0$. *Then the integral curve* $\varphi : [0,T) \to \mathcal{M}_{\mathbb{R}}$ *of* u, *starting from the identity, is given by*

$$\varphi(t,x) = x + \int_{-\infty}^{x} \left(\left(\frac{t u_0'(y)}{2} + 1 \right)^2 - 1 \right) dy. \tag{4.15}$$

In particular, the maximal time of existence of the solution is given by

$$T^*(u_0) = \begin{cases} \infty & \inf u_0' \ge 0 \\ -\frac{2}{\inf u_0'} & \inf u_0' < 0. \end{cases}$$

Proof: The proof follows a similar line as (4.17), only it is much simpler: $\mathcal{U}_{\mathbb{R}}$ in this case is an open subset of an inner-product space, and thus the geodesics there are straight lines, and in particular, the geodesic starting from

$$f(0,\cdot) = \Psi(\mathrm{Id}) = 0,$$

in direction

$$f_t(0,\cdot) = T\Psi_{\mathrm{Id}}(u_0) = u_0'$$

is simply

$$f(t,x) = t u_0'(x).$$

■

Comment:

1. Theorem 4.16 implies that the right-invariant $\dot{H}^1$ metric on S^1 has constant positive sectional curvature, and Theorem 4.18 implies that right-invariant $\dot{H}^1$ metric on $\mathbb{R}$ has zero curvature.

2. Consistently with what we have seen regarding the Christoffel symbol, the solution map (4.15) implies that indeed $\mathcal{M}_{\mathbb{R}}$ is the correct space to consider, and that we cannot simply consider the smaller space $\mathrm{Diff}_{W^{\infty,1}}(\mathbb{R})$, since, if u_0 is not identically zero, we obtain that for any $t > 0$, $\lim_{x\to\infty} \varphi(t,x) - x > 0$. In other words, there is no exponential map of the right-invariant $\dot{H}^1$ metric on $\mathrm{Diff}_{W^{\infty,1}}(\mathbb{R})$.

3. Everything that was done here can be done also for maps with finite (e.g., Sobolev) regularity, as is evident from the solution maps.

Chapter 5

Geodesic distance

In this chapter we consider the geodesic distance of weak right-invariant Riemannian metrics on $\mathrm{Diff}(M)$, where M is a closed Riemannian manifold; see Definition 2.36 for the definition of the geodesic distance.

In particular, we will investigate for which metrics the geodesic distance collapses (fails to separate points), and for which it induces a true metric on $\mathrm{Diff}(M)$. Since the geodesic distance is strictly infinite between elements that are not path-connected (and these manifolds, are, in general, not, for example if M is closed and oriented), we will consider only the connected component of the identity map; with a slight abuse of notation, we will still denote this connected component by $\mathrm{Diff}(M)$.

In the definition of the geodesic distance, only the norm induced by the Riemannian metric plays a role, so instead of specifying an inner product on the Lie algebra $\mathfrak{X}(M)$, we will specify a norm, $\|\cdot\|_A$. Then the norm on some $\xi \in T_\varphi \mathrm{Diff}(M)$ is given by right-translation,

$$\|\xi\|_{\varphi,A} := \|\xi \circ \varphi^{-1}\|_A,$$

as usual. We will denote the geodesic distance induced by $\|\cdot\|_A$ by dist_A, and the length of a curve $\varphi : [0,1] \to \mathrm{Diff}(M)$ by $\mathrm{Len}_A(\varphi)$.

Notice that the right invariance immediately implies that

$$\mathrm{dist}_A(\varphi, \psi) = \mathrm{dist}_A(\mathrm{Id}, \psi \circ \varphi^{-1}),$$

and thus it is sufficient to consider distances from the identity in order to investigate collapse/non-collapse of the metric.

As before, we will be mainly interested in Sobolev norms $A = W^{s,p}$. For an integer s, these are defined similar as the H^k metrics in Section 4.1. However, it will be illuminating to consider also fractional s; we will not delve too much on their definition here, and we will mainly use the fact that Sobolev embeddings work the same way for fractional s. See, e.g., [JM19b, Section 2] for explicit definitions (although other notions of fractional Sobolev spaces are likely to work as well).

5.1 Non-vanishing geodesic distance

The simplest criterion that guarantees non-collapse of dist_A was observed in [BBHM13, Theorem 4.1]:

Theorem 5.1 *If $\|\cdot\|_A$ controls $\|\cdot\|_{L^\infty}$, that is, if there exists $C > 0$ such that*

$$\|u\|_{L^\infty} \le C\|u\|_A,$$

then dist_A separates points.

Proof: Let $\mathrm{Id} \neq \varphi_1 \in \mathrm{Diff}(M)$, and let $\varphi : [0,1] \to \mathrm{Diff}(M)$ be a path from Id to $\varphi_1 = \varphi(1,\cdot)$. Since $\varphi_1 \neq \mathrm{Id}$, then there exists $x \in M$ such that $\varphi_1(x) \neq x$. We therefore have that $\varphi(\cdot, x)$ is a path from x to $\varphi(x)$, and thus

$$\begin{aligned}
\mathrm{dist}_M(x,\varphi_1(x)) &\le \int_0^1 |\partial_t\varphi(t,x)|\,dt \le \int_0^1 \|\partial_t\varphi(t,\cdot)\|_{L^\infty}\,dt \\
&= \int_0^1 \|\partial_t\varphi(t,\varphi^{-1}(t,\cdot))\|_{L^\infty}\,dt \\
&\le C\int_0^1 \|\partial_t\varphi\circ\varphi^{-1}\|_A\,dt = C\,\mathrm{Len}_A(\varphi).
\end{aligned}$$

Taking an infimum over all paths from Id to φ_1 we obtain the result. ∎

In other words, this result tells us that if the norm controls the movement of a single point, then the distance does not collapse. A second criterion that guarantees non-collapse is a control over the L^1 norm of the divergence of a vector field (i.e., the infinitesimal volume change), as was shown by Michor and Mumford [MM05, Theorem 5.7]:

Theorem 5.2 *Assume there exists $C > 0$ such that*

$$\|u\|_{L^1} + \|\operatorname{div}_\mu(u)\|_{L^1} \le C\|u\|_A,$$

where $\mu = \mathrm{Vol}_g$ is a volume form on M. Then dist_A separates points.

Proof: Let $\mathrm{Id} \ne \varphi_1 \in \mathrm{Diff}(M)$, and let $\varphi : [0,1] \to \mathrm{Diff}(M)$ be a path from Id to $\varphi_1 = \varphi(1,\cdot)$. Denote $\psi(t,\cdot) := \varphi^{-1}(t,\cdot)$, and $u = \partial_t\varphi \circ \varphi^{-1}$. Differentiating $\psi(t,\varphi(t,x)) = x$ with respect to t we obtain that

$$\partial_t\psi = -D\psi \circ u,$$

where D is the spatial differentiation.

Let $\varphi_1 \ne \mathrm{Id}$. First, assume that φ_1 is not volume preserving; then there exists a real-valued function f on M such that

$$\int_M f(\psi(1,x))\,\mathrm{dVol}_g(x) \ne \int_M f(x)\,d\mu(x).$$

We have

$$\begin{aligned}
&\int_M f(\psi(1,x)) - f(x)\,d\mu(x)\\
&\quad= \int_0^1 \int_M \partial_t f(\psi(t,x))\,d\mu(x)\,dt\\
&\quad= \int_0^1 \int_M (df \circ \psi)(\partial_t\psi(t,x))\,d\mu(x)\,dt\\
&\quad= -\int_0^1 \int_M (df \circ \psi)(D\psi(t,u(t,x)))\,d\mu(x)\,dt.
\end{aligned}$$

Now, notice that

$$\operatorname{div}_\mu(f(\psi(t,\cdot))\cdot u) = f(\psi(t,\cdot))\cdot \operatorname{div}_\mu(u(t,\cdot)) + df_{\psi(t,\cdot)}(D\psi(t,u(t,\cdot))),$$

and thus we have

$$
\begin{aligned}
&\left|\int_M f(\psi(1,x)) - f(x)\,\mathrm{dVol}_g(x)\,dt\right| \\
&\qquad = \left|\int_0^1 \int_M f(\psi(t,x))\cdot \mathrm{div}_\mu(u(t,\cdot)) - \mathrm{div}_\mu(f(\psi(t,x))\cdot u)\,d\mu(x)\,dt\right| \\
&\qquad = \left|\int_0^1 \int_M f(\psi(t,x))\cdot \mathrm{div}_\mu(u(t,\cdot))\,d\mu(x)\,dt\right| \\
&\qquad \le \sup|f| \int_0^1 \int_M |\,\mathrm{div}_\mu(u(t,\cdot))|\,d\mu\,dt \\
&\qquad \le C\sup|f| \int_0^1 \|u\|_A\,dt = C\sup|f|\,\mathrm{Len}_A(\varphi).
\end{aligned}
$$

Again, taking the infimum over all paths φ completes the proof for the case φ_1 is not volume preserving. Now, if φ_1 is volume preserving; in this case, we simply need to change the volume form, that is, to choose a positive function $\rho \in C^\infty(M)$ such that

$$\int_M f(\psi(1,x))\,\rho(x)\,d\mu(x) \neq \int_M f(x)\,\rho(x)\,d\mu(x).$$

For some $f \in C^\infty(M)$. This can always be done (verify!). Now the proof continuous in a similar manner, only we take the the divergence with respect to the volume form $\rho\mu$ instead of μ. Since

$$\mathrm{div}_{\rho\mu}(u) = \mathrm{div}_\mu(u) + d\rho(u),$$

we obtain similar estimates, involving the C^1 norm on ρ and the L^1 norm of u. ■

Note that this proof shows that the right-invariant L^2 metric on the group of volume preserving diffeomorphisms $\mathrm{Diff}_\mu(M)$ separates points — in this case $\mathrm{div}_\mu(u) = 0$ so only control of the L^1 norm of u is needed.

5.2 Vanishing geodesic distance

In this section we will show that the mechanisms described above — controlling the movement of a point and controlling volume change — are essentially the only ones that play a role. That is, if $\|\cdot\|_A$ does not control them, the geodesic distance collapses. The collapse of the geodesic distance for right-invariant metrics was first observed by Eliashberg and Polterovich [EP93] in the context of Hamiltonian symplectomorphisms for some bi-invariant metrics. For the full diffeomorphism group it was first observed by Michor and Mumford [MM05] for the L^2 case, and then generalized in a sequence of works [BBHM13, BBM13, JM19b, BHP20], culminating in a full classification for Sobolev metrics in [JM19a].

5.2.1 General considerations

In order to show collapse of the geodesic distance, one should construct a sequence of paths φ^n from Id to some fixed φ_1 such that $\mathrm{Len}(\varphi^n) \to 0$. The paths are usually constructed by specifying the vector fields u^n whose flows are φ^n; a technical difficulty in achieving that is verifying that all these flows indeed hit φ_1 at time $t = 1$. In [EP93], Eliashberg and Polterovich found an ingenious way of bypassing this problem, via the use of "*displacement energy*", defined below. Here we follow [BHP20], which generalized their method so to adapt it to right-invariant metrics (see also [She17]).

Definition 5.3 *The* ***displacement energy*** *of a set $V \subset M$ with respect to* dist_A *is defined by*

$$E_A(V) := \inf \{ \mathrm{dist}_A(\mathrm{Id}, \varphi) \, : \, \varphi \in \mathrm{Diff}(M), \varphi(V) \cap V = \varnothing \} .$$

That is, displacement energy measures the cost of moving a set V to a disjoint set. Obviously, if $\mathrm{dist}_A(\mathrm{Id}, \varphi) = 0$, then there is some open set V such that $E_A(V) = 0$. Remarkably, the converse is also true:

Proposition 5.4 *Assume that for every $\varphi \in \mathrm{Diff}(M)$, the left multiplication operator $L_\varphi : \mathrm{Diff}(M) \to \mathrm{Diff}(M)$, $L_\varphi(\psi) = \varphi \circ \psi$ is smooth and Lipschitz with respect to* dist_A*, that is*

$$|L_\varphi| := \inf \left\{ C > 0 \ : \ d(\varphi\varphi_0, \varphi\varphi_1) \le C d(\varphi_0, \varphi_1), \ \forall \varphi_0, \varphi_1 \in \mathrm{Diff}(M) \right\} < \infty. \tag{5.1}$$

Then, there exists $\varphi \in \mathrm{Diff}(M)$, $\varphi \neq \mathrm{Id}$, such that $\mathrm{dist}_A(\mathrm{Id}, \varphi) = 0$ if any only if there exists an open set V such that $E_A(V) = 0$.

This makes our lives much easier — now we need to construct a sequence of vector field u^n, with $\mathrm{dist}_A(\mathrm{Id},\varphi^n) \to 0$, that displace some open set V, without worrying about the endpoint of φ^n. The condition that the left multiplication (for a fixed φ) is Lipschitz is satisfied in virtually for all metrics of interest, and in particular for all Sobolev metrics that we discussed. We will not prove this Lipschitz property here.

Proof: In order to shorten notation, we will write d instead of dist_A throughout the proof. We start with a bound of $d(\mathrm{Id},[\varphi_0,\varphi_1])$, where $[\varphi_0,\varphi_1] := \varphi_0^{-1}\varphi_1^{-1}\varphi_0\varphi_1$ is the commutator of φ_0 and φ_1. We have:

$$d(\mathrm{Id},[\varphi_0,\varphi_1]) \le (1+|L_{\varphi_0^{-1}}|)d(\mathrm{Id},\varphi_1). \tag{5.2}$$

Indeed,

$$\begin{aligned} d(\mathrm{Id},[\varphi_0,\varphi_1]) &= d(\mathrm{Id},\varphi_0^{-1}\varphi_1^{-1}\varphi_0\varphi_1) = d(\varphi_1^{-1}\varphi_0^{-1},\varphi_0^{-1}\varphi_1^{-1}) \\ &\le d(\varphi_1^{-1}\varphi_0^{-1},\varphi_0^{-1}) + d(\varphi_0^{-1},\varphi_0^{-1}\varphi_1^{-1}) \\ &\le d(\varphi_1^{-1},\mathrm{Id}) + |L_{\varphi_0^{-1}}|d(\mathrm{Id},\varphi_1^{-1}) = (1+|L_{\varphi_0^{-1}}|)d(\mathrm{Id},\varphi_1). \end{aligned}$$

Now, let V be an open set with $E_A(V)=0$, and denote by $G_V \subset \mathrm{Diff}(M)$ the subgroup of diffeomorphisms supported on V (that is, $\varphi(x)=x$ for $x \notin V$). Denote by $D_V \subset \mathrm{Diff}(M)$ the set of diffeomorphisms φ that displace V, that is $\varphi(V)\cap V = \varnothing$. Note that this is not a subgroup but it is inverse-invariant.

A simple calculation shows that for $\varphi,\psi \in G_V$ and $\alpha \in D_V$, φ and $\alpha^{-1}\psi\alpha$ commute, that is

$$\alpha^{-1}\psi^{-1}\alpha\varphi\alpha^{-1}\psi\alpha = \varphi. \tag{5.3}$$

This is obtained by showing that their (effective) action on each $x \in M$ is the same (check the three cases $\alpha x \in V$, $x \notin V$ and $\alpha x \notin V$, $x \notin V$ and $x \in V$ separately).

Now let $\varphi_0,\varphi_1 \in G_V$ whose commutator $[\varphi_0,\varphi_1] \ne \mathrm{Id}$ (these exist since G_V is not abelian). Rewriting (5.3) with $\varphi = \varphi_0^{-1}\varphi_1\varphi_0$ and $\psi=\varphi_1$, we obtain

$$[\varphi_1,\varphi_0] = [\varphi_1,\beta],$$

where $\beta = [\varphi_0^{-1},\alpha]$. We can now estimate, using (5.2)

$$\begin{aligned} d(\mathrm{Id},[\varphi_1,\varphi_0]) &= d(\mathrm{Id},[\varphi_1,\beta]) \le (1+|L_{\varphi_1^{-1}}|)d(\mathrm{Id},\beta) \\ &= (1+|L_{\varphi_1^{-1}}|)d(\mathrm{Id},[\varphi_0^{-1},\alpha]) \le (1+|L_{\varphi_1^{-1}}|)(1+|L_{\varphi_0^{-1}}|)d(\mathrm{Id},\alpha). \end{aligned}$$

Taking the infimum over all $\alpha \in D_V$, we obtain that $d(\mathrm{Id}, [\varphi_1, \varphi_0]) = 0$, since $E_A(V) = 0$. ∎

Proposition 5.5 *Under the assumption* (5.1), *the set*

$$\mathrm{Diff}^{d=0}(M) := \{\varphi \in \mathrm{Diff}(M) \ : \ \mathrm{dist}_A(\mathrm{Id}, \varphi) = 0\}$$

is a normal subgroup of $\mathrm{Diff}(M)$.

Proof: The fact that $\mathrm{Diff}^{d=0}(M)$ is a subgroup only uses right-invariance: Let $\varphi, \psi \in \mathrm{Diff}^{d=0}(M)$, then

$$\mathrm{dist}_A(\mathrm{Id}, \varphi\psi) \le \mathrm{dist}_A(\mathrm{Id}, \psi) + \mathrm{dist}_A(\psi, \varphi\psi) = \mathrm{dist}_A(\mathrm{Id}, \psi) + \mathrm{dist}_A(\mathrm{Id}, \varphi) = 0.$$

For normality, let now $\alpha \in \mathrm{Diff}(M)$ and $\varphi \in \mathrm{Diff}^{d=0}(M)$, then

$$\mathrm{dist}_A(\mathrm{Id}, \alpha\varphi\alpha^{-1}) = \mathrm{dist}_A(\alpha, \alpha\varphi) \le |L_\alpha| \mathrm{dist}_A(\mathrm{Id}, \varphi) = 0.$$

∎

The normality of $\mathrm{Diff}^{d=0}(M)$ is important because of the following classical statement about the group structure of $\mathrm{Diff}(M)$ [Eps70]:

Theorem 5.6 *Given a manifold* M, *the connected component of the identity of* $\mathrm{Diff}_c(M)$ *is a simple group.*

Combining all the above results, we obtain

Corollary 5.7 *If there exists an open set* $V \subset M$ *such that* $E_A(V) = 0$, *then* dist_A *vanishes identically on* $\mathrm{Diff}(M)$.

5.2.2 Constructions

We first consider the case $\dim M = 1$, that is, $M = S^1$ or $M = \mathbb{R}$. In this case, controlling volume change (L^1 control of u_x) is a stronger condition than a pointwise control of u. Thus our aim is to show that whenever we do not have a pointwise control of u, the geodesic distance collapses.

A more formal way of saying "not having a pointwise control of u" is by the notion of capacity:

*Definition 5.8 Let $\|\cdot\|_A$ be a norm on $\mathfrak{X}(M)$. The **capacity of a point** $x_0 \in M$ is defined as*

$$\inf\{\|u\|_A \ : \ u \in \mathfrak{X}(M), |u(x_0)| = 1\}.$$

Note that if $\|u\|_{L^\infty} \le C\|u\|_A$, then all points have positive capacity.

Theorem 5.9 Let $\dim M = 1$, Let $\|\cdot\|_A$ be a translation invariant norm (i.e., $\|u(\cdot - x_0)\|_A = \|u(\cdot)\|_A$) and assume that there exists a point x_0 with zero capacity. Furthermore, assume that for each $\varphi \in \mathrm{Diff}(M)$, the left-multiplication operator L_φ is smooth and Lipschitz with respect to dist_A. Then $\mathrm{dist}_A \equiv 0$ on $\mathrm{Diff}(M)$. In particular, this is true for $A = W^{s,p}$ whenever $sp \le 1$.

Proof: Let $v_n \in \mathfrak{X}(M)$ be a sequence of vector fields with $v_n(x_0) = 1$ and $\|v_n\|_A \to 0$. Define time-dependent vector fields by $u_n(t,x) = v_n(x - t)$, and let $\varphi_n(t,x)$ be their flow. We then have that

$$\varphi_n(t, x_0) = x_0 + t.$$

In particular, for some small enough $\varepsilon > 0$, we have that $\varphi_n(2\varepsilon, (x_0, x_0 + \varepsilon)) \cap (x_0, x_0 + \varepsilon) = \varnothing$, that is, φ_n displaces $(x_0, x_0 + \varepsilon)$. Now,

$$\mathrm{dist}_A(\mathrm{Id}, \varphi_n) \le \int_0^{2\varepsilon} \|u_n(t,\cdot)\|_A \, dt = 2\varepsilon \|v_n\|_A \to 0.$$

This show that $E_A((x_0, x_0 + \varepsilon)) = 0$, and by Corollary 5.7 this completes the proof. ∎

Note that the translation invariance assumption can be relaxed to assuming that $\|u \circ \varphi\|_A \le C_\varphi \|u\|_A$, for $\varphi \in \mathrm{Diff}(M)$ and that the constant C_φ can be chosen uniformly on compact subsets of $\mathrm{Diff}(M)$. This assumption holds for any reasonable norm I am familiar with.

For higher dimensional M, the same proof holds if some hypersurface has zero capacity (again, we can work in local coordinates, and translate a neighborhood of the hypersurface a bit). For Sobolev norms W^{sp}, hypersurfaces have zero capacity if and only if $sp \le 1$. In particular, this shows vanishing geodesic distance for the L^2 metric on any manifold.

This result can be improved to norms in which points have zero capacity (which is equivalent, for Sobolev norms, to $sp \le \dim M$), if in addition large volume changes are undetected by the norm (that is, the norm does not control the L^1 norm of the divergence of the vector field). For Sobolev metrics, the result reads

Theorem 5.10 *Let $A = W^{s,p}$ for $sp \le \dim M$ and $s < 1$. Then $\mathrm{dist}_A \equiv 0$ on $\mathrm{Diff}(M)$.*

The proof of this claim is much more elaborate, and requires a combination of squeezing parts of an open set V, transporting it outside of V using vector fields of small capacity, and then expanding again. See [JM19a] for details. A schematic video of the idea for $\dim M = 2$ (hitting a specific target diffeomorphism) can be seen in this link: `https://www.math.toronto.edu/rjerrard/geo_dist_diffeo/vanishing.html`.

Finally, we note that the classification of vanishing/non-vanishing geodesic distance is quite open for other diffeomorphism groups, for example volume-preserving diffeomorphisms and Hamiltonian symplectomorphisms; for some recent developments in these cases see [BHP20].

5.3 Diameter

Assuming $\|\cdot\|_A$ is a norm on (M), the geodesic distance it induces on $\mathrm{Diff}(M)$ is non-degenerate. One of the next natural questions is whether the resulting metric space is bounded or not. One might expect, given the complexity of the manifolds, that the resulting metric space will always be of infinite diameter; however, we have already seen one example of a bounded metric space:

recall that the right invariant $\dot{H}^1$-metric on $\mathrm{Rot}(\mathrm{S}^1) \setminus \mathrm{Diff}(S^1)$ is isometric to an open subset of a sphere, that has a bounded diameter (Section 4.7). The study of finiteness/infiniteness of diameter was initiated by Shnirelman in the late 1980's for volumorphisms, in the context of the incompressible Euler equation; it was later investigated in the context of Hamiltonian symplectomorphisms by Eliashberg and Ratiu. For the full diffeomorphism group this question was studied in [BM21], which we follow here.

Recall that if $\|u\|_A$ controls $\|\operatorname{div}_g(u)\|_L^1$, then there is no collapse; we now show that controlling $\|\operatorname{div}_g(u)\|_{L^\infty}$ results in infinite diameter:

Proposition 5.11 *Assume that there exists $C > 0$ such that $\|\operatorname{div}_g(u)\|_{L^\infty} \leq C\|u\|_A$. Then the metric space $(\mathrm{Diff}(M), \mathrm{dist}_A)$ has infinite diameter. For Sobolev metrics $A = W^{s,p}$, this holds whenever $s > 1 + \frac{\dim M}{p}$.*

Proof: Let $\varphi_1 \in \mathrm{Diff}(M)$, and let $\varphi : [0,1] \to \mathrm{Diff}(M)$ be a path from Id to φ_1. To shorten notation we will write $\varphi_t(\cdot) = \varphi(t,\cdot)$. As usual, we write $\partial_t \varphi_t = u_t \circ \varphi_t$ for $u_t \in \mathfrak{X}(M)$. For each t, we denote by $D\varphi_t$ the spacial derivative of φ_t, and by $\psi_t = |D\varphi_t|$ the Jacobian determinant of φ_t with respect to the metric g of M. A direct calculation shows that

$$\partial_t \psi_t = \operatorname{div}_g(u_t) \circ \varphi_t \cdot \psi_t.$$

We therefore have, for any $x \in M$,

$$\begin{aligned}
\log |D\varphi_1|(x) &= \int_0^1 \partial_t(\log \psi_t(x))\, dt = \int_0^1 \operatorname{div}_g(u_t)(\psi_t(x))\, dt \\
&\leq \int_0^1 \|\operatorname{div}_g(u_t)\|_{L^\infty}\, dt \leq C \int_0^1 \|u_t\|_A\, dt \\
&= \mathrm{Len}_A(\varphi).
\end{aligned}$$

Taking the infimum over all paths from Id to φ_1 we obtain

$$\log |D\varphi_1|(x) \leq C \mathrm{dist}_A(\mathrm{Id}, \varphi_1).$$

Thus, diffeomorphisms φ_1 with arbitrary large Jacobian determinants at a point (for example, ones that squeezes a ball of radius 1 in M to a ball of radius ε, with $\varepsilon \to 0$), are arbitrarily far away from Id with respect to dist_A, and thus the diameter is infinite. ∎

Proving that dist_A has a finite diameter is trickier — one must find, for each $\varphi \in \mathrm{Diff}(M)$, a path from Id with a uniformly bounded length. The best result so far, for Sobolev metrics, is

Theorem 5.12 *The diameter of* $(\mathrm{Diff}(M), \mathrm{dist}_{W^{s,p}})$ *is infinite for* $s \geq 1 + \frac{\dim M}{p}$*. If* $s < 1 + \frac{1}{p}$*, the diameter of* $(\mathrm{Diff}(S^n), \mathrm{dist}_{W^{s,p}})$ *is finite.*

We will not detail the proof, but focus on some of the main components of the finiteness proof:

- **Uniform fragmentation property of spheres:** For a fixed finite atlas of S^n, there exists a number N such that every $\varphi \in \mathrm{Diff}(S^n)$ can be written as $\varphi = \varphi_N \circ \ldots \circ \varphi_1$, where each φ_i is supported in one of the coordinate charts in the atlas (in fact, it can be shown that $N \leq 6$).

- **Localization:** By right-invariance,

$$\mathrm{dist}_A(\mathrm{Id}, \varphi) \leq \sum_{i=1}^{N-1} \mathrm{dist}(\varphi_i \circ \ldots \circ \varphi_1, \varphi_{i+1} \circ \ldots \circ \varphi_1) = \sum_{i=1}^{N-1} \mathrm{dist}(\mathrm{Id}, \varphi_{i+1}).$$

  Thus, it is enough to prove that the diameter is finite for diffeomorphism that are supported in a single coordinate chart. Since changing the metric g on each chart results in an equivalent distance function, and thus does not affect finiteness/infiniteness of the diameter, the diameter $(\mathrm{Diff}(S^n), \mathrm{dist}_{W^{s,p}})$ is finite if the diameter of $(\mathrm{Diff}_c(B_1(\mathbb{R}^n)), \mathrm{dist}_{W^{s,p}})$ is finite, where $B_1(\mathbb{R}^n)$ is the open flat unit ball in $\mathbb{R}^n$.

- **Reduction to a two-parameter family of maps:** Write $B_1(\mathbb{R}^n) = \{(r,\theta) \,:\, r \in [0,1), \theta \in S^{n-1}\}$ in polar coordinate, and let $\lambda \in \mathbb{N}$ and $\delta \in (0,1)$. Consider the maps $\Psi_{\lambda,\delta} \in \mathrm{Diff}_c(B_1(\mathbb{R}^n))$ that satisfy

$$\Psi_{\lambda,\delta}(r,\theta) = (\lambda r, \theta), \qquad r \in \left[0, \frac{1-\delta}{\lambda}\right], \theta \in S^{n-1}.$$

That is, $\Psi_{\lambda,\delta}$ radially expends a ball of radius $\frac{1-\delta}{\lambda}$ to a ball of radius $1-\delta$. Let $\varphi \in \mathrm{Diff}_c(B_1(\mathbb{R}^n))$. Then there exists $\delta > 0$ such that $\operatorname{supp}\varphi \subset B_{1-\delta}(\mathbb{R}^n)$. We therefore have that, for every $\lambda \in \mathbb{N}$,

$$\varphi^\lambda(r,\theta) := \frac{1}{\lambda}\varphi(\lambda r,\theta) = \Psi^{-1}_{\lambda,\delta}(r,\theta) \circ \varphi \circ \Psi_{\lambda,\delta}(r,\theta),$$

and thus, by right-invariance,

$$\mathrm{dist}_A(\mathrm{Id},\varphi) = 2\mathrm{dist}_A(\mathrm{Id},\Psi_{\lambda,\delta}) + \mathrm{dist}_A(\mathrm{Id},\varphi^\lambda).$$

Now, given time dependent vector field u_t, whose flow at time $t = 1$ is φ, the flow of

$$u_t^\lambda(r,\theta) = \frac{1}{\lambda}u_t(\lambda r,\theta)$$

at time $t = 1$ is φ^λ. Thus, simple rescaling properties of the Sobolev norms $W^{s,p}$ shows that

$$\mathrm{dist}_{W^{s,p}}(\mathrm{Id},\varphi^\lambda) \le \lambda^{(s-1)-\frac{n}{p}}\mathrm{dist}_{W^{s,p}}(\mathrm{Id},\varphi).$$

Assume $(s-1)p < n$. Taking $\lambda \to \infty$, we obtain that

$$\mathrm{dist}_{W^{s,p}}(\mathrm{Id},\varphi) \le \limsup_{\lambda\to\infty} 2\mathrm{dist}_{W^{s,p}}(\mathrm{Id},\Psi_{\lambda,\delta}).$$

In other words, we obtain that if $(s-1)p < n$ and $\mathrm{dist}_{W^{s,p}}(\mathrm{Id},\Psi_{\lambda,\delta})$ is uniformly bounded for every λ and δ, then the diameter of $(\mathrm{Diff}_c(B_1(\mathbb{R}^n)), \mathrm{dist}_{W^{s,p}})$ is finite.

- When $(s-1)p < 1$, a lengthy but straight-forward calculation shows that the affine homotopy in the radial direction between Id and $\Psi_{\lambda,\delta}$ has a uniformly bounded length in δ and λ, thus completing the finiteness proof.

The sketch above does not cover the infiniteness of the diameter in the critical case $s = 1 + \frac{\dim M}{p}$; this requires a more elaborate argument that in Proposition 5.11, that, in particular, involves isometries to infinite dimensional spheres, similar to the ones used in Section 4.7 that allows us to obtain explicit lower bounds for the diameter.

Presumably, the diameter of $(\mathrm{Diff}(S^n), \mathrm{dist}_{W^{s,p}})$ is finite whenever $(s-1)p < n$. As the proof above shows, the only component missing is constructing a "smarter" path to $\Psi_{\lambda,\delta}$ of length that is independent of λ and δ.

Finally, I do not know for which manifolds, except spheres, the uniform fragmentation property discussed above holds (and for which it does not). For those it does not, it might be that the diameter is never finite — this is an interesting open question.

Chapter 6

Metric and geodesic completeness

In this section we will give a very brief overview of results regarding metric and geodesic completeness of right-invariant metrics on diffeomorphism groups. Namely, the strongest result to-date is of Bruveris–Vialard [BV17]:

Theorem 6.1 *Let M be a closed manifold, and $k > \frac{\dim M}{2} + 1$. If G^k be a right-invariant Sobolev metric of order k on $\mathrm{Diff}_{H^k}(M)$ then*

1. $(\mathrm{Diff}_{H^k}(M), \mathrm{dist}_{G^k})$ *is a complete metric space;*
2. $(\mathrm{Diff}_{H^k}(M), G^k)$ *is geodesically complete — geodesic continue to exist for all time;*
3. *Any two elements in the same connected component of $\mathrm{Diff}_{H^k}(M)$ can be joined by a minimizing G-geodesic.*

By a Sobolev metric of order k we mean as defined in Section 4.1. Note that such G is a strong metric on $\mathrm{Diff}_{H^k}(M)$. Thus, by the Hopf–Rinow theorem (Theorem 2.46), the metric completeness property (1) implies the geodesic completeness property (2).[1]

[1] Though note that there exists works earlier than [BV17] that proved geodesic completeness directly.

The regularity results we proved in Section 4.6 (see Corollary 4.10 and the discussion prior to it) for right-invariant metrics immediately implies the following corollary:

Corollary 6.2 *Under the assumptions of the above theorem,* $(\mathrm{Diff}_{H^{k+l}}(M), G^k)$ *for* $l > 0$ *and* $(\mathrm{Diff}(M), G^k)$ *are geodesically complete.*

Note that neither $(\mathrm{Diff}_{H^{k+l}}(M), \mathrm{dist}_{G^k})$ nor $(\mathrm{Diff}(M), \mathrm{dist}_{G^k})$ are metrically complete, since it is not difficult to construct a dist_{G^k}-Cauchy sequence in $\mathrm{Diff}(M)$ that converges pointwise to a map $\varphi \in \mathrm{Diff}_{H^k}(M) \setminus \mathrm{Diff}_{H^{k+1}}(M)$. These give another example in which the finite dimensional Hopf-Rinow theorem fails in infinite dimensions.

Note that negative results in this direction are less general: We have seen that solutions to the Burgers equation (that is, geodesics of the right-invariant L^2 metric) on $\mathrm{Diff}(S^1)$ breaks down after finite time; this is also true for the H^1 metric (Camassa–Holm), where, unlike the Burgers equation, the exponential map behaves nice locally. As far as I know, for H^k metrics in the range $1 < k \le \frac{\dim M}{2} + 1$, the geodesic completeness is unknown.

We will now describe the general framework of the proof of metric completeness in Theorem 6.1, focusing on the soft arguments. For simplicity, we will show that $(\mathrm{Diff}_{H^k}(\mathbb{R}^d), \mathrm{dist}_{G^k})$ is complete. The basic idea is to use the fact that $\mathrm{Diff}_{H^k}(\mathbb{R}^d)$ is an open subset of the Hilbert space $H^k(\mathbb{R}^d, \mathbb{R}^d)$, which is complete.

- We have two Riemannian metrics on $\mathrm{Diff}_{H^k}(\mathbb{R}^d)$: the metric G^k, and also the trivial Riemanninan metric H^k on $H^k(\mathbb{R}^d, \mathbb{R}^d)$ (trivial as it does not depend on the base point). In detail, we have that for $\eta, \xi \in T_\varphi \mathrm{Diff}_{H^k}(\mathbb{R}^d) \cong H^k(\mathbb{R}^d, \mathbb{R}^d)$

$$H^k_\varphi(\eta, \xi) = \sum_{i=0}^k \int_{\mathbb{R}^d} \langle \nabla^i \eta, \nabla^i \xi \rangle \, dx,$$
$$G^k_\varphi(\eta, \xi) = \sum_{i=0}^k \int_{\mathbb{R}^d} \langle \nabla^i (\eta \circ \varphi^{-1}), \nabla^i (\xi \circ \varphi^{-1}) \rangle \, dx = H_k(\eta \circ \varphi^{-1}, \xi \circ \varphi^{-1}).$$

  Both are strong metrics on $\mathrm{Diff}_{H^k}(\mathbb{R}^d)$, and are pointwise equivalent, that is

$$C_\varphi^{-1} \| \cdot \|_{H^k_\varphi} \le \| \cdot \|_{G^k_\varphi} \le C_\varphi \| \cdot \|_{H^k_\varphi}$$

for some $C_\varphi > 0$.

- The key analytic estimate is to show that this equivalent is uniform locally — that is, that given $r > 0$, there exists a constant $C > 0$ such that
$$C^{-1}\|\cdot\|_{H^k_\varphi} \le \|\cdot\|_{G^k_\varphi} \le C\|\cdot\|_{H^k_\varphi}$$
for all $\varphi \in \mathrm{Diff}_{H^k}(\mathbb{R}^d)$ with $\mathrm{dist}_{G^k}(\mathrm{Id},\varphi) < r$.

- This implies, that for $r > 0$, there exists a constant $C_r > 0$ such that
$$\|\varphi_0 - \varphi_1\|_{H^k} \le C_r \mathrm{dist}_{G^k}(\varphi_0,\varphi_1) \tag{6.1}$$
for all $\varphi_0,\varphi_1 \in \mathrm{Diff}_{H^k}(\mathbb{R}^d)$ with $\mathrm{dist}_{G^k}(\mathrm{Id},\varphi_i) < r$. Indeed, $\mathrm{dist}_{G^k}(\varphi_0,\varphi_1) < 2r$, and it follows that every path φ of length $< 2r$ between φ_0 and φ_1 is satisfies $\mathrm{dist}_{G^k}(\mathrm{Id},\varphi_t) < 2r$ for all t — otherwise,
$$\begin{aligned}\mathrm{Len}_{G^k}(\varphi) &\ge \mathrm{dist}_{G^k}(\varphi_0,\varphi_t) + \mathrm{dist}_{G^k}(\varphi_t,\varphi_1)\\ &\ge 2\mathrm{dist}_{G^k}(\mathrm{Id},\varphi_t) - \mathrm{dist}_{G^k}(\mathrm{Id},\varphi_0) - \mathrm{dist}_{G^k}(\mathrm{Id},\varphi_1) > 2r,\end{aligned}$$
in contradiction. Therefore, one can use the above bounds to bound the length of φ with respect to H^k.

- Since $\mathrm{Diff}_{H^k}(\mathbb{R}^d)$ is an open subset of the Hilbert space $H^k(\mathbb{R}^d,\mathbb{R}^d)$, there exists $\varepsilon > 0$ such that $\mathrm{Id} + B_\varepsilon(0) \subset \mathrm{Diff}_{H^k}(\mathbb{R}^d)$, where $B_\varepsilon(0)$ is the $H^k(\mathbb{R}^d,\mathbb{R}^d)$ ball.

- Let $(\varphi^n)_{n\in\mathbb{N}}$ be a G^k-Cauchy sequence in $\mathrm{Diff}_{H^k}(\mathbb{R}^d)$. By looking at the tail of the sequence, we can assume that $\mathrm{dist}_{G^k}(\varphi^n,\varphi^m) < \frac{1}{2}\varepsilon/C_\varepsilon$ for all n,m. By right-invariance, φ^n converges if and only if $\varphi^n \circ (\varphi^1)^{-1}$ converges, and thus, we can assume without loss of generality that $\varphi^1 = \mathrm{Id}$.

- Now, by (6.1) we have that $(\varphi^n)_{n\in\mathbb{N}}$ is an H^k-Cauchy sequence in $H^k(\mathbb{R}^d,\mathbb{R}^d)$, and thus converges to some function $\varphi^* \in H^k(\mathbb{R}^d,\mathbb{R}^d)$. Note that
$$\|\mathrm{Id} - \varphi^*\|_{H^k} = \lim_{n\to\infty}\|\varphi^1 - \varphi^n\|_{H^k} \le C_\varepsilon \limsup_{n\to\infty} \mathrm{dist}_{G^k}(\varphi^1,\varphi^n) \le \frac{1}{2}\varepsilon.$$
Thus $\mathrm{Id} - \varphi^* \in B_\varepsilon(0)$ and therefore $\varphi^* \in \mathrm{Diff}_{H^k}(\mathbb{R}^d)$.

- Since both H^k and G^k are strong metrics on $\mathrm{Diff}_{H^k}(\mathbb{R}^d)$, they induce the same (manifold) topology, and thus $\varphi^n \to \varphi^*$ also with respect to dist_{G^k}. Therefore $(\mathrm{Diff}_{H^k}(\mathbb{R}^d),\mathrm{dist}_{G^k})$ is complete.

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