Representation theory of real groups
(UNPOLISHED DRAFT)

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Chapter 1

Introduction

These notes are unpolished, might contain errors, imprecisions, etc.

My motivating end-point will be establishing the Langlands classification.

Probably I'll give wrong credits, or no credits (should put better references). Insert names of Harish-Chandra, Gelfand, Langlands, Milicic, Casselman, and others...

Chapter 2

Basics

2.1 Basic topological representation theory

In this section, G is a Lie group, and $K \subset G$ is a compact subgroup, intersecting each connected component of G. We denote by $\int_G, \int_K$ the corresponding Haar integrals (assume that G is unimodular for simplicity).

2.1.1 The definition of a representation

Definition 2.1.1.

1. A **representation** of G is a Frechet space $\mathcal{V}$, equipped with a continuous action map $G \times \mathcal{V} \to \mathcal{V}$. We usually write $\pi : G \to GL(\mathcal{V})$ for the resulting homomorphism.
2. A **unitary representation** of G is a Hilbert space, which is also a representation of G, and such that each $\pi(g)$ is unitary.

Claim 2.1.2.

An action $G \times \mathcal{V} \to \mathcal{V}$ is continuous if and only if the following two properties are satisfied:

1. *The map $G \times \mathcal{V} \to \mathcal{V}$ is continuous in the first variable.*
2. *For every compact $\Omega \subset G$ and continuous seminorm σ on $\mathcal{V}$, there exists a continuous seminorm τ on $\mathcal{V}$ such that $\sigma(gv) \leq \tau(v)$ for every $g \in \Omega$ and $v \in \mathcal{V}$ (this is continuity in the second variable, locally uniform w.r.t. the first variable).*

Proof. easy (write down?) □

Claim 2.1.3. *Suppose that $\mathcal{V}$ is Banachable. Then an action $G \times \mathcal{V} \to \mathcal{V}$ is continuous if and only if the map $G \times \mathcal{V} \to \mathcal{V}$ is continuous in each variable separately.*

Proof. This follows from the previous claim, using the uniform boundeness principle. □

Claim 2.1.4. *Suppose that $\mathcal{V}$ is Hilbertable. Then an action $G \times \mathcal{V} \to \mathcal{V}$ is continuous if and only if the following properties are satisfied:*

1. *The map $G \times \mathcal{V} \to \mathcal{V}$ is weakly continuous in the first variable, in the sense that the map $G \to \mathbb{C}$ given by $g \mapsto \alpha(gv)$ is continuous, for every $v \in \mathcal{V}$ and $\alpha \in \mathcal{V}^*$.*

2. *For every compact $\Omega \subset G$ one has $sup||\pi(g)|| < \infty$ (in particular, the map $G \times \mathcal{V} \to \mathcal{V}$ is continuous in the second variable).*

Proof. We need to verify the continuity of $G \times \mathcal{V} \to \mathcal{V}$ in the first variable. Let $\mathcal{V}_0 \subset \mathcal{V}$ denote the subset of vectors v for which $g \mapsto gv$ is continuous.

First, notice that $\mathcal{V}_0$ is closed in $\mathcal{V}$; Indeed, if $v_n \to v$ and $v_n \in \mathcal{V}_0$, then we can write

$$gv - g_0 v = g(v - v_n) + (gv_n - g_0 v_n) + g_0(v_n - v),$$

and use condition 3.

Next, we will show that $\mathcal{V}_0$ is dense in $\mathcal{V}$ w.r.t. the weak topology. We will consider $\mathcal{V}^*$ with the operator norm. For $f \in C_c(G)$, let us consider

$$b_f(v, \alpha) = \int_G f(g)\alpha(gv),$$

where $v \in \mathcal{V}$ and $\alpha \in \mathcal{V}^*$. Notice that

$$|b_f(v, \alpha)| \leq C \cdot ||f||_{L^1} \cdot ||\alpha|| \cdot ||v||,$$

where C is a constant that depends only on $supp(f)$. In particular, $\alpha \mapsto b_f(v, \alpha)$ is continuous, and so defines a vector "$\pi(f)v$" $\in \mathcal{V}$ (here we use the Hilbertability). We now notice that for an approximation of identity f_n, we will have "$\pi(f_n)v$" $\to v$ w.r.t. the weak topology. Indeed, given $\alpha \in \mathcal{V}^*$, we have

$$|\alpha(\text{"}\pi(f_n)\text{"}v - v)| = |\int_G f_n(g)\alpha(gv - v)| \leq sup_{g \in supp(f_n)}||\alpha(gv) - \alpha(v)||,$$

which tends to 0 as $n \to \infty$.

Since $\mathcal{V}_0$ is closed in $\mathcal{V}$ and also dense in $\mathcal{V}$ w.r.t. the weak topology, we obtain $\mathcal{V}_0 = \mathcal{V}$ (by the Hahn-Banach theorem). □

Corollary 2.1.5. *Let $\mathcal{V}$ be a Hilbertable representaiton of G. Then $\mathcal{V}^*$, equipped with the operator norm and the standard G-action on the dual, is also a G-representaiton.*

Remark 2.1.6. For a Banachable representation the corollary is flase; This can be remedied by passing to continuous vectors in the dual (those, for which the orbit maps are continuous).

2.1.2 Smooth vectors

Definition 2.1.7. Let $v \in \mathcal{V}$. Then v is called **smooth** if the map $G \to \mathcal{V}$ given by $g \mapsto gv$ is smooth.

We denote by $\mathcal{V}^\infty \subset \mathcal{V}$ the subspace of smooth vectors. Given $v \in \mathcal{V}^\infty$, we have a linear map $\mathfrak{g} \to \mathcal{V}$, which we denote by $X \mapsto Xv$, characterized by $e^X v - v = Xv + o(||X||)$. More individualistically in X, we have $Xv = \lim_{t \to 0} \frac{e^{tX} v - v}{t}$. In this way, we obtain an action of the Lie algebra $\mathfrak{g}$ on $\mathcal{V}^\infty$ (and so an action of the algebra $U(\mathfrak{g})$).

We equip $\mathcal{V}^\infty$ with the locally convex topology given by seminorms of the form $\sigma_U(v) := \sigma(Uv)$, where σ is a continuous seminorm on $\mathcal{V}$ and $U \in U(\mathfrak{g})$.

Lemma 2.1.8. *The subspace $\mathcal{V}^\infty$ is itself (with the locally convex topology defined above) a G-representation; In other words, it is complete and the action map $G \times \mathcal{V}^\infty \to \mathcal{V}^\infty$ is continuous.*

Proof. We consider the map $\mathcal{V}^\infty \to C^\infty(G; \mathcal{V})$ given by $v \mapsto (g \mapsto gv)$. One checks quite easily that it is a closed embedding intertwining the G-action. Hence, $\mathcal{V}^\infty$ is a Frechet representation since $C^\infty(G; \mathcal{V})$ is. □

Lemma 2.1.9. *One has $\pi(C_c^\infty(G))) \cdot \mathcal{V} \subset \mathcal{V}^\infty$.*

Proof. For $f \in C_c^\infty(G)$ and $v \in \mathcal{V}$, It is not hard to check that $\pi(f)v$ has a derivative given by $X(\pi(f)v) = \pi(L_X f)v$. complete? □

Corollary 2.1.10. $\mathcal{V} = Cl(\mathcal{V}^\infty)$.

Proof. We use the previous lemma, using a smooth approximation of identity. □

2.1.3 K-finite vectors

Definition 2.1.11. Let $v \in \mathcal{V}$. Then v is called:

1. K**-finite**, if Kv spans a finite-dimensional subspace of $\mathcal{V}$.
2. **weakly analytic**, if for every $\alpha \in \mathcal{V}^*$, the function $G \to \mathbb{C}$ given by $g \mapsto \alpha(gv)$ is analytic.

Let us denote by $\mathcal{V}^{[K]}/\mathcal{V}^\omega$ the subspaces of $\mathcal{V}$ consisting of K-finite/weakly analytic vectors. Also, for $\alpha \in K^\vee$, let us denote by $\mathcal{V}^{[K,\alpha]}$ the α-isotypical part of $\mathcal{V}^{[K]}$.

Definition 2.1.12. We say that $\mathcal{V}$ is **admissible**, if $Hom(E, \mathcal{V}^{[K]})$ is finite-dimensional for every finite-dimensional representation E of K. Equivalently, if $\mathcal{V}^{[K,\alpha]}$ is finite-dimensional for every $\alpha \in K^\vee$.

Lemma 2.1.13. *One has $\pi(C(K)^{[K]}) \cdot \mathcal{V} \subset \mathcal{V}^{[K]}$.*

Proof. Clear. □

Lemma 2.1.14. $\mathcal{V} = Cl(\mathcal{V}^{[K]})$.

Proof. Taking into account the previous lemma, we use a continuous approximation of identity on K, and then a K-finite approximation of it (for the supremum norm). $\square$

Remark 2.1.15. To contrast the previous lemma, notice that one can also pose a stronger question, whether for a given $v \in \mathcal{V}$, one has a convergence of $\sum_{\alpha \in S} pr_\alpha v$ to v, where $S \subset \hat{K}$ is a finite subset, and pr_α denotes the projection onto $\mathcal{V}^{[K,\alpha]}$ given by $\int_K \chi_\alpha^{-1}(k)\pi(k)$. An even stronger thing would be to have absolute convergence, meaning that in addition $\sum_{\alpha \in S} \sigma(pr_\alpha v)$ is convergent for every continuous seminorm σ on $\mathcal{V}$. One can show that this is so for $v \in \mathcal{V}^\infty$ (but, of course, not for arbitrary $v \in \mathcal{V}$). (add this?)

Lemma 2.1.16. $\mathcal{V}^{[K,\alpha]} = Cl(\mathcal{V}^{[K,\alpha],\infty})$.

Proof. We denote by $pr_\alpha : \mathcal{V} \to \mathcal{V}$ the projection operator given by $\int_K \chi_\alpha^{-1}(k)\pi(k)$. Its image is $\mathcal{V}^{[K,\alpha]}$. Furthermore, $pr_\alpha(\mathcal{V}^\infty) \subset \mathcal{V}^\infty$, since the integral involved can be interpreted as an integral inside $\mathcal{V}^\infty$ (with its own topology).

Let $v \in \mathcal{V}^{[K,\alpha]}$ and let $v_n \to v$ with $v_n \in \mathcal{V}^\infty$. Then

$$v = pr_\alpha v = \lim pr_\alpha v_n$$

and we are done. $\square$

Corollary 2.1.17. *If $\mathcal{V}^\infty$ is admissible (in particular, if $\mathcal{V}$ is admissible), then $\mathcal{V}^{[K]} \subset \mathcal{V}^\infty$.*

Proof. Since $\mathcal{V}^{[K,\alpha],\infty}$ is finite-dimensional, it is closed, and hence the previous lemma forces $\mathcal{V}^{[K,\alpha]} = \mathcal{V}^{[K,\alpha],\infty}$. $\square$

Remark 2.1.18. If $\mathcal{V}$ is not admissible, the conclusion of the previous corollary does not necessarily hold, even if K is "big" in G. For example, we can consider the action of G on $C(G)$ by left translations (where $C(G)$ has the Frechet topology of uniform convergence on compacts). Then one has plenty of K-invariant vectors, which are not smooth (simply continuous but non-smooth functions on $K \backslash G$).

Claim 2.1.19. *Let $v \in \mathcal{V}^{[K],\infty}$. Suppose that $U(\mathfrak{g})^K$ acts finitely on v. Then $v \in \mathcal{V}^\omega$.*

Proof. Let $\alpha \in \mathcal{V}^*$. Then Set $f(g) := \alpha(gv)$. We want to show that f is analytic.

We will exhibit an element $D \in U(\mathfrak{g})^K$ for which R_D is an elliptic differential operator on G. We have $p(D)v = 0$ for some monic polynomial p by the assumption of finiteness, and hence $R_{p(D)}f = 0$. Since $p(D)$ is elliptic, by elliptic regularity we obtain that f is analytic.

To exhibit such D, let us find a $Ad(K)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, fix an orthonormal basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ w.r.t. $\langle \cdot, \cdot \rangle$. Then $D := X_1^2 + \ldots +$

$X_n^2 \in U(\mathfrak{g})^K$. Indeed, it can be interpreted as the image of $Id_{\mathfrak{g}}$ under the K-equivariant map

$$End(\mathfrak{g}) \cong \mathfrak{g} \otimes \mathfrak{g}^* \overset{\text{via}\langle\cdot,\cdot\rangle}{\cong} \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\text{mult.}} U(\mathfrak{g}).$$

Clearly, R_D is an elliptic differential operator on G. □

Corollary 2.1.20. *Suppose that $\mathcal{V}$ is admissible. Then $\mathcal{V}^{[K]} \subset \mathcal{V}^{\omega}$.*

Proof. This is clear, since $U(\mathfrak{g})^K$ preserves the K-types, hence acts finitely on K-finite vectors if $\mathcal{V}$ is admissible. □

Lemma 2.1.21. *Let $U \subset \mathcal{V}^{\omega} \cap \mathcal{V}^{\infty}$ be a subspace stable under $U(\mathfrak{g})$. Then $Cl(U)$ is stable under G°.*

Proof. Let $v \in U$. By standard arguments, it is enough to see that $e^X v \in Cl(U)$ for every $X \in \mathfrak{g}$. Fix $\alpha \in \mathcal{V}^*$ which vanishes on U. The function $\mathfrak{g} \to \mathbb{C}$ given by $X \mapsto \alpha(e^X v)$ is analytic. Moreover, its partial derivatives at 0 are of the form $\alpha(Zv)$ for $Z \in U(\mathfrak{g})$, hence vanish. Thus, our function is zero. We get that for a given $X \in \mathfrak{g}$, $e^X v$ is annihilated by all $\alpha \in \mathcal{V}^*$ which annihilate U. By the Hahn-Banach theorem we get $e^X v \in Cl(U)$.

□

Claim 2.1.22. *Suppose that $\mathcal{V}$ is admissible. Then one has an isomorphism of partial orders, between closed subspaces $\mathcal{U} \subset \mathcal{V}$ stable under G, and subspaces $U \subset \mathcal{V}^{[K]}$ stable under K and $\mathfrak{g}$, given by the correspondence $\mathcal{U} \mapsto \mathcal{U}^{[K]}$ and $U \mapsto Cl(U)$.*

Proof. The subspace $Cl(U) \subset \mathcal{V}$ is closed under G° by lemmas above. Since it is also closed under K, it is closed under $KG^{\circ} = G$. Let us show now that $U = Cl(U)^{[K]}$. For $v \in Cl(U)^{[K,\alpha]}$, we take $v_n \to v$ with $v_n \in U$. By replacing v_n with $pr_{\alpha} v_n$, we can assume $v_n \in U^{[K,\alpha]}$. Since $U^{[K,\alpha]}$ is finite-dimensional, we obtain $v \in U^{[K,\alpha]}$. □

2.1.4 $(\mathfrak{g}, K)$-modules

Definition 2.1.23. An algebraic representation of K is a locally finite representation V of K, such that for every finite-dimensional K-invariant subspace $W \subset V$, the action of K on W is continuous (thus smooth).

For an algebraic representation V of K, we have the isotypical components $V^{[\alpha]}$, and $V = \oplus_{\alpha \in \hat{K}} V^{[\alpha]}$.

Definition 2.1.24. A $(\mathfrak{g}, K)$-module is a vector space V equipped with an action of $\mathfrak{g}$ (denote $\pi_{\mathfrak{g}} : U(\mathfrak{g}) \to End(V)$) and an algebraic action of K (denote $\pi_K : K \to GL(V)$), such that:

1. $\pi_K(k) \circ \pi_{\mathfrak{g}}(U) \circ \pi_K(k^{-1}) = \pi_{\mathfrak{g}}(Ad(k)U)$ for $k \in K, U \in U(\mathfrak{g})$.

2. $d\pi_K = \pi_{\mathfrak{g}}|_{\mathfrak{k}}$.

We have an obvious notion of a morphism of $(\mathfrak{g}, K)$-modules, and $(\mathfrak{g}, K)$-modules form an abelian category $\mathcal{M}(\mathfrak{g}, K)$.

Example 2.1.25. *Let $\mathcal{V}$ be an admissible G-representation. Then $\mathcal{V}^{[K]}$ is a $(\mathfrak{g}, K)$-module (the underlying $(\mathfrak{g}, K)$-module). Two admissible G-representations are said to be* ***infinitesimally equivalent****, if their underlying $(\mathfrak{g}, K)$-modules are isomorphic.*

Example 2.1.26. *In the usual example of $G = SL_2(\mathbb{R})$ acting on $M = \mathbb{P}(\mathbb{R}^2)$, the representations $L^2(M)$ and $C(M)$ are infinitesimally equivalent.*

Lemma 2.1.27. *Assume that the inclusion $K \to G$ induces an isomorphism of fundamental groupoids. Then forgetful functor*

$$\{\textit{f.d. } G\textit{-rep.}\} \to \{\textit{f.d. } (\mathfrak{g}, K)\textit{-mod.}\}$$

is an equivalence of categories.

Proof. The functor is clearly faithful, and easily seen to be full. Let us show that it is essentially surjective. So, let V be a f.d. $(\mathfrak{g}, K)$-module. Denote $H = GL(V)$, $\alpha : \mathfrak{g} \to \mathfrak{h}, \beta : K \to H$. We want to show that there exists $G \to H$ whose derivative is α and whose restriction to K is β.

Since $KG^\circ = G$ and G° is normal in G, it is enough to show that there exists $\gamma : G^\circ \to H$, whose derivative is α. Indeed, then we will have:

1. $\gamma|_{K \cap G^\circ} = \beta$ (because, by assumption, $K \cap G^\circ$ is connected, and $\gamma|_{K \cap G^\circ}, \beta$ have the same derivative by the second $(\mathfrak{g}, K)$-module axiom).

2. $\gamma(kgk^{-1}) = \beta(k)\gamma(g)\beta(k)^{-1}$ for $k \in K, g \in G^\circ$ (by the second $(\mathfrak{g}, K)$-module axiom).

Combining the two, by simple group theory there will be a unique $\widetilde{\gamma} : G \to H$ such that $\widetilde{\gamma}|_{G^\circ} = \gamma$ and $\widetilde{\gamma}|_K = \beta$.

Consider the universal cover $\pi : G' \to G^\circ$. We have $\gamma' : G' \to H$ whose derivative is α. It is enough to show that γ' is trivial on $\pi^{-1}(e)$.

Let $K' \subset G'$ be the inverse image of $K \cap G^\circ$ under π. Notice that $\gamma'|_{(K')^\circ}$ has derivative $\alpha|_{\mathfrak{k}}$, and thus by the first $(\mathfrak{g}, K)$-module axiom its derivative is the derivative of β, implying that $\gamma'|_{(K')^\circ}$ factors through $\pi : (K')^\circ \to K$. In particular, γ' is trivial on $\pi^{-1}(e) \cap (K')^\circ$. It is thus left to show that $\pi^{-1}(e) \subset K'$ sits in the trivial connected component. Indeed, this follows from $K \to G$ being an isomorphism on $\pi_1(\cdot; e)$. □

Thus, it is starting to seem that $(\mathfrak{g}, K)$-modules are a good tool to capture the algebraic properties of topological representations.

2.1.5 Examples

Let G act transitively on a manifold M, and let μ be a G-invariant Radon measure on M.

We have the G-representation $C(M)$ of continuous functions on M, with the Frechet topology of uniform convergence on compacts. We also have the unitary G-representation $L^2(M)$.

One has $C(M)^\infty = C^\infty(M)$, the space of smooth functions on M, with the Frechet topology of uniform convergence of all iterated derivatives on compacts.

Assume that M is compact. Then $L^2(M)^\infty = C^\infty(M)$ (as Frechet representations). This is basically a case of Sobolev's embedding theorem.

Examples for $SL_2(\mathbb{R})$

Let us consider $G = SL_2(\mathbb{R}), K = SO_2(\mathbb{R})$. We have an action of G on $M := \mathbb{P}(\mathbb{R}^2)$, and hence on functions on this space. We think of M as the unit circle modulo ± 1. The standard measure $d\theta$ on M is G-invariant (because it is K-invariant). The space $L^2(M)^{[K]}$ is the span of $\phi_{2n} := e^{i\cdot 2n\cdot\theta}$, $n \in \mathbb{Z}$.

More generally, let us fix $\lambda \in \mathbb{C}, \epsilon \in \{0,1\}$ and consider the space $\mathcal{P}_{\lambda,\epsilon}$ of smooth functions $f : \mathbb{R}^2 - \{0\} \to \mathbb{C}$ satisfying $f(cv) = \mathrm{sgn}(c)^\epsilon |c|^{-\lambda-1} f(v)$ for $c \in \mathbb{R}^\times, v \in \mathbb{R}^2 - \{0\}$. This is a Frechet G-representation; The topology is given by uniform convergence of iterated derivatives on compacts, and the G-action is via the natural G-action on $\mathbb{R}^2 - \{0\}$. This is called the **principal series** of representations. One can give an infinitesimally equivalent Hilbertable variant, by considering on $\mathcal{P}_{\lambda,\epsilon}$ the inner product $(f_1, f_2) \mapsto \int_S f_1 \cdot \bar{f}_2 d\theta$ and taking completion w.r.t. it (here $S \subset \mathbb{R}^2 - \{0\}$ is the unit circle).

Let us write very explicitly the $(\mathfrak{g}, K)$-module underlying $\mathcal{P}_{\lambda,\epsilon}$. Let us denote by $e_n^\lambda \in \mathcal{P}_{\lambda,\epsilon}$ the element which on the unit circle is given by $\begin{pmatrix} c \\ s \end{pmatrix} \mapsto (c+is)^n$. For $\epsilon = 0$ (resp. $\epsilon = 1$), the module $\mathcal{P}_{\lambda,\epsilon}^{[K]}$ has basis e_n^λ with $n \in 2\mathbb{Z}$ (resp. $n \in 1 + 2\mathbb{Z}$).

We have the following basis of $\mathfrak{g}_\mathbb{C}$:

$$H_c = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ X_c = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}, \ Y_c = \frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}.$$

Notice that iH_c is a basis element for $\mathfrak{k}$, and we have

$$[H_c, X_c] = 2X_c, \ [H_c, Y_c] = -2Y_c, \ [X_c, Y_c] = H_c$$

(i.e. H_c, X_c, Y_c is an $\mathfrak{sl}_2$-triple).

Then, one has:

$$H_c e_n^\lambda = n e_n^\lambda, \ X_c e_n^\lambda = \frac{1}{2} i(-(\lambda+1) - n) e_{n+2}^\lambda, \ Y_c e_n^\lambda = \frac{1}{2} i(\lambda + 1 - n) e_{n-2}^\lambda.$$

For example, for $\mathcal{P}_{-1,0}$, one notices that $\mathcal{P}_{-1,0}^{[K]} / \mathbb{C} e_0^{-1}$ admits a direct summand consisting of the span of $e_2^{-1}, e_4^{-1}, \ldots$. Let us describe a representation which this summand underlies in independent terms.

In analogy with the principal series, we can consider a complex variant. Thus, something like smooth functions $f : \mathbb{C}^2 - \{0\} \to \mathbb{C}$, satisfying $f(cv) = c^{-n}f(v)$ for $c \in \mathbb{C}^\times, v \in \mathbb{C}^2 - \{0\}$. Clearly G acts on those, but there are too many of them, so we restrict ourselves to holomorphic f's. Then there are too few of them, so we pass to an orbit of G, and consider f defined on the subspace of $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ for which $\Im(\frac{z_1}{z_2}) > 0$. Those, up to homothety, can be identified with the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0)\}$.

Then, for each n, we obtain a representation of G on the space of holomorphic functions on $\mathbb{H}$, induced by the usual action of G on $\mathbb{H}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

but with an automorphy factor:

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix} f)(z) = (-cz+a)^{-n} f(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} z).$$

Let us find K-finite vectors in this representation. For this, it is convenient to use the Cayley transform $C = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$, which identifies $\mathbb{H}$ with $\mathbb{D} = \{w \in \mathbb{C} \mid |w| < 1\}$. The inverse transform is $C^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$. Let us denote $\gamma = a + ic, \delta = d - ib$. Then one computes:

$$C \begin{pmatrix} a & b \\ c & d \end{pmatrix} C^{-1} = \frac{1}{2}\begin{pmatrix} \overline{\gamma+\delta} & \overline{\gamma-\delta} \\ \gamma-\delta & \gamma+\delta \end{pmatrix}.$$

Hence, the action of G on functions on $\mathbb{D}$ corresponding to its action on functions on $\mathbb{H}$ is:

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix} h)(w) = h(\frac{1}{2}\begin{pmatrix} (\delta+\gamma)w + \overline{\delta-\gamma} \\ (\delta-\gamma)w + \overline{\delta+\gamma} \end{pmatrix}) = \left(\frac{1}{2}(\delta-\gamma)w + \frac{1}{2}\overline{\delta+\gamma}\right)^{-n} h(\frac{(\delta+\gamma)w + \overline{\delta-\gamma}}{(\delta-\gamma)w + \overline{\delta+\gamma}}).$$

In particular, if for $\gamma = c+is$ of absolute value 1 we denote $T_\gamma = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ (thus identifying K with the complex numbers of absolute value 1) we obtain:

$$(T_\gamma h)(w) = \gamma^n h(\gamma^2 w);$$

a very nice expression, indicating that he Cayley transform is "good for K".

So, the K-finite vectors of type $\gamma \mapsto \gamma^m$ are identified now with holomorphic solutions on $\mathbb{D}$ of the following equation:

$$h(\gamma^2 w) = \gamma^{m-n} h(w).$$

We assume $n > 0$. The solutions are easily found; There exists a non-zero solution if $m \in n + 2\mathbb{Z}_{\geq 0}$, in which case the solution is unique up to scalar, given by $h_m(w) = w^{(m-n)/2}$ (so, simply $1, w, w^2, \ldots$).

We see that the underlying $(\mathfrak{g}, K)$-module of the representation $\mathcal{D}_n$ described has a basis $e_n, e_{n+2}, \ldots$, where $T_\gamma e_k = \gamma^k e_k$.

Going back to the principal series notice that consider, for example, $\mathcal{P}^{[K]}_{-1,0}$. One sees from the formulas above that as a K-module it is the same as $\mathcal{D}^{[K]}_2$. One can also calculate that these are in fact isomorphic as $(\mathfrak{g}, K)$-modules. Moreover, one observes that the same $(\mathfrak{g}, K)$-module appears as a submodule in $\mathcal{P}^{[K]}_{1,0}$.

We thus described the discrete series, and observed that it embeds into principal series.

2.2 Structure of real reductive algebraic groups

2.2.1 Compact real forms

By Galois descent, real forms of a complex affine algebraic group $\boldsymbol{G}$ are in bijection with anti-algebraic involutions $\sigma : \boldsymbol{G}(\mathbb{C}) \to \boldsymbol{G}(\mathbb{C})$, so by real forms we will simply mean such σ's. Note that $\boldsymbol{G}(\mathbb{C})^\sigma$ (which is the set of $\mathbb{R}$-points of $\boldsymbol{G}$, equipped with the real form corresponding to σ) is a Lie group with finitely many connected components.

Definition 2.2.1. Let $\boldsymbol{G}$ be a complex affine algebraic group.

1. A **compact real form** of $\boldsymbol{G}$ is a real form σ such that $\boldsymbol{G}(\mathbb{C})^\sigma$ is compact and intersects each connected component of $\boldsymbol{G}(\mathbb{C})$.

2. A **geometric compact real form** of $\boldsymbol{G}$ is a compact Lie subgroup $\widetilde{K} \subset \boldsymbol{G}(\mathbb{C})$ such that $Lie(\widetilde{K})$ is a real form of $Lie(\boldsymbol{G}(\mathbb{C}))$ and such that $\widetilde{K}$ intersects each connected component of $\boldsymbol{G}(\mathbb{C})$.

Theorem 2.2.2. *Let $\boldsymbol{G}$ be a complex affine algebraic group. Then $\boldsymbol{G}$ admits a compact real form if and only if $\boldsymbol{G}$ is reductive (meaning that its neutral component is reductive).*

Proof. Is there a good reference? □

Hypothesis: From now on, through all the text, $\boldsymbol{G}$ denotes a complex reductive algebraic group.

Claim 2.2.3. *Any holomorphic (anti-holomorphic) homomorphism between connected reductive complex algebraic groups is algebraic (anti-algebraic).*

Proof. See [4, Proposition D.2.1] or [1, Lemma 3.1]. □

Claim 2.2.4. *Associating to a compact real form σ the geometric compact real form $\widetilde{K} := \boldsymbol{G}(\mathbb{C})^\sigma$ gives a bijection between compact real forms and geometric compact real forms.*

Proof. Clearly, given a compact real form σ, the subgroup $\widetilde{K} = \boldsymbol{G}(\mathbb{C})^{\sigma}$ is a geometric compact real form. This correspondence is injective, since if for two compact real forms σ_1, σ_2 we have $\widetilde{K} = \boldsymbol{G}(\mathbb{C})^{\sigma_1} = \boldsymbol{G}(\mathbb{C})^{\sigma_2}$, then σ_1, σ_2 induce the same map on $Lie(\boldsymbol{G}(\mathbb{C}))$ (specifically, the map which acts as 1 on $Lie(\widetilde{K})$ and as -1 on $i \cdot Lie(\widetilde{K})$), hence are equal on $\boldsymbol{G}(\mathbb{C})^{\circ}$, and hence on $\boldsymbol{G}(\mathbb{C}) = \boldsymbol{G}(\mathbb{C})^{\circ}\widetilde{K}$. Later (corollary 2.2.8) we will see that this correspondence is in fact bijective, i.e. that every geometric compact real form comes from a compact real form. □

2.2.2 Cartan decomposition for the complex group

Let $\widetilde{K} \subset \boldsymbol{G}(\mathbb{C})$ be a geometric compact real form. Denote $\widetilde{\mathfrak{s}} := i\widetilde{\mathfrak{k}}$ (so $\mathfrak{g}_{\mathbb{C}} = \widetilde{\mathfrak{k}} \oplus \widetilde{\mathfrak{s}}$).

Theorem 2.2.5 (Cartan decomposition)**.** *The map $\widetilde{K} \times \widetilde{\mathfrak{s}} \to \boldsymbol{G}(\mathbb{C})$ given by $(k, X) \mapsto k \cdot exp(X)$ is a diffeomorphism.*

Proof. We assume that we know this theorem for GL_n with its standard geometric compact real form arising from the compact real form $\sigma_0(M) := (\overline{M}^t)^{-1}$. In other words, $U(n) \times i\mathfrak{u}(n) \to GL_n(\mathbb{C})$ given by $(k, X) \mapsto k \cdot exp(X)$ is a diffeomorphism. We embed $\boldsymbol{G}$ into GL_n algebraically, and by the unitary trick (finding an invariant inner product for $\widetilde{K}$), we can assume that $\widetilde{K}$ embeds into $U(n)$. Thus, our map factors via $\widetilde{K} \times \widetilde{\mathfrak{s}} \to U(n) \times i\mathfrak{u}(n) \to GL_n(\mathbb{C})$. Thus, our map is an injective immersion, and its image is closed. Since the domain and codomain have the same dimension, we deduce that our map is the embedding of a union of connected components. Since $\widetilde{K}$ intersects each connected component of $\boldsymbol{G}(\mathbb{C})$, we deduce that our map is a diffeomorphism. □

Corollary 2.2.6. *The inclusion $\widetilde{K} \to \boldsymbol{G}(\mathbb{C})$ is a homotopy equivalence, and $\widetilde{K}$ is a maximal compact subgroup of $\boldsymbol{G}(\mathbb{C})$.*

Proof. If a subgroup L contains $\widetilde{K}$ properly, then it contains some element $exp(X)$ for $X \in \widetilde{\mathfrak{s}}$, and hence contains the non-compact closed subset $exp(X)^{\mathbb{Z}} = exp(\mathbb{Z} \cdot X)$. □

Corollary 2.2.7. *Fix an algebraic embedding $\boldsymbol{G} \subset GL_n$ such that $\widetilde{K} \subset U(n)$. Then $\boldsymbol{G}$ is stable under $M \mapsto (\overline{M}^t)^{-1}$, and for the resulting real form σ_c of $\boldsymbol{G}$ we have $\widetilde{K} = U(n) \cap \boldsymbol{G}(\mathbb{C}) = \boldsymbol{G}(\mathbb{C})^{\sigma_c}$.*

Proof. This follows from $\widetilde{K}$ being maximal compact in $\boldsymbol{G}(\mathbb{C})$. □

Corollary 2.2.8. *Any geometric compact real form comes from a compact real form.*

2.2.3 Real forms and Cartan involutions

The following theorem is a main tool:

Theorem 2.2.9. *Let σ be a real form of $\boldsymbol{G}$.*

1. *There exists a compact real form σ_c of $\boldsymbol{G}$, such that $\sigma \circ \sigma_c = \sigma_c \circ \sigma$.*

2. *Given two compact real forms σ_c^1, σ_c^2 of $\boldsymbol{G}$ commuting with σ, there exists $g \in \boldsymbol{G}(\mathbb{C})^\sigma$ such that $\sigma_c^2 = i_g^{-1} \circ \sigma_c^1 \circ i_g$, where $i_g : \boldsymbol{G}(\mathbb{C}) \to \boldsymbol{G}(\mathbb{C})$ is given by $i_g(h) := ghg^{-1}$.*

3. *Suppose given a compact real form σ_c of $\boldsymbol{G}$ commuting with σ. Suppose furthermore given a reductive algebraic group $\boldsymbol{G}'$, a real form σ' of $\boldsymbol{G}'$, and an embedding $\boldsymbol{G} \to \boldsymbol{G}'$ intertwining the two real forms σ and σ'. Then there exists a compact real form σ_c' of $\boldsymbol{G}'$ commuting with σ', such that σ_c' preserves $\boldsymbol{G}$ and is equal to σ_c when restricted to $\boldsymbol{G}$.*

Proof. See [1]. □

Given a compact real form σ_c commuting with the real form σ, we denote $\theta := \sigma_c \circ \sigma$ (the **Cartan involution**). Then θ is an algebraic involution of $\boldsymbol{G}$, which commutes with σ and with σ_c.

From now on, we assume that $\boldsymbol{G}$ is a complex reductive algebraic group, σ is a real form of $\boldsymbol{G}$, and σ_c is a compact real form of $\boldsymbol{G}$ commuting with σ. We denote $G := \boldsymbol{G}(\mathbb{R}) := \boldsymbol{G}(\mathbb{C})^\sigma$, $\widetilde{K} := \boldsymbol{G}(\mathbb{C})^{\sigma_c}$, $K := \boldsymbol{G}(\mathbb{R}) \cap \widetilde{K} = \boldsymbol{G}(\mathbb{R})^{\sigma_c} = \boldsymbol{G}(\mathbb{R})^\theta$. We write $\mathfrak{g} := Lie(\boldsymbol{G}(\mathbb{R}))$ and $\mathfrak{g}_\mathbb{C} := Lie(\boldsymbol{G}(\mathbb{C}))$. We have $\mathfrak{k} := \mathfrak{g} \cap \widetilde{\mathfrak{k}} = \mathfrak{g}^{\sigma_c} = \mathfrak{g}^\theta$. Also, denote $\widetilde{\mathfrak{s}} := i \cdot \widetilde{\mathfrak{k}}$, , $\mathfrak{s} := \mathfrak{g} \cap \widetilde{\mathfrak{s}} = \mathfrak{g}^{\sigma_c,-1} = \mathfrak{g}^{\theta,-1}$. We have $\mathfrak{g}_\mathbb{C} = \widetilde{\mathfrak{k}} \oplus \widetilde{\mathfrak{s}}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$.

One can imagine $\mathfrak{g}_\mathbb{C}$ breaking into:

$\sigma_c \setminus \sigma$	1	-1
1	$\mathfrak{k}$	$i\mathfrak{s}$
-1	$\mathfrak{s}$	$i\mathfrak{k}$

Notice that $\mathfrak{k} \oplus \mathfrak{s}$ is the Lie algebra of the real form $G = \boldsymbol{G}(\mathbb{C})^\sigma$, $\mathfrak{k} \oplus i\mathfrak{s}$ is the Lie algebra of the compact real form $\widetilde{K} = \boldsymbol{G}(\mathbb{C})^{\sigma_c}$, and $\mathfrak{k} \oplus \imath\mathfrak{k}$ is the Lie algebra of the complexification of K, which is $K_\mathbb{C} = \boldsymbol{G}(\mathbb{C})^\theta$.

2.2.4 Cartan decomposition

Theorem 2.2.10 (Cartan decomposition)**.** *The map $K \times \mathfrak{s} \to G$ given by $(k, X) \mapsto k \cdot exp(X)$ is a diffeomorphism.*

Proof. This is clear by using the diffeomorphism $\widetilde{K} \times \widetilde{\mathfrak{s}} \to \boldsymbol{G}(\mathbb{C})$ and taking σ-invariants of everything. □

Corollary 2.2.11. *The inclusion $K \to G$ is a homotopy equivalence (in particular, K intersects each component of G), and K is maximal compact in G.*

Claim 2.2.12. *There exist algebraic embeddings $\boldsymbol{G} \subset GL_n$ such that σ is the restriction to $\boldsymbol{G}(\mathbb{C})$ of $M \mapsto \overline{M}$ and σ_c is the restriction to $\boldsymbol{G}(\mathbb{C})$ of $M \mapsto (\overline{M}^t)^{-1}$.*

Proof. Choose first a real embedding of $(\boldsymbol{G}, \sigma)$ into $(GL_n, M \mapsto \overline{M})$ (so σ is the restriction of $M \mapsto \overline{M}$). By item 3 of theorem 2.2.9 we can find a compact real form σ'_c of GL_n which preserves $\boldsymbol{G}$ and is equal to σ_c when restricted to $\boldsymbol{G}$. By item 2 of theorem 2.2.9, we can find $g \in GL_n(\mathbb{R})$ such that $\sigma'_c = i_g^{-1} \circ \tau \circ i_g$ where $\tau(M) = (\overline{M}^t)^{-1}$. This exactly means that after changing coordinates via g, we will get the desired. □

Hypothesis: From now on,through all the text, we assume that $\boldsymbol{G}$ is connected (but still, G might be not connected).

2.2.5 Invariant bilinear forms

Claim 2.2.13. *The Lie algebra $\mathfrak{g}_{\mathbb{C}}$ admits a $\boldsymbol{G}(\mathbb{C})$-invariant non-degenerate symmetric bilinear form $B(\cdot,\cdot)$, which is real negative-definite on $\mathfrak{k}$, real positive-definite on $\mathfrak{p}$, and for which $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal.*

Proof. By averaging, we find a negative-definite form on $\widetilde{\mathfrak{k}}$, invariant w.r.t. $\widetilde{K}$. We then consider the extension by $\mathbb{C}$-bilinearity of this form to $\mathfrak{g}_{\mathbb{C}}$, call it $\widetilde{B}$. It is invariant w.r.t. $\widetilde{K}$ and thus w.r.t. $\widetilde{\mathfrak{k}}$. Thus, it is clearly invariant w.r.t. $\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \widetilde{\mathfrak{k}}$. Restricting to $\mathfrak{g}$ and taking the real part, we obtain a real form which is $\mathfrak{g}$-invariant, negative-definite on $\mathfrak{k}$, positive-definite on $\mathfrak{p}$, and for which $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal. Extending this form by $\mathbb{C}$-bilinearity to $\mathfrak{g}_{\mathbb{C}}$, we obtain a form B which $\mathfrak{g}_{\mathbb{C}}$-invariant, and hence $\boldsymbol{G}(\mathbb{C})$-invariant, since $\boldsymbol{G}(\mathbb{C})$ is assumed connected. This form B is as desired. □

Claim 2.2.14. *The Lie algebra $\mathfrak{g}$ admits a G-invariant non-degenerate real symmetric bilinear form $B(\cdot,\cdot)$, which is negative-definite on $\mathfrak{k}$ and positive-definite on $\mathfrak{s}$, and for which $\mathfrak{k}$ and $\mathfrak{s}$ are orthogonal. The form $B(\cdot,\theta\cdot)$ is positive-definite and K-invariant. The operators $ad(X) = [X,\cdot]$ are skew-symmetric for $X \in \mathfrak{k}$ and symmetric for $X \in \mathfrak{s}$ (w.r.t. $B(\cdot,\theta\cdot)$).*

Proof. We consider the restriction of B from claim 2.2.13 to $\mathfrak{g}$. □

2.2.6 Iwasawa decomposition

Maximal abelian subspaces

Let $\mathfrak{a} \subset \mathfrak{s}$ be a maximal abelian subspace. Since $ad(X)$ are symmetric w.r.t. the form $B(\cdot,\theta\cdot)$ for $X \in \mathfrak{s}$, the operators $ad(X)$ for $X \in \mathfrak{a}$ are mutually diagnolizable. Thus, we can write

$$\mathfrak{g} = \mathfrak{g}^{\mathfrak{a},0} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^{\mathfrak{a},\alpha}$$

where

$$\mathfrak{g}^{\mathfrak{a},\alpha} = \{Y \in \mathfrak{g} \mid [X,Y] = \alpha(X)Y \ \forall X \in \mathfrak{a}\}$$

and $R \subset \mathfrak{a}^*$ is the subset of **restricted roots** (by definition, $\alpha \in \mathfrak{a}^*$ which are non-zero and for which $\mathfrak{g}^{\mathfrak{a},\alpha} \neq 0$). We further have $\mathfrak{g}_\emptyset := \mathfrak{g}^{\mathfrak{a},0} = \mathfrak{a} \oplus \mathfrak{k}_\emptyset$ where $\mathfrak{k}_\emptyset := C_{\mathfrak{k}}(\mathfrak{a})$.

Notice that we have isomorphisms $\theta : \mathfrak{g}^{\mathfrak{a},\alpha} \to \mathfrak{g}^{\mathfrak{a},-\alpha}$; In particular, $-R = R$.

Let us denote $\mathfrak{a}^{reg} := \mathfrak{a} - \cup_{\alpha \in R} Ker(\alpha)$.

Lemma 2.2.15. *For $H' \in \mathfrak{a}^{reg}$ one has $C_{\mathfrak{g}}(H') = C_{\mathfrak{g}}(\mathfrak{a})$ $(= \mathfrak{g}^{\mathfrak{a},0})$ (in particular, $C_{\mathfrak{s}}(H') = \mathfrak{a}$).*

Proof. Clear. □

Lemma 2.2.16. *Let $H_1, H_2 \in \mathfrak{s}$. Then there exists $k \in K^\circ$ such that $[Ad(k)H_1, H_2] = 0$.*

Proof. Consider the function $f : K^\circ \to \mathbb{R}$ given by $f(g) = B(Ad(g)H_1, H_2)$. Let k_0 be a minimum point for f. Then for every $Y \in \mathfrak{k}$ we have

$$0 = \frac{d}{dt}|_{t=0} B(Ad(exp(tY)k_0)H_1, H_2) = B([Y, Ad(k_0)H_1], H_2) = B(Y, [Ad(k_0)H_1, H_2])$$

and thus $[Ad(k_0)H_1, H_2] = 0$ (since B is non-degenerate on $\mathfrak{k}$). □

Claim 2.2.17. *Let $\mathfrak{a}_1, \mathfrak{a}_2 \subset \mathfrak{s}$ be two maximal abelian subspaces. Then there exists $k \in K^\circ$ such that $Ad(k)\mathfrak{a}_1 = \mathfrak{a}_2$.*

Proof. Let $H_i \in \mathfrak{a}_i^{reg}$, and find $k \in K^\circ$ such that $Ad(k)H_1 = H_2$. Then $Ad(k)\mathfrak{a}_1 = Ad(k)C_{\mathfrak{s}}(H_1) = C_{\mathfrak{s}}(Ad(k)H_1) = C_{\mathfrak{s}}(H_2) = \mathfrak{a}_2$. □

Iwasawa decomposition

We choose $H' \in \mathfrak{a}^{reg}$. We denote $R^+ = \{\alpha \in R \mid \alpha(H') > 0\}$, and similarly R^-, so that $R^- = -R^+$ and $R = R^+ \cup R^-$. We define $\mathfrak{n} = \oplus_{\alpha \in R^+} \mathfrak{g}^{\mathfrak{a},\alpha}$ and similarly $\mathfrak{n}^-$ (so $\mathfrak{n}^- = \theta(\mathfrak{n})$). Thus, we have

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{k}_\emptyset \oplus \mathfrak{n} \oplus \mathfrak{n}^-.$$

Notice that $\mathfrak{n}$ and $\mathfrak{n}^-$ are nilpotent Lie subalgebras of $\mathfrak{g}$.

Lemma 2.2.18. *We have*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Proof. To show independence, assume $X + Y + Z = 0$ with $X \in \mathfrak{k}, Y \in \mathfrak{a}, Z \in \mathfrak{n}$. Then applying θ we obtain $X - Y + \theta(Z) = 0$ and substracting we obtain $2Y + (Z - \theta(Z)) = 0$. From here we get (by seperating w.r.t. the $\mathfrak{a}$-action) $Y = 0, Z = 0$.

To show that $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, it is enough to show that $\mathfrak{n}^- \subset \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$. But clearly, by writing $X = -\theta(X) + (X + \theta(X))$, we see that $\mathfrak{n}^- \subset \mathfrak{n} + \mathfrak{k}$.

□

Lemma 2.2.19. *The embedding of 2.2.12 can be made such that all the elements of* $\mathfrak{a}$ *are diagonal matrices, and all the elements of* $\mathfrak{n}$ *are nilpotent upper-triangular matrices.*

Proof. Since $\mathfrak{a}$ is a space of commuting symmetric matrices, they can be simultaneously diagnolized. Choose $H' \in \mathfrak{a}^{reg}$, and reorder the current basis such that the eigenvalues of H' do not increase. Then for $X \in \mathfrak{g}^{\mathfrak{a},\alpha} \subset \mathfrak{n}$, from the relation $[H', X] = \alpha(H')X$, since $\alpha(H') > 0$, we easily see that X is nilpotent upper-triangular. $\square$

Lemma 2.2.20. *There exist closed connected subgroup* $A, N \subset G(\mathbb{R})$ *such that* $Lie(A) = \mathfrak{a}$ *and* $Lie(N) = \mathfrak{n}$*. The map* $exp : \mathfrak{a} \to A$ *is a Lie group isomorphism, while* $exp : \mathfrak{n} \to N$ *is a manifold isomorphism.*

Proof. We will embed $G \subset GL_n$ as in the previous lemma.

Then $\mathfrak{a}$ is a sub Lie algebra of the diagonal, and the claim follows from the corresponding one for the diagonal. Similarly, $\mathfrak{n}$ is a sub Lie algebra of the unipotent upper triangular matrices, and the claim follows from the structure of subgroups of connected unipotent groups. $\square$

Theorem 2.2.21. *The map* $K \times A \times N \to G(\mathbb{R})$ *given by* $(k, a, n) \mapsto kan$ *is a diffeomorphism.*

Proof. Let us show that $K \times A \times N \to G(\mathbb{R})$ is a local isomorphism. By translation, it is enough to prove it at points $(1, a, 1)$. At those points, the differential is the map $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \to \mathfrak{g}$ given by $(X, Y, Z) \mapsto Ad(a^{-1})X + Y + Z = Ad(a^{-1})(X + Y + Ad(a)Z)$, which is an isomorphism by the Iwasawa decomposition on the Lie algebra level.

Let us show that $K \times A \times N \to G(\mathbb{R})$ is injective. We will think in terms of the embedding into matrices as in lemmas 2.2.12 and 2.2.19. If $kan = k_1 a_1 n_1$, then $k_1^{-1}k = a_1 n_1 n^{-1} a^{-1}$ is an upper-triangular orthogonal matrix with positive diagonal entries, hence equal to e. Thus $k = k_1$. We are left with $an = a_1 n_1$, which clearly implies $a = a_1$ and finally $n = n_1$.

We will finally want to show that the image of $K \times A \times N \to G(\mathbb{R})$ is closed. Then we will get that this map is an embedding of a union of connected components. But since K intersects each connected component of $G(\mathbb{R})$, we will get that our map is a diffeomorphism as desired. Since K is compact, it is easy to see that it is enough to show that the image of $A \times N \to G(\mathbb{R})$ is closed. Thinking in terms of the matrix embedding, this is an easy exercise. $\square$

2.2.7 The root system

The coroots

Let $\alpha \in R$, and $0 \neq X_\alpha \in \mathfrak{g}^{\mathfrak{a},\alpha}$. Set $Y_\alpha := \theta X_\alpha$ and $H_\alpha := [X_\alpha, Y_\alpha]$. Notice that $H_\alpha \in \mathfrak{s} \cap \mathfrak{g}^{\mathfrak{a},0} = \mathfrak{a}$. Also, notice that

$$B(H, H_\alpha) = \alpha(H) B(X_\alpha, Y_\alpha)$$

for $H \in \mathfrak{a}$. Plugging in $H = H_\alpha$, we see that $\alpha(H_\alpha) > 0$. We can thus normalize the initial choice of X_α, so that $\alpha(H_\alpha) = 2$. The triple $(X_\alpha, Y_\alpha, H_\alpha)$ becomes an $\mathfrak{sl}_2$-triple. Notice that the formula above also shows that H_α does not depend on the choice of X_α, because it can be described as the unique element lying in the one-dimensional space of $\mathfrak{a}$ corresponding to $Sp\{\alpha\}$ under the form B, and normalized by the condition $\alpha(H_\alpha) = 2$. The element H_α is called the **coroot** corresponding to the root α.

We denote $\mathfrak{a}_{cent} := \mathfrak{a} \cap \mathfrak{z}(\mathfrak{g}) = \cap_{\alpha \in R} Ker(\alpha)$, and $\mathfrak{a}_{ss} := \mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}]$.

Lemma 2.2.22. *One has* $\mathfrak{a}_{ss} = Sp\{H_\alpha\}_{\alpha \in R^+}$.

Proof. We have

$$\mathfrak{g}_\emptyset \cap [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{k}_\emptyset, \mathfrak{k}_\emptyset] \oplus \bigoplus_{\alpha \in R^+} [\mathfrak{g}^{\mathfrak{a},\alpha}, \mathfrak{g}^{\mathfrak{a},-\alpha}].$$

From the formula

$$B(H, [X, Y]) = \alpha(H) B(X, Y)$$

for $X \in \mathfrak{g}^{\mathfrak{a},\alpha}$ and $Y \in \mathfrak{g}^{\mathfrak{a},-\alpha}$, we see that the $\mathfrak{a}$-component of $[X, Y]$ w.r.t. $\mathfrak{g}_\emptyset = \mathfrak{k}_\emptyset \oplus \mathfrak{a}$ lies in $Sp\{H_\alpha\}$. From this the claim is clear.

□

Corollary 2.2.23 (of the proof of the lemma)**.** *Denoting by* $p : \mathfrak{g}_\emptyset \to \mathfrak{a}$ *the projection corresponding to* $\mathfrak{g}_\emptyset = \mathfrak{k}_\emptyset \oplus \mathfrak{a}$*, one has* $p([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_\emptyset) = \mathfrak{a}_{ss}$.

Lemma 2.2.24. *One has* $\mathfrak{a} = \mathfrak{a}_{cent} \oplus \mathfrak{a}_{ss}$.

Proof. Since $\mathfrak{g}$ is reductive, we have $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$. Since $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{g}_\emptyset$, we get $\mathfrak{g}_\emptyset = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_\emptyset$. Now the claim follows from $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{k}_\emptyset \oplus \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{a}$ and the corollary above. □

The Weyl group

Denote $W := N_K(\mathfrak{a})/C_K(\mathfrak{a})$. Notice that W acts faithfully on $\mathfrak{a}$ by orthogonal automorphisms.

Lemma 2.2.25. *The Lie algebra of both* $N_K(\mathfrak{a})$ *and* $C_K(\mathfrak{a})$ *is* $\mathfrak{k}_\emptyset$.

Corollary 2.2.26. *The group* W *is finite.*

Lemma 2.2.27. *Let* $H' \in \mathfrak{a}^{reg}$*. Then* $C_K(\mathfrak{a}) = C_K(H')$.

Proof. It is enough to show that $C_{\widetilde{K}}(H') = C_{\widetilde{K}}(\mathfrak{a})$. Notice that both have Lie algebra $\mathfrak{k}_\emptyset \oplus i\mathfrak{s}$. Thus, it is enough to show that $C_{\widetilde{K}}(H')$ is connected. But $C_{\widetilde{K}}(H') = C_{\widetilde{K}}(iH')$ where $iH' \in \widetilde{\mathfrak{k}}$, hence the connectedness follows from the fact that in a compact connected Lie group C, the centralizer of a vector $X \in Lie(C)$ is connected (here we use the assumption that $\boldsymbol{G}(\mathbb{C})$ is connected!). This follows from the fact that the centralizer of X is equal to the centralizer of $\overline{\{exp(tX)\}_{t \in \mathbb{R}}}$, which is a torus, and the classical result that the centralizer of a torus in a connected compact Lie group is connected.

□

The elements s_α

Thinking of W as a subgroup of the group of orthogonal automorphisms of $\mathfrak{a}$, we would like to show that s_α, the orthogonal reflection through $Ker(\alpha)$, lies in W, and in fact in the subgroup corresponding to $N_{K^\circ}(\mathfrak{a}) \subset N_K(\mathfrak{a})$ (we will see in a bit that this subgroup is in fact the whole W).

Take $0 \neq X_\alpha \in \mathfrak{g}^{\mathfrak{a},\alpha}$. Set $Y_\alpha := \theta(X_\alpha)$ and $H_\alpha := [X_\alpha, Y_\alpha]$. Notice that

$$B(H, H_\alpha) = \alpha(H)B(X_\alpha, \theta(X_\alpha)),$$

so in particular, plugging in $H = H_\alpha$, we deduce that $\alpha(H_\alpha) > 0$. Thus, by normalizing the initial choice of X_α appropriately, we can assume that $\alpha(H_\alpha) = 2$. Thus, we get an embedding $\mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{g}$ by mapping the standard H, X, Y to $H_\alpha, X_\alpha, Y_\alpha$. This integrates to a morphism $i_\alpha : SL_2(\mathbb{R}) \to G(\mathbb{R})$ (this is due to a general fact - a morphism from a covering group of $SL_2(\mathbb{R})$ to a matrix group always factors via $SL_2(\mathbb{R})$). One now notices that the standard θ of $SL_2(\mathbb{R})$ is compatible with our θ, and one deduces that $i_\alpha(SO_2(\mathbb{R})) \subset K^\circ$. Now one easily checks that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SO_2(\mathbb{R})$ is mapped to the desired s_α.

Weyl chambers

The connected components of $\mathfrak{a}^{reg}$ are called Weyl chambers. Those are convex open cones, given by the non-empty intersections $\cap_{\alpha \in R}\, \alpha^{-1}(\epsilon_\alpha \mathbb{R}_{>0})$ where $\epsilon_\alpha = \pm 1$.

Claim 2.2.28. *The action of W on the set of Weyl chambers is free.*

Proof. Suppose that $g \in N_K(\mathfrak{a})$ sends a Weyl chamber to itself. By averaging an arbitrary element of the Weyl chamber w.r.t. powers of g, we obtain an element in the Weyl chamber fixed by g. By lemma 2.2.27, we get $g \in C_K(\mathfrak{a})$, as desired. □

Claim 2.2.29. *The action of the subgroup $W' := \langle s_\alpha \rangle_{\alpha \in R} \subset W$ on the set of Weyl chambers is transitive.*

Proof. Let $H_1, H_2 \in \mathfrak{a}^{reg}$. Let $w \in W'$ be such that wH_2 is closest to H_1 as possible. We claim that $w(H_2)$ is then in the same Weyl chamber as H_1. Indeed, if not, then $\alpha(H_1) > 0$ and $\alpha(w(H_2)) < 0$ for some $\alpha \in R$. But then clearly $s_\alpha(w(H_2))$ is closer to H_1 than wH_2, in cotradiction to the choice of w. □

Corollary 2.2.30 (of the two claims)**.** *The group W is generated by the subset $\{s_\alpha\}_{\alpha \in R}$, and it acts simply transitively on the set of Weyl chambers (in particular, we also deduce that $N_{K^\circ}(\mathfrak{a}) \to W$ is surjective, since the s_α were realized as element in $N_{K^\circ}(\mathfrak{a})$).*

The simple roots

complete

2.2.8 The polar decomposition

Let us fix, once and for all, a Weyl chamber $\mathfrak{a}^{--} \subset \mathfrak{a}$, and denote by $\mathfrak{a}^{-}$ the closure of $\mathfrak{a}^{--}$ in $\mathfrak{a}$. We take the $H' \in \mathfrak{a}^{reg}$ to determine the positive roots lying in $\mathfrak{a}^{++} := -\mathfrak{a}^{--}$, so $R^+ \subset R$ is given by $\{\alpha \mid \alpha(H) < 0 \ \forall H \in \mathfrak{a}^{--}\}$. We denote by $A^{--}, A^{-} \subset A$ the corresponding subsets of A (under exp).

Recall the notation $K_\emptyset = C_K(\mathfrak{a})$. Denote by

$$m : K/K_\emptyset \times A \times K \to G(\mathbb{R})$$

the map given by $m([k_1], a, k_2) := k_1 a k_1^{-1} k_2$. Denote by $m^-,\ m^{--}$ the restrictions of m to the domains where we replace A by $A^-,\ A^{--}$.

Lemma 2.2.31. *The map m^- is surjective.*

Proof. By the Cartan decomposition, $G(\mathbb{R}) = exp(\mathfrak{s}) \cdot K$. Since any element of $\mathfrak{s}$ is conjugate via an element of K° to an element of $\mathfrak{a}$, we have $G(\mathbb{R}) = K^\circ \cdot exp(\mathfrak{a}) \cdot K = K^\circ A K$. Since every element of A can be moved to A^- by an element of W, we get $G(\mathbb{R}) = K^\circ A^- K$. □

Claim 2.2.32. *The map m^{--} is an open embedding with dense image.*

Proof. Since m^- is surjective and A^{--} is dense in A^-, it is clear that m^{--} has dense image.

Let us show that m^{--} is injective. Indeed, if $k_1 a k_1^{-1} k_2 = k_1' a' (k_1')^{-1} k_2'$, then by the Cartan decomposition with get $k_1 a k_1^{-1} = k_1' a' (k_1')^{-1}$ and $k_2 = k_2'$. Then we obtain $(k_1')^{-1} k_1$ sends $log(a) \in \mathfrak{a}^{reg}$ into $\mathfrak{a}^{reg}$, hence sends $C_{\mathfrak{s}}(log(a)) = \mathfrak{a}$ into the centralizer of an element in $\mathfrak{a}^{reg}$, i.e. into $\mathfrak{a}$. In other words, $(k_1')^{-1} k_1 \in N_K(\mathfrak{a})$. Now, as an element of the Weyl group, $(k_1')^{-1} k_1$ preserves a Weyl chamber, hence is trivial in the Weyl group, i.e. belongs to $K_\emptyset$. From this, we also deduce that $a = a'$.

It is easy to calaculate that the dimensions of the domain and codomain of m^{--} are equal. Hence, we are left to show that m^{--} is a submersion. For this, it is enough to show that the map $m_1 : K \times A \times K \to G$ given by $(k_1, a, k_2) \mapsto k_1 a k_2$ is a submersion at all points of $K \times A^{reg} \times K$. We calculate that the differential of m_1 at a point (k_1, a, k_2) is modeled by the map $\mathfrak{g} \oplus \mathfrak{a} \oplus \mathfrak{k} \to \mathfrak{g}$ given by $(X, Y, Z) \mapsto Ad(k_2^{-1}) Ad(a^{-1})(X + Y) + Z$, so it is surjective if and only if the map $(X, Y, Z) \mapsto Ad(a^{-1})X + Y + Z$ is surjective. In other words, we want to show that $Ad(a^{-1})\mathfrak{k} + \mathfrak{a} + \mathfrak{k} = \mathfrak{g}$ for $a \in A^{reg}$. By the Iwasawa decomposition, it is enough to show that $\mathfrak{g}^{\mathfrak{a},\alpha} \subset Ad(a^{-1})\mathfrak{k} + \mathfrak{k}$ for $\alpha \in R^+$. But given $0 \neq X \in \mathfrak{g}^{\mathfrak{a},\alpha}$, $Ad(a^{-1})$ acts on X and on $\theta(X)$ with different eigenvalues, hence $X + \theta(X)$ and $Ad(a^{-1})(X + \theta(X))$ are not linearly dependent. Thus $Sp(X, \theta(X)) = Sp(X + \theta(X), Ad(a^{-1})(X + \theta(X))$, and thus we can express X as a linear combination of $X + \theta(X) \in \mathfrak{k}$ and $Ad(a^{-1})(X + \theta(X)) \in Ad(a^{-1})\mathfrak{k}$, as desired. □

2.2.9 Parabolics, Levis

The Levis G_I

Fix a set of simple roots $I \subset R^s$. Denote $\mathfrak{a}_{cent,I} := \cap_{\alpha \in I} Ker(\alpha)$, and set $G_I := Z_G(\mathfrak{a}_{cent,I})$.

Notice that G_I is an algebraic subgroup of G, stable under σ and σ_c. We claim that $G_I(\mathbb{C})^{\sigma_c}$ intersects each connected component of $G_I(\mathbb{C})$ (so that the restriction σ_c is a compact form of G_I). Indeed, via the nice embedding of G into matrices, G_I can be described as G intersected with a particular block-diagonal subgroup. Using that, we see that the Cartan decomposition of an element in G_I consists of elements in G_I; Hence, we deduce that G_I has its own Cartan decomposition, so that the inclusion of $G_I(\mathbb{C})^{\sigma_c}$ into $G_I(\mathbb{C})$ is an homotopy equivalence.

Hence, G_I has "the same status" as G. We have the maximal compact $K_I = G_I(\mathbb{R}) \cap K = G_I(\mathbb{R})^{\sigma_c}$. We have $\mathfrak{s}_I = \mathfrak{s} \cap \mathfrak{g}_I = \mathfrak{g}_I^{\sigma_c,-1}$. Since $\mathfrak{a} \subset \mathfrak{s}_I \subset \mathfrak{s}$ and $\mathfrak{a}$ is already maximal abelian in $\mathfrak{s}$, we can take $\mathfrak{a}$ as a maximal abelian subspace in $\mathfrak{s}_I$, so G and G_I have "the same $\mathfrak{a}$".

$R_I = R \cap \sum_{\beta \in I} \mathbb{R}\beta$ is the root system of (G_I, σ). One has

$$\mathfrak{g}_I = \mathfrak{k}_\emptyset \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in R_I} \mathfrak{g}^{\mathfrak{a},\alpha}.$$

Notice that $\mathfrak{a}_{cent,I} = \mathfrak{a} \cap \mathfrak{z}(\mathfrak{g}_I)$, i.e. the notation is compatible. The coroots are compatible, i.e. for $\alpha \in R_I$, the coroots H_α for G and G_I are the same. We have $\mathfrak{a}_{ss,I} = \mathfrak{a} \cap [\mathfrak{g}_I, \mathfrak{g}_I] = Sp_{\alpha \in R_I^+}\{H_\alpha\}$ and $\mathfrak{a} = \mathfrak{a}_{cent,I} \oplus \mathfrak{a}_{ss,I}$.

One has $\mathfrak{n}_I = \bigoplus_{\alpha \in R_I^+} \mathfrak{g}^{\mathfrak{a},\alpha}$ (where $R_I^+ = R^+ \cap R_I$). One has N_I - the connected subgroup of $G_I(\mathbb{R})$ whose Lie algebra is $\mathfrak{n}_I$. Similarly, one has $\mathfrak{n}_I^-$ and N_I^-.

The parabolics

One has a Lie subalgebra $\mathfrak{n}_{(I)} \subset \mathfrak{n}$, defined by $\mathfrak{n}_{(I)} = \bigoplus_{\alpha \in R^+ - R_I^+} \mathfrak{g}^{\mathfrak{a},\alpha}$. One has the corresponding connected subgroup of $G(\mathbb{R})$ whose Lie algebra is $\mathfrak{n}_{(I)}$. Similarly, one has $\mathfrak{n}_{(I)}^-$ and $N_{(I)}^-$.

The subgroup $G_I(\mathbb{R})$ normalizes $N_{(I)}$, and the product $P_{(I)} := G_I(\mathbb{R}) \cdot N_{(I)}$ is a **standard parabolic subgroup**. Its Lie algebra is $\mathfrak{p}_{(I)} = \mathfrak{g}_I \oplus \mathfrak{n}_{(I)}$. Similarly, one has $P_{(I)}^-$ (which is a **semi standard parabolic subgroup**) and $\mathfrak{p}_{(I)}^-$.

Philosophy of cusp forms

One should study (representations of) G inductively, by relating to the G_I's; But not via $G_I \to G$ directly; rather, via $G_I \leftarrow P_{(I)} \to G$.

2.3 Scale and moderate growth

2.3.1 Scale

Let E be a faithful algebraic representation of $G(\mathbb{C})$, and fix a $\widetilde{K}$-invariant inner product on E. We define as usual $||g||_E := sup_{v\in E-\{0\}} \frac{||gv||}{||v||}$. We define

$$s(g) := 1 + log\ max\{||g||_E, ||g^{-1}||_E\}$$

(a "**scale function**").

Then $s(g)$ is a continuous function on g with values in $\mathbb{R}_{\geq 1}$, $s(e) = 1$, $s(k_1 g k_2) = s(g)$ for $k_1, k_2 \in K$, and $s(g_1 g_2) \leq s(g_1) + s(g_2)$. Also, $s(e^H) \sim 1 + ||H|| \quad (H \in \mathfrak{a})$, where $||H||$ is any norm on $\mathfrak{a}$; Similarly, $s(e^Y) \sim 1 + log(1 + ||Y||) \quad (Y \in \mathfrak{n}^-)$, where $||Y||$ is any norm on $\mathfrak{n}^-$.

For different E's, the resulting scale functions are equivalent.

Lemma 2.3.1. *One has*

$$s(\ell_I(g)) \leq s(g) \quad (g \in G).$$

Proof. In some orthonormal coordinates on E, $\ell_I(g)$ is block diagonal, while $n_{(I)}(g)$ block unipotent upper triangular. Then $s(\ell_I(g)) \leq s(\ell_I(g) n_{(I)}(g)) = s(g)$. □

Notice that s_I, a scale function for G_I, can be taken to be just the restriction of s; Hence we will only use the notation s.

2.3.2 Moderate growth

Definition 2.3.2. Let $\mathcal{V}$ be a Frechet representation of G. We say that $\mathcal{V}$ is **of moderate growth**, if for every continuous seminorm σ on $\mathcal{V}$ there exists a continuous seminorm σ' on $\mathcal{V}$ and $d \in \mathbb{Z}_{\geq 0}$ such that

$$\sigma(gv) \preceq e^{ds(g)} \sigma'(v), \quad g \in G, v \in \mathcal{V}.$$

Claim 2.3.3. *Suppose that $\mathcal{V}$ is a Banachable representation of G. Then $\mathcal{V}$ is of moderate growth.*

Proof. Let $\Omega \subset G$ be a compact subset such that $G = \cup_{n\geq 1} \Omega^n$. By continuity, there exists $C > 0$ such that

$$||gv|| \leq C||v||$$

for all $g \in \Omega, v \in V$. Iterating, we obtain $||gv|| \leq C^{c(g)}||v||$ for all $g \in G, v \in V$, where $c(g)$ denotes the first $n \geq 1$ for which $g \in \Omega^n$. Clearly it is enough now to show that for some choice of Ω, we have

$$c(g) \preceq s(g), \quad g \in G.$$

And indeed, let us choose

$$\Omega = \{g \in G \mid s(g) \leq 2\}.$$

Then Ω is compact. Moreover, Ω is closed under inversion, contains K and contains an open neighborhood of $e \in G$ – hence $G = \cup_{n \geq 1} \Omega^n$. Now, given $g \in G$, present $g = kexp(X)$ with $k \in K, X \in \mathfrak{s}$. Then we have $g = kexp(X/n)^n$ for every n, and since $s(exp(X/n)) \sim s(exp(X))/n = s(g)/n$ we get $c(g) \preceq s(g)$. □

2.4 $(\mathfrak{g}, K)$-modules continued

2.4.1 About $Z(\mathfrak{g})$

Let us recall that $Z(\mathfrak{g})$ is "quite big". Namely, $\mathrm{gr}(Z(\mathfrak{g})) \subset \mathrm{gr}(U(\mathfrak{g}))$ is identified with $S(\mathfrak{g})^G \subset S(\mathfrak{g})$. Furthermore, by using the form B, $S(\mathfrak{g})^G \subset S(\mathfrak{g})$ is identified with $Pol(\mathfrak{g})^G \subset Pol(\mathfrak{g})$. Fixing a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ (i.e. a maximal *ad*-semisimple abelian subalgebra), $Pol(\mathfrak{g})^G$ is identified, via restriction, with $Pol(\mathfrak{h})^{W(\mathfrak{h}, \mathfrak{g}_{\mathbb{C}})}$ (where $W(\mathfrak{h}, \mathfrak{g}_{\mathbb{C}})$ is a suitable Weyl group).

Decomposing $\mathfrak{g} = \mathfrak{n}_{(I)} \oplus \mathfrak{g}_I \oplus \mathfrak{n}^-_{(I)}$, we obtain $Z(\mathfrak{g}) \subset Z(\mathfrak{g}_I) \oplus (\mathfrak{n}_{(I)} U(\mathfrak{g}) \cap U(\mathfrak{g}) \mathfrak{n}^-_{(I)})$. Projecting, we obtain an algebra homomorphism

$$hc'_I : Z(\mathfrak{g}) \to Z(\mathfrak{g}_I).$$

The $()'$ stands for it being not normalized. This morphism hc'_I is *gr*-finite, and in particular finite.

2.4.2 Various finiteness

Definition 2.4.1. A $(\mathfrak{g}, K)$-module V is:

1. **finitely generated**, if it is finitely generated as a $U(\mathfrak{g})$-module.
2. Z**-finite**, if there exists an ideal of finite codimension $I \subset Z(\mathfrak{g})$ such that $IV = 0$.
3. **admissible**, if $\dim V^{[\alpha]} < \infty$ for all $\alpha \in \hat{K}$.

Lemma 2.4.2 ("Schur's lemma")**.** *Let V be an irreducible $(\mathfrak{g}, K)$-module. Then $End_{\mathfrak{g},K}(V) = \mathbb{C} \cdot Id$.*

Proof. Notice that V, being irreducible, is finitely generated, and hence of at most countable dimension. Hence, a well-known lemma insures that for any linear operator $T : V \to V$, there exists $c \in \mathbb{C}$ such that $T - c \cdot Id$ is not invertible (i.e. has a non-trivial kernel or cokernel), and the rest is as in the "usual" Schur's lemma. □

Corollary 2.4.3. *Let V be an irreducible $(\mathfrak{g}, K)$-module. Then $Z(\mathfrak{g})$ acts on V via a character (in particular, V is Z-finite).*

Lemma 2.4.4. *The morphism of algebras $p : S(\mathfrak{g})^G \to S(\mathfrak{s})^K$ corresponding to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ is finite.*

Proof. Using the form B, we can identify this map with the map $Pol(\mathfrak{s})^G \to Pol(\mathfrak{s})^K$ given by restriction, so we need to show that this map is finite.

By ..., the restriction map $Pol(\mathfrak{s})^K \to Pol(\mathfrak{a})^W$ is injective (in fact bijective, by Chevalley's restriction theorem which we will mention later on). Since $Pol_{\mathbb{C}}(\mathfrak{a})^W \to Pol(\mathfrak{a})$ is injective as well, it is enough to see that the restriction map $Pol(\mathfrak{g})^G \to Pol(\mathfrak{a})$ is finite.

Embedding $\mathfrak{a}_{\mathbb{C}} \subset \mathfrak{h}$ for some Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$, it is enough to see that the restriction map $Pol(\mathfrak{g})^G \to Pol(\mathfrak{h})$ is finite. This follows from the complex Chevalley restriction theorem. □

Theorem 2.4.5 (Harish-Chandra)**.** *Let V be a finitely generated $(\mathfrak{g}, K)$-module. Then for every $\alpha \in \hat{K}$, $V^{[\alpha]}$ is a finitely generated $Z(\mathfrak{g})$-module.*

Proof. Let us fix a finite-dimensional K-invariant subspace $V_0 \subset V$ such that $V = U(\mathfrak{g})V_0$. Let us consider the filtration $V_n = U(\mathfrak{g})^{\leq n}V_0$. Then V_n are K-invariant, and we have $V_{n+1} = V_n + \mathfrak{s}V_n$.

To show that $V^{[\alpha]}$ is a finitely generated $Z(\mathfrak{g})$-module, it is enough to show that $\mathrm{gr}(V)^{[\alpha]}$ is a finitely generated $\mathrm{gr}(Z(\mathfrak{g})) \cong S(\mathfrak{g})^G$-module. Notice that $S(\mathfrak{g})^G$ acts on $\mathrm{gr}(V)$ via $p : S(\mathfrak{g})^G \to S(\mathfrak{s})^K$ from the lemma above. Since by the lemma above p is finite, it is enough to show that $\mathrm{gr}(V)^{[\alpha]}$ is a finitely generated $S(\mathfrak{s})^K$-module.

It is sufficient to see that $Hom_K(E_\alpha, \mathrm{gr}(V))$ is a finitely generated $S(\mathfrak{s})^K$-module (because we have a surjective $S(\mathfrak{s})^K$-morphism $E_\alpha \otimes Hom_K(E_\alpha, \mathrm{gr}(V)) \to \mathrm{gr}(V)^{[\alpha]}$ given by sending $e \otimes \phi \mapsto \phi(e)$).

Denote $F := Hom(E_\alpha, \mathrm{gr}(V))$. Then F admits a K-action and a compatible $S(\mathfrak{s})$-action (in the sense that $Ad(k)X$ acts the same as $k \circ X \circ k^{-1}$). One has $F^K = Hom_K(E_\alpha, \mathrm{gr}(V))$. Since F is a finitely generated $S(\mathfrak{s})$-module, and since $S(\mathfrak{s})$ is Noetherian, we can find a finite dimensional subspace $F_0 \subset F^K$ such that $F^K \subset S(\mathfrak{s})F_0$. Applying the projector p on K-invariants, we obtain

$$F^K \subset S(\mathfrak{s})^K F_0.$$

□

Claim 2.4.6. *One has the following implications for $(\mathfrak{g}, K)$-modules:*

finite length $\Rightarrow$ *f.g. + adm.* $\Leftrightarrow$ *f.g. + Z-fin.* $\Rightarrow$ *adm. + Z-fin.*

Proof. Suppose that V is finitely generated and admissible, and let us show that V is Z-finite. Since V is finitely generated, we can find a finite sum of K-types $V_0 \subset V$ which generates V. Since V is admissible, V_0 is finite dimensional. Since $Z(\mathfrak{g})$ preserves V_0, it acts via a finite quotient on V_0, and hence on $V = U(\mathfrak{g})V_0$.

Suppose that V is finitely generated and Z-finite, and let us show that V is admissible. Let $\alpha \in \hat{K}$. Since V is finitely generated, by 2.4.5 we have that

$V^{[\alpha]}$ is a finitely generated $Z(\mathfrak{g})$-module. Since V is Z-finite, this implies that $V^{[\alpha]}$ is a finite dimensional space. Thus, V is admissible.

Finally, if V has finite length, then it is finitely generated and admissible - we reduce immediately to the case when V is irreducible, in which we finite generation is clear, and from corollary 2.4.3 Z-finiteness is known, hence admissibility by an implication we already have shown.

□

Remark 2.4.7. Suppose that we know that there are only finitely many irreducible $(\mathfrak{g}, K)$-modules with a given infinitesimal character. Then, we claim an admissible and Z-finite $(\mathfrak{g}, K)$-module V has finite length. Indeed, one reduces to the case when $Z(\mathfrak{g})$ acts on V by a character χ. Then, one can choose a big enough finite subset $S \subset \hat{K}$ such that every irreducible $(\mathfrak{g}, K)$-module with infinitesimal character χ has a K-type from S. Then, the functor $\mathcal{M}(\mathfrak{g}, K)^{\mathrm{adm}}_{\chi} \to Vect$ given by $W \mapsto \oplus_{\alpha \in S} W^{[\alpha]}$ is exact and faithful, which implies that modules in $\mathcal{M}(\mathfrak{g}, K)^{\mathrm{adm}}_{\chi}$ have finite length.

Later, we will be able to show that the are finitely many irreducible $(\mathfrak{g}, K)$-modules with a given infinitesimal character, and thus all the four conditions in the claim above will turn out to be equivalent.

Definition 2.4.8. A finitely generated and Z-finite $(\mathfrak{g}, K)$-module is called an **Harish-Chandra module** ("Harish-Chandra" to be abbreviated "HC").

2.4.3 Matrix coefficients and globalization

Definition 2.4.9. Let V be a $(\mathfrak{g}, K)$-module. We denote by $\widetilde{V}$ the subspace of V^* consisting of K-finite functionals. It is a $(\mathfrak{g}, K)$-module. One can write $V^* = \oplus_{\alpha \in \hat{K}} (V^{[\alpha]})^*$.

Remark 2.4.10. The contravariant functor $V \mapsto \widetilde{V}$ clearly preserves Z-finiteness and admissability. Moreover, it is easy to see that a $(\mathfrak{g}, K)$-module V is of finite length if and only if V and $\widetilde{V}$ are HC modules. Indeed, if $\widetilde{V}$ is HC, then it is Noetherian, and hence V is Artinian.

Definition 2.4.11. Let V be a $(\mathfrak{g}, K)$-module. A **matrix coefficients** map for V is a $(\mathfrak{g} \oplus \mathfrak{g}, K \times K)$-morphism $m : \widetilde{V} \otimes V \to C^\infty(G)$ satisfying $m(\widetilde{v} \otimes v)(e) = \langle \widetilde{v}, v \rangle$.

Lemma 2.4.12. *Let V be an admissible $(\mathfrak{g}, K)$-module, and let $m : \widetilde{V} \otimes V \to C^\infty(V)$ be a matrix coefficient map. Then all the functions in $Im(m)$ are analytic.*

Proof. Let us consider $C^\infty(G)$ as a G-representation for the right G-action. The vector $m(\widetilde{v} \otimes v)$ lies in the admissible $(\mathfrak{g}, K)$-submodule $m(\widetilde{v} \otimes V) \subset C^\infty(G)^{[K]}$. Hence it is weakly analytic. Considering the continuous functional $\alpha(f) := f(e)$, we obtain that $\alpha(R_g m(\widetilde{v} \otimes v)) = m(\widetilde{v} \otimes v)$ is analytic. □

Claim 2.4.13. *Let V be an admissible $(\mathfrak{g}, K)$-module. Then there exists at most one matrix coefficients map for V.*

Proof. Given a matrix coefficients map m, notice that $(R_U m(\widetilde{v} \otimes v))(k) = \langle \widetilde{v}, kUv \rangle$ (where $U \in U(\mathfrak{g}), k \in K$). Hence, given two matrix coefficients maps m_1, m_2, we see that $m_1 - m_2$ has a vanishing Taylor series at all points of K, and hence it must be zero, since it is analytic and K intersects each connected component of G. □

Definition 2.4.14. Let V be an admissible $(\mathfrak{g}, K)$-module. A **Globalization** of V is an admissible G-representaiton $\mathcal{V}$ together with an isomorphism of $(\mathfrak{g}, K)$-modules $V \cong \mathcal{V}^{[K]}$.

Lemma 2.4.15. *Let V be an admissible $(\mathfrak{g}, K)$-module and let $\mathcal{V}$ be a globalization of V. Then the restriction map $\mathcal{V}^* \to V^*$ is injective, and contains $\widetilde{V}$.*

Proof. The map is injective since V is dense in $\mathcal{V}$. Given $\xi \in \widetilde{V}$, we can assume that $\xi \in \widetilde{V}^{[\alpha^\vee]}$ for some $\alpha \in \hat{K}$. Then $\xi \circ pr_\alpha \in \mathcal{V}^*$ restricts to ξ on V. □

Lemma 2.4.16. *Let V be an admissible $(\mathfrak{g}, K)$-module. Suppose that V admits a globalization. Then V admits a matrix coefficients map.*

Proof. Embedding V in a globalization $\mathcal{V}$, and thus by the previous lemma embedding $\widetilde{V}$ in $\mathcal{V}^*$, we define $m(\widetilde{v} \otimes v) := \widetilde{v}(gv)$ and check that it is indeed a matrix coefficients map. □

Lemma 2.4.17. *Let V be a $(\mathfrak{g}, K)$-module of finite length. Suppose that V admits a matrix coefficients map. Then V admits a globalization.*

Proof. Choose generators $\xi_1, \ldots, \xi_d$ of $\widetilde{V}$ as a $U(\mathfrak{g})$-module. Consider the map $\iota : V \to C^\infty(G)^d$ given by $v \mapsto (m(\xi_1 \otimes v), \ldots, m(\xi_d \otimes v)$. Then one easily sees that ι is a $(\mathfrak{g}, K)$-module embedding. Thus, the closure of the image of ι is a globalization of V. □

Theorem 2.4.18. *Let V be an admissible $(\mathfrak{g}, K)$-module. Then there exists a matrix coefficients map for V.*

Proof. We omit the proof. □

Remark 2.4.19. This theorem is crucial for us. We use the existence of matrix coefficients to establish asymptotics of matrix coefficients and then Casselman's submodule theorem.

If we would have established Casselman's submodule theorem independently, we would deduce that Harish-Chandra modules admit (Hilbertable) globalizations, and hence matrix coefficients maps (then the existence of matrix coefficients maps for admissible modules follows).

To establish Casselman's submodule theorem independently, one approach is to first establish the subquotient theorem, deducing that irreducible modules admit (Hilbertable) globalization. From this one deduces that irreducible modules admit matrix coefficients maps, and then one is able to establish Casselman's submodule theorem as we do.

Another approach would be to establish Casselman's submodule theorem algebraically (there are proofs by Gabber, Beilinson-Bernstein).

2.5 Unitary representations

Lemma 2.5.1. *Let $\mathcal{V}$ be a G-representation. Let $v \in \mathcal{V}^{[K],\infty}$ be $Z(\mathfrak{g})$-finite. Then $v \in \mathcal{V}^{\omega}$.*

Proof. The proof is similar to the one when we merely assumed that v is $U(\mathfrak{g})^K$-finite. Namely, it is enough to find $D \in U(\mathfrak{k})Z(\mathfrak{g})$ such that R_D is an elliptic differential operator on G.

Namely, we fix an orthonormal basis (X_i) of $\mathfrak{k}$ w.r.t. $-B$, and an orthonormal basis (Y_j) of $\mathfrak{s}$ w.r.t. B. Then the dual basis of the concatenation (X_i, Y_j) w.r.t. B is $(-X_i, Y_j)$. Hence, arguing as before, we see that $C := -\sum X_i^2 + \sum Y_j^2 \in Z(\mathfrak{g})$ (the **Casimir** element). Hence, $D := \sum X_i^2 + \sum Y_j^2$ satisfies $D \in U(\mathfrak{k}) + Z(\mathfrak{g}) \subset U(\mathfrak{k})Z(\mathfrak{g})$ and R_D is elliptic. □

Claim 2.5.2. *Let $\mathcal{V}$ be an irreducible G-representation. If $Z(\mathfrak{g})$ acts on $\mathcal{V}^{[K],\infty}$ via a finite quotient (in particular, if it acts via scalars), then $\mathcal{V}$ is admissible.*

Proof. $\mathcal{V}^{[K],\infty}$ is non-zero (since it is dense in $\mathcal{V}$ by the results above). Let $E \subset \mathcal{V}^{[K],\infty}$ be a non-zero finite-dimensional K-invariant subspace. By the above, $E \subset \mathcal{V}^{\omega}$ and so $\mathcal{E} := Cl(U(\mathfrak{g})E)$ is G°-invariant. Since it is also K-invariant, it is G-invariant. Hence, since $\mathcal{V}$ is irreducible, we obtain $\mathcal{E} = \mathcal{V}$.

Since $U(\mathfrak{g})E$ is finitely generated and Z-finite, it is admissible. Hence, $\mathcal{V}^{[K]} = U(\mathfrak{g})E$. Indeed, given $v \in \mathcal{V}^{[K,\alpha]}$, we have $v_n \in U(\mathfrak{g})E$ such that $v_n \to v$. Then, replacing v_n by $pr_{\alpha}v_n$, we can assume that $v_n \in (U(\mathfrak{g})E)^{[\alpha]}$. Since the latter is finite-dimensinoal, we must have $v \in (U(\mathfrak{g})E)^{[\alpha]}$. □

Lemma 2.5.3 (Schur's lemma)**.** *Let $\mathcal{V}$ be an irreducible unitary representation of G. Let $T \in End_G(\mathcal{V})$ (so, a continuous endomorphism commuting with $\pi(g)$'s). Then $T \in \mathbb{C} \cdot Id_{\mathcal{V}}$.*

Proof. Notice that $End_G(\mathcal{V}) \subset End(\mathcal{V})$ is a C^*-subalgebra. It is easy, by reducing to a commutative subalgebra and using the Gelfand theorem, to see that if a C^*-algebra A is not $\mathbb{C}$, then there exist two non-zero self-adjoint commuting $T, S \in A$ such that $TS = 0$. If $T, S \in End_G(\mathcal{V})$ are such, then clearly $Ker(T)$ is a non-trivial subrepresentation. □

We have also a strengthening:

Lemma 2.5.4 (Strengthening of Schur's lemma)**.** *Let $\mathcal{V}$ be an irreducible unitary representation of G. Let $\mathcal{V}_0 \subset \mathcal{V}$ be a dense subspace closed under the $\pi(g)$'s. Let $T, S : \mathcal{V}_0 \to \mathcal{V}$ be operators, such that*

$$(Tv, w) = (v, Sw) \quad \forall v, w \in \mathcal{V}_0$$

(notice that S is determined by T, if exists). If T commutes with all the $\pi(g)$'s, then $T \in \mathbb{C} \cdot Id_{\mathcal{V}}$.

Proof. Recall von Neumann's bicommutant theorem, which says that for a subalgebra $A \subset End(\mathcal{V})$ closed under adjoints, the closure in the strong operator topology of A is the double commutant $End_{End_A(\mathcal{V})}(\mathcal{V})$.

Let us take $A \subset End(\mathcal{V})$ to be the subspace spanned by the $\pi(g)$'s. Then By Schur's lemma above, we obtain that the double commutant of A is $End(\mathcal{V})$. Hence, the closure of A in the strong operator topology is $End(\mathcal{V})$.

Assume now that for some $v \in \mathcal{V}_0$, the vectors v, Tv are linearly independent (we will obtain a contradiction). We can find $R \in End(\mathcal{V})$ such that $R(v) = R(Tv) = v$. By the above, we can find a sequence $R_i \in A$ such that $||R_i(v) - R(v)|| \to 0$ and $||R_i(Tv) - R(Tv)|| \to 0$. Then, for all $w \in \mathcal{V}_0$:

$$(Tv, w) = (v, Sw) = \lim(R_i v, Sw) = \lim(TR_i v, w) = \lim(R_i Tv, w) = (RTv, w) = (v, w).$$

Since $\mathcal{V}_0$ is dense in $\mathcal{V}$, we obtain $Tv = v$, a contradiction.

Thus, $Tv \in \mathbb{C}\cdot v$ for all $v \in \mathcal{V}_0$. It easy to see that this implies $T \in \mathbb{C}\cdot Id_{\mathcal{V}}$. □

Claim 2.5.5. *Let $\mathcal{V}$ be an irreducible unitary representation of G. Then elements of $Z(\mathfrak{g})$ act on $\mathcal{V}^\infty$ by scalars.*

Proof. Let $U \in Z(\mathfrak{g})$. We apply the above strengthening of Schur's lemma to $\mathcal{V}_0 := \mathcal{V}^\infty$ and $T := \pi(U)$. Notice that indeed T has an adjoint S as in the lemma; It is given by $\pi(U^t)$, where $(\cdot)^t : U(\mathfrak{g}) \to U(\mathfrak{g})$ is the anti-involution given by $X \mapsto -X$ for $X \in \mathfrak{g}$. □

Corollary 2.5.6 (Harish-Chandra)**.** *Let $\mathcal{V}$ be an irreducible unitary representation of G. Then $\mathcal{V}$ is admissible.*

Remark 2.5.7. By the previous corollary, we obtain a well-defined map, from the set of isomorphism classes of irreducible unitary G-representations, to the set of isomorphism classes of irreducible $(\mathfrak{g}, K)$-modules admitting a $(\mathfrak{g}, K)$-invariant inner product. One can see that this is a bijection. Hence, $(\mathfrak{g}, K)$-modules are adequate in this sense as well.

Remark 2.5.8. To contrast the previous corollary, one should note that there exist irreducible Banachable G-representations which are not admissible (see [8]). This is based on the fact that there exist bounded endomorphisms of Banach spaces of dimension bigger than 1, which have no non-trivial invariant subspaces.

2.6 Parabolic induction

2.6.1 Representations

Let $\mathcal{W}$ be a G_I-representation. We define a G-representation $pind_I(\mathcal{W})$ as follows. As a space, it is the space of continuous functions $f : G \to \mathcal{W}$ satisfying

$f(gmn) = \Delta_{(I)}(m)^{-1/2}m^{-1}f(g)$ for $g \in G, m \in G_I, n \in N_{(I)}$. Notice that $pind_I(\mathcal{W})$ is a closed subspace of $C(G;\mathcal{W})$, and we endow it with the induced topology.

Lemma 2.6.1. *If $\mathcal{W}$ is an admissible G_I-representation, then $pind_I(\mathcal{W})$ is an admissible G-representation.*

Let us denote by $C(G//P_{(I)})$ the space of continuous functions $\phi : G \to \mathbb{C}$ satisfying $\phi(gmn) = \Delta_{(I)}(m)^{-1}\phi(g)$ for $g \in G, m \in G_I, n \in N_{(I)}$. Then we have a G-invariant functional $\int_{G//P_{(I)}} : C(G//P_{(I)}) \to \mathbb{C}$.

Claim 2.6.2. *Let $\mathcal{W}$ be an admissible Hilbertable representation of G_I. Then one has a natural isomorphism*

$$\widetilde{pind_I(\mathcal{W})}^{[K]} \cong pind_I(\mathcal{W}^*)^{[K]}.$$

Proof. Let us remark that we assume that $\mathcal{W}$ is Hilbertable since we have then a nice dual representation $\mathcal{W}^*$ (in fact, for Banachable representations one can do this similarly).

For $f \in pind_I(\mathcal{W}), h \in pind_I(\mathcal{W}^*)$, notice that the function $\phi : G \to \mathbb{C}$ given by $\phi(g) = \langle h(g), f(g)\rangle$ lies in $C(G//P_{(I)})$.

We can thus define a G-invariant pairing

$$pind_I(\mathcal{W}^*) \otimes pind_I(\mathcal{W}) \to \mathbb{C}$$

by

$$(h, f) \mapsto \int_{G//P_\emptyset} \langle h(g), f(g)\rangle.$$

Restricting to K-finite vectors, we obtain a $(\mathfrak{g}, K)$-invariant pairing

$$pind_I(\mathcal{W}^*)^{[K]} \otimes pind_I(\mathcal{W})^{[K]} \to \mathbb{C}.$$

This pairing is non-degenerate; Indeed, given a non-zero K-finite vector, it is enough to show that it pairs non-trivially with some continuous vector (because we can then average); To find such a continuous vector, we construct a suitable bump function add details?). Hence, it is easy to infer from admissability that the resulting comparison map

$$pind_I(\mathcal{W}^*)^{[K]} \to \widetilde{pind_I(\mathcal{W})}^{[K]}$$

is an isomorphism of $(\mathfrak{g}, K)$-modules. □

2.6.2 $(\mathfrak{g}, K)$-modules

Let us consider the functor

$$\mathcal{M}(\mathfrak{g}_I, K_I) \leftarrow \mathcal{M}(\mathfrak{g}, K) : pres_I$$

given by

$$\mathbb{C}_{-\rho_{(I)}} \otimes (V/\mathfrak{n}_{(I)}V) \leftarrow V.$$

We denote by

$$pind_I : \mathcal{M}(\mathfrak{g}_I, K_I) \to \mathcal{M}(\mathfrak{g}, K)$$

the right adjoint of $pres_I$. It exists, since $pres_I$ commutes with small colimits and the categories of $(\mathfrak{g}, K)$-modules are locally presentable.

Lemma 2.6.3. *The functor $pres_I$ preserves finite generation and Z-finiteness. The functor $pind_I$ preserves admissability and Z-finiteness.*

Proof. The functor $pres_I$ preserves finite generation, since we have $\mathfrak{g} = \mathfrak{n}_{(I)} \oplus \mathfrak{g}_I \oplus \mathfrak{k}$ (it is also clear that the twist preserves finite generation).

The functor $pind_I$ preserves admissability: For a K-module E, we have the $(\mathfrak{g}, K)$-module $V_E := U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} E$. Given an admissible $(\mathfrak{g}_I, K_I)$-module W, we have $Hom_K(E, pind_I(W)) = Hom_{\mathfrak{g},K}(V_E, pind_I(W)) = Hom_{\mathfrak{g}_I,K_I}(pres_I(V_E), W)$. Since V_E is finitely generated, by what we just showed also $pres_I(V_E)$ is finitely generated. Thus, since $pres_I(V_E)$ is finitely generated and W is admissible, it is clear that the latter Hom space is finite dimensional. Hence $Hom_K(E, pind_I(W))$ is finite dimensional, as desired.

The functor $pres_I$ preserves Z-finiteness: Recall the morphism $hc'_I : Z(\mathfrak{g}) \to Z(\mathfrak{g}_I)$. It is clear that for $U \in Z(\mathfrak{g})$ and $v \in V$, one has $Uv - hc'_I(U)v \in \mathfrak{n}_{(I)}V$. Hence, it is clear that if V is $Z(\mathfrak{g})$-finite, then $V/\mathfrak{n}_{(I)}V$ is $Z(\mathfrak{g}_I)$-finite (it is also clear that the twist preserves Z-finiteness).

The functor $pind_I$ preserves Z-finiteness: Let W be a Z-finite $(\mathfrak{g}_I, K_I)$-module. It is enough to see that there exists a finite-dimensional quotient of $Z(\mathfrak{g})$, by which $Z(\mathfrak{g})$ acts on $Hom_{\mathfrak{g}_I,K_I}(V, pind_I(W))$ for every $(\mathfrak{g}, K)$-module V. Here, the action of $Z(\mathfrak{g})$ is via its action on $pind_I(W)$. But, we can also interpret the action of $Z(\mathfrak{g})$ via its action on V. Now, $Hom_{\mathfrak{g},K}(V, pind_I(W)) \cong Hom_{\mathfrak{g}_I,K_I}(pres_I(V), W)$, and we can interpret the action of $Z(\mathfrak{g})$, via this identification, as where z acts by $\alpha(z)$ acting on $pres_I(V)$, where $\alpha : Z(\mathfrak{g}) \to Z(\mathfrak{g}_I)$ is the above-mentioned finite morphism. Finally, we may interpret this as the action where z acts by $\alpha(z)$ on W. Since W is Z-finite, this action factors via a finite-dimensional quotient of $Z(\mathfrak{g}_I)$ and hence, via α, via a finite-dimensional quotient of $Z(\mathfrak{g})$. ◻

2.6.3 Relation

Claim 2.6.4. *Let $\mathcal{W}$ be an admissible representation of G_I. Then one has a natural isomorphism of $(\mathfrak{g}, K)$-modules*

$$pind_I(\mathcal{W})^{[K]} \cong pind_I(\mathcal{W}^{[K_I]}).$$

Proof. Given a map of $(\mathfrak{g}, K)$-modules $\Phi : V \to pind_I(\mathcal{W})^{[K]}$, define a map $\phi : V \to \mathcal{W}^{[K_I]}$ by $v \mapsto \Phi(v)(e)$. Then one verifies that $\phi(\ell v) = \ell\phi(v)$ for $\ell \in K_I$, that $\phi(Xv) = 0$ for $X \in \mathfrak{n}_{(I)}$, and that $\phi(Xv) = X\phi(v) + \rho_{(I)}(X)\phi(v)$ for $X \in \mathfrak{g}_I$. This means that ϕ induces a map of $(\mathfrak{g}_I, K_I)$-modules $\phi : pres_I(V) \to \mathcal{W}^{[K_I]}$.

Conversely, given a map of $(\mathfrak{g}_I, K_I)$-modules $\phi : pres_I(V) \to \mathcal{W}^{[K_I]}$, define a map of $(\mathfrak{g}, K)$-modules $\Phi : V \to pind_I(\mathcal{W})^{[K]}$ as follows. Notice that restriction to K gives an identification of functions $f \in pind_I(\mathcal{W})^{[K]}$ and continuous functions $f_1 : K \to \mathcal{W}$ which are K-finite and satisfying $f_1(k\ell) = \ell^{-1} f_1(k)$ for $k \in K, \ell \in K_I$. Let us define $\Phi(v)(k) = \phi([k^{-1}v])$. Then one easily verifies that this gives rise to a well defined $\Phi(v) \in pind_I(\mathcal{W})^{[K]}$ by the above. One then checks that this Φ is a $(\mathfrak{g}, K)$-module morphism.

Finally, one checks that the above assignments are mutually inverse.

□

Claim 2.6.5. *Let W be an admisisble $(\mathfrak{g}_I, K_I)$-module, admitting a Hilbertable globalization. Then one has an isomorphism of $(\mathfrak{g}, K)$-modules*

$$\widetilde{pind_I(W)} \cong pind_I(\widetilde{W}).$$

Proof. Let $\mathcal{W}$ be a Hilbertable globalization of W. We have:

$$\widetilde{pind_I(W)} \cong \widetilde{pind_I(\mathcal{W})}^{[K]} \cong pind_I(\mathcal{W}^*)^{[K]} \cong pind_I(\widetilde{W}).$$

□

Remark 2.6.6. After we will establish Casselman's submodule theorem, we will know that every HC $(\mathfrak{g}, K)$-module admits a Hilbertable globalization.

Remark 2.6.7. It is interesting to note that the duality of the previous claim is not clear at all on the algebraic level, i.e. it is not clear how to define it if we don't use globalizations.

Chapter 3

K-finite Matrix coefficients

3.1 K-bifinite functions

Let V be an admissible $(\mathfrak{g}, K)$-module. Then the functions in $Im(m_V)$ are analytic, left K-finite, right K-finite, and $Z(\mathfrak{g})$-finite. We would like to study the growth of functions in $Im(m_V)$ at infinity. Since $G = KA^-K$ and A^- is much simpler than G, we would like to restrict functions to A^- and study their growth. To facilitate this, let us notice:

Lemma 3.1.1. *Let $f \in C^\infty(G)$ be a K-bifinite and $Z(\mathfrak{g})$-finite function. Then there exists a HC $(\mathfrak{g}, K)$-module V such that $f \in Im(m_V)$.*

Lemma 3.1.2. *Let V be a HC $(\mathfrak{g}, K)$-module, and let $v \in V, \widetilde{v} \in \widetilde{V}$. Then there exists a finite-dimensional $(K \times K)$-representation $(E, \tau = \tau_1 \times \tau_2)$, and a function $f \in C^\infty(G; E)$ which is K-biequivariant (i.e. $f(k_1 g k_2) = \tau(k_1, k_2^{-1}) f(g)$) and $Z(\mathfrak{g})$-finite, and such that $m_V(\widetilde{v} \otimes v)$ is obtained by composing f with some functional in E^*.*

3.2 Radial coordinates

3.2.1 The map Π_τ

Recall the polar decomposition, which gives an open dense embedding $K/K_\emptyset \times A^{--} \times K \to G$ (given by $([k_1], a, k_2) \mapsto k_1 a k_1^{-1} k_2$); Denote by G^{reg} the image of this embedding.

Lemma 3.2.1. *Let $X \in \mathfrak{g}^{\mathfrak{a}, \alpha}$ and $a \in A$ such that $\alpha(a) \neq 0$. Then*

$$X = \frac{1}{a^{-\alpha} - a^{\alpha}} \cdot {}^{a^{-1}}(X + \theta X) - \frac{a^{\alpha}}{a^{-\alpha} - a^{\alpha}} \cdot (X + \theta X).$$

Lemma 3.2.2. *Let $a \in A^{reg}$. Then the map*

$$\mathfrak{g} \leftarrow \mathfrak{a} \oplus (\mathfrak{k} \oplus \mathfrak{k})/\mathfrak{k}_\emptyset$$

given by

$$H + {}^{a^{-1}}W_1 + W_2 \leftarrow\!\shortmid (H, W_1, W_2)$$

is an isomorphism (here, $\mathfrak{k}_\emptyset$ is embedded in $\mathfrak{k} \oplus \mathfrak{k}$ anti-diagonally).

Proof. The previous lemma shows that this map is surjective, and one calcaultes that the dimensions are the same. □

Let us denote by

$$\Pi_a : U(\mathfrak{g}) \rightleftarrows U(\mathfrak{a}) \otimes U(\mathfrak{k}) \otimes_{U(\mathfrak{k}_\emptyset)} U(\mathfrak{k}) : \Gamma_a$$

the isomorphism given by

$${}^{a^{-1}}V_1 \cdot U \cdot V_2 \leftarrow\!\shortmid U \otimes V_1 \otimes V_2,$$

which is indeed an isomorphism by the PBW theorem and the above lemma.

Let us denote by $\mathcal{R} \subset C^\infty(A^{reg})$ the subalgebra generated by $\frac{1}{a^{-\alpha}-a^{\alpha}}$ and $\frac{a^\alpha}{a^{-\alpha}-a^{\alpha}}$, where $\alpha \in R^+$.

Lemma 3.2.3. *There exists a unique linear map*

$$\Pi : U(\mathfrak{g}) \to \mathcal{R} \otimes U(\mathfrak{a}) \otimes U(\mathfrak{k}) \otimes_{U(\mathfrak{k}_\emptyset)} U(\mathfrak{k})$$

such that $\Pi(U)(a) = \Pi_a(U)$ *for all* $a \in A^{reg}$.

Proof. Uniqueness is clear.

Let us show the existence of $\Pi(U)$ for $U \in U(\mathfrak{g})$. If $U \in U(\mathfrak{a})U(\mathfrak{k})$, this is clear. By the PBW theorem applied to $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$, we can assume that the existence of $\Pi(U')$ is shown for all $U' \in U(\mathfrak{g})^{\leq n-1}$, and assume that $U = XV$ for $X \in \mathfrak{n}$ and $V \in U(\mathfrak{g})^{\leq n-1}$. We can assume that $X \in \mathfrak{g}^{\mathfrak{a},\alpha}$ for $\alpha \in R^+$.

We have

$$U = XV = \frac{1}{a^{-\alpha} - a^{\alpha}} {}^{a^{-1}}(X + \theta X)V - \frac{a^\alpha}{a^{-\alpha} - a^\alpha}(V(X + \theta X) + [X + \theta X, V]),$$

from which the existence of $\Pi(U)$ is clear (notice that $V, [X+\theta X, V] \in U(\mathfrak{g})^{\leq n-1}$). □

Now, fix a finite-dimensional $(K \times K)$-representation $(E, \tau = \tau_1 \times \tau_2)$, and consider the space $C^\infty_\tau(G; E)$ of smooth functions from G to E, which are $(K \times K)$-equivariant (i.e. $f(k_1 a k_2) = \tau(k_1, k_2^{-1}) f(a)$). Notice that by the polar decomposition we have an identification

$$Res : C^\infty_\tau(G^{reg}; E) \rightleftarrows C^\infty(A^{--}; E^{K_\emptyset}) : Ext,$$

where the map Res is simply restricting to A^{--}.

Let us denote, for two subspaces $E_1, E_2 \subset E$,

$$\mathcal{D}_{E_1,E_2} := \mathcal{R} \otimes Hom(E_1, E_2) \otimes U(\mathfrak{a}).$$

We consider $\mathcal{D}_{E_1,E_2}$ as a subspace of the space of differential operators on A, from E_1 to E_2. If $E_1 = E_2$, then this is a subalgebra (a calculation shows that $\mathcal{R}$ is closed under differentiation along invariant vector fields).

Let us denote by
$$\widetilde{\Pi}_\tau : U(\mathfrak{g}) \to \mathcal{D}_{E^{K_\emptyset},E}$$
the composition
$$U(\mathfrak{g}) \xrightarrow{\Pi} \mathcal{R} \otimes U(\mathfrak{a}) \otimes U(\mathfrak{k}) \otimes_{U(\mathfrak{k}_\emptyset)} U(\mathfrak{k}) \to \mathcal{R} \otimes Hom(E^{K_\emptyset}, E) \otimes U(\mathfrak{a}),$$
where the second map is
$$f \otimes U \otimes V_1 \otimes V_2 \mapsto f \otimes \tau_1(V_1)\tau_2(V_2^t) \otimes U.$$

Claim 3.2.4. *Let $f \in C^\infty_\tau(G; E)$ and $U \in U(\mathfrak{g})$. Then $(R_U f)|_{A^{reg}} = \widetilde{\Pi}_\tau(U)(f|_{A^{reg}})$.*

Proof. Let us write $\Pi(U) = \sum \phi^i \otimes U_1^i \otimes V_1^i \otimes V_2^i$.

Then
$$(R_U f)(a) = \sum \phi^i(a)\cdot(L_{(V_1^i)^t} R_{U_1^i} R_{V_2^i} f)(a) = \sum \phi^i(a)\tau_1(V_1^i)\tau_2(V_2^i)^t(R_{U_1^i} f)(a) = (\widetilde{\Pi}_\tau(U) f|_{A^{reg}})(a).$$
□

Notice that we have an embedding map $\mathcal{D}_{E^{K_\emptyset},E^{K_\emptyset}} \to \mathcal{D}_{E^{K_\emptyset},E}$.

Lemma 3.2.5. *The image of*
$$U(\mathfrak{g})^K \to U(\mathfrak{g}) \xrightarrow{\widetilde{\Pi}_\tau} \mathcal{D}_{E^{K_\emptyset},E}$$
sits in $\mathcal{D}_{E^{K_\emptyset},E^{K_\emptyset}}$. Moreover, the resulting map
$$\Pi_\tau : U(\mathfrak{g})^K \to \mathcal{D}_{E^{K_\emptyset},E^{K_\emptyset}}$$
is an algebra homomorphism.

Proof. Let $U \in U(\mathfrak{g})^K$ and $f \in C^\infty(A^{reg}; E^{K_\emptyset})$. Then by the previous claim we have $\widetilde{\Pi}_\tau(U)f = (R_U Ext(f))|_{A^{reg}}$. Since $R_U Ext(f) \in C^\infty_\tau(G^{reg}; E)$, we have $(R_U Ext(f))|_{A^{reg}} \in C^\infty(A^{reg}; E^{K_\emptyset})$.

The map Π_τ is an algebra homomorphism, since (for $f \in C^\infty(A^{reg}; E^{K_\emptyset})$):
$$\Pi_\tau(U_1U_2)f = Res(R_{U_1U_2} Ext(f)) = Res(R_{U_1} R_{U_2} Ext(f)) =$$
$$= \Pi_\tau(U_1) Res(R_{U_2} Ext(f)) = \Pi_\tau(U_1)\Pi_\tau(U_2)f.$$
□

We denote $\mathcal{D} := \mathcal{D}_{E^{K_\emptyset},E^{K_\emptyset}}$.

Let us summarize. We have an identification
$$Res : C^\infty_\tau(G^{reg}; E) \rightleftarrows C^\infty(A^{--}; E^{K_\emptyset}) : Ext.$$

We have actions by differential operators, of $U(\mathfrak{g})^K$ on the left space, and of $\mathcal{D}$ on the right space. The "polar parts" homomorphism

$$\Pi_\tau : U(\mathfrak{g})^K \to \mathcal{D}$$

that we constructed satisfies:

$$Res(R_U f) = \Pi_\tau(U) Res(f).$$

Example ($SL_2(\mathbb{R})$, of course)

We have the usual basis H, X, Y of $\mathfrak{g}$, and the Casimir element $C = H^2 + 2H + 4YX$. Let us also consider the element $W = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let us denote $W_t := {}^{e^{-tH}}W$. One has

$$X = \frac{1}{1-e^{-4t}}(e^{-2t} \cdot W_t - W), \quad Y = \frac{1}{e^{4t}-1}(e^{2t} \cdot W_t - W).$$

One computes

$$[W, W_t] = (e^{-2t} - e^{2t}) \cdot H.$$

Let us compute $\Pi_{e^{tH}}(C)$:

$$C = H^2+2H+4YX = H^2+2H-\frac{4}{(1-e^{-4t})(1-e^{4t})}(W_t^2+W^2-e^{-2t}W_tW-e^{2t}WW_t)$$

$$H^2 - 2\frac{1+e^{-4t}}{1-e^{-4t}}H - \frac{4}{(1-e^{-4t})(1-e^{4t})}(W_t^2 + W^2 - (e^{2t}+e^{-2t})W_tW)$$

Thus,

$$\Pi_{e^{tH}}(C) = (H^2-2\frac{1+e^{-4t}}{1-e^{-4t}}H)\otimes 1\otimes 1-\frac{4}{(1-e^{-4t})(1-e^{4t})}(1\otimes W^2\otimes 1+1\otimes 1\otimes W^2-(e^{2t}+e^{-2t})1\otimes W\otimes W).$$

Thus,

$$\Pi_{sph}(C) = H^2 - 2\frac{1+e^{-4t}}{1-e^{-4t}}H.$$

3.2.2 Finiteness and the resulting PDE

Claim 3.2.6. *Let $I \subset Z(\mathfrak{g})$ be an ideal such that $Z(\mathfrak{g})/I$ is finite-dimensional. Then $\mathcal{D}/\mathcal{D} \cdot \Pi_\tau(I)$ is finitely-generated as an $\mathcal{R}$-module.*

Proof. Let us denote by $p : Z(\mathfrak{g}) \to U(\mathfrak{a})$ the composition of $hc'_\emptyset : Z(\mathfrak{g}) \to Z(\mathfrak{g}_\emptyset)$ and $Z(\mathfrak{g}_\emptyset) \cong Z(\mathfrak{k}_\emptyset) \otimes U(\mathfrak{a}) \to U(\mathfrak{a})$ induced by the augmentation morphism $Z(\mathfrak{k}_\emptyset) \to \mathbb{C}$. The map p preserves the filtrations, and is gr-finite.

Notice that $\widetilde{\Pi}_\tau$ respects the filtrations. Furthermore, clearly $\widetilde{\Pi}_\tau$ strictly decreases the filtration on elements in $U(\mathfrak{g})\mathfrak{k}$, and from the formula in the proof of lemma 3.2.3, also on elements in $\mathfrak{n}U(\mathfrak{g})$. Also, notice that for $U \in U(\mathfrak{a})$ one has simply $\widetilde{\Pi}_\tau(U) = 1 \otimes Id \otimes U$. From these, it is clear that the associated graded map

$$gr\Pi_\tau : grZ(\mathfrak{g}) \to gr\mathcal{D}$$

sends $U \in Z(\mathfrak{g})$ to $1 \otimes Id \otimes gr(p)(U)$.

To show that $\mathcal{D}/\mathcal{D} \cdot \Pi_\tau(I)$ is a finitely-generated $\mathcal{R}$-module, it is enough to show that $gr(\mathcal{D}/\mathcal{D} \cdot \Pi_\tau(I)) \cong gr(\mathcal{D})/gr(\mathcal{D} \cdot \Pi_\tau(I))$ is. Notice that

$$gr(\mathcal{D}) \cdot gr(\Pi_\tau)(gr(I)) \subset gr(\mathcal{D}) \cdot gr(\Pi_\tau gr(I)) \subset gr(\mathcal{D} \cdot \Pi_\tau(I)),$$

so it is enough to show that $gr(\mathcal{D})/gr(\mathcal{D}) \cdot gr(\Pi_\tau)(gr(I))$ is a finitely-generated $\mathcal{R}$-module. Notice that $gr(\mathcal{D}) \cdot gr(\Pi_\tau)(gr(I)) \cong \mathcal{R} \otimes End(E^{K_\emptyset}) \otimes gr(p)(gr(I))$, so $gr(\mathcal{D})/gr(\mathcal{D}) \cdot gr(\Pi_\tau)(gr(I)) \cong \mathcal{R} \otimes End(E^{K_\emptyset}) \otimes U(\mathfrak{a})/gr(p)(gr(I))$, and so is a finitely generated $\mathcal{R}$-module since $gr(p)$ is finite. □

Corollary 3.2.7. *Let $f \in C^\infty_\tau(G^{reg}; E)$, and suppose that f is $Z(\mathfrak{g})$-finite. Then denoting $h_1 := Res(f) \in C^\infty(A^{--}; E^{K_\emptyset})$, we can find functions $h_2, \dots, h_r \in C^\infty(A^{--}; E^{K_\emptyset})$ such that the vector $h = (h_1, \dots, h_r) \in C^\infty(A^{--}; (E^{K_\emptyset})^r)$ satisfies differential equations*

$$R_H h = M(H) \cdot h,$$

where $M : \mathfrak{a} \to \mathcal{R} \otimes End((E^{K_\emptyset})^r)$ is some linear map.

Proof. Let $I \subset Z(\mathfrak{g})$ be an ideal of finite codimension which annihilates f. By the previous claim, we can choose $d_1 := 1, d_2, \dots, d_r \in \mathcal{D}$ which span the $\mathcal{R}$-module $\mathcal{D}/\mathcal{D} \cdot \Pi_\tau(I)$. Denote $h_i := d_i f$. □

Let us now introduce the coordinates $z_\alpha := a^\alpha$ on A_{ss}, which identify it with $(0,\infty)^{R^s}$. These identify A^{--}_{ss} with $(0,1)^{R^s}$. Notice that the functions in $\mathcal{R}$ extend, in these z-coordinates, to holomorphic functions on D^{R^s}, where $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Thus, the function $h \in C^\infty((0,1)^{R^s}; (E^{K_\emptyset})^r)$ from the previous corollary, restricted to A^{--}_{ss} and viewed in the z-coordinates, satisfies differential equations of the form

$$z_\alpha \partial_{z_\alpha} h = M_\alpha h,$$

where M_α are holomorphic $End((E^{K_\emptyset})^r)$-valued functions on D^{R^s}.

Example 3.2.8. *For $SL_2(\mathbb{R})$, we have $z = a^\alpha = e^{2t}$. The ring $\mathcal{R}$ is generated by $\frac{z}{1-z^2}, \frac{z^2}{1-z^2}$. The map $\Pi_{sph} : Z(\mathfrak{g}) \to \mathcal{D}$ is given by*

$$\frac{1}{4}C \mapsto (z\partial_z)^2 - \frac{1+z^{-2}}{1-z^{-2}} \cdot z\partial_z$$

. Considering the ideal $J_c = (C - 4c) \subset Z(\mathfrak{g})$, we see that $\mathcal{D}/\mathcal{D}\Pi_{sph}(J_c)$ is generated over $\mathcal{R}$ by $1, z\partial_z$.

If a K-biinvariant function f satisfies $\frac{1}{4}Cf = cf$, setting $h = h_1 := f|_{A^{--}}$ and $h_2 = z\partial_z h_1$, we obtain the equation

$$z\partial_z \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ c & \frac{1+z^{-2}}{1-z^{-2}} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

3.3 PDE with regular singularities

Let $D \subset \mathbb{C}$ denote the open unit disc, $D^* := D - \{0\}$, $H = \{z \in \mathbb{C} \mid Re(z) < 0\}$. We have the universal cover $H \to D^* : w \mapsto e^w$.

Let $p \in \mathbb{Z}_{\geq 0}$. Let E be a finite-dimensional complex vector space. Let $G_1, \dots, G_p$ be holomorphic $End(E)$-valued functions on D^p. We consider the PDE

$$z_i \frac{\partial}{\partial z_i} f = G_i f, \ 1 \leq i \leq p,$$

where f is a holomorphic E-valued function on (an open subset of) $X := (D^*)^p$.

We use the universal cover $\widetilde{X} := H^p \to X$ given by

$$(w_1, \dots, w_p) \mapsto (e^{w_1}, \dots, e^{w_p}).$$

Our PDE on $\widetilde{X}$ becomes

$$\frac{\partial}{\partial w_i} \widetilde{f} = \widetilde{G}_i \widetilde{f}, \ 1 \leq i \leq p.$$

Lemma 3.3.1. *Any solution of our PDE on an open connected non-empty subset of $\widetilde{X}$ extends (uniquely) to a solution on the whole $\widetilde{X}$.*

Proof. complete □

Fix a point $x \in \widetilde{X}$, and identify the space of solutions with the space $E_0 \subset E$ of their values at x, getting the universal solution $F : E_0 \times \widetilde{X} \to E$ (where

$$(w_1, \dots, w_p) \mapsto F(v, w_1, \dots, w_p)$$

is the solution having value v at x). Then sending

$$\widetilde{f}(w_1, \dots, w_p) \mapsto \widetilde{f}(w_1, \dots, w_i + 2\pi i, \dots, w_p)$$

gives us a transformation on solutions, encoded by a map $M_i \in Aut(E_0)$, so that

$$F(v, w_1, \dots, w_i + 2\pi i, \dots, w_p) = F(M_i v, w_1, \dots, w_p).$$

These maps $M_1, \dots, M_p$ commute, and one can see that one can find then commuting maps $R_1, \dots, R_p \in End(E_0)$ such that $e^{-2\pi i R_i} = M_i$ (why?). We notice now that the function

$$F'(v, w_1, \dots, w_p) := F(e^{w_1 R_1 + \dots + w_p R_p} v, w_1, \dots, w_p)$$

is invariant under adding $2\pi i$ to the w_i's. We have:

$$F(v, w_1, \dots, w_p) := F'(e^{-(w_1 R_1 + \dots + w_p R_p)} v, w_1, \dots, w_p).$$

From this and Jordan theory, we conclude that, fixing some $v \in E_0$, there exist $v_1, \dots, v_k \in E_0$, $\lambda_{i,j} \in \mathbb{C}$ with $1 \leq i \leq p, 1 \leq j \leq k$, and $p_j \in Pol_{\mathbb{C}}(w_1, \dots, w_p)$ such that

$$F(v, w_1, \dots, w_p) = \sum_{1 \leq j \leq k} e^{\lambda_{1,j} w_1 + \dots + \lambda_{p,j} w_p} p_j(w_1, \dots, w_p) F'(v_i, w_1, \dots, w_p).$$

Let us denote by $f'(v, z_1, \dots, z_p)$ the function on $E_0 \times X$ corresponding to F' on $E_0 \times \widetilde{X}$.

Lemma 3.3.2. *Fix $v \in E_0$. There exists $(d_1, \dots, d_p) \in \mathbb{Z}^p_{\geq 0}$ such that $z_1^{d_1} \cdots z_p^{d_p} f'$ extends to a holomorphic function on $E_0 \times D^p$.*

Proof. complete ☐

Corollary 3.3.3. *Let f be a solution of our PDE on an open subset in X. Then f extends to a multivalued solution on the whole X. Moreover, f can be written as a finite sum of functions of the form*

$$z_1^{\lambda_1} \dots z_p^{\lambda_p} p(log z_1, \dots, log z_p) h(z_1, \dots, z_p),$$

where p is a polynomial and h is a holomorphic function on D^p with values in E.

3.4 Asymptotics of K-bifinite Z-finite functions

A lot of trivial inaccuracies with the ρ-shift are below...

We now combine the two previous sections, to obtain the existence of an asymptotic development of K-finite matrix coefficient.

Definition 3.4.1. Let us say that a function $f \in C^\infty(A^{--})$ is **regular at** ∞, if it can be written as the sum of an absolutely convergent series

$$\sum_{\mu \in \mathbb{Z}_{\geq 0} R^s} c_\mu \cdot a^\mu$$

(where $c_\mu \in \mathbb{C}$ are some coefficients).

Claim 3.4.2. *Let $f \in C^\infty(G)$ be K-bifinite and $Z(\mathfrak{g})$-finite. Then f can be written as a finite sum of functions of the form*

$$a^\lambda \cdot p(log(a)) \cdot h,$$

where $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $p \in Po_{\mathbb{C}}l(\mathfrak{a})$ and h is a function regular at ∞.

Proof. On A^{--}_{ss} we deduce it from the theory above. The $Z(\mathfrak{g})$-finiteness gives us $U(\mathfrak{a}_{cent})$-finiteness, from which it is easy to deduce the needed form on the whole A^{--}. ☐

Definition 3.4.3.

1. A subset $S \subset \mathfrak{a}^*_{\mathbb{C}}$ is called **conical**, if there exists a finite subset $S_0 \subset \mathfrak{a}^*_{\mathbb{C}}$ such that $S \subset S_0 + \mathbb{Z}_{\geq 0} R^s$.

2. Let $S \subset \mathfrak{a}^*_{\mathbb{C}}$ be a conical subset. We denote by $S^{min} \subset S$ the subset of $\geq_{R^s}$-minimal elements (it is a finite subset, non-empty if S is).

3. Let $S \subset \mathfrak{a}_{\mathbb{C}}^*$ be a conical subset. We denote by $S \subset S^{cl}$ the subset $S^{cl} := S + \mathbb{Z}_{\geq 0} R^s = S^{min} + \mathbb{Z}_{\geq 0} R^s$ (it is again a conical subset).

Corollary 3.4.4. *Let $f \in C^\infty(G)$ be K-bifinite and $Z(\mathfrak{g})$-finite. Then there exists a conical subset $S \subset \mathfrak{a}_{\mathbb{C}}^*$ and polynomials $p_\mu \in Pol_{\mathbb{C}}(\mathfrak{a})$ (for $\mu \in S$) such that f is equal to the sum of the absolutely convergent series*

$$\sum_{\mu \in S} p_\mu(loga) \cdot a^{\mu+\rho}.$$

One can show that the set $supp(f)$ of $\mu \in S$ for which $p_\mu \neq 0$ is a conical subset which does not depend on the choices. One can also show that $p_\mu, \mu \in supp(f)$ don't depend on the choices (we will write $p_\mu(f)$ for these p_μ).

Lemma 3.4.5. *Let $f \in C^\infty(G)$ be K-bifinite and $Z(\mathfrak{g})$-finite. Let $\lambda \in \mathfrak{a}^*$. Then the property*

$$|f(a)| \preceq a^\lambda s(a)^d \quad (a \in A^{--,\epsilon})$$

for some $d \in \mathbb{Z}_{\geq 0}$ and some/all $\epsilon > 0$ is equialent to the property

$$Re(supp(f|_{A^{--}})) \geq \lambda.$$

3.5 The support of a HC $(\mathfrak{g}, K)$-module

For subspaces $F \subset \widetilde{V}, E \subset V$, we denote

$$supp_F(E) := \cup_{\widetilde{v} \in F, v \in E} supp(m_{\widetilde{v},v}).$$

We also denote $supp(V) := supp_{\widetilde{V}}(V)$.

Claim 3.5.1. *Let V be a HC $(\mathfrak{g}, K)$-module, and let $\widetilde{V}' \subset \widetilde{V}$ be a finitely generated $(\mathfrak{g}, K)$-submodule. Then $supp_{\widetilde{V}'}(V)$ is conical.*

Proof. Let $E \subset V, F \subset \widetilde{V}'$ be finite-dimensional K-invariant subspaces, s.t. $V = U(\mathfrak{g})E, \widetilde{V}' = U(\mathfrak{g})F$. Denote $S := \cup_{\widetilde{v} \in F, v \in E} supp(m_{\widetilde{v},v})$. We will show that $supp_{\widetilde{V}'}(V) \subset S^{cl}$. Assume inductively that $supp(m_{\widetilde{v},v}) \subset S^{cl}$ for all $\widetilde{v} \in U(\mathfrak{g})^m F, v \in U(\mathfrak{g})^n E$. Let now $\widetilde{v} \in U(\mathfrak{g})^m F, v \in U(\mathfrak{g})^n E$ and $X \in \mathfrak{g}$ - we would like to show that $supp(m_{\widetilde{v},Xv}) \subset S^{cl}$. If $X \in \mathfrak{k}$, this is clear, since $\mathfrak{k}U(\mathfrak{g})^n E \subset U(\mathfrak{g})^n E$. If $X \in \mathfrak{a}$, this is clear, since $m_{\widetilde{v},Xv} = R_X m_{\widetilde{v},v}$, and clearly $supp(R_X f) \subset supp(f)$ when $X \in \mathfrak{a}$. Finally, let us assume that $X \in \mathfrak{g}^{\mathfrak{a},\alpha}$, where $\alpha \in R^+$. Then by a calculation we did above:

$$m_{\widetilde{v},Xv}(a) = (R_X m_{\widetilde{v},v})(a) = \frac{1}{a^{-\alpha} - a^\alpha} \cdot (L_{X+\theta X} m_{\widetilde{v},v})(a) - \frac{a^\alpha}{a^{-\alpha} - a^\alpha} \cdot (R_{X+\theta X} m_{\widetilde{v},v})(a) =$$

$$= (a^\alpha + a^{3\alpha} + \cdots) \cdot m_{(X+\theta X)\widetilde{v},v}(a) - (a^{2\alpha} + a^{4\alpha} + \cdots) \cdot m_{\widetilde{v},(X+\theta X)v}(a).$$

Thus, we see that in this case $supp(m_{\widetilde{v},Xv}) \subset S^{cl} + \alpha$.

Similarly one handles the examination of $m_{X\widetilde{v},v}$ for $X \in \mathfrak{g}$. □

Claim 3.5.2. *Let V be a HC $(\mathfrak{g}, K)$-module, and let $\widetilde{V}' \subset \widetilde{V}$ be a finitely generated $(\mathfrak{g}, K)$-submodule. Then*

$$supp_{\widetilde{V}'}(\mathfrak{g}^{\mathfrak{a},\alpha}v) \subset supp_{\widetilde{V}'}(v) + \alpha$$

for $v \in V$ and $\alpha \in R^+$. As a corollary,

$$supp_{\widetilde{V}'}(\mathfrak{n}^k V) \subset supp_{\widetilde{V}'}(V) + kR^+$$

for $k \geq 0$.

Proof. Let $X \in \mathfrak{g}^{\mathfrak{a},\alpha}$ and $\widetilde{v} \in \widetilde{V}'$. Notice that

$$m_{\widetilde{v},Xv}(a) = (R_X m_{\widetilde{v},v})(a) = -L_{{}^aX} m_{\widetilde{v},v} = -a^{\alpha} L_X m_{\widetilde{v},v} = -a^{\alpha} m_{X\widetilde{v},v}.$$

Thus, clearly $supp(m_{\widetilde{v},Xv}) \subset supp_{\widetilde{V}'}(v) + \alpha$. □

Claim 3.5.3. *Let V be a HC $(\mathfrak{g}, K)$-module and $\widetilde{V}' \subset \widetilde{V}$ a finitely generated $(\mathfrak{g}, K)$-submodule. Then $supp_{\widetilde{V}'}(V)^{min} \subset wt(pres_{\emptyset}(V))$.*

Proof. Let $\lambda \in supp_{\widetilde{V}'}(V)^{min}$. There exist $\widetilde{v} \in \widetilde{V}', v \in V$ such that $\lambda \in supp(m_{\widetilde{v},v})$. Let us consider the map $V \to Pol_{\mathbb{C}}(\mathfrak{a}) \cdot a^{\lambda+\rho}$ given by $v \mapsto p_{\lambda}(m_{\widetilde{v},v})a^{\lambda}$. Since $supp(m_{\widetilde{v},\mathfrak{n}v}) \subset supp_{\widetilde{V}'}(V)^{cl} + R^+$, it is clear that $p_{\lambda}(m_{\widetilde{v},\mathfrak{n}v}) = 0$. One hence sees that the map considered factors throuth an $\mathfrak{a}$-map $pres_{\emptyset}(V) \to Pol_{\mathbb{C}}(\mathfrak{a}) \cdot a^{\lambda+\rho}$. Thus, since v maps to a non-zero element, and on the target $\mathfrak{a}$ acts via generalized eigenweight λ, we conclude that v must have a non-zero λ-component. □

Corollary 3.5.4. *Let V be a HC $(\mathfrak{g}, K)$-module. Then $supp(V)$ is conical; More precisely,*

$$supp(V) \subset wt(pres_{\emptyset}(V))^{cl}.$$

We have

$$supp(V)^{min} \subset wt(pres_{\emptyset}(V)).$$

3.6 Casselmans submodule theorem, existence of globalizations

Theorem 3.6.1 (Casselmans submodule theorem)**.** *Let V be a HC $(\mathfrak{g}, K)$-module. Then if $V \neq 0$, we have $pres_{\emptyset}(V) \neq 0$ (i.e., $V/\mathfrak{n}V \neq 0$).*

Proof. If $V \neq 0$, then $supp(V) \neq \emptyset$, and thus $supp(V)^{min} \neq \emptyset$. Since $supp(V)^{min} \subset wt(pres_{\emptyset}(V))$, we have $wt(pres_{\emptyset}(V)) \neq \emptyset$, and hence $pres_{\emptyset}(V) \neq 0$. □

Corollary 3.6.2. *Let V be an irreducible HC $(\mathfrak{g}, K)$-module. Then there exists an irreducible $(\mathfrak{g}_{\emptyset}, K_{\emptyset})$-module W, and an injection $V \to pind_{\emptyset}(W)$. In particular, V admits a Hilbert globalization - a subrepresentation of $pind_{\emptyset}^{Hilb}(\mathcal{W})$, where $\mathcal{W}$ is W considered as a $G_{\emptyset}$-representation.*

We would like now to generalize the corollary to arbitrary HC $(\mathfrak{g}, K)$-modules (not necessarily irreducible).

Claim 3.6.3. *Let V be a HC $(\mathfrak{g}, K)$-module. Then $\cap_{m\geq 0}\mathfrak{n}^m V = 0$.*

Proof. Let $v \in \cap_{m\geq 0}\mathfrak{n}^m V$, and let $\widetilde{v} \in \widetilde{V}$. We have $supp(m_{\widetilde{v},v}) \subset supp(V) + m \cdot R^+$ for every $m \geq 1$. This forces $supp(m_{\widetilde{v},v}) = \emptyset$. This means $m_{\widetilde{v},v} = 0$. Since this holds for every $\widetilde{v} \in \widetilde{V}$, we deduce $v = 0$. □

Lemma 3.6.4. *Let $J \subset Z(\mathfrak{g})$ be an ideal of finite codimension. Then there exists a finite subset $S \subset \mathfrak{a}^*_{\mathbb{C}}$ such that for every HC $(\mathfrak{g}, K)$-module V annihilated by J, we have $wt(V/\mathfrak{n}V) \subset S$.*

Proof. If $JV = 0$ then $hc'_{\emptyset}(J)(V/\mathfrak{n}V) = 0$. Hence we have an ideal of finite codimension in $U(\mathfrak{a})$ (namely, $Z(\mathfrak{g}_{\emptyset})hc'_{\emptyset}(J) \cap U(\mathfrak{a})$) which annihilates $V/\mathfrak{n}V$. From this the claim is clear.

□

Lemma 3.6.5. *Let V be a HC $(\mathfrak{g}, K)$-module. Then there exists $k \in \mathbb{Z}_{\geq 1}$, such that $\mathfrak{n}^k V$ contains no non-zero $(\mathfrak{g}, K)$-submodules.*

Proof. Let $J \subset Z(\mathfrak{g})$ be an ideal of finite codimension annihilating V. Let $S \subset \mathfrak{a}^*_{\mathbb{C}}$ be corresponding to J as in the previous lemma. Notice that $wt(\mathfrak{n}^m V/\mathfrak{n}^{m+1}V) \subset wt(\mathfrak{n}^{m-1}V/\mathfrak{n}^m V) + R^+$. Hence, we can find k such that $S \cap wt(\mathfrak{n}^m V/\mathfrak{n}^{m+1}V) = \emptyset$ for all $m \geq k$. Let now $W \subset V$ be a non-zero $(\mathfrak{g}, K)$-submodule - we want to show that $W \not\subset \mathfrak{n}^k V$. By passing to a submodule, we can assume that W is finitely generated (so a HC $(\mathfrak{g}, K)$-module). Then $W/\mathfrak{n}W \neq 0$. Since $\cap_{m\geq 1}\mathfrak{n}^m V = 0$, there exists $m \geq 0$ such that $W \not\subset \mathfrak{n}^{m+1}V$ - let m_0 be the smallest such m. Then we have a non-zero map of $\mathfrak{a}$-modules $W/\mathfrak{n}W \to \mathfrak{n}^{m_0}V/\mathfrak{n}^{m_0+1}V$. Since $wt(W/\mathfrak{n}W) \subset S$, we obtain that $\mathfrak{n}^{m_0}V/\mathfrak{n}^{m_0+1}V$ has an $\mathfrak{a}$-weight in S. This implies $m_0 < k$. Thus, we see that $W \not\subset \mathfrak{n}^k V$. □

Let E be a finite-dimensional $P_{\emptyset}$-representation. Define $ind'(E)$ analogously to before, as the space of continuous functions $f : G \to E$, satisfying $f(gp) = p^{-1}f(g)$ for $p \in P_{\emptyset}, g \in G$. Then $ind'(E)$ is admissible and pre-Hilbertizable. One has Frobenius reciprocity: For a $(\mathfrak{g}, K)$-module W, one has

$$Hom_{(\mathfrak{g},K)}(W, ind'(E)) \cong Hom_{(\mathfrak{p}_{\emptyset},K_{\emptyset})}(W, E).$$

Claim 3.6.6. *Let V be a HC $(\mathfrak{g}, K)$-module. Then there exists a finite-dimensional $P_{\emptyset}$-representation E, and an injection of $(\mathfrak{g}, K)$-modules $V \to ind'(E)$.*

Proof. By the previous lemma, let $k \in \mathbb{Z}_{\geq 1}$ be such that $\mathfrak{n}^k V$ does not contain non-zero $(\mathfrak{g}, K)$-submodules. The $(\mathfrak{p}_{\emptyset}, K_{\emptyset})$-module $V/\mathfrak{n}^k V$ can be lifted to a $P_{\emptyset}$-representation. Then the morphism $V \to ind'(V/\mathfrak{n}^k V)$ corresponding to the projection $V \to V/\mathfrak{n}^k V$ is injective, because its kernel is a $(\mathfrak{g}, K)$-submodule which sits in $\mathfrak{n}^k V$. □

Corollary 3.6.7. *Let V be a HC $(\mathfrak{g}, K)$-module. Then V admits a Hilbert globalization.*

3.7 An estimate on matrix coefficients

Lemma 3.7.1. *Let V be a HC $(\mathfrak{g}, K)$-module. Then there exist $\lambda \in \mathfrak{a}^*$ and $d \in \mathbb{Z}_{\geq 0}$ such that for all $v \in V, \widetilde{v} \in \widetilde{V}$ one has*

$$|m_{\widetilde{v},v}(a)| \preceq a^{\lambda} s(a)^d, \quad a \in A^-.$$

Proof. It is not hard to see that it is enough to provide a bound on A^-_{ss} (complete why).

We have a Hilbert globalization $\mathcal{V}$, so in particular a globalization of moderate growth, by claim 2.3.3. Then, considering the continuous seminorm $w \mapsto |\langle \widetilde{v}, w\rangle|$, we have a continuous seminorm σ and $d \in \mathbb{Z}_{\geq 0}$ such that

$$|\langle \widetilde{v}, gv\rangle| \preceq e^{ds(g)} \sigma(v), \quad g \in G.$$

Notice now that

$$e^{ds(a)} \sim e^{d'||log(a)||}$$

(for some d') and we can find λ such that

$$||log(a)|| \preceq \lambda(log(a)), \quad a \in A^-_{ss}.$$

∎

Claim 3.7.2. *Let V be a HC $(\mathfrak{g}, K)$-module, and let $\lambda \in \mathfrak{a}^*$. Suppose that $Re(wt(pres_{\emptyset}(V))) \geq \lambda$. Then there exists $d \in \mathbb{Z}_{\geq 0}$ such that for all $v \in V, \widetilde{v} \in \widetilde{V}$ one has*

$$|m_{\widetilde{v},v}(a)| \preceq a^{\lambda+\rho} \cdot s(a)^d, \quad a \in A^-.$$

We will prove the claim after a few lemmas.

Let us denote $d_\alpha := min_{\lambda' \in wt(V/\mathfrak{n}_{(\emptyset)}V)}(Re(\lambda')(\Omega_\alpha))$ for $\alpha \in R^s$. We can assume that $\lambda + \rho = \lambda_V$, where $\lambda_V(\Omega_\alpha) = d_\alpha$ for $\alpha \in R^s$, and $(\lambda_V)|_{\mathfrak{a}_{cent}} = (\lambda + \rho)|_{\mathfrak{a}_{cent}}$.

Given $\mu \in \mathfrak{a}^*$ and $d \in \mathbb{Z}_{\geq 0}$, we say that $v \in V$ is (μ, d)-good if one has

$$|m_{\widetilde{v},v}(a)| \leq D a^{\mu} \cdot s(a)^d, \quad a \in A^-$$

for some $D \in \mathbb{R}_{>0}$. We say that a subspace $V_0 \subset V$ is μ-good if there exists $d \in \mathbb{Z}_{\geq 0}$ such that every $v \in V_0$ is (μ, d)-good. We want to show that V is λ_V-good.

By lemma 3.7.1, there exists $\mu \in \mathfrak{a}^*$ such that V is μ-good. We can assume that $\mu|_{\mathfrak{a}_{cent}} = (\lambda_V)|_{\mathfrak{a}_{cent}}$.

For the next few lemmas, let us fix $\alpha \in R^s$, and denote $I = R^s - \{\alpha\}$. For $\mu \in \mathfrak{a}^*$ and $r \in \mathbb{R}$, we denote $\mu_r := \mu + (r - \mu(\Omega_\alpha)\alpha$ (i.e. μ_r coincides with μ on $\mathfrak{a}_{cent} + \sum_{\beta \neq \alpha} \mathbb{R} \cdot \Omega_\beta$, and is equal to r on Ω_α).

Lemma 3.7.3. *Suppose that V is μ-good. Then $\mathfrak{n}_{(I)}V$ is $(\mu + \alpha)$-good.*

Proof. For $X \in \mathfrak{g}^{\mathfrak{a},\beta}$ we have

$$m_{\widetilde{v},Xv}(a) = -a^{\beta} m_{X\widetilde{v},v}(a).$$

If $\mathfrak{g}^{\mathfrak{a},\beta} \subset \mathfrak{n}_{(I)}$, we have $\beta \geq_{R^s} \alpha$, hence $\beta \geq \alpha$, i.e. $a^{\beta} \preceq a^{\alpha},\ a \in A^{-}$. $\square$

Lemma 3.7.4. *Let $f(t), h(t) \in C^1(\mathbb{R}_{\geq 0})$. Suppose that*

$$|h(t)| \leq Ce^{rt}(1+t)^d$$

for some $r \in \mathbb{R}, d \in \mathbb{Z}_{\geq 0}$ and $C \in \mathbb{R}_{>0}$. Suppose also that one has $\partial_t f - cf = h$ for some $c \in \mathbb{C}$. Then, if $r \leq Re(c)$ one has

$$|f(t)| \leq D(|f(0)| + C)e^{Re(c)t}(1+t)^{d+1},$$

and if $r > Re(c)$ one has

$$|f(t)| \leq D(|f(0)| + C)e^{rt}(1+t)^d;$$

Both for some $D \in \mathbb{R}_{>0}$ which depends only on r, c.

Proof. One has

$$f(t) = e^{ct}\left(f(0) + \int_0^t e^{-cs}h(s)\right).$$

Thus:

$$|f(t)| \leq e^{Re(c)t}\left(|f(0)| + C\int_0^t e^{(r-Re(c))s}(1+s)^d\right) \leq$$

$$e^{Re(c)t}\left(|f(0)| + C\int_0^t e^{(r-Re(c))s}\right)(1+t)^d.$$

If $r \leq Re(c)$, we estimate

$$\int_0^t e^{(r-Re(c))s} \leq t$$

and obtain the desired estimate. If $r > Re(c)$, we estimate

$$\int_0^t e^{(r-Re(c))s} \leq \frac{1}{r - Re(c)} e^{(r-Re(c))t}$$

and obtain the desired estimate.

$\square$

Lemma 3.7.5. *Let $v \in V$ and $c \in \mathbb{C}$. Denote $w := \Omega_\alpha v - cv$. Suppose that v is (μ, d)-good and w is (μ_r, d)-good. Then v is $(\mu_{min(Re(c),r)}, d+1)$-good.*

Proof. Fix $\widetilde{v} \in \widetilde{V}$ and let us denote

$$f(H,t) := m_{\widetilde{v},v}(ae^{-t\Omega_\alpha}),\ h(H,t) := m_{\widetilde{v},w}(ae^{-t\Omega_\alpha})$$

where $a \in exp(\mathfrak{a}_{cent} - \sum_{\beta \neq \alpha} \mathbb{R}_{\geq 0} \Omega_\beta)$ and $t \in \mathbb{R}_{\geq 0}$. We have

$$|f(a,0)| \leq D a^\mu s(a)^d, \quad |h(a,t)| \leq D a^\mu e^{-rt} s(a)^d (1+t)^d$$

for some $D \in \mathbb{R}_{>0}$. We have $\partial_t f + cf = h$. Hence, by lemma 3.7.4 (where we plug in $C := D a^\mu s(a)^d$), we obtain that if $-r \leq -Re(c)$,we have

$$|f(a,t)| \leq D' \cdot a^\mu s(a)^d \cdot e^{-Re(c)t} (1+t)^{d+1}$$

while if $-r > -Re(c)$ we have

$$|f(a,t)| \leq D' \cdot a^\mu s(a)^d \cdot e^{-rt} (1+t)^d$$

(both for some $D' \in \mathbb{R}_{>0}$).

□

Lemma 3.7.6. *If V is μ-good, then V is μ_{d_α}-good.*

Proof. Let us write $\mu = \mu_r$ (so $r := \mu(\Omega_\alpha)$).

First, notice that since $\Omega_\alpha \in \mathfrak{a}_{cent,I}$, and since $V/\mathfrak{n}_{(I)}V$ is a HC $(\mathfrak{g}_I, K_I)$-module, the action of Ω_α is finite on $V/\mathfrak{n}_{(I)}V$. Moreover, it is easy to see that the eigenvalues of this action are exactly $\lambda(\Omega_\alpha)$, as λ runs over $wt(V/\mathfrak{n}_{(\emptyset)}V)$. By lemma 3.7.3, $\mathfrak{n}_{(I)}V$ is μ_{r+1}-good. Denoting by $V_c \subset V$ the subspace which is the preimage of the generalized eigenspace of Ω_α in $V/\mathfrak{n}_{(I)}V$ with eigenvalue c, by repeated use of lemma 3.7.5, we see that V_c is $\mu_{min(Re(c),r+1)}$-good. Hence, V is $\mu_{min(d_\alpha,r+1)}$-good. Thus, replacing μ_r with μ_{r+1}, after a finite number of steps we will arrive to the desired conclusion. □

Proof (of claim 3.7.2). By repeated use of lemma 3.7.6, we see that V is λ_V-good. □

Chapter 4

Intertwining integrals - minimal parabolic case

4.1 Principal series

Let us notice that all HC $(\mathfrak{g}_\emptyset, K_\emptyset)$-modules are finite-dimensional. The irreducible HC $(\mathfrak{g}_\emptyset, K_\emptyset)$-modules are classified by pairs (λ, ϵ) where $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $\epsilon \in K^\vee$. We denote by $\mathcal{E}_{\lambda,\epsilon}$ the corresponding irreducible finite-dimensional $G_\emptyset$-representation. We denote $\mathcal{P}_{\lambda,\epsilon} := pind_\emptyset(\mathcal{E}_{\lambda,\epsilon})$ (the (continuous) **principal series**). Let us also denote $\mathcal{P}_\lambda := \mathcal{P}_{\lambda,triv}$ (the **spherical** principal series).

4.2 Intertwining

Let $\mathcal{E}$ be a finite-dimensional $G_\emptyset$-representation, and fix $w \in N_K(\mathfrak{a})$. We denote by ${}^w\mathcal{E}$ the representation on $\mathcal{U}$ given by $g *^{\text{new}} u := ({}^{w^{-1}}g) *^{\text{old}} u$. We define, formally (i.e. ignoring convergence) a G-equivariant morphism

$$I_w : pind_I(\mathcal{E}) \to pind_I({}^w\mathcal{E})$$

as follows:

$$I_w(f)(g) := \int_{N_{(\emptyset)}/N_{(\emptyset)} \cap {}^w N_{(\emptyset)}} f(gxw).$$

We also define formally a map

$$J_w : pind_I(\mathcal{E}) \to \mathcal{E}$$

by

$$J_w(f) := \int_{{}^{w^{-1}}N_{(\emptyset)}/{}^{w^{-1}}N_{(\emptyset)} \cap N_{(\emptyset)}} f(x) \left(= \int_{{}^{w^{-1}}N_{(\emptyset)} \cap N^-_{(\emptyset)}} f(x) \right).$$

Notice that $J_w(f) = I_w(f)(w^{-1})$ and $I_w(f)(g) = J_w((gw)^{-1}f)$. Also notice that J_w depends only on the class of w in W, while I_w depends "slightly" on the actual representative w.

Lemma 4.2.1. *Suppose that $J_w(f)$ converges absolutely for every $f \in pind_{\emptyset}(\mathcal{E})^{[K]}$. Then there exists a unique $(\mathfrak{g}, K)$-morphism*

$$I_w : pind_{\emptyset}(\mathcal{E})^{[K]} \to pind_{\emptyset}({}^w\mathcal{E})^{[K]},$$

satisfying

$$I_w(f)(e) = J_w(w^{-1}f).$$

Proof. Recall that $pind_{\emptyset}(\mathcal{E})^{[K]} \cong pind_{\emptyset}(\mathcal{E}^{[K]})$, so that a morphism as desired is the same as a morphism of $(\mathfrak{g}_{\emptyset}, K_{\emptyset})$-modules

$$pres_{\emptyset} pind_{\emptyset}(\mathcal{E}^{[K]}) \to {}^w\mathcal{E}^{[K]}.$$

We obtain such a morphism by sending $f \mapsto J_w(w^{-1}f)$. □

For example:

Lemma 4.2.2. *Suppose that $\mathfrak{a}$ acts on $\mathcal{E}$ via a character $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Suppose that the integral*

$$\int_{N^-_{(\emptyset)}} r(x)^{-Re(\lambda)-\rho}$$

converges.

1. *For every $w \in N_K(\mathfrak{a})$ and $f \in pind_{\emptyset}(\mathcal{E})^{[K]}$, the integral $J_w(f)$ converges absolutely.*
2. *The map $pind_{\emptyset}(\mathcal{E})^{[K]} \to \mathcal{E}$ given by $f \mapsto J_{w_0}(f)$ is surjective.*

Proof.

1. First, we want to see that if the above integral converges, then the same integral, but over ${}^{w^{-1}}N_{(\emptyset)} \cap N^-_{(\emptyset)}$, also converges. Indeed, we can express the integral over $N^-_{(\emptyset)}$ as an integral of $r(xy)^{-Re(\lambda)-\rho}$ over

$$({}^{w^{-1}}N^-_{(\emptyset)} \cap N^-_{(\emptyset)}) \times ({}^{w^{-1}}N_{(\emptyset)} \cap N^-_{(\emptyset)}).$$

Noticing that $x \in K \cdot {}^{w^{-1}}r({}^w x) \cdot {}^{w^{-1}}N_{(\emptyset)}$, we obtain

$$\int_{x \in {}^{w^{-1}}N^-_{(\emptyset)} \cap N^-_{(\emptyset)}} \int_{y \in {}^{w^{-1}}N_{(\emptyset)} \cap N^-_{(\emptyset)}} r(xy)^{-Re(\lambda)-\rho} =$$

$$= \int_{x \in {}^{w^{-1}}N^-_{(\emptyset)} \cap N^-_{(\emptyset)}} \int_{y \in {}^{w^{-1}}N_{(\emptyset)} / {}^{w^{-1}}N_{(\emptyset)} \cap N_{(\emptyset)}} r(xy)^{-Re(\lambda)-\rho} =$$

$$= \int_{x \in {}^{w^{-1}}N^-_{(\emptyset)} \cap N^-_{(\emptyset)}} \int_{y \in {}^{w^{-1}}N_{(\emptyset)} / {}^{w^{-1}}N_{(\emptyset)} \cap N_{(\emptyset)}} r({}^{w^{-1}}r({}^w x) y)^{-Re(\lambda)-\rho} =$$

$$= \int_{x \in {}^{w^{-1}}N^-_{(\emptyset)} \cap N^-_{(\emptyset)}} \int_{y \in {}^{w^{-1}}N_{(\emptyset)} / {}^{w^{-1}}N_{(\emptyset)} \cap N_{(\emptyset)}} r(x)^{\cdots} r(y)^{-Re(\lambda)-\rho}$$

(where in the last passage we jumped ${}^{w^{-1}}r({}^{w}x)$ over y). Thus the integral that we assume converging, separates into an integral of a positive quantity in x, times the integral we want to make sure is converging.

Second, notice that for $f \in pind_{\emptyset}(\mathcal{E}_{\lambda,\epsilon})^{[K]}$, we have $||f(g)|| \preceq r(g)^{-Re(\lambda)-\rho}$. Hence, lemma 4.2.1 implies that we have the desired well-defined morphism I_w.

2. Let $\zeta \in \mathcal{E}^*$. We would like to show that there exists $f \in pind_{\emptyset}(\mathcal{E})^{[K]}$ such that $\langle \zeta, J_{w_0}(f) \rangle \neq 0$.

   Notice that for every $f \in pind_{\emptyset}(\mathcal{E})$ one has

$$||f(g)|| \preceq r(g)^{-Re(\lambda)-\rho} \quad (g \in G),$$

   and thus $J_{w_0}(f)$ converges absolutely.

   Let us fix a function $u \in C_c^{\infty}(N_{(\emptyset)}^-)$ for which $\int_{N_{(\emptyset)}^-} u = 1$ and let us fix an element $v \in \mathcal{E}$ for which $\langle \zeta, v \rangle \neq 0$. Define an element $f_0 \in pind_{\emptyset}(\mathcal{E})$ by $f_0(xmn) = u(x) \cdot \Delta(m)^{-1} mv$ for $(x, m, n) \in N_{(\emptyset)}^- \times G_{\emptyset} \times N_{(\emptyset)}$, and $f_0(g) = 0$ if $g \notin N_{(\emptyset)}^- P_{(\emptyset)}$. Then one has

$$\langle \zeta, J_{w_0}(f_0) \rangle = \langle \zeta, v \rangle \neq 0.$$

   We now consider elements $\phi * f_0 \in pind_{\emptyset}(\mathcal{E})$ where $\phi \in C(K)$. Denote $S = supp(f_0|_{N_{(\emptyset)}^-})$. Given $\epsilon > 0$, we can find $e \in U \subset K$ such that $|f_0(kx) - f_0(x)| < \epsilon$ for $k \in U, x \in S$. Then, we fix $\phi \in C(K)$ such that ϕ takes real non-negative values, $\int_K \phi = 1$ and $supp(\phi) \subset U$. Then we have

$$||J_{w_0}(\phi * f_0) - J_{w_0}(f_0)|| = ||\int_{N_{(\emptyset)}^-} \int_K \phi(k) f_0(k^{-1}x) - f_0(x)|| =$$

$$= ||\int_{KS \cap N_{(\emptyset)}^-} \int_U \phi(k)(f_0(k^{-1}x) - f_0(x))|| \leq \epsilon \cdot \mathrm{meas}(KS \cap N_{(\emptyset)}^-).$$

   Thus, we can find $\phi \in C(K)$ such that $\langle \zeta, J_{w_0}(\phi * f_0) \rangle \neq 0$.

   Next, for $\phi_1, \phi_2 \in C(K)$ we have

$$||J_{w_0}(\phi_1 * f_0) - J_{w_0}(\phi_2 * f_0)|| = ||\int_{N_{(\emptyset)}^-} \int_K (\phi_1(k) - \phi_2(k)) f_0(k^{-1}x)|| \leq$$

$$\leq ||\phi_1 - \phi_2||_{\sup} \cdot C \cdot \int_{N_{(\emptyset)}^-} r(x)^{-Re(\lambda)-\rho},$$

   where $C > 0$ is such that

$$||f_0(g)|| \leq C r(g)^{-Re(\lambda)-\rho} \quad (g \in G).$$

From here, taking $\phi_2 := \phi$ and $\phi_1 = \psi * \phi$ where $\psi \in C(K)^{[K]}$ is suitable, we obtain that

$$\langle \zeta, J_{w_0}(\phi_1 * f_0) \rangle \neq 0$$

(we use Stone-Weierstrass). Since $\phi_1 \in C(K)^{[K]}$, we see that $\phi_1 * f_0 \in pind_{\emptyset}(\mathcal{E})^{[K]}$.

$\square$

Example 4.2.3. *Let us consider the integral*

$$\int_{N^-_{(\emptyset)}} r(x)^{-Re(\lambda)-\rho}$$

in the example of $SL_2(\mathbb{R})$. *We identify* $N^-_{(\emptyset)}$ *with* $\mathbb{R}$ *via* $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \leftarrow\!\shortmid x$. *Using Gram-Schmidt orthogonalization we obtain*

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1/r & -x/r \\ x/r & 1/r \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix} \begin{pmatrix} 1 & x/r^2 \\ 0 & 1 \end{pmatrix}$$

where we have denoted $r := \sqrt{1+x^2}$. *We also identify* $\mathfrak{a}^*_{\mathbb{C}}$ *with* $\mathbb{C}$ *via* $\lambda \mapsto \lambda(H)$. *We thus obtain* $r(x)^{\lambda} = (1+x^2)^{\lambda/2}$. *Hence, the integral is*

$$\int_{\mathbb{R}} (1+x^2)^{-\frac{Re(\lambda)+1}{2}} dx.$$

This integral converges absolutely when $Re(\lambda) > 0$.

4.3 Lemmas about r

Lemma 4.3.1. *Let* E *be an algebraic representation of* $\boldsymbol{G}(\mathbb{C})$, *and* $0 \neq v \in E^{\mathfrak{a},\lambda}$ *satisfying* $\mathfrak{n}_{(\emptyset)} v = 0$.

1. *Let* $g \in G^{\circ}$ *and* $a \in A^-$. *The we have* $r(a^{-1}ga)^{\lambda} \leq r(g)^{\lambda}$.
2. *Let* $g \in N^-_{(\emptyset)}$. *Then* $r(g)^{\lambda} \geq 1$, *with equality if and only if* $gv = v$.
3. *Let* $a \in A^{--}$. *Then there exists* $0 < c < 1$ *such that* $r(a^{-t}ga^t)^{\lambda} \leq \sqrt{1 + c^t r(g)^{2\lambda}}$ *for all* $g \in N^-_{(\emptyset)}$ *and* $t \geq 0$.

Proof. We choose a $\widetilde{K}$-invariant inner product on E, and can assume $||v|| = 1$. Notice that we have $||gv||^2 = r(g)^{2\lambda}$ for all $g \in G$. Let us write $gv = \sum_{i \geq 1} v_i$ with $v_i \in E^{\mathfrak{a},\lambda_i}$ (all λ_i different). The different v_i are orthogonal.

(1) Notice that since $g \in G^{\circ}$, we have $gv \in U(\mathfrak{g})v = U(\mathfrak{n}^-_{\emptyset})U(\mathfrak{k}_{\emptyset})v$, so $\lambda - \lambda_i \geq_{R^+} 0$ and thus $\lambda - \lambda_i \geq 0$. Then

$$||a^{-1}gav||^2 = ||\sum_i a^{\lambda-\lambda_i} v_i||^2 = \sum_i a^{\lambda-\lambda_i} ||v_i||^2 \leq \sum_i ||v_i||^2 = ||\sum_i v_i||^2 = ||gv||^2.$$

(2) Since $g \in N^-_{(\emptyset)}$, we can assume that $v_1 = v$. Then $||gv||^2 = 1 + \sum_{i \neq 1} ||v_i||^2$ from which the claim is clear.

(3) Set $c = max_{\alpha \in R^s}\ a^\alpha$ ($0 < c < 1$ because $a \in A^{--}$). We can assume that $v_1 = v$. We obtain:

$$||a^{-t} g a^t||^2 = 1 + \sum_{i \neq 1} a^{t(\lambda - \lambda_i)} ||v_i||^2 \leq 1 + c^t \sum_{i \neq 1} ||v_i||^2 \leq 1 + c^t ||gv||^2.$$

□

Lemma 4.3.2.

1. *For dominant $\lambda \in \mathfrak{a}^*$, we have*

$$r(a^{-1} g a)^\lambda \leq r(g)^\lambda$$

for all $g \in G^\circ$ and $a \in A^-$.

2. *For dominant $\lambda \in \mathfrak{a}^*$, we have*

$$r(g)^\lambda \geq 1$$

for $g \in N^-_{(\emptyset)}$.

3. *Let $a \in A^{--}$. Then there exists $0 < c < 1$ such that*

$$r(a^{-t} x a^t)^\rho \leq \sqrt[4]{1 + c^t r(x)^{4\rho}}$$

for all $x \in N^-_{(\emptyset)}$ and $t \geq 0$.

4. *Let $g \in N^-_{(\emptyset)}$. If $r(g)^\rho = 1$, then $g = e$.*

Proof.

(1) Clearly, the set of λ for which the inequality holds is closed under $\mathbb{R}_{\geq 0}$-span. Notice that if λ is the restriction to $\mathfrak{a}$ of highest weight for $\mathfrak{a} \oplus i\mathfrak{b}$ (in the notations of section 4.3.1), then by part 1 of lemma 4.3.1 we have the desired inequality. Using corollary 4.3.6, we obtain the desired inequality for all dominant λ.

(2) The proof is the same as in (1), using part 2 of lemma 4.3.1.

(3) We use part 3 of lemma 4.3.1 with $E := \bigwedge^{\dim \mathfrak{n}_{(\emptyset)}} \mathfrak{g}_{\mathbb{C}}$, and a non-zero vector v in the one-dimensional subspace corresponding to $\mathfrak{n}_{(\emptyset)}$. Notice that $\mathfrak{n}_{(\emptyset)} v = 0$ and $v \in E^{\mathfrak{a}, 2\rho}$.

(4) We use part 2 of lemma 4.3.1, applied to E, v as in part (3) above. It is easy to calculate that the stabilizer of v in $\mathfrak{g}$ is $\mathfrak{p}_{(\emptyset)}$. Thus, the intersection of $N^-_{(\emptyset)}$ with the stabilizer of v in G is finite. Since $N^-_{(\emptyset)}$ does not contain non-trivial finite subgroups, we deduce that $N^-_{(\emptyset)}$ intersects this stabilizer trivially. □

4.3.1 Complex roots

Note to self: this subsection is quite messy

Let $\mathfrak{b} \subset \mathfrak{k}_\emptyset$ be a maximal abelian subalgebra.

Lemma 4.3.3. *The complexification $\mathfrak{h} := (\mathfrak{a} \oplus \mathfrak{b})_{\mathbb{C}}$ is a Cartan subalgebra in $\mathfrak{g}_{\mathbb{C}}$.*

Proof. The adjoint action of this subalgebra is semisimple, because the operators from $\mathfrak{a}$ are symmetric and those from $\mathfrak{b}$ are skew-symmetric (w.r.t. the form $B(\cdot, \theta \cdot)$). If an element $X \in \mathfrak{g}$ commutes with $\mathfrak{a} \oplus \mathfrak{b}$, then so does θX, and thus the components of X w.r.t. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Then from the definitions of $\mathfrak{a}$ and $\mathfrak{b}$ we see that those components lie in $\mathfrak{b}$ and $\mathfrak{a}$ respectively. □

We thus can consider the **complex root system**, corresponding to $\mathfrak{h}$ acting on $\mathfrak{g}_{\mathbb{C}}$.

One can (and we always will) choose positive complex roots in such a way that their restriction to $\mathfrak{a}$ is either zero, or a positive root.

The real form $\mathfrak{a} \oplus i\mathfrak{b}$ of $\mathfrak{h}$ is the "correct" one, in the sense that it is the $\mathbb{R}$-span of the lattice of integral weights, i.e. the weights which are derivatives of characters of the complex torus whose Lie algebra is $\mathfrak{h}$.

The form $B(\cdot, \cdot)$ is positive definite on $\mathfrak{a} \oplus i\mathfrak{b}$, and is preserved under the Weyl group (because it is invariant under $G(\mathbb{C})$). Furthermore, $\mathfrak{a}$ is orthogonal to $i\mathfrak{b}$. Let us denote by $\lambda = \lambda_{\mathfrak{a}} + \lambda_{\mathfrak{b}}$ the decomposition corresponding to the decomposition $(\mathfrak{a} \oplus i\mathfrak{b})^* = \mathfrak{a}^* \oplus (i\mathfrak{b})^*$. We also denote by $B(\cdot, \cdot)$ the form on $(\mathfrak{a} + i\mathfrak{b})^*$ corresponding to the form $B(\cdot, \cdot)$ on $\mathfrak{a} + i\mathfrak{b}$, and $||\lambda||^2 := B(\lambda, \lambda)$.

Let us denote by $L_{\mathfrak{a}} \subset \mathfrak{a}_{cent}$ and by $L_{\mathfrak{b}} \subset \mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) \cap i\mathfrak{b}$ the integral lattices. Then there exists $m \in \mathbb{Z}_{\geq 1}$ such that a complex weight $\lambda \in (\mathfrak{a} \oplus i\mathfrak{b})^*$ is integral if $\lambda(L_{\mathfrak{a}}) \subset m\mathbb{Z}$, $\lambda(L_{\mathfrak{b}}) \subset m\mathbb{Z}$, and $\lambda(H'_\gamma) \in m\mathbb{Z}$ for every complex root $\gamma \in (\mathfrak{a} \oplus i\mathfrak{b})^*$ (we denote by $H'_\gamma \in \mathfrak{a} \oplus i\mathfrak{b}$ the corresponding complex coroot).

Lemma 4.3.4. *Let $\gamma \in (\mathfrak{a} \oplus i\mathfrak{b})^*$ be a complex root. Then $\frac{||\gamma_{\mathfrak{a}}||^2}{||\gamma||^2}$ is rational.*

Proof. By applying σ, we see that $\sigma(\gamma) = \gamma_{\mathfrak{a}} - \gamma_{\mathfrak{b}}$ is a root as well. Thus

$$\frac{2B(\gamma_{\mathfrak{a}} + \gamma_{\mathfrak{b}}, \gamma_{\mathfrak{a}} - \gamma_{\mathfrak{b}})}{||\gamma_{\mathfrak{a}} + \gamma_{\mathfrak{b}}||^2} \in \mathbb{Z}.$$

Hence

$$\frac{||\gamma_{\mathfrak{a}}||^2 - ||\gamma_{\mathfrak{b}}||^2}{||\gamma_{\mathfrak{a}}||^2 + ||\gamma_{\mathfrak{b}}||^2} \in \mathbb{Q}.$$

Adding 1, we obtain

$$\frac{||\gamma_{\mathfrak{a}}||^2}{||\gamma||^2} = \frac{||\gamma_{\mathfrak{a}}||^2}{||\gamma_{\mathfrak{a}}||^2 + ||\gamma_{\mathfrak{b}}||^2} \in \mathbb{Q}.$$

□

Corollary 4.3.5. *There exists $m \in \mathbb{Z}_{\geq 1}$ and a lattice $L \subset \mathfrak{a}_{ss}$ with the following property. Let $\lambda \in \mathfrak{a}^*$ be satisfying $\lambda(H_\alpha) \in m\mathbb{Z}_{\geq 0}$ for all $\alpha \in R^+$, and $\lambda(L) \in m\mathbb{Z}$. Then $\lambda + 0 \in (\mathfrak{a} \oplus i\mathfrak{b})^*$ is a dominant and integral complex weight.*

Proof. For a positive complex root γ, we have

$$2\frac{B(\lambda,\gamma)}{||\gamma||^2} = 2\frac{B(\lambda,\gamma_{\mathfrak{a}})}{||\gamma_{\mathfrak{a}}||^2} \cdot \frac{||\gamma_{\mathfrak{a}}||^2}{||\gamma||^2} \in m\mathbb{Z}_{\geq 1}.$$

□

Corollary 4.3.6. *The $\mathbb{R}_{\geq 0}$-span of the restrictions to $\mathfrak{a}$ of dominant and integral complex weights is equal to the set of dominant weights.*

Proof. Notice that for a dominant complex weight $\lambda \in (\mathfrak{a} \oplus i\mathfrak{b})^*$, the restriction $\lambda|_{\mathfrak{a}}$ is a dominant weight. Indeed, for a positive root $\alpha \in \mathfrak{a}^*$, we can find a positive complex root $\gamma \in (\mathfrak{a} \oplus i\mathfrak{b})^*$ such that $\gamma|_{\mathfrak{a}} = \alpha$, and then $\alpha = \frac{1}{2}(\gamma + \sigma(\gamma))$, and $\sigma(\gamma)$ is also a positive complex root. Hence, $B(\lambda,\alpha) = \frac{1}{2}(B(\lambda,\gamma) + B(\lambda,\sigma(\gamma))$, so is positive. □

4.4 The convergence of an integral - 1

Claim 4.4.1. *Let $\lambda \in \mathfrak{a}^*$ be dominant. Then*

$$\int_{N^-_{(\emptyset)}} r(x)^{-2\rho-\lambda}$$

converges.

Proof. Let us first notice that

$$\int_{N^-_{(\emptyset)}} r(x)^{-2\rho}$$

converges for "formal" reasons - it is an expression for

$$\int_{G//P_{(\emptyset)}} r(g)^{-2\rho}.$$

Now, we consider the integral

$$\int_{N^-_{(\emptyset)}} r(a^{-1}xa)^{\lambda} r(x)^{-2\rho-\lambda}$$

for $a \in A^-$. Letting $log(a) \to \infty$, the integrand converges to $r(x)^{-2\rho-\lambda}$ pointwise. On the other hand, the integrand can be esimated using lemma 4.3.2:

$$r(a^{-1}xa)^{\lambda} r(x)^{-2\rho-\lambda} \leq r(x)^{-2\rho},$$

and hence is convergent. Hence, by the dominated convergence theorem our integral, the integral of the pointwise limit, converges. □

4.5 Harish-Chandra's homomorphism hc

There is a G-invariant pairing between $\mathcal{P}_\lambda$ and $\mathcal{P}_{-\lambda}$, given by

$$\langle f_1, f_2\rangle := \int_{G//P_{(\emptyset)}} f_1(g)f_2(g).$$

We have a K-invariant vector $\xi_\lambda \in \mathcal{P}_\lambda$ given by

$$\xi_\lambda(g) := r(g)^{-\lambda-\rho}.$$

Clearly, the subspace of K-invariant vectors in $\mathcal{P}_\lambda$ is one-dimensional, spanned by ξ_λ.

Since the action of $U(\mathfrak{g})^K$ preserves K-invariancy, we obtain a map

$$hc : U(\mathfrak{g})^K \to Fun(\mathfrak{a}^*_{\mathbb{C}}),$$

given by

$$hc(U)(\lambda) := \frac{U\xi_\lambda}{\xi_\lambda} = (U\xi_\lambda)(e).$$

Using the Iwasawa decomposition, we have $U(\mathfrak{g}) = U(\mathfrak{a}) \oplus (U(\mathfrak{g})\mathfrak{k} + \mathfrak{n}_\emptyset U(\mathfrak{g}))$. Let us denote by $pr : U(\mathfrak{g}) \to U(\mathfrak{a})$ the resulting projection. We will think of $U(\mathfrak{a})$ as embedded into $Fun(\mathfrak{a}^*_{\mathbb{C}})$, by interpreting elements of $\mathfrak{a}$ as linear functionals on $\mathfrak{a}^*_{\mathbb{C}}$ (thus, the image of the embedding consists of the polynomial functions).

Lemma 4.5.1. *For $U \in U(\mathfrak{g})^K$, one has*

$$hc(U)(\lambda) = pr(U)(\lambda + \rho).$$

Proof. We want to show that $(U\xi_\lambda)(e) = pr(U)(\lambda+\rho)\xi_\lambda(e)$.

Let $W \in \mathfrak{k}$ and $V \in U(\mathfrak{g})$. Then $VW\xi_\lambda = 0$ because ξ_λ is K-invariant. Let now $X \in \mathfrak{n}_\emptyset$ and $V \in U(\mathfrak{g})$. Then

$$(XV\xi_\lambda)(e) = lim_{t\to 0}\frac{(V\xi_\lambda)(e^{-tX}) - (V\xi_\lambda)(e)}{t} = 0.$$

Finally, notice that for $H \in \mathfrak{a}$ we have

$$(H\xi_\lambda)(a) = lim_{t\to 0}\frac{\xi_\lambda(e^{-tH}a) - \xi_\lambda(a)}{t} = \left(lim_{t\to 0}\frac{e^{t(\lambda+\rho)(H)} - 1}{t}\right)\cdot\xi_\lambda(a) = (\lambda+\rho)(H)\xi_\lambda(a).$$

Let us write $H(\lambda+\rho)$ instead of $(\lambda+\rho)(H)$ by the convention above of looking at $U(\mathfrak{a})$ as an algebra of functions on $\mathfrak{a}^*_{\mathbb{C}}$, and we can iterate the above calculation to obtain, for $V \in U(\mathfrak{a})$:

$$(V\xi_\lambda)(a) = V(\lambda+\rho)\xi_\lambda(a).$$

Summing up the above findings, we get the desired.

□

Thus, we can consider hc as an algebra homomorphism

$$hc : U(\mathfrak{g})^K \to U(\mathfrak{a}) \cong Pol_{\mathbb{C}}(\mathfrak{a}^*_{\mathbb{C}})$$

(having formula $hc(U) = pr(U)(\cdot + \rho)$).

Claim 4.5.2. *The image of hc is contained in* $U(\mathfrak{a})^W$.

Proof. It is enough to show that for $U \in U(\mathfrak{g})^K$, $w \in W$ and $\lambda \in \mathfrak{a}^*$ for which $\lambda - \rho$ is dominant, we have $hc(U)(w\lambda) = hc(U)(\lambda)$ (because the set of such λ's is Zariski-dense in $\mathfrak{a}^*_{\mathbb{C}}$).

By claim 4.4.1 and lemma 4.2.2, we have a morphism

$$I_w : \mathcal{P}_\lambda^{[K]} \to \mathcal{P}_{w\lambda}^{[K]},$$

which is non-zero on ξ_λ (because $I_w(\xi_\lambda)(e)$ is given by a positive integral). Hence, $I_w(\xi_\lambda)$, being K-invariant, must be a non-zero multiple of $\xi_{w\lambda}$. Then by $I_w(U\xi_\lambda) = UI_w(\xi_\lambda)$ we obtain the desired equality. □

We will now want to see that $hc : U(\mathfrak{g})^K \to U(\mathfrak{a})^W$ is onto. For this, we need:

Theorem 4.5.3 (Chevalley's restriction theorem)**.** *The restriction map*

$$Pol_{\mathbb{C}}(\mathfrak{s})^K \to Pol_{\mathbb{C}}(\mathfrak{a})^W$$

is an isomorphism.

We can reformulate the above theorem as follows:

Corollary 4.5.4. *The map* $\mathfrak{s} \to \mathfrak{a}$ *given by* $\mathfrak{s} \hookrightarrow \mathfrak{g} \xrightarrow{pr} \mathfrak{a}$ *induces an isomorphism*

$$Sym_{\mathbb{C}}(\mathfrak{s})^K \to Sym_{\mathbb{C}}(\mathfrak{a})^W.$$

Proof. Since w.r.t. the form $B(\cdot, \theta\cdot)$ the subspace $\mathfrak{a}$ is orthogonal to $\mathfrak{k} + \mathfrak{n}_\emptyset$, we see that $pr : \mathfrak{g} \to \mathfrak{a}$ is the orthogonal projection. Hence, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{s} & \overset{\sim}{\text{———}} & \mathfrak{s}^* \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle \text{restriction}} \\ \mathfrak{a} & \overset{\sim}{\text{———}} & \mathfrak{a}^* \end{array}$$

where the horizontal identifications are via $B(\cdot, \theta\cdot)$. Since $B(\cdot, \theta\cdot)$ is K-invariant, the upper (resp. lower) horizontal identification is K-equivariant (resp. W-equivariant). Now the corollary follows quite clearly.

□

Claim 4.5.5. *The map* $hc : U(\mathfrak{g})^K \to U(\mathfrak{a})^W$ *is onto.*

Proof. Consider the situation

$$\begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{U \mapsto pr(U)(\cdot+\rho)} & U(\mathfrak{a}) \\ \uparrow & & \uparrow \\ U(\mathfrak{g})^K & \xrightarrow{hc} & U(\mathfrak{a})^W \end{array} .$$

All the spaces have filtrations and the maps are compatible with them. When passing to the associated graded, the situation becomes

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{S(pr)} & S(\mathfrak{a}) \\ \uparrow & & \uparrow \\ S(\mathfrak{g})^K & \longrightarrow & S(\mathfrak{a})^W \end{array} ,$$

where $S(pr)$ is the map of symmetric algebras induced by the map of vector spaces $pr : \mathfrak{g} \to \mathfrak{a}$. To show that hc is surjective, it is enough to show that the bottom map in the last diagram is surjective. And indeed, precomposing with the inclusion $S(\mathfrak{s})^K \to S(\mathfrak{g})^K$, we obtain an isomorphism, by the previous corollary. □

4.6 Harish-Chandra's functions Ξ_λ

We form now the matrix coefficient

$$\Xi_\lambda(g) := \langle g\xi_\lambda, \xi_{-\lambda} \rangle.$$

The function Ξ_λ is K-biinvariant. From the definitions we get

$$\Xi_\lambda(g) = \int_K r(g^{-1}k)^{-\lambda-\rho}$$

and also, for $a \in A$:

$$\Xi_\lambda(a) = a^{\lambda+\rho} \int_{N^-_{(\emptyset)}} r(a^{-1}xa)^{-\lambda-\rho} r(x)^{\lambda-\rho}.$$

Lemma 4.6.1.

$$\Xi_\lambda(g_1 g_2^{-1}) = \int_K r(g_1 k)^{\lambda-\rho} r(g_2 k)^{-\lambda-\rho}.$$

Proof.

$$\Xi_\lambda(g_1 g_2^{-1}) = \langle g_1 g_2^{-1} \xi_\lambda, \xi_{-\lambda} \rangle = \langle g_2^{-1}\xi_\lambda, g_1^{-1}\xi_\lambda \rangle = \int_K r(g_1 k)^{\lambda-\rho} r(g_2 x)^{-\lambda-\rho}.$$

□

Corollary 4.6.2. *For every compact $\Omega \subset G$, there exists $C > 0$ such that*

$$\Xi_\lambda(g_1 g_2) \leq C \cdot \Xi_\lambda(g_2)$$

for all $g_2 \in G, g_1 \in \Omega$.

Corollary 4.6.3. $\Xi_\lambda(g^{-1}) = \Xi_{-\lambda}(g)$.

Claim 4.6.4. *For $\lambda, \lambda' \in \mathfrak{a}_{\mathbb{C}}^*$ we have $\Xi_\lambda = \Xi_{\lambda'}$ if and only if $\lambda' \in W\lambda$.*

Proof. Notice that for $U \in U(\mathfrak{g})^K$ we have $R_U \Xi_\lambda = hc(U)\Xi_\lambda$. Indeed:

$$(R_U \Xi_\lambda)(g) = \langle gU\xi_\lambda, \xi_{-\lambda}\rangle = hc(U)\langle g\xi_\lambda, \xi_{-\lambda}\rangle = hc(U)\Xi_\lambda(g).$$

Suppose that $\Xi_\lambda = \Xi_{\lambda'}$. Then we have $hc(U)(\lambda) = hc(U)(\lambda')$ for all $U \in U(\mathfrak{g})^K$. By what we saw, this means that $p(\lambda) = p(\lambda')$ for all W-invariant polynomials $p \in Pol_{\mathbb{C}}(\mathfrak{a}_{\mathbb{C}}^*)$. This yields $\lambda' \in W\lambda$.

Conversely, let us show that $\Xi_{w\lambda} = \Xi_\lambda$ for $w \in W$. The differnce, $f := \Xi_{w\lambda} - \Xi_\lambda$, is an analytic function on G, which is K-biinvariant, and such that $(R_U f)(e) = 0$ for all $U \in U(\mathfrak{g})^K$. If we show that $(R_U f)(e) = 0$ for all $U \in U(\mathfrak{g})$, then we are obviously done (because f will have Taylor series 0 at e, hence will be equal to 0 on G°, and since it is K-biinvariant, it will be equal to 0 on G). But, given $U \in U(\mathfrak{g})$ and $k \in K$, we have

$$(R_{Ad(k)U} f)(e) = (R_k R_U R_{k^{-1}} f)(e) = (R_k R_U f)(e) = (R_U f)(k) =$$

$$= (L_k R_U f)(e) = (R_U L_k f)(e) = (R_U f)(e).$$

Hence, if we consider the average

$$U' = \int_K Ad(k)U,$$

we have $U' \in U(\mathfrak{g})^K$ and so $(R_U f)(e) = (R_{U'} f)(e) = 0$. □

For $\lambda \in \mathfrak{a}^*$, let us denote by λ^{max} (resp. λ^{min}) the unique dominant (resp. antidominant) element in $W\lambda$.

Claim 4.6.5. *Let $\lambda \in \mathfrak{a}^*$. Then we have*

$$a^{\lambda^{max}+\rho} \preceq \Xi_\lambda(a) \preceq a^{\lambda^{min}+\rho} s(a)^d \quad (a \in A^-),$$

for some $d \in \mathbb{Z}_{\geq 0}$.

Proof. Let us show the left estimate. Since, by claim 4.6.4, $\Xi_\lambda = \Xi_{\lambda^{max}}$, we can replace λ by λ^{max} and so assume that λ is dominant. We then have, using lemma 4.3.2:

$$\Xi_\lambda(a) = a^{\lambda+\rho} \int_{N^-_{(\emptyset)}} r(a^{-1}xa)^{-\lambda-\rho} r(x)^{\lambda-\rho} \geq a^{\lambda+\rho} \int_{N^-_{(\emptyset)}} r(x)^{-\lambda-\rho} r(x)^{\lambda-\rho} = a^{\lambda+\rho} \int_{N^-_{(\emptyset)}} r(x)^{-2\rho}.$$

Let us now show the right estimate. Again, we can assume that λ itself is antidominant. Let $V \subset \mathcal{P}_\lambda^{[K]}$ be the $(\mathfrak{g}, K)$-submodule generated by ξ_λ. By 3.4.5, in order to establish the estimate on $A^{--,\epsilon}$, it is enough to show that $Re(supp(V)) \geq \lambda$, and for this it is enough to see that for every $\mu \in wt(pres_\emptyset(V))$, we have $Re(\mu) \geq \lambda$. For such μ, we can find an irreducible representation E of $K_\emptyset$, such that E_μ is a quotient of $pres_\emptyset(V)$. Then we obtain a non-zero morphism $V \to pind_\emptyset(E_\mu)$. Examining the (non-zero) image of ξ_λ under this morphism, we see that E is the trivial representation, and furthermore $hc(U)(\lambda) = hc(U)(\mu)$ for every $U \in U(\mathfrak{g})^K$. Since hc is onto $Pol_{\mathbb{C}}(\mathfrak{a}^*)^W$, we obtain that $\mu \in W\lambda$. Since λ is antidominant, we get $Re(\mu) = \mu \geq \lambda$.

Thus, we have established the estimate on $A^{--,\epsilon}$. Extending it to A^- is facilitated by corollary 4.6.2. Indeed, one easily finds compact $\Omega \subset G$ such that $A^- \subset \Omega \cdot A^{--,\epsilon}$ (take Ω to be a big closed ball around e in A), and then deduces the estimate on A^- from that on $A^{--,\epsilon}$. □

4.7 The convergence of an integral - 2

Claim 4.7.1. *Let $\lambda \in \mathfrak{a}^*$ be dominant and regular. Let $d \in \mathbb{Z}_{\geq 0}$. Then*

$$\int_{N^-_{(\emptyset)}} r(x)^{-\rho-\lambda}$$

converges.

Proof. Let us notice that $\lambda - \epsilon\rho$ is dominant for small enough $\epsilon > 0$. Then $r(x)^{\lambda-\epsilon\rho} \geq 1$ for $x \in N^-_{(\emptyset)}$ by part (2) of lemma 4.3.2, i.e. $r(x)^{-\lambda} \leq r(x)^{-\epsilon\rho}$, and thus it is enough to show the convergence of

$$\int_{N^-_{(\emptyset)}} r(x)^{-\rho-\epsilon\rho}.$$

Let us fix $a \in A^{--}$. The right estimate of claim 4.6.5 gives, for $\lambda = 0$:

$$a^{t\rho} \int_{N^-_{(\emptyset)}} r(a^{-t}xa^t)^{-\rho} r(x)^{-\rho} = \Xi_0(a^t) \preceq a^{t\rho} s(a^t)^d \quad (t \in \mathbb{R}_{\geq 0}).$$

Rewriting:

$$\int_{N^-_{(\emptyset)}} r(a^{-t}xa^t)^{-\rho} r(x)^{-\rho} \preceq (1+t)^d \quad (t \in \mathbb{R}_{\geq 0}).$$

We can find a compact subset $\Omega \subset N^-_{(\emptyset)}$ such that $a^t\Omega a^{-t}$ sits in the interior of $a^s\Omega a^{-s}$ for $t < s$, and $\cup_{t\geq 0} a^t\Omega a^{-t} = N^-_{(\emptyset)}$. For $t \geq 1$, let us denote $\Omega_t := a^t\Omega a^{-t}$.

There exists $C_1 > 1$ such that $r(x)^\rho \geq C_1$ for $x \in N^-_{(\emptyset)} - \Omega_0$. Thus, for $x \notin \Omega_n$, since $a^{-n}xa^n \notin \Omega_0$, we have

$$r(a^{-n}xa^n)^\rho \geq C_1.$$

By part (3) of lemma 4.3.2, there exists $0 < c < 1$ such that

$$r(a^{-n}xa^n)^\rho \leq \sqrt[4]{1 + c^n r(x)^{4\rho}}$$

for all $x \in N^-_{(\emptyset)}$. In particular, for $x \notin \Omega_n$, we get

$$r(x)^{-\epsilon\rho} \leq (C_1^4 - 1)^{-\epsilon/4} c^{\epsilon n/4}.$$

There exists $C_2 > 0$ such that $r(x)^\rho \leq C_2$ for $x \in \Omega_1$. Thus, for $x \in \Omega_{n+1}$, since $a^{-n}xa^n \in \Omega_1$, we have

$$r(a^{-n}xa^n)^{-\rho} \geq C_2^{-1}.$$

We get:

$$\int_{\Omega_{n+1}-\Omega_n} r(x)^{-\rho-\epsilon\rho} \preceq c^{\epsilon n/4} \int_{\Omega_{n+1}-\Omega_n} r(x)^{-\rho} r(a^{-n}xa^n)^{-\rho} \leq$$

$$\leq c^{\epsilon n/4} \int_{N^-_{(\emptyset)}} r(x)^{-\rho} r(a^{-n}xa^n)^{-\rho} \preceq c^{\epsilon n/4}(1+n)^d.$$

From this, the convergence of the integral is clear.

□

4.8 A formula of (Harish-Chandra and) Langlands

Claim 4.8.1. *Let E be a HC $(\mathfrak{g}_\emptyset, K_\emptyset)$-module. Suppose that $\mathfrak{a}$ acts on E strictly via a character $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, and suppose that $Re(\lambda)$ is regular and antidominant. Then for every $f \in pind_\emptyset(E), h \in pind_\emptyset(\widetilde{E})$ one has*

$$a^{-\lambda-\rho} \cdot \langle h, af \rangle \xrightarrow{log(a) \to -\infty} \langle \int_{N^-_{(\emptyset)}} h(x), f(e) \rangle.$$

Here $a \in A^{--}$ and $log(a) \to -\infty$ means that $\alpha(log(a)) \to -\infty$ for all $\alpha \in R^s$.

Proof. We have

$$a^{-\lambda-\rho}\langle h, af \rangle = a^{-\lambda-\rho} \int_{N^-_{(\emptyset)}} \langle h(x), f(a^{-1}x) \rangle =$$

$$= \int_{N^-_{(\emptyset)}} \langle h(x), f(a^{-1}xa) \rangle \xrightarrow{log(a) \to -\infty} \int_{N^-_{(\emptyset)}} \langle h(x), f(e) \rangle = \langle \int_{N^-_{(\emptyset)}} h(x), f(e) \rangle.$$

For the last equality, we notice that the integral converges by what we saw above. As for the limit procedure, the convergence is clear pointwise for the integrand, and in order to verify it for the integral, we will invoke the dominated convergence theorem. By the K-finiteness of f and h, it is enough to estimate the following integrand:

$$r(x)^{Re(\lambda)-\rho}r(a^{-1}xa)^{-Re(\lambda)-\rho}.$$

Since $Re(\lambda)$ is antidominant and regular, we can find $1 > \epsilon > 0$ such that $-Re(\lambda) - \epsilon\rho$ is dominant. We then have

$$r(x)^{Re(\lambda)-\rho}r(a^{-1}xa)^{-Re(\lambda)-\rho} =$$

$$= r(x)^{-(1+\epsilon)\rho} \cdot \left(r(a^{-1}xa)^{-(1-\epsilon)\rho}\right) \cdot \left(r(x)^{-(-Re(\lambda)-\epsilon\rho)} \cdot r(a^{-1}xa)^{-Re(\lambda)-\epsilon\rho}\right) \leq$$

$$\leq r(x)^{-(1+\epsilon)\rho}$$

(the middle brackets are ≤ 1 by lemma ... since $a^{-1}xa \in N^-_{(\emptyset)}$ and $(1-\epsilon)\rho$ is dominant; The right brackets are ≤ 1 by lemma ... since $-Re(\lambda) - \epsilon\rho$ is dominant and $a \in A^-, x \in N^-_{(\emptyset)}$). This last expression doesn't depend on a and is integrable by ...

□

Remark 4.8.2. Let $f \in C^\infty_{K,Z}(G)$ and let $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Suppose that

$$a^{-\lambda-\rho}f(a) \xrightarrow{log(a)\to-\infty} c$$

for some non-zero c. Then it is not hard to see that $\lambda \in supp(f)$ and for every $\mu \in supp(f)$, one has $Re(\mu) \geq Re(\lambda)$.

4.9 A more precise relation between $supp(V)$ and $pres_\emptyset(V)$

Claim 4.9.1. *Let E be an irreducible HC $(\mathfrak{g}_\emptyset, K_\emptyset)$-module, on which $\mathfrak{a}$ acts via $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Suppose that $Re(\lambda)$ is antidominant and regular. Then for every non-zero submodule $V \subset pind_\emptyset(E)$, we have $\lambda \in supp(V)^{min}$.*

Proof. Denoting by $V_e \subset E$ the image of V under $h \mapsto h(e)$, we see that V_e is a non-zero $(\mathfrak{g}_\emptyset, K_\emptyset)$-submodule. Since E is irreducible, we get $V_e = E$. Thus, since $J_- : pind_\emptyset(\widetilde{E}) \to \widetilde{E}$ is non-zero, we deduce that there exists $h \in V$ and $f \in pind_\emptyset(\widetilde{E})$ such that $\langle J_-(f), h(e)\rangle \neq 0$. Then, by claim 5.7.1 and remark 4.8.2, we see that $\lambda \in supp(V)^{min}$. □

Lemma 4.9.2. *Let $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, and assume that $Re(\lambda)$ is antidominant and regular. Let V be a HC $(\mathfrak{g}, K)$-module. Then if $\lambda \in wt(pres_\emptyset(V))$, then $\lambda \in supp(V)$.*

Proof. Suppose that $\lambda \in wt(pres_\emptyset(V))$. We can then find an irreducible representation E of $K_\emptyset$, such that the $(\mathfrak{g}_\emptyset, K_\emptyset)$-module E_λ is a quotient module of $pres_\emptyset(V)$. We thus obtain a non-zero morphism $V \to pind_\emptyset(E_\lambda)$. Denoting by V_0 the image of this moprhism, it is enough to see that $\lambda \in supp(V_0)$, and this follows from claim 4.9.1. □

We would like now to tensor with algebraic representations of $\boldsymbol{G}(\mathbb{C})$, in order to generalize the previous lemma to arbitrary λ's.

Lemma 4.9.3. *Let V be a HC $(\mathfrak{g}, K)$-module, and E an irreducible algebraic representation of $\boldsymbol{G}(\mathbb{C})$ with lowest weight $\nu \in \mathfrak{a}^*$. Then $wt(pres_\emptyset(V)) + \nu \subset wt(pres_\emptyset(V \otimes E))$.*

Proof. Let $\lambda \in wt(pres_\emptyset(V))$, and let $v \in V$ be a corresponding weight vector (modulo $\mathfrak{n}V$). Let $e \in E$ be a lowest weight vector, with lowest weight $\nu \in \mathfrak{a}^*$.

Notice that $(H - (\lambda + \nu + \rho)(H))(v \otimes e) \in (\mathfrak{n}V) \otimes e \subset \mathfrak{n}(V \otimes E) + V \otimes \mathfrak{n}E$, so $\lambda + \nu \in pres_\emptyset(\frac{V \otimes E}{V \otimes \mathfrak{n}E})$ (notice that $\frac{V \otimes E}{V \otimes \mathfrak{n}E}$ is a quotient $\mathfrak{p}_\emptyset$-module of $V \otimes E$, so that we can take $pres_\emptyset$ of it). Hence, since $wt(pres_\emptyset(\frac{V \otimes E}{V \otimes \mathfrak{n}E})) \subset wt(pres_\emptyset(V \otimes E))$, we obtain $\lambda + \nu \in wt(pres_\emptyset(V \otimes E))$.

□

Lemma 4.9.4. *Let V be a HC $(\mathfrak{g}, K)$-module, and E an algebraic representation of $\boldsymbol{G}(\mathbb{C})$. Then $supp(V \otimes E) \subset supp(V) + wt(E)$.*

Proof. The matrix coefficients of $V \otimes E$ are spanned my those of the form $m_{v \otimes e, \zeta \otimes \delta}$, where e is a weight vector, and the claim follows easily. □

Claim 4.9.5. *Let V be a HC $(\mathfrak{g}, K)$-module. Then $wt(pres_\emptyset(V)) \subset supp(V)^{cl}$.*

Proof. Fix $\lambda \in wt(pres_\emptyset(V))$. There exists an algebraic irreducible representation of $\boldsymbol{G}(\mathbb{C})$, with lowest weight $\nu \in \mathfrak{a}^*$, such that $Re(\lambda) + \nu$ is regular and antidominant (add reference to something above..). By the first lemma above, we have $\lambda + \nu \in wt(pres_\emptyset(V \otimes E))$. By 4.9.2, we have then $\lambda + \nu \in supp(V \otimes E)$. By the second lemma above, we have now $\lambda + \nu \in supp(V) + wt(E)$. Thus, we obtain

$$\lambda \in supp(V) + wt(E) - \nu \in supp(V)^{cl}.$$

□

Corollary 4.9.6 (Milicic)**.** *Let V be a HC $(\mathfrak{g}, K)$-module. Then $wt(pres_\emptyset(V))^{min} = supp(V)^{min}$.*

Proof. We saw that $supp(V)^{min} \subset wt(pres_\emptyset(V)) \subset supp(V)^{cl}$, from which the claim follows easily. □

4.10 Further finiteness

Lemma 4.10.1. *Let E be an irreducible HC $(\mathfrak{g}_\emptyset, K_\emptyset)$-module, on which $\mathfrak{a}$ acts via $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Suppose that $Re(\lambda)$ is dominant and regular. Then $pind_\emptyset(E)$ is finitely generated.*

Proof. Recall that $pind_\emptyset(E) \to E$ given by

$$f \mapsto J_{w_0}(f) = \int_{N_\emptyset^-} f(x)$$

absolutely converges for every f and is non-zero for some f. Fix f for which $J_{w_0}(f) \neq 0$. We will show that f generates $pind_\emptyset(E)$. Let $V \subset pind_\emptyset(E)$ be the submodule generated by f.

Let $W \subset \widetilde{E}$ be a non-zero sub $(\mathfrak{g}, K)$-module. Let $h \in W$ be non-zero. Then there exists $k \in K$ such that $(kh)(e) = h(k^{-1}) \neq 0$. Thus, the $(\mathfrak{g}_\emptyset, K_\emptyset)$-submodule of $\widetilde{E}$ obtained by considering the values at e of elements in W is non-zero. Since $\widetilde{E}$ is irreducible, this submodule is the whole $\widetilde{E}$. In particular, we can find $h \in W$ such that $\langle h(e), J_{w_0}(f)\rangle \neq 0$. Then claim 5.7.1 shows that $m_{h,f} \neq 0$. This implies that h is not orthogonal to V, and thus W is not orthogonal to V. Since this is true for every non-zero W, one has $V = pind_\emptyset(E)$. □

Lemma 4.10.2. *Let E be an irreducible HC $(\mathfrak{g}_\emptyset, K_\emptyset)$-module. Then $pind_\emptyset(E)$ is finitely generated.*

Proof. Notice that one has a "projection formula" morphism

$$Z \otimes pind_\emptyset(E) \to pind_\emptyset(Z/\mathfrak{n}_\emptyset Z \otimes E)$$

where Z is a $(\mathfrak{g}, K)$-module and E is a $(\mathfrak{g}_\emptyset, K_\emptyset)$-module.

Let $\lambda \in \mathfrak{a}^*_\mathbb{C}$ be the weight by which $\mathfrak{a}$ acts on E. We can find an algebraic representation Z of $\boldsymbol{G}(\mathbb{C})$ such that there exists $\mu \in wt(Z/\mathfrak{n}_{(\emptyset)}Z)$ for which $Re(\lambda) - \mu$ is dominant and regular. Let F be an irreudcible $(\mathfrak{g}_\emptyset, K_\emptyset)$-quotient of $Z/\mathfrak{n}_{(\emptyset)}Z$ on which $\mathfrak{a}$ acts by μ. We obtain a $(\mathfrak{g}, K)$-morphism

$$Z \otimes pind_\emptyset(F^* \otimes E) \to pind_\emptyset(E).$$

This morphism is surjective why??. Hence, since by the previous lemma the module $pind_\emptyset(F^* \otimes E)$ is finitely generated and since tensoring with finite-dimensional modules preserves finite generation, we obtain that $pind_\emptyset(E)$ is finitely generated. □

Lemma 4.10.3. *Let E be an irreducible HC $(\mathfrak{g}_\emptyset, K_\emptyset)$-module. Then $pind_\emptyset(E)$ is of finite length.*

Proof. We know that $pind_\emptyset(E)$ is finitely generated. Moreover, since $\widetilde{pind_\emptyset(E)} = pind_\emptyset(\widetilde{E})$ is finitely generated, it satisfies the ascending chain condition, and hence the module $pind_\emptyset(E)$ satisfies the descending chain condition. This shows that $pind_\emptyset(E)$ has finite length. □

Theorem 4.10.4 (Harish-Chandra)**.** *Let $\chi \in Hom(Z(\mathfrak{g}), \mathbb{C})$. Then there exists finitely many isomorphism classes of irreducible $(\mathfrak{g}, K)$-modules on which $Z(\mathfrak{g})$ acts by χ.*

Proof. There are finitely many isomorphism classes of irreducible $(\mathfrak{g}_\emptyset, K_\emptyset)$-modules E on which $Z(\mathfrak{g}_\emptyset)$ acts by a given character. Recall that $Z(\mathfrak{g})$ acts then on $pind_\emptyset(E)$ by the character obtained from the given one by precomposing with $hc_\emptyset : Z(\mathfrak{g}) \to Z(\mathfrak{g}_\emptyset)$, a finite morphism. since each such $pind_\emptyset(E)$ is of finite length and every irreducible $(\mathfrak{g}, K)$-module is a submodule of such an induction, the theorem follows. □

Corollary 4.10.5. *Let V be a $(\mathfrak{g}, K)$-module which is admissible and $Z(\mathfrak{g})$-finite. Then V has finite length.*

Proof. We can assume without loss of generality that $Z(\mathfrak{g})$ acts on V by a character, say χ. Since by the previous theorem there are finitely many isomoprhism classes of irreducible $(\mathfrak{g}, K)$-modules on which $Z(\mathfrak{g})$ acts by χ, we can find a finite subset $S \subset K^\vee$ such that every irreducible $(\mathfrak{g}, K)$-module on which $Z(\mathfrak{g})$ acts by χ admits some element of S as a K-type. Thus, the functor

$$\mathcal{M}(\mathfrak{g}, K)_{adm,\chi} \to Vect_{\text{f.d.}}, \quad W \mapsto \oplus_{\alpha \in S} W^{[\alpha]}$$

is exact and faithful. This implies the claim. □

Remark 4.10.6. Thus, we now know that all the four finiteness conditions in ... are equivalent.

4.11 Examples for $SL_2(\mathbb{R})$

4.11.1 Embedding discrete series into principal series

Recall the discrete series representation D_n (where $n \in \mathbb{Z}_{>0}$), consisting of holomorphic functions on the upper half plane, on which G acts by:

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix} f)(z) = (-cz + a)^{-n} f(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} z).$$

We have found the K-types: the functions w^m, $m \in \mathbb{Z}_{\geq 0}$ in the $\mathbb{D}$-model (w^m has K-type $n + 2m$). Converting back to the $\mathbb{H}$-model, we obtain the functions:

$$\phi_m(z) = \frac{(z - i)^m}{(z + i)^{n+m}}.$$

Notice that N acts as follows:

$$(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} f)(z) = f(z - x).$$

Thus, it is clear how to produce an N-invariant functional:

$$\ell(f) = \int_{\mathbb{R}} f(x + i) dx.$$

If $n > 1$, we see that ℓ converges absolutely on K-finite vectors.

Let us calculate the A-equivariancy of this functional:

$$\ell(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f) = a^{-n} \int_{\mathbb{R}} f(a^{-2}x + a^{-2}i) = a^{2-n} \int_{\mathbb{R}} f(x + a^{-2}i) = a^{2-n} \int_{\mathbb{R}} f(x + i) = a^{2-n} \ell(f).$$

And the $K_\emptyset$-equivariancy:

$$\ell(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} f) = (-1)^n \ell(f).$$

Thus, we can interpret ℓ as a functional

$$D_n^{[K]}/\mathfrak{n}D_n^{[K]} \to (sgn^n)_{(2-n)\rho},$$

or

$$pres_{\emptyset}(D_n^{[K]}) \to (sgn^n)_{(1-n)\rho}.$$

This gives us an embedding

$$D_n^{[K]} \to \mathcal{P}^{[K]}_{1-n,(-1)^{-n}}.$$

some mistake with indices..

Chapter 5

Intertwining integrals

5.1 Intertwining

Let $\mathcal{U}$ be an admissible G_I-representaiton. Let $f \in pind_I(\mathcal{U})$ and $\zeta \in \mathcal{U}^*$. We will write

$$J_-(f;\zeta) := \int_{N^-_{(I)}} \langle \zeta, f(x) \rangle.$$

5.2 Lemmas about r

Generalizing part 3 of lemma 4.3.1, we have:

Lemma 5.2.1. *Let E be an algebraic representation of $\boldsymbol{G}(\mathbb{C})$, and $0 \neq v \in E^{\mathfrak{a},\lambda}$ satisfying $\mathfrak{n}_{(\emptyset)} v = 0$. Assume additionally that $\mathfrak{n}^-_I v = 0$. Let $a \in A^{--,(I)}$. Then there exists $0 < c < 1$ such that $r(a^{-t} g a^t)^\lambda \leq \sqrt[4]{1 + c^t r(g)^{4\lambda}}$ for all $g \in N^-_{(I)}$ and $t \geq 0$.*

Proof. (notations as in ...) Set $c = max_{\alpha \in R^s - I}\ a^\alpha$ ($0 < c < 1$ because $a \in A^{--,(I)}$). We can write $gv = \sum v_i$ as above, with $v_1 = v$ and $\lambda - \lambda_i \in \mathbb{Z}_{\geq 0} wt(\mathfrak{n}_{(I)}) - \{0\}$ for $i \neq 1$. We obtain:

$$||a^{-t} g a^t||^2 = 1 + \sum_{i \neq 1} a^{t(\lambda - \lambda_i)} ||v_i||^2 \leq 1 + c^t \sum_{i \neq 1} ||v_i||^2 \leq 1 + c^t ||gv||^2.$$

□

Lemma 5.2.2.

1. *Fix $a \in A^{--,(I)}$. Then there exists $0 < c < 1$ such that*

$$r(a^{-t} x a^t)^{\rho_{(I)}} \leq \sqrt[4]{1 + c^t r(x)^{4\rho_{(I)}}}$$

for all $x \in N^-_{(I)}$, and $t \geq 0$.

2. Let $g \in N^-_{(I)}$. If $r(g)^{\rho_{(I)}} = 1$, then $g = e$.

Proof.
(1) We use ... with $E := \bigwedge^{\dim \mathfrak{n}_{(I)}} \mathfrak{g}_{\mathbb{C}}$, and a non-zero vector v in the one-dimensional subspace corresponding to $\mathfrak{n}_{(I)}$. Then $(\mathfrak{n}_{(\emptyset)} + \mathfrak{n}^-_I)v = 0$ and $v \in E^{\mathfrak{a},\rho_{(I)}}$.

(2) We use ..., applied to E, v as in part (1) above. It is easy to calculate that the stabilizer of v in $\mathfrak{g}$ is $\mathfrak{p}_{(I)}$. Thus, the intersection of $N^-_{(I)}$ with the stabilizer of v in G is finite. Since $N^-_{(I)}$ does not contain non-trivial finite subgroups, we deduce that $N^-_{(I)}$ intersects this stabilizer trivially. □

5.3 The convergence of an integral - 3

Claim 5.3.1. *Let $\lambda \in \mathfrak{a}^*$ be dominant and $(R^s - I)$-regular. Let $d \in \mathbb{Z}_{\geq 0}$. Then*

$$\int_{N^-_{(I)}} r(x)^{-\rho-\lambda} s(x)^d$$

converges.

Proof. Let us notice that $\lambda - \epsilon\rho_{(I)}$ is dominant for small enough $\epsilon > 0$. Then $r(x)^{\lambda-\epsilon\rho_{(I)}} \geq 1$ for $x \in N^-_{\emptyset}$ by part (2) of lemma 4.3.2, i.e. $r(x)^{-\lambda} \leq r(x)^{-\epsilon\rho_{(I)}}$, and thus it is enough to show the convergence of

$$\int_{N^-_{(I)}} r(x)^{-\rho-\epsilon\rho_{(I)}} s(x)^d.$$

Let us fix $a \in A^{--,(I)}$, which furthermore satisfies $a^\alpha = 1$ for $\alpha \in I$.

We first show that, up to a scalar (not depending on t), the integrals

$$\int_{N^-_{(\emptyset)}} r(a^{-t}xa^t)^{-\rho} r(x)^{-\rho}, \quad \int_{N^-_{(I)}} r(a^{-t}xa^t)^{-\rho} r(x)^{-\rho}$$

are equal (here $t \geq 0$). Indeed, we can write

$$\int_{N^-_{(\emptyset)}} r(a^{-t}xa^t)^{-\rho} r(x)^{-\rho} = \int_{N^-_{(I)}} \int_{N^-_I} r(a^{-t}xya^t)^{-\rho} r(xy)^{-\rho} =$$

$$= \int_{N^-_{(I)}} \int_{N^-_I} \left[r(a^{-t}\kappa_I(y)^{-1}x\kappa_I(y)a^t)^{-\rho} r(y)^{-\rho}\right] \left[r(\kappa_I(y)^{-1}x\kappa_I(y))^{-\rho} r(y)^{-\rho}\right] =$$

$$= \int_{N^-_{(I)}} \int_{N^-_I} r(a^{-t}xa^t)^{-\rho} r(x)^{-\rho} r(y)^{-2\rho}.$$

In the second passage, we used a lying in the center of G_I, so in particular commuting with $\kappa_I(y)$. In the third passage, we changed variables on $N^-_{(I)}$, replacing x by $\kappa_I(y)^{-1}x\kappa_I(y)$.

From here, we conclude using the estimate 4.6.5, that

$$\int_{N^-_{(I)}} r(a^{-t}xa^t)^{-\rho} r(x)^{-\rho} \preceq (1+t)^d$$

for $t \geq 0$.

We can find a compact subset $\Omega \subset N^-_{(I)}$ such that $a^t \Omega a^{-t}$ sits in the interior of $a^s \Omega a^{-s}$ for $t < s$, and $\cup_{t \geq 0} a^t \Omega a^{-t} = N^-_{(I)}$. For $t \geq 1$, let us denote $\Omega_t := a^t \Omega a^{-t}$.

There exists $C_1 > 1$ such that $r(x)^{\rho_{(I)}} \geq C_1$ for $x \in N^-_{(I)} - \Omega_0$. Thus, for $x \notin \Omega_n$, since $a^{-n}xa^n \notin \Omega_0$, we have

$$r(a^{-n}xa^n)^{\rho_{(I)}} \geq C_1.$$

By ..., there exists $0 < c < 1$ such that

$$r(a^{-n}xa^n)^{\rho_{(I)}} \leq \sqrt[4]{1 + c^n r(x)^{4\rho_{(I)}}}$$

for all $x \in N^-_{(I)}$. In particular, for $x \notin \Omega_n$, we get

$$r(x)^{-\epsilon\rho_{(I)}} \leq (C_1^4 - 1)^{-\epsilon/4} c^{\epsilon n/4}.$$

There exists $C_2 > 0$ such that $r(x)^{\rho} \leq C_2$ for $x \in \Omega_1$. Thus, for $x \in \Omega_{n+1}$, since $a^{-n}xa^n \in \Omega_1$, we have

$$r(a^{-n}xa^n)^{-\rho} \geq C_2^{-1}.$$

We get:

$$\int_{\Omega_{n+1}-\Omega_n} r(x)^{-\rho-\epsilon\rho_{(I)}} s(x)^d \preceq c^{\epsilon n/4} \int_{\Omega_{n+1}-\Omega_n} r(x)^{-\rho} r(a^{-n}xa^n)^{-\rho} \leq$$

$$\leq c^{\epsilon n/4} \int_{N^-_{(I)}} r(x)^{-\rho} r(a^{-n}xa^n)^{-\rho} \preceq c^{\epsilon n/4}(1+n)^d.$$

From this, the convergence of the integral is clear.

□

5.4 The weak inequality; tempered and square-integrable HC $(\mathfrak{g}, K)$-modules

Definition 5.4.1. Let V be a HC $(\mathfrak{g}, K)$-module.

1. Let $\lambda \in \mathfrak{a}^*$. We say that V **satisfies the weak (resp. strong) λ-inequality**, if there exists $d \in \mathbb{Z}_{\geq 0}$ such that (resp. for every $d \in \mathbb{Z}_{<0}$ and) for all $v \in V, \widetilde{v} \in \widetilde{V}$, one has

$$|m_{\widetilde{v},v}(g)| \preceq \Xi_\lambda(g) \cdot s(g)^d \quad (g \in G).$$

2. We say that V is **tempered** (resp. **square integrable**), if it satisfies the weak (resp. strong) 0-inequality.

Lemma 5.4.2. *Let V be a HC $(\mathfrak{g}, K)$-module. Then V is tempered (resp. square integrable) if and only if for every $\lambda \in supp(V)$ one has $Re(\lambda) \geq 0$ (resp. $Re(\lambda) > 0$).*

Proof. By ..., one has $Re(\lambda) \geq 0$ for all $\lambda \in supp(V)$ if and only if there exists $d \in \mathbb{Z}_{\geq 0}$ such that

$$|m_{\widetilde{v},v}(a)| \preceq a^{\rho} s(a)^d \quad (a \in A^-).$$

Since $G = KA^-K$ and since (enlarging d if needed)

$$a^{\rho} \preceq \Xi_0(a) \preceq a^{\rho} s(a)^d \quad (a \in A^-),$$

one easily sees that this is equivalent to

$$|m_{\widetilde{v},v}(g)| \preceq \Xi_{\lambda}(g) \cdot s(g)^d \quad (g \in G).$$

As for square integrability, complete.. □

5.5 Parabolic induction and the weak inequality

Claim 5.5.1. *Let U be a tempered HC $(\mathfrak{g}_I, K_I)$-module and let $\chi \in G_I^{\vee,unr}$. Then $pind_I(U_{\chi})$ satisfies the weak $(-d\chi)$-inequality.*

Proof. There exists $d \in \mathbb{Z}_{\geq 0}$ such that

$$|m_{\widetilde{u},u}(g)| \preceq |\chi(g)| \cdot \Xi_I(g) \cdot s(g)^d \quad (g \in G_I)$$

for all $u \in U_{\chi}, \widetilde{u} \in \widetilde{U}_{\chi}$.

Let $f \in pind_I(U_{\chi}), h \in pind_I(\widetilde{U}_{\chi^{-1}})$.

Using the K-finiteness of f, h, we easily reduce the estimation of

$$\langle h, gf \rangle = \int_K |\langle h(k), f(g^{-1}k) \rangle|$$

to the estimation of

$$\int_K |\langle \zeta, f(\ell_I(g^{-1}k)) \rangle|$$

for $f \in pind_I(U_{\chi}), \zeta \in \widetilde{U}_{\chi}^{-1}$.

We have:

$$\int_K |\langle \zeta, f(\ell_I(g^{-1}k)) \rangle| = \int_K \Delta_{(I)}^{-1/2}(\ell_I(g^{-1}k)) \cdot |m_{\zeta, f(e)}((\ell_I(g^{-1}k)^{-1})| \preceq$$

$$\preceq \int_K \Delta_{(I)}^{-1/2}(\ell_I(g^{-1}k)) \cdot |\chi^{-1}(\ell_I(g^{-1}k))| \cdot \Xi_I(\ell_I(g^{-1}k)) \cdot s(\ell_I(g^{-1}k))^d =$$

$$\preceq \int_{K\times K_I} \Delta_{(I)}^{-1/2}(\ell_I(g^{-1}k))\cdot|\chi^{-1}(\ell_I(g^{-1}k))|\cdot r(\ell_I(g^{-1}k)k_I)^{-\rho_I}s(g)^d =$$

$$= \int_{K\times K_I} \Delta_{(I)}^{-1/2}(\ell_I(g^{-1}kk_I))\cdot|\chi^{-1}(\ell_I(g^{-1}kk_I))|\cdot r(g^{-1}kk_I)^{-\rho_I}s(g)^d =$$

$$= \int_K r(g^{-1}k)^{-\rho-d\chi}s(g)^d = \Xi_{-d\chi}(g)s(g)^d.$$

□

Corollary 5.5.2. *Let U be a tempered HC $(\mathfrak{g}_I, K_I)$-module. Then $pind_I(U)$ is a tempered HC $(\mathfrak{g}, K)$-module.*

5.6 Convergence and non-vanishing of $J_-(f;\zeta)$

Claim 5.6.1. *Let U be a tempered HC $(\mathfrak{g}_I, K_I)$-module and let $\chi \in G_I^{\vee,unr}$ be (I)-positive. Then for every $f \in pind_I(U_\chi)$ and $\zeta \in \widetilde{U}_{\chi^{-1}}$, the integral $J_-(f;\zeta)$ converges absolutely. Moreover, for every non-zero ζ there exists f such that $J_-(f;\zeta) \neq 0$.*

Proof. Throughout the discussion, we fix $\zeta \in \widetilde{U}_{\chi^{-1}}$.

By ..., there exists $d \in \mathbb{Z}_{\geq 0}$ such that for every $u \in U_\chi$ we have

$$|\langle\zeta, gu\rangle| \preceq |\chi(g)|\cdot\Xi_I(g)\cdot s(g)^d, \quad g \in G_I.$$

For the sake of showing the non-vanishing claim, we will need the following elaboration. Let us define, for $f \in pind_I(\mathcal{U}_\chi)$:

$$\sigma(f) := sup_{k\in K, g\in G_I}\frac{|\langle\zeta, gf(k)\rangle|}{|\chi(g)|\cdot\Xi_I(g)\cdot s(g)^d}.$$

It is easy to see that if f is K-finite, then $\sigma(f) < \infty$ (see fact 1 below for a more precise claim). We will show that $J_-(f;\zeta)$ converges absolutely whenever $\sigma(f) < \infty$, and more precisely

$$\int_{N_{(I)}^-} |\langle\zeta, f(x)\rangle| \preceq \sigma(f) \quad (f \in pind_I(\mathcal{U}_\chi)).$$

We have:

$$\int_{N_{(I)}^-} |\langle\zeta, f(x)\rangle| = \int_{N_{(I)}^-} |\langle\zeta, \Delta_{(I)}^{-1/2}(\ell_I(x))\ell_I(x)^{-1}f(\kappa(x))\rangle| \leq$$

$$\leq \int_{N_{(I)}^-} \Delta_{(I)}^{-1/2}(\ell_I(x))\cdot\sigma(f)\cdot|\chi(\ell_I(x)^{-1})|\cdot\Xi_I(\ell_I(x)^{-1})\cdot s(\ell_I(x)^{-1})^d \leq$$

$$\leq \sigma(f) \int_{N^-_{(I)} \times K_I} \Delta_{(I)}^{-1/2}(\ell_I(x)) \cdot |\chi(\ell_I(x)^{-1})| \cdot r(\ell_I(x)k_I)^{-\rho_I} s(x)^d =$$

$$= \sigma(f) \int_{N^-_{(I)} \times K_I} \Delta_{(I)}^{-1/2}(\ell_I(x)) \cdot |\chi(\ell_I(x)^{-1})| \cdot r(xk_I)^{-\rho_I} s(x)^d =$$

(because the first, second and fourth multipliers are invariant under K_I-conjugation)

$$= \sigma(f) \int_{N^-_{(I)}} \Delta_{(I)}^{-1/2}(\ell_I(x)) \cdot |\chi(\ell_I(x)^{-1})| \cdot r(x)^{-\rho_I} s(x)^d = \sigma(f) \int_{N^-_{(I)}} r(x)^{-\rho - d|\chi|} s(x)^d,$$

and the last integral converges by claim 5.3.1.

We now would like to establish the non-vanishing claim. We have the following facts:

1. If there exists a subset $S \subset G$ such that $SP_{(I)} = G$ and $f(S)$ is contained in U and spans a finite-dimensional subspace, then $\sigma(f) < \infty$.
2. Let μ be a compactly supported measure on G. Then
$$\sigma(\pi(\mu)f) \leq C_{supp(\mu)} \cdot ||\mu|| \cdot \sigma(f),$$
where $C_{supp(\mu)}$ is a constant depending only on $supp(\mu)$.
3. If $supp(f|_{N^-_{(I)}})$ is compact in $N^-_{(I)}$ and $\sigma(f) < \infty$, then for an approximation of identity $\phi_n \to \delta_e$ on K, one has $J_-(\pi(\phi_n)f; \zeta) \to J_-(f; \zeta)$.

For fact 1, denoting $E = Sp\ f(S) \subset U$ (so, by the assumption, it is a finite-dimensional subspace), we notice that $f(Cl(S)) \subset Clf(S) \subset E$, so we can assume that S is closed. Then, we can find a compact subset $S' \subset S$ such that still $S'P_{(I)} = G$. Hence, we can assume that S is compact.

Notice that for every $k \in K$, we can write $k \in sP_{(I)}$ for some $s \in S$, and so we have $\kappa(s)^{-1}k \in P_{(I)}$, thus $\kappa(s)^{-1}k \in K_I$. In other words, we see that $K = \kappa(S)K_I$.

By projecting $S \subset G$ along the decomposition $G \cong K \times A \times N_I \times N_{(I)}$, we see that we can find a compact subset $\Omega \subset G_I$ such that $s \in \kappa(s)\Omega N_{(I)}$ for every $s \in S$. Thus, $\kappa(S) \subset S\Omega^{-1}N_{(I)}$.

Summing up the above, we have $K = \kappa(S)K_I \subset S\Omega^{-1}K_I N_{(I)}$. Let us rename $\Omega := \Omega^{-1}K_I$, so that Ω is a compact subset of G_I, and $K \subset S\Omega N_{(I)}$. We have, for $k \in K$ and $m \in G_I$:

$$|\langle mf(k), \zeta\rangle| = |\langle mf(s\omega n_{(I)}), \zeta\rangle| = \Delta_{(I)}^{-1/2}(\omega)|\langle m\omega^{-1}f(s), \zeta\rangle|.$$

The factor $\Delta_{(I)}^{-1/2}(\omega)$ is bounded, and writing $f(s) = \sum f_i(s)e_i$ where (e_i) is a basis for E, the $f_i(s)$ are also bounded. Thus the expression is majorized by

$$\sum_i |\langle m\omega^{-1}e_i, \zeta\rangle|.$$

Since the summands are finitely many, and each one is majorized by $\Xi_{I,\lambda}(m\omega^{-1})s(m\omega^{-1})^d$, our expression is majorized by

$$\Xi_{I,\lambda}(m\omega^{-1})s(m\omega^{-1})^d.$$

By lemma ..., $\Xi_{I,\lambda}(m\omega^{-1})$ is majorized by $\Xi_{I,\lambda}(m)$, and since $s(m\omega^{-1})^d \leq (s(m)+s(\omega^{-1}))^d$, the expression $s(m\omega^{-1})^d$ is majorized by $s(m)^d$. This shows that $\sigma(f)<\infty$.

For fact 2, let us estimate

$$|\langle m(\pi(\mu)f)(k),\zeta\rangle| = |\int_G \langle mf(g^{-1}k),\zeta\rangle d\mu|.$$

The integrand is equal to

$$\Delta_{(I)}^{-1/2}(\ell_I(g^{-1}k))|\langle m\ell_I(g^{-1}k)^{-1}f(\kappa(g^{-1}k)),\zeta\rangle|.$$

The factor $\Delta_{(I)}^{-1/2}(\ell_I(g^{-1}k))$ is bounded (only depending on $supp(\mu)$), while the second factor is

$$\leq \sigma(f)\cdot\Xi_{I,\lambda}(m\ell_I(g^{-1}k)^{-1})s(m\ell_I(g^{-1}k)^{-1})^d.$$

Since $\ell_I(g^{-1}k)^{-1}$ lies in a compact depending only on $supp(\mu)$, by ... we obtain that our integrand is majorized by

$$\sigma(f)\cdot\Xi_{I,\lambda}(m)s(m)^d,$$

with a constant depending only on $supp(\mu)$.

For fact 3, denote $S := supp(f|_{N^-_{(I)}})$. Notice that $supp(f)=SP_{(I)}$. Fix a precompact open $V\subset N^-_{(I)}$ containing S. Then there exists an open $U\subset K$ such that $US\subset VP_{(I)}$. Then, if $f(k^{-1}x)\neq 0$ (for $k\in U, x\in N^-_{(I)}$), we have $k^{-1}x\in SP_{(I)}$ and so $x\in kSP_{(I)}\subset VP_{(I)}$, hence $x\in V$. Thus, whenever $supp(\phi_n)\subset U$, we have

$$J_-(\pi(\phi_n)f-f;\zeta)=\int_{N^-_{(I)}}\int_K \phi_n(k)\langle f(k^{-1}x)-f(x),\zeta\rangle =$$

$$=\int_V\int_{supp(\phi_n)}\phi_n(k)\langle f(k^{-1}x)-f(x),\zeta\rangle.$$

This latter integral clearly tends to 0 as $n\to\infty$.

Let us finally establish the non-vanishing claim. Assume that $\zeta\neq 0$. We can choose $u\in U$ such that $\zeta(u)\neq 0$, and a function $f_0\in C_c(N^-_{(I)})$ for which $\int_{N^-_{(I)}} f_0(x)=1$. Then there is a unique $f\in pind_I(\mathcal{U})$ such that $f(x)=f_0(x)u$ for $x\in N^-_{(I)}$ and $f(g)=0$ for $g\notin N^-_{(I)}P_{(I)}$. By fact 1 above, $\sigma(f)<\infty$ (taking

$S = (G - N^-_{(I)} P_{(I)}) \cup supp(f_0))$. Notice that $J_-(f;\zeta) = \langle u, \zeta\rangle \neq 0$. By fact 3 above, we can find $\phi \in C(K)$ such that $J_-(\pi(\phi)f;\zeta) \neq 0$. By fact 1 above, for $\chi \in C(K)$:

$$|J_-(\pi(\chi)\pi(\phi)f;\zeta) - J_-(\pi(\phi)f;\zeta)| \preceq ||(\chi * \phi - \phi|| \cdot \sigma(f).$$

Notice now that by taking a χ to be a K-finite supremum-approximation of an approximation of identity, we can make $||(\chi * \phi - \phi||$ to be as small as desired. Hence, we can find such χ for which $J_-(\pi(\chi)\pi(\phi)f;\zeta) \neq 0$; Since $\pi(\chi)\pi(\phi)f$ is K-finite, we are finally done. □

5.7 A formula of Langlands

Claim 5.7.1. *Let U be a tempered HC $(\mathfrak{g}_I, K_I)$-module on which $\mathfrak{a}_{cent,I}$ acts by a weight $\mu \in \mathfrak{a}^*_{\mathbb{C}}$ strictly, and let $\chi \in G_I^{\vee,unr}$ be (I)-negative. Then for every $f \in pind_I(U_\chi)$ and $h \in pind_I(\widetilde{U}_{\chi^{-1}})$ one has*

$$a^{-(\mu + d\chi + \rho_{(I)})} \cdot \langle h, af\rangle \xrightarrow{log(a)\to -\infty} J_-(h; f(e)).$$

Here, $a \in A_{cent,I}$ and $log(a) \to -\infty$ means that $\alpha(log(a)) \to -\infty$ for all $\alpha \in R^s - I$.

Proof. We have

$$a^{-(\mu+d\chi+\rho_{(I)})}\langle h, af\rangle = a^{-(\mu+d\chi+\rho_{(I)})} \int_{N^-_{(I)}} \langle h(x), f(a^{-1}x)\rangle =$$

$$= \int_{N^-_{(I)}} \langle h(x), f(a^{-1}xa)\rangle \xrightarrow{log(a)\to -\infty} \int_{N^-_{(I)}} \langle h(x), f(e)\rangle = J_-(h; f(e)).$$

Here, the convergence is clear pointwise for the integrand, and in order to verify it for the integral, we will invoke the dominated convergence theorem. By the K-finiteness of f and h, it is enough to estimate the following integrand:

$$\Delta_{(I)}^{-1/2}(\ell_I(x))\Delta_{(I)}^{-1/2}(\ell_I(a^{-1}xa)) \cdot |m_{h(e),f(e)}(\ell_I(x)\ell_I(a^{-1}xa)^{-1})| \preceq$$

$$\preceq \Delta_{(I)}^{-1/2}(\ell_I(x))\Delta_{(I)}^{-1/2}(\ell_I(a^{-1}xa)) \cdot |\chi|^{-1}(\ell_I(a^{-1}xa)\ell_I(x)^{-1}) \cdot \Xi_I(\ell_I(x)\ell_I(a^{-1}xa)^{-1}) \cdot s(\ell_I(x)\ell_I(a^{-1}xa)^{-1})^d \preceq$$

(we use ... to rewrite Ξ_I via an integral over K_I; We also estimate $s(\ell_I(a^{-1}xa)\ell_I(x)) \leq s(axa^{-1}) + s(x)$, and then use $s(x) \sim 1 + log(1 + ||log(x)||)$ to see that $s(a^{-1}xa) \preceq s(x)$)

$$\preceq (\Delta_{(I)}^{-1/2}|\chi|)(\ell_I(x))(\Delta_{(I)}^{-1/2}|\chi|^{-1})(\ell_I(a^{-1}xa) \cdot \int_{K_I} r(\ell_I(a^{-1}xa)k_I)^{-\rho_I} r(\ell_I(x)k_I)^{-\rho_I} s(x)^d =$$

(since K_I normalizes $N_{(I)}$, we have $r(\ell_I(x)k_I) = r(xk_I) = r(k_I^{-1}xk_I)$, and similarly, since a commutes with the elements of K_I, we have $r(\ell_I(a^{-1}xa)k_I) = r(a^{-1}k_I^{-1}xk_Ia)$, which, since K_I normalizes $N_{(I)}^-$ (and we are dealing with an integrand over $N_{(I)}^-$), allows us to eliminate the integration over K_I)

$$\overset{\text{same } N_{(I)}^- \text{ integral}}{\equiv} r(x)^{d|\chi|-\rho_{(I)}} r(a^{-1}xa)^{-d|\chi|-\rho_{(I)}} \cdot r(a^{-1}xa)^{-\rho_I} r(x)^{-\rho_I} s(x)^d =$$

$$= r(a^{-1}xa)^{-\rho-d|\chi|} r(x)^{-\rho+d|\chi|} s(x)^d.$$

Now, fix $\epsilon > 0$ so small so that $-d|\chi| - \epsilon\rho_{(I)}$ and $\rho - \epsilon\rho_{(I)}$ are both dominant. Then by lemma 4.3.2 (parts 1 and 2), we obtain

$$r(a^{-1}xa)^{-\rho-d|\chi|} r(x)^{-\rho+d|\chi|} =$$

$$= r(a^{-1}xa)^{-\rho+\epsilon\rho_{(I)}} \cdot \left(r(a^{-1}xa)^{-\epsilon\rho_{(I)}-d|\chi|} r(x)^{\epsilon\rho_{(I)}+d|\chi|} \right) \cdot r(x)^{-\rho-\epsilon\rho_{(I)}} \leq r(x)^{-\rho-\epsilon\rho_{(I)}}.$$

We obtain that our integrand is further estimated:

$$\leq r(x)^{-\rho-\epsilon\rho_{(I)}} s(x)^d,$$

which does not depend on a, and whose integral over $N_{(I)}^-$ converges by claim 5.3.1. □

Chapter 6

The Langlands classification

6.1 The results

The results of this chapter are:

Claim 6.1.1. *Let U be an irreducible tempered HC $(\mathfrak{g}_I, K_I)$-module. Let $\chi \in G_I^{\vee,unr}$ be (I)-positive. Then $pind_I(U_\chi)$ admits a unique irreducible quotient.*

Definition 6.1.2.

1. By a **Langlands datum** (I, U, χ) we mean a subset $I \subset R^s$, (an isomorphism class of) an irreducible tempered HC $(\mathfrak{g}_I, K_I)$-module U, and an (I)-positive $\chi \in G_I^{\vee,unr}$. We denote by $LD(G)$ the set of Langlands data.

2. By the **Langlands quotient** associated to the Langlands datum (I, U, χ) we mean (the isomorphism class of) the unique irreducible quotient of $pind_I(U_\chi)$. Thus, we have a map $q : LD(G) \to Irr(G)$.

Claim 6.1.3 (Langlands classification)**.** *The map*

$$q : LD(G) \to Irr(G)$$

is a bijection.

Claim 6.1.4. *Let V be an irreducible tempered HC $(\mathfrak{g}, K)$-module. Then there exists $I \subset R^s$ and an irreducible square integrable HC $(\mathfrak{g}_I, K_I)$-module U, such that V is isomorphic to a submodule of $pind_I(U)$.*

Corollary 6.1.5. *Let V be an irreducible tempered HC $(\mathfrak{g}, K)$-module. Then V is unitarizable.*

Corollary 6.1.6. *Let V be an irreducible tempered HC $(\mathfrak{g}, K)$-module. Then $\widetilde{V}$ is also tempered.*

Corollary 6.1.7. *Let V be an irreducible HC $(\mathfrak{g}, K)$-module. Then there exists $I \subset R^s$ and an irreducible square integrable HC $(\mathfrak{g}_I, K_I)$-module U, such that V is isomorphic to a submodule of $pind_I(U)$.*

6.2 Proofs

6.2.1 Proof of uniqueness of irreducible quotient

Proof (of claim 6.1.1). By ..., we have a well-defined

$$J_-(f;\zeta) : pind_I(U_\chi) \otimes \widetilde{U}_{\chi^{-1}} \to \mathbb{C}.$$

Under $J_-(f) = 0$ we will understand $J_-(f;\zeta) = 0$ for all ζ (so we have $Ker(J_-) \subset pind_I(U_\chi)$). By ..., $Ker(J_-) \neq pind_I(U_\chi)$.

Let us denote by $Ker(J_-)'$ the subspace consisting of v for which the $(\mathfrak{g}, K)$-submodule generated by v is contained in $Ker(J_-)$. Then a $(\mathfrak{g}, K)$-submodule of $pind_I(U_\chi)$ is contained in $Ker(J_-)$ if and only if it is contained in $Ker(J_-)'$.

It is enough to show that every proper submodule of $pind_I(U_\chi)$ is contained in $Ker(J_-)'$ or, equivalently in $Ker(J_-)$. Equivalently, for $f \in pind_I(U_\chi)$ such that $f \notin Ker(J_-)$, we want to show that the submodule $V \subset pind_I(U_\chi)$ generated by f is equal to the whole $pind_I(U_\chi)$.

Denoting by $V^\perp \subset pind_I(\widetilde{U}_{\chi^{-1}})$ the orthogonal complement, we need to show that $V^\perp = 0$.

Since $V^\perp$ is a $(\mathfrak{g}, K)$-submodule, we easily verify that $V_e^\perp := \{h(e) : h \in V^\perp\}$ is a $(\mathfrak{g}_I, K_I)$-submodule in $\widetilde{U}_{\chi^{-1}}$. If $V^\perp \neq 0$, then $V_e^\perp \neq 0$, because for $h \in V^\perp$, $h(k) = (k^{-1}h)(e)$, and if $h|_K = 0$ then $h = 0$. Since U is irreducible, we see that if $V^\perp \neq 0$, then $V_e^\perp = \widetilde{U}_{\chi^{-1}}$, which implies (since $f \notin Ker(J_-)$) that there exists $h \in V^\perp$ for which $J_-(f; h(e)) \neq 0$. But by ..., $J^-(f; h(e))$ is the limit of values in $\mathbb{C}^\times \cdot \langle a^{-1}f, h\rangle$, which are zero since h is orthogonal to V, and hence to the closure of V (in which $a^{-1}f$ is contained). This contradiction shows that $V^\perp = 0$. □

6.2.2 Proof of injectivity of q

Proof (of injectivity in claim 6.1.3). Suppose that V is a quotient both of $pind_I(U_\chi)$ and of $pind_{I'}(U'_\chi)$.

By ..., $pind_{I'}(\widetilde{U}'_{\chi'})$ satisfies the weak $(-d\chi')$-inequality, and hence so does $\widetilde{V}$. By ..., since $(-d\chi')$ is antidominant, we have

$$|m_{v,\widetilde{v}}(a^{-1})| \preceq a^{-d\chi'+\rho} s(a)^d, \quad a \in A^-,$$

for $v \in V, \widetilde{v} \in \widetilde{V}$.

On the other hand, by ..., we can find $v \in V, \widetilde{v} \in \widetilde{V}$ such that

$$|m_{v,\widetilde{v}}(a^{-1})| \succeq a^{-d\chi+\rho_{(I)}}, \quad log(a) \in \mathfrak{a}_{cent,I}^{(I)\text{-remote}},$$

where

$$\mathfrak{a}_{cent,I}^{(I)\text{-remote}} = \{H \in \mathfrak{a}_{cent,I} \mid \alpha(H) \leq -R \ \forall \alpha \in R^s - I\}$$

for some fixed big-enough R.

Combining these two estimates, we obtain

$$a^{d\chi'-d\chi} \preceq s(a)^d, \quad log(a) \in \mathfrak{a}_{cent,I}^{(I)\text{-remote}}.$$

This forces $d\chi' - d\chi$ to be non-positive on $\mathfrak{a}_{cent,I}^{(I)\text{-remote}}$, and hence on $\mathfrak{a}_{cent,I}^{(I)\text{-negative}}$, where

$$\mathfrak{a}_{cent,I}^{(I)\text{-negative}} = \{H \in \mathfrak{a}_{cent,I} \mid \alpha(H) \leq 0 \ \forall \alpha \in R^s - I\}.$$

Notice now that $H_\alpha \in \mathfrak{a}_{cent,I}^{(I)\text{-negative}}$ for $\alpha \in I$. Thus, we get $d\chi'(H_\alpha) \leq d\chi(H_\alpha) = 0$ for $\alpha \in I$. This forces $\alpha \in I'$ for $\alpha \in I$, i.e. $I' \subset I$. By symmetry, we get $I = I'$.

Now, once we know $I = I'$, we obtain by the above that both $d\chi' \leq d\chi$ and $d\chi \leq d\chi'$ on $\mathfrak{a}_{cent,I}^{(I)\text{-negative}}$. Thus $d\chi' = d\chi$ on $\mathfrak{a}_{cent,I}^{(I)\text{-negative}}$. This implies $d\chi' = d\chi$ on $\mathfrak{a}_{cent,I}$, and this implies $\chi' = \chi$.

It is left now to see that U is isomorphic to U'.

complete

□

6.2.3 Langlands geometric lemmas

To prove the surjectivity of q, we will need the following material. Our reference is [5].

Let E be a finite-dimensional Euclidean vector space over $\mathbb{R}$. Let $(v_i)_{i\in\Sigma}$ be a basis for E, and $(w_i)_{i\in\Sigma}$ the dual basis. We have the positive cone

$$E^{pos} := \sum \mathbb{R}_{\geq 0} v_i$$

and the dominant cone

$$E^{dom} := \sum \mathbb{R}_{\geq 0} w_i.$$

Define the Langlands retraction

$$L : E \to E^{dom}$$

by sending v to the closest vector to it in E^{dom}. This is well-defined since E^{dom} is non-empty, closed and convex. It is also not hard to see that L is continuous.

For $v \in E$, define $I_v \subset \Sigma$ by

$$I_v = \{i \in \Sigma \mid (v_i, v) = 0\}.$$

Lemma 6.2.1. *Let $v \in E$. Then $L(v)$ can be characterized as the unique element $w \in V^{dom}$ satisfying*

$$v - w \in -\sum_{i \in I_w} \mathbb{R}_{\geq 0} v_i.$$

Proof. Let us denote $I = I_w$ for brevity. Let us write

$$v = \sum_{i \in I} c_i v_i + \sum_{i \notin I} c_i w_i.$$

Notice that $w \in \sum_{i \notin I} \mathbb{R}_{>0} w_i$.

Notice that $w = L(v)$ if and only if for every $w' \in E$ for which $w + \epsilon w' \in E^{dom}$ for small ϵ, one has

$$dist(v, w + \epsilon w')^2 \geq dist(v, w)^2$$

for small ϵ. One has:

$$dist(v, w + \epsilon w')^2 = dist(v, w)^2 + \epsilon^2 ||w'||^2 - 2\epsilon(v - w, w'),$$

so the condition for $w = L(v)$ is

$$(v - w, w') \leq 0$$

for all the above w'. It is enough to check for $w' \in \sum_{i \notin I} \mathbb{R} w_i$ and $w' = w_i$ for $i \in I$. The condition for the first yields

$$(\sum_{i \notin I} c_i w_i - w, w') \leq 0$$

for all $w' \in \sum_{i \notin I} \mathbb{R} w_i$, hence $w = \sum_{i \notin I} c_i w_i$ (and the condition is equivalent to this). The condition for the second, $w' = w_i$ for some $i \in I$, gives then $c_i \leq 0$.

To conclude, we see that $w = L(v)$ if and only if $w = \sum_{i \notin I} c_i w_i$ and $c_i \leq 0$ for $i \in I$. This makes the lemma clear.

□

For $I \subset \Sigma$, let us define

$$C_I := -\sum_{i \in I} \mathbb{R}_{\geq 0} v_i + \sum_{i \notin I} \mathbb{R}_{\geq 0} w_i.$$

Also, denote by $P_I \in End(E)$ the orthogonal projection with kernel $\sum_{i \in I} \mathbb{R} v_i$.

Lemma 6.2.2.

1. *One has* $L|_{C_I} = P_I|_{C_I}$.
2. *The* C_I*'s cover* E *(as we run over* $I \subset \Sigma$*).*

Proof. Both claims are clear by the previous lemma. □

From now on, suppose that $(v_i, v_j) \leq 0$ for $i \neq j$. Define the partial order $v \leq w$ if $w - v \in E^{pos}$.

Lemma 6.2.3. *L is order preserving, i.e. $v \leq w$ implies $L(v) \leq L(w)$.*

Proof. First, notice that it is enough to assume that $v, w \in C_I$ for some $I \subset \Sigma$ (indeed, we see this by walking along the line segment between v and w). On C_I, L is given by P_I. Hence, it is enough to show that P_I is order preserving. In other words, we want to show that $P_I(E^{pos}) \subset E^{pos}$. For that, it is enough to show that $P_I(v_i) \in E^{pos}$ for every $i \in \Sigma$. If $i \in I$, $P_I(v_i) = 0$ and the claim is clear. If $i \notin I$, then since $(v_i - P_I(v_i), v_j) = (v_i, v_j)$ for all $j \in I$, we have $(v_i - P_I(v_i), v_j) \leq 0$ for all $j \in I$. Thus, since $v_i - P_I(v_i) \in \sum_{j \in I} \mathbb{R} v_j$, we have $v_i - P_I(v_i) \in \sum_{j \in I} \mathbb{R}_{\leq 0} v_j$. Hence $P_I(v_i) \in v_i + \sum_{j \in I} \mathbb{R}_{\geq 0} v_j \subset E^{pos}$. □

Lemma 6.2.4. *Let $v \in E$. Then $L(v)$ can be characterized as the unique $\leq$-minimal element of the set*

$$\{w \in E^{dom} \mid w \geq v\}.$$

Proof. By lemma ..., we clearly have $v \leq L(v)$. Furthermore, if $v \leq w$ and $w \in E^{dom}$, then $L(v) \leq L(w) = w$ by the previous lemma. □

Lemma 6.2.5. *Let $S \subset E$ be a finite subset. Let $v \in S$ be such that $L(v)$ is a $\leq$-maximal element in $L(S)$. Then for every $v' \in S$ satisfying $v' - v \in \sum_{i \in I_{L(v)}} \mathbb{R} v_i$, we have $L(v') = L(v)$.*

Proof. Let us denote by L' the retraction L for $\sum_{i \in I_{L(v)}} \mathbb{R} v_i$ with its basis v_i, and write $M' = Id - L'$. Then

$$v' = (v' - L(v)) + L(v) \geq M'(v' - L(v)) + L(v)$$

and thus, applying L, we obtain

$$L(v') \geq L(M'(v' - L(v)) + L(v)) = L(v).$$

From the property of v, we obtain $L(v') = L(v)$. □

6.2.4 Proof of surjectivity of q

Proof (of surjectivity in claim 6.1.3). Let V be an irreducible HC $(\mathfrak{g}, K)$-module. We want to show that V can be embedded into $pind_I(U_\chi)$ where U is tempered and χ is (I)-negative (then $\widetilde{V} = q(I, \widetilde{U}, \chi^{-1})$ and substituting $\widetilde{V}$ for V gives the desired).

This is equivalent to finding $I \subset R^s$, a quotient U of $pres_I(V)$, and $\chi \in G_I^{\vee, unr}$ such that χ is (I)-negative and $U_{\chi^{-1}}$ is tempered.

For $\mu \in wt(pres_\emptyset(V))$ and $I \subset R^s$, we consider $\chi \in G_I^{\vee, unr}$ defined by $d\chi|_{\mathfrak{a}_{cent,I}} = \mu|_{\mathfrak{a}_{cent,I}}$, and consider the summand U of $pres_I(V)$ on which $\mathfrak{a}_{cent,I}$ acts via generalized character $d\chi$. We will find μ and I such that χ is (I)-negative and $U_{\chi^{-1}}$ is tempered.

We use $E := \mathfrak{a}_{ss}^*$ with its Euclidean form B. We use the basis $v_\alpha := \alpha$ of E, and the dual basis w_α. Let us denote by $S \subset E$ the set of $-Re(\mu_{ss})$ with $\mu \in wt(pres_\emptyset(V))$. Let us choose $\mu \in wt(pres_\emptyset(V))$ such that for the corresponding $v := -Re(\mu_{ss}) \in S$, $L(v)$ is a maximal element in $L(S)$, and set $I := I_{L(v)}$.

Let us denote $w := -Re(d\chi_{ss}) \in E$. Notice that $(w_\alpha, w) = (w_\alpha, v)$ for $\alpha \notin I$, and also $(v_\alpha, w) = 0$ for $\alpha \in I$. This forces $w = L(v)$. In particular, $(v_\alpha, w) > 0$ for $\alpha \notin I$. This translates to $Re(d\chi)(H_\alpha) < 0$ for $\alpha \notin I$, i.e. $Re(d\chi)$ is (I)-negative.

Furthermore, for $v' \in S$ such that $(w_\alpha, v') = (w_\alpha, v)$ when $\alpha \notin I$, by ... we obtain $L(v') = L(v) = w$, showing that $v' \leq L(v') = w$. This translates to: For $\mu' \in wt(pres_\emptyset(V))$ for which $\mu'|_{\mathfrak{a}_{cent,I}} = \mu|_{\mathfrak{a}_{cent,I}}$, we have $Re(\mu' - d\chi)(\Omega_\alpha) \geq 0$ for $\alpha \in R^s$. In other words, we obtain $Re(\mu' - d\chi) \geq_I 0$.

□

6.3 Example: $SL_2(\mathbb{R})$

6.3.1 Notations

We have the standard basis of $\mathfrak{g}$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then H is a basis element of $\mathfrak{a}$, X is a basis element of $\mathfrak{n}$, and Y is a basis element of $\mathfrak{n}^-$.

We identify $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}$, via $\lambda \mapsto \lambda(H)$.

We also have a basis of $\mathfrak{g}_{\mathbb{C}}$ given by

$$H_c = -iX + iY,\ X_c = \frac{1}{2}(-iH + X + Y),\ Y_c = \frac{1}{2}(iH + X + Y).$$

Notice that iH_c is a basis element for $\mathfrak{k}$.

6.3.2 Principal series

The irreducible representations of $K_\emptyset = \{\pm 1\}$ are the trivial and sign representations - we will parametrize them by $\epsilon \in \{0, 1\}$. Thus, the irreducible HC $(\mathfrak{g}_\emptyset, K_\emptyset)$-modules are parametrized by $(\lambda, \epsilon) \in \mathbb{C} \times \{0, 1\}$, and so are the principal series; we have

$$\mathcal{P}_{\lambda,\epsilon} = \{\text{smooth } f : G/N \to \mathbb{C} \mid f(g \cdot c \cdot e^{tH}) = c^\epsilon e^{-(\lambda+1)t} f(g) \quad \forall g \in G, t \in \mathbb{R}, c \in K_\emptyset\}.$$

We can identify $G/N \cong \mathbb{R}^2 - \{0\}$, via $gN \mapsto g \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then

$$\mathcal{P}_{\lambda,\epsilon} = \{\text{smooth } f : \mathbb{R}^2 - \{0\} \to \mathbb{C} \mid f(sv) = |s|^{-(\lambda+1)} sgn(s)^\epsilon f(v) \quad \forall v \in \mathbb{R}^2 - \{0\}, s \in \mathbb{R}^\times\}.$$

Let us denote by $e_n^\lambda : \mathbb{R}^2 - \{0\} \to \mathbb{C}$ the function given by

$$e_n^\lambda(\begin{pmatrix} s \cdot cos(\theta) \\ s \cdot sin(\theta) \end{pmatrix}) := s^{-(\lambda+1)} e^{in\theta}, \quad s \in \mathbb{R}_+^\times,\ \theta \in \mathbb{R}/2\pi\mathbb{R}.$$

$\mathcal{P}^{[K]}_{\lambda,\epsilon}$ has a basis $e^\lambda_n,\ n \in 2\mathbb{Z}$ when $\epsilon = 0$, and $e^\lambda_n,\ n \in 2\mathbb{Z}+1$ when $\epsilon = 1$. One has:

$$H_c e^\lambda_n = n e^\lambda_n,\ X_c e^\lambda_n = \frac{1}{2} i(-(\lambda+1) - n) e^\lambda_{n+2},\ Y_c e^\lambda_n = \frac{1}{2} i(\lambda + 1 - n) e^\lambda_{n-2}.$$

6.3.3 Decomposition of principal series

The case $\epsilon = 0$:

If $\lambda \notin 2\mathbb{Z}+1$, then $\mathcal{P}^{[K]}_{\lambda,0}$ is irreducible. Let's assume that $\lambda \in 2\mathbb{Z}+1$. If $\lambda \in \{1, 3, \cdots\}$, then $\mathcal{P}^{[K]}_{\lambda,0}$ has a finite dimensional quotient module - the span of e^λ_n with $-(\lambda-1) \le n \le \lambda - 1$, and the kernel is the direct sum of the two "tails". If $\lambda \in \{-1, -3, \cdots\}$, then $\mathcal{P}^{[K]}_{\lambda,0}$ has a finite dimensional submodule - the span of e^λ_n with $\lambda + 1 \le n \le -(\lambda+1)$, and the cokernel is the direct sum of the two "tails".

The case $\epsilon = 1$:

If $\lambda \notin 2\mathbb{Z}$, then $\mathcal{P}^{[K]}_{\lambda,1}$ is irreducible. Let's assume that $\lambda \in 2\mathbb{Z}$. If $\lambda \in \{2, 4, \cdots\}$, then $\mathcal{P}^{[K]}_{\lambda,1}$ has a finite dimensional quotient module - the span of e^λ_n with $-(\lambda - 1) \le n \le \lambda - 1$, and the kernel is the direct sum of the two "tails". If $\lambda \in \{-2, -4, \cdots\}$, then $\mathcal{P}^{[K]}_{\lambda,1}$ has a finite dimensional submodule - the span of e^λ_n with $\lambda + 1 \le n \le -(\lambda + 1)$, and the cokernel is the direct sum of the two "tails". Finally, if $\lambda = 0$, then $\mathcal{P}^{[K]}_{\lambda,1}$ is the direct sum of two "tails" - that with H_c-weights $\{1, 3, \cdots\}$, and that with H_c-weights $\{-1, -3, \cdots\}$.

6.3.4 Irreducible $(\mathfrak{g}, K)$-modules

The center $Z(\mathfrak{g})$ is the polynomial algebra in the Casimir element

$$Z = \frac{1}{8}(H_c^2 + 2H_c + 4Y_c X_c).$$

One calculates that Z acts on $\mathcal{P}^{[K]}_{\lambda,\epsilon}$ by $\frac{1}{8}(\lambda^2 - 1)$. Thus, $\mathcal{P}^{[K]}_{\lambda,\epsilon}$ and $\mathcal{P}^{[K]}_{\lambda',\epsilon'}$ have no common irreducible constituents, unless $\lambda' \in \{\lambda, -\lambda\}$ and $\epsilon' = \epsilon$. One can see by direct calculation, or using the calculation of character (add?), that in the latter case, the Jordan-Holder contents of the two modules is the same.

This allows us to summarize (in view of Casselman's submodule theorem), what are the irreducible $(\mathfrak{g}, K)$-modules:

1. $\mathcal{P}^{[K]}_{\lambda,\epsilon}$ for $\epsilon = 0$ and $\lambda \notin 2\mathbb{Z}+1$, or $\epsilon = 1$ and $\lambda \notin 2\mathbb{Z}$.
2. A finite dimensional module L_k of dimension k, for $k \in \mathbb{Z}_{\ge 1}$.
3. A module D^+_k with H_c-weights $\{k, k+1, \ldots\}$ for $k \in \mathbb{Z}_{\ge 1}$.
4. A module D^-_k with H_c-weights $\{-k, -(k+1), \ldots\}$ for $k \in \mathbb{Z}_{\ge 1}$.

6.3.5 Tempered irreducible $(\mathfrak{g}, K)$-modules

Let us determine which irreducible $(\mathfrak{g}, K)$-modules V are tempered/square integrable.

Recall that V is temp. (resp. s.i.) if for all $\mu \in wt(pres_{\emptyset}(V))$, one has $Re(\mu) \geq 0$ (resp. $Re(\mu) > 0$).

Notice that $\mu \in wt(pres_{\emptyset}(V))$ if and only if one has a non-zero morphism $V \to \mathcal{P}^{[K]}_{\mu,\epsilon}$ for some ϵ.

Let us consider the weights in $wt(pres_{\emptyset}(V))$ for irreducible representations V according to the list of the previous subsection:

1. The weights of $\mathcal{P}^{[K]}_{\lambda,\epsilon}$ are $\lambda, -\lambda$.
2. The weights of L_k are $-k$.
3. The weights of D^+_k are $k-1$.
4. The weights of D^-_k are $k-1$.

We see thus that:

1. $\mathcal{P}^{[K]}_{\lambda,\epsilon}$ are tempered when $Re(\lambda) = 0$, and never square integrable.
2. L_k are not tempered.
3. D^+_k are square integrable for $k \geq 2$, and D^+_1 is tempered (but not square integrable).
4. D^-_k are square integrable for $k \geq 2$, and D^-_1 is tempered (but not square integrable).

6.3.6 Langlands classification

We have identifications $G^{\vee,unr}_{\emptyset} \cong \mathfrak{a}^*_{\mathbb{C}} \cong \mathbb{C}$ (the first via $\chi \mapsto d\chi$, and the second as before). Via this identification, $\mu \in \mathbb{C}$ corresponds to a $(\emptyset)$-positive character, if $\mu \in \mathbb{R}_{>0}$.

The tempered irreducible HC $(\mathfrak{g}_{\emptyset}, K_{\emptyset})$-modules correspond to (λ, ϵ) with $\lambda \in i\mathbb{R}$. Thus, the parabolic inductions of positive twists of tempered representations, appearing in the Langlands classification, are $\mathcal{P}^{[K]}_{\lambda+\mu,\epsilon}$, with $\lambda \in i\mathbb{R}$ and $\mu \in \mathbb{R}_{>0}$.

The Langlands quotient of such $\mathcal{P}^{[K]}_{\lambda+\mu,\epsilon}$ is simply $\mathcal{P}^{[K]}_{\lambda+\mu,\epsilon}$ unless:

1. $\lambda = 0$, $\epsilon = 0$, and $\mu \in \{1, 3, \cdots\}$, in which case the Langlands quotient is L_{μ}.
2. $\lambda = 0$, $\epsilon = 1$, and $\mu \in \{2, 4, \cdots\}$, in which case the Langlands quotient is L_{μ}.

Chapter 7

Character

Chapter 8

Further spherical stuff

8.1 Spherical irreducible $(\mathfrak{g}, K)$-modules

An irreducible $(\mathfrak{g}, K)$-module V is called **spherical**, if $V^K \neq 0$.

Claim 8.1.1. *For each $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, $\mathcal{P}^{[K]}_\lambda$ has exactly one spherical irreducible subquotient (i.e. any Jordan-Holder filtration) - call it S_λ. We have $S_\lambda \cong S_{\lambda'}$ if and only if $\lambda' \in W\lambda$. Every sphercial irreducible $(\mathfrak{g}, K)$-module is isomorphic to one of the S_λ.*

Proof. That $\mathcal{P}^{[K]}_\lambda$ has exactly one spherical irreducible subquotient follows from $V \mapsto V^K$ being exact, and $\dim \mathcal{P}^K_\lambda = 1$.

One has $S_\lambda \ncong S_{\lambda'}$ if $\lambda' \notin W\lambda$ because $U(\mathfrak{g})^K$ acts differently.

One has $S_{w\lambda} \cong S_\lambda$ because, denoting by λ^{max} the dominant element in $W\lambda$, one has an non-zero intertwining morphism $\mathcal{P}^{[K]}_{\lambda^{max}} \to \mathcal{P}^{[K]}_\lambda$. complete

Given a spherical irreducible $(\mathfrak{g}, K)$-module V, one can find an embedding $V \to pind_\emptyset(E_\lambda)$ where E is an irreducible $(\mathfrak{g}_\emptyset, K_\emptyset)$-module. This implies that $pind_\emptyset(E_\lambda)^K \neq 0$ and hence E must be the trivial module. Hence we have an embedding of V to $\mathcal{P}^{[K]}_\lambda$, and thus $V \cong S_\lambda$. □

Corollary 8.1.2 (Gelfand property of (G, K))**.** *For every spherical irreducible $(\mathfrak{g}, K)$-module V, one has $\dim V^K = 1$.*

8.2 The Harish-Chandra transform

Completely analogously to the above

$$hc : U(\mathfrak{g})^K \to Fun(\mathfrak{a}^*_{\mathbb{C}}),$$

we can define

$$\mathcal{H} : C^\infty_c(K\backslash G/K) \to Fun(\mathfrak{a}^*_{\mathbb{C}}).$$

Namely,

$$\mathcal{H}(\phi)(\lambda) := \frac{\pi(\phi)\xi_\lambda}{\xi_\lambda}.$$

We call it the **Harish-Chandra transform**.

Claim 8.2.1. *The map $\mathcal{H}$ is an injective algebra homomorphism (where the source is an algebra under convolution, and the target is an algebra under point-wise multiplication).*

Proof. That $\mathcal{H}$ is an algebra homomorphism is clear. To show that it is injective, notice that if $\mathcal{H}(\phi) = 0$, then ϕ acts by zero on every spherical irreducible admissible quasi-simple representation of G. but ϕ clearly acts by zero also on every non-spherical irreducible representation of G. In particular, we see that ϕ acts by zero on every finite-dimensional representation of $G(\mathbb{C})$ (restricted to G). Equivalently, the inner product of ϕ with any matrix coefficient of such a representation is zero. By the Stone-Weierstrass theorem we see that $\phi = 0$. □

Claim 8.2.2. *We have*

$$\mathcal{H}(\phi)(\lambda) = \int_G \phi(g)\Xi_\lambda(g) = \int_A a^\lambda \cdot \left[a^\rho \int_{N_{(\emptyset)}} \phi(an) \right].$$

Proof. For the first equality, notice that

$$\mathcal{H}(\phi)(\lambda) = \langle \pi(\phi)\xi_\lambda, \xi_{-\lambda} \rangle = \int_G \phi(g)\Xi_\lambda(g).$$

For the second equality:

$$\int_G \phi(g)\Xi_\lambda(g) = \int_{G\times K} \phi(g) r(g^{-1}k)^{-\lambda-\rho} = \int_G \phi(g) r(g^{-1})^{-\lambda-\rho} = \int_{A\times N_{(\emptyset)}} \phi(an) a^{\lambda+\rho}.$$

□

Define the **Abel transform**

$$\mathcal{A} : C_c^\infty(K\backslash G/K) \to Fun(A)$$

by the formula inspired by the above:

$$\mathcal{A}(\phi)(a) := a^\rho \int_{N_{(\emptyset)}} \phi(an).$$

We see that $\mathcal{H}(\phi)$ is the Fourier transform of $\mathcal{A}(\phi)$.

Theorem 8.2.3 (The real version of the p-adic Satake isomorphism (is there a better name? Payley-Wiener type?))**.** *The transform $\mathcal{A}$ induces an algebra isomorphism*

$$\mathcal{A} : C_c^\infty(K\backslash G/K) \to C_c^\infty(A)^W$$

(where the target is an algebra under convolution).

Proof. Notice that $Im(\mathcal{H}) \subset Fun(\mathfrak{a}_{\mathbb{C}}^*)^W$ because $\Xi_{w\lambda} = \Xi_\lambda$. Also, notice that for $\phi \in C_c^\infty(K\backslash G/K)$, $\mathcal{A}(\phi)$ is smooth and has compact support. Thus, all points in the above theorem are clear/already established, except the surjectivity. We **omit** the proof of the surjectivity (it is not immediate). □

One would like also variants of this theorem for L^2 and Schwartz spaces (maybe add formulations).

Chapter 9

Square integrable representations

9.1 The results

Chapter 10

The Langlands parameters

10.1 The Langlands dual group

10.2 The Langlands parameters

10.3 The Langlands correspondence

Chapter 11

More precise information on asymptotics

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