Representation theory of compact groups
UNPOLISHED DRAFT

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1 Introduction

These notes are not polished yet; They are not very organized, and contain some mistakes. My main source was the book "Representations of Compact Lie Groups" by Brocker and Dieck.

Consider the circle group $S^1 \cong \mathbb{R}/\mathbb{Z} \cong \mathbb{C}_1^\times$ - it is a compact group. One studies the space $L^2(S^1)$ of square-integrable complex-valued fucntions on the circle. We have an action of S^1 on $L^2(S^1)$ given by

$$(gf)(x) = f(g^{-1}x).$$

We can suggestively write

$$L^2(S^1) = \int_{x \in S^1}^{\wedge} \mathbb{C} \cdot \delta_x,$$

where

$$g\delta_x = \delta_{gx}.$$

Thus, the "basis" of delta functions is "permutation", or "geometric". Fourier theory describes another, "spectral" basis. Namely, we consider the functions

$$\chi_n : x \pmod 1 \mapsto e^{2\pi i n x}$$

where $n \in \mathbb{Z}$. The main claim is that these functions form a Hilbert basis for $L^2(S^1)$, hence we can write

$$L^2(S^1) = \widehat{\bigoplus_{n \in \mathbb{Z}}} \mathbb{C} \cdot \chi_n,$$

where

$$g\chi_n = \chi_n^{-1}(g)\chi_n.$$

In other words, we found a basis of eigenfunctions for the translation operators.

What we would like to do in the course is to generalize the above picture to more general compact groups. For a compact group G, one considers $L^2(G)$ with the action of $G \times G$ by translation on the left and on the right:

$$((g_1, g_2)f)(x) = f(g_1^{-1} x g_2).$$

When the group is no longer abelian, we will not be able to find enough eigenfunctions for the translation operators to form a Hilbert basis. Eigenfunctions can be considered as spanning one-dimensional invariant subspaces, and what we will be able to find is "enough" finite-dimensional "irreducible" invariant subspaces (which can not be decomposed into smaller invariant subspaces). This we will do for a general compact group (the Peter-Weyl theorem). The second problem is to "parametrize" these "iireducible" building blocks and "understand" them. This we will do for a connected compact Lie group (theory of highest weight, Weyl's character formula).

2 Preliminaries

2.1 Topological groups

Definition 2.1. Definitions of a **topological group** (a set equipped with a Hausdorff topology and a group structure, such that the multiplication and inverse maps are continuous), a **morphism** between topological groups (a continuous group homomorphism).

We are "mostly" interested in locally compact topological groups. Examples of locally compact topological groups:

1. Discrete groups - such as $\mathbb{Z}$, finitely generated groups, fundamental groups; In particular, finite groups (those are the compact discrete groups) - such as S_n. For a linear algebraic group $\mathbb{G}$ over a finite field F, we have the finite group $\mathbb{G}(F)$ - such as $GL_n(\mathbb{F}_q)$.

2. Lie groups (A set equipped with the structure of a smooth manifold and the structure of a group, such that the multiplication and inverse maps are continuous). for a linear algebraic group $\mathbb{G}$ over $\mathbb{R}$, we have the Lie group $\mathbb{G}(\mathbb{R})$ (or for $\mathbb{G}$ over $\mathbb{C}$, we have $\mathbb{G}(\mathbb{C})$) - such as $GL_n(\mathbb{R})$ (non-compact) or $O_n(\mathbb{R})$ (compact).

3. Totally disconnected groups (the topology admits a basis consisting of open-compact subsets); In particular, profinite groups (those are the compact totally disconnected groups) - such as infinite Galois groups. For a linear algebraic group $\mathbb{G}$ over the field of p-adics $\mathbb{Q}_p$, we have the totally disconnected group $\mathbb{G}(\mathbb{Q}_p)$ - such as $GL_n(\mathbb{Q}_p)$. We also obtain compact totally disconnected groups by taking $\mathbb{Z}_p$-points of algebraic groups defined over $\mathbb{Z}_p$ - such as $GL_n(\mathbb{Z}_p)$.

4. Adelic groups. For a linear algebraic group $\mathbb{G}$ over $\mathbb{Q}$, we have $\mathbb{G}(\mathbb{A}_{\mathbb{Q}})$.

People also study some smooth but not locally compact groups, such as $Diff(S^1)$ (the group of diffeomorphisms of S^1) or $\Omega(S^1)$ (the groups of loops in S^1).

In this course we will be interested in representation theory of compact groups, especially connected compact Lie groups.

2.2 G-spaces

Let G be a locally compact topological group.

Definition 2.2. A **G-space** is a locally compact topological space X, equipped with a G-action, i.e. a continuous map $a : G \times X \to X$ satisfying $a(e, x) = x$ and $a(g_1, a(g_2, x)) = a(g_1g_2, x)$. We usually write simply gx instead of $a(g, x)$. A **morphism** between two G-spaces X, Y is a continuous map $\phi : X \to Y$ satisfying $\phi(gx) = g\phi(x)$.

Example 2.3. *Let E be a Euclidean space (a finite-dimensional real vector space equipped with an inner product). Let $O(E) \subset GL(E)$ be the subgroup of orthogonal transformations, and let $S(E) \subset E$ be the subset of unit-length vectors. Then $O(E)$ is a locally compact topological group, $S(E)$ is a locally compact topological space, and we have the natural action of $O(E)$ on $S(E)$.*

Example 2.4 (Homogenous G-spaces)**.** *Let $H \subset G$ be a closed subgroup. Then we equip G/H with the quotient topology (the finest topology for which the quotient map $G \to G/H$ is continuous), and the natural G-space structure ($a(g_1, g_2H) = g_1g_2H$).*

Remark 2.5. Let X be a G-space. Assume that the action of G on X is transitive (i.e. for any $x, y \in X$ there exists $g \in G$ such that $gx = y$). Let $x_0 \in X$ and set $H = \{g \in G \mid gx_0 = x_0\}$. Then we have a bijective G-space morphism $G/H \to X$, given by $gH \mapsto gx_0$. One can show that if G is separable (i.e. admits a dense countable subset), then this is in fact a homeomorphism (i.e. its inverse is also continuous) - so a G-space isomorphism.

Example 2.6. *In the example above, the action of $O(E)$ on $S(E)$ is transitive. In standard coordinates O_n is the group of orthogonal $n \times n$ matrices. Set $x_0 = (1, 0, \dots, 0)^t$. Then $H = Stab_{O_n}(x_0) \cong O_{n-1}$ is the subgroup of block diagonal matrices of type $(1, n-1)$ whose first component is 1, so that we are dealing with the homogenous space O_n/O_{n-1} (abusing notation...).*

In some sense, the theory of representations of G is a "qunatization" of the theory of G-spaces. To describe the "quantization procedure" $X \mapsto L^2(X)$, we will review topological vector spaces, measures and Haar measures.

2.3 Topological vector spaces

All vector spaces are over $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.7. Definition of a **topological vector space (TVS)** (a vector space equiped with a Hausdorff topology, for which addition and multiplication by scalar are continuous).

Example 2.8. *Let V be finite-dimensional. Then one can show that V admits a unique topology making it a TVS. It is the "standard one", inherited by any linear isomorphism $V \cong \mathbb{R}^d$.*

Example 2.9. *Let us be given an inner product $\langle \cdot, \cdot \rangle$ on a vector space V. Then we can equip V with the topology induced by the metric $d(v, w) := ||w - v||$ where $||u|| := \sqrt{\langle u, u \rangle}$. This makes V a TVS. The pair $(V, \langle \cdot, \cdot \rangle)$ is called a* ***Hilbert space*** *if V is complete w.r.t. the metric d.*

2.4 Representations of topological groups

Let G be a topological group. For a topological vector space V (over $\mathbb{R}$ or $\mathbb{C}$), we denote by $Aut(V)$ the group of automorphisms of V as a topological vector space (i.e. continuous linear self-maps admitting a continuous linear inverse).

Definition 2.10.

1. A **representation of** G (or a G**-representation**) is a pair (V, π) consisting of a topological vector space V and a homomorphism $\pi : G \to Aut(V)$ such that the resulting map $act : G \times V \to V$ given by $act(g, v) := \pi(g)(v)$ is continuous.

2. A **morphism** between G-representations (V_1, π_1) and (V_2, π_2) is a continuous linear map $T : V_1 \to V_2$ satisfying $T \circ \pi_1(g) = \pi_2(g) \circ T$ for all $g \in G$.

Definition 2.11. A representation (V, π) of G on a Hilbert space is called **unitary**, if $\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$ for all $v, w \in V$ and $g \in G$.

Remark 2.12. If V is finite-dimensional, the continuity requirement can be stated as follows: choosing a basis $e_1, \ldots, e_n$ for V and writing $\pi(g)e_i = \sum_j a_{ij}(g)e_j$, the functions a_{ij} on G should be continuous.

Remark 2.13. If V is an Hilbert space and $\pi(g)$ is unitary for every $g \in G$, the continuity requirement can be stated as follows: For every $v \in V$, the map $o_v : G \to V$ given by $o_v(g) = \pi(g)v$ is continuous at e.

2.5 Measures and Haar measures

Let X be a locally compact topological space. Denote by $C(X)$ the vector space of continuous complex-valued functions on X. For a compact subset $K \subset X$, denote by $C_K(X) \subset C(X)$ the subspace of functions vanishing outside of K. Denote

$$C_c(X) := \cup_K C_K(X) \subset C(X)$$

(in this union, K runs over all compact subsets of X) - this is the space of continuious functions with compact support on X. Define a topology on $C_K(X)$ using the supremum norm. Define a topology on $C_c(X)$ as the finest for which all the inclusions $C_K(X) \subset C_c(X)$ are continuous.

Definition 2.14 (Radon measure)**.** Let X be a locally compact topological space.

1. A **signed measure** on X is a continuous linear functional μ on $C_c(X)$. Concretely, the continuity means that for every compact subset $K \subset X$ there exists $C > 0$ such that $|\mu(f)| \leq C \cdot sup_{x\in K}|f(x)|$ for every $f \in C_K(X)$.

2. A **measure** on X is a signed measure μ on X satisfying $\mu(f) \geq 0$ for $f \geq 0$.

3. A **nowhere vanishing** measure on X is a measure μ on X satisfying $\mu(f) > 0$ for $0 \neq f \geq 0$.

Let G be a locally compact topological group. For $g \in G$ and $f \in C_c(G)$, we define $(L_g f)(x) = f(g^{-1}x)$ and $(R_g f)(x) = f(xg)$. A measure μ on G is called left-G-invariant (right-G-invariant) if $\mu(L_g f) = \mu(f)$ $(\mu(R_g f) = \mu(f))$ for every $g \in G, f \in C_c(G)$.

Theorem 2.15 (Haar measure)**.** *Let G be a locally compact topological group. There exists a non-zero left-G-invariant measure on G. Moreover, each such two differ by a positive scalar multiple. Those measures are nowhere vanishing. Analogous claims hold for right-G-invariant measures.*

Example 2.16. *A Haar measure on $\mathbb{R}$ is got by*

$$f \mapsto \int_{\mathbb{R}} f(x)dx.$$

Example 2.17. *Consider the locally compact topological group $\mathbb{C}_1^\times$ - the complex numbers of length 1 (with complex multiplication as group law). It can also be denoted S^1 or $SO(2)$. To describe a Haar measure on it, consider the map $\phi : \mathbb{R} \to \mathbb{C}_1^\times$ given by $\phi(x) = e^{2\pi ix}$. Then one can see that*

$$f \mapsto \int_0^1 f(\phi(x))dx$$

is a Haar measure on $\mathbb{C}_1^\times$.

Remark 2.18 (The modulus function)**.** Given a left Haar measure μ on G and an element $g \in G$, we get a new left Haar measure by $f \mapsto \mu(R_{g^{-1}}f)$; So, it must differ from μ by a scalar, which we denote by $\Delta_G(g)$. Then $\Delta_G : G \to \mathbb{R}_{>0}$ is a continuous group homomorphism. It is trivial if and only if left and right Haar measures coincide.

Remark 2.19. Left and right Haar measures must coincide in the following cases: If G is abelian - clear. If G is compact - then $\Delta_G(G)$ must be a compact subgroup of $\mathbb{R}_+^\times$, hence $\{1\}$. If G is discrete - then the counting measure is left and right Haar measure.

Remark 2.20. In the case when G is compact, we can (and usually will) normalize the Haar measure by requiring that the measure of $\mathbb{1}_G$ is 1. In the case when G is discrete, we can normalize the Haar measure by requiring that the measure of $\mathbb{1}_{\{e\}}$ is 1. Notice that for a finite group, these normalizations do not coincide - their quotient is an important number, the cardinality of the finite group (i.e. we get some canonical "volume").

Similarly to above, we can define G-invariant measures on a G-space X.

Theorem 2.21. *Let $H \subset G$ be a closed subgroup. Then G/H admits a non-zero G-invariant measure if and only if $\Delta_G|_H = \Delta_H$. In that case, each such two differ by a positive scalar multiple, and those measures are nowhere vanishing.*

2.6 The representations $L^2(X)$

Given a locally compact topological space X and a nowhere vanishing measure μ on X, we consider on $C_c(X)$ the inner product

$$\langle f_1, f_2 \rangle := \mu(f_1 \cdot \overline{f_2}) = \int_X f_1(x)\overline{f_2(x)} d\mu.$$

The completion of $C_c(X)$ w.r.t. to the resulting norm $||f|| := \sqrt{\langle f, f \rangle}$ is a Hilbert space (complete inner product space), denoted by $L^2(X, \mu)$.

Let G be a locally compact topological group and X a homogenous G-space equipped with a G-invariant measure μ. Then $L^2(X) = L^2(X, \mu)$ is naturally a unitary G-representation, by extending by completion the G-action on $C_c(X)$.

Basic problem 2.22. *To study the unitary G-representation $L^2(G/H)$ for interesting pairs (G, H).*

For example, unitary representations such as $L^2(SL_2(\mathbb{R})/SL_2(\mathbb{Z}))$ or $L^2(SL_2(\mathbb{A})/SL_2(\mathbb{Q}))$ occupy (arguably) a central place in mathematics. As a simpler example, the study of the $O(E)$-representation $L^2(S(E))$ is called the study of **spherical harmonics**. Another very important example is the following; Assume that left and right Haar measures on G coincide. Consider G as a $G \times G$-space, via $(g_1, g_2)g = g_1 g g_2^{-1}$. Then $L^2(G)$ (where the measure is a Haar measure), as a $(G \times G)$-representation, is the basic object to study regarding G.

As we will formalize later (as a consequence of the Peter-Weyl theorem), when dealing with compact groups, we can safely restrict ourselves to finite-dimensional representations, so we will next concentrate on those.

3 Finite-dimensional representations of compact groups

We assume that all vector spaces are over $\mathbb{C}$.

Definition 3.1. Denote by $Rep^{fd}(G)$ the category of finite-dimensional representations of G, and by $Hom_G(\cdot,\cdot)$ the Hom-spaces in this category.

3.1 Constructions

Definition 3.2. A **subrepresentation** of a G-representation $(V,\pi) \in Rep^{fd}(G)$ is a subspace $W \subset V$ invariant under the operators $\pi(g)$ for $g \in G$. Then W itself naturally becomes a G-representation (with $\tau : G \to Aut(W)$ given by $\tau(g) := \pi(g)|_W$).

Example 3.3. *$0 \subset V$ and $V \subset V$ are examples of subrepresentations.*

Example 3.4. *Another example of a subrepresentation is $V^G = \{v \in V \mid \pi(g)v = v\ \forall g \in G\}$ - the subspace of G-**invariants**. The action of G on V^G is trivial.*

Remark 3.5. One should also define quotient representations, and give the example of coinvariants V_G. One should state that for compact G, the natural map $V^G \to V_G$ is an isomorphism.

Example 3.6. *Given a G-morphism $T : (W,\tau) \to (V,\pi)$, the subspaces $Ker(T) \subset W$ and $Im(T) \subset V$ are subrepresentations.*

We have the following standard functorial constructions:

1. **Trivial representation:** $(\mathbb{C},\pi) \in Rep^{fd}(G)$ where $\pi(g) = id$ for every $g \in G$.
2. **Dual (or contra-gradient):** For $(V,\pi) \in Rep^{fd}(G)$, we define $(V^*,\tau) \in Rep^{fd}(G)$ by $\tau(g) := \pi(g^{-1})^t$ (here $(\cdot)^t$ is the transpose).
3. **Complex conjugate:** For $(V,\pi) \in Rep^{fd}(G)$, we define $(\overline{V},\tau) \in Rep^{fd}(G)$ by $\overline{V}$ being the same topological abelian group as V, but with complex scalar action twisted by conjugation, and $\tau = \pi$.
4. **Direct sum:** For $(V_1,\pi_1),(V_2,\pi_2) \in Rep^{fd}(G)$, we define $(V_1 \oplus V_2,\tau) \in Rep^{fd}(G)$ by $\tau(g) := \pi_1(g) \oplus \pi_2(g)$.
5. **Tensor product:** For $(V_1,\pi_1),(V_2,\pi_2) \in Rep^{fd}(G)$, we define $(V_1 \otimes V_2,\tau) \in Rep^{fd}(G)$ by $\tau(g) := \pi_1(g) \otimes \pi_2(g)$.
6. ***Hom*:** For $(V_1,\pi_1),(V_2,\pi_2) \in Rep^{fd}(G)$, we define $(Hom(V_1,V_2),\tau) \in Rep^{fd}(G)$ by $\tau(g)(T) := \pi_2(g) \circ T \circ \pi_1(g^{-1})$.

Remark 3.7. For $V,W \in Rep^{fd}(G)$, we should not confuse $Hom_G(V,W)$ with $Hom(V,W)$. One has in fact $Hom_G(V,W) = Hom(V,W)^G$. Notice that we also have $Hom_G(\mathbb{C},V) \cong V^G$.

Lemma 3.8. *One has a functorial isomorphism of G-representations $Hom(V_1,V_2) \cong V_2 \otimes V_1^*$.*

3.2 Complete reducibility and Schur's lemma

Definition 3.9. A representation $(V, \pi) \in Rep^{fd}(G)$ is called **irreducible** if $V \neq 0$ and V contains no subrepresentations except V and 0.

Claim 3.10 (Schur's lemma)**.**

1. *Let $V \in Rep^{fd}(G)$ be irreducible. Then $End_G(V) = \mathbb{C} \cdot Id_V$.*
2. *Let $V, W \in Rep^{fd}(G)$ be irreducible. If V is isomorphic to W, then $Hom_G(V, W)$ is one-dimensional, and every non-zero element in $Hom_G(V, W)$ is an isomorphism. If V is non-isomorphic to W, then $Hom_G(V, W) = 0$.*

Proof. Let V, W be irreducible. Let $T : V \to W$ be non-zero. Then $Im(T)$ is a non-zero subrepresentation of W, hence $Im(T) = W$, so T is surjective. Similarly, $Ker(T)$ is a subrepresentation of V which is not the whole of V, hence $Ker(T) = 0$, so T is injective. Thus, T is an isomorphism.

This shows that if V, W are non-isomorphic, then $Hom_G(V, W) = 0$.

Let $T \in End_G(V)$. We know that T has an eigenvalue (since $V \neq 0$), say λ. Then $S := T - \lambda \cdot Id_V$ has non-trivial kernel, hence is not an isomorphism. So by what we saw, $S = 0$, i.e. $T = \lambda \cdot Id_V$.

If V, W are isomorphic, $Hom_G(V, W) \cong Hom_G(V, V)$ (by composing with any isomoprhism), hence the dimension is 1.

□

Lemma 3.11. $(V, \pi) \in Rep^{fd}(G)$. *Then there exists a unique projection $Av_G \in End_G(V)$ onto V^G (a projection is an operator P satisfying $P^2 = P$; it is onto it's image, on which it acts as identity).*

Proof. We construct

$$Av_G(v) := \int_G \pi(g) v d\mu$$

(here, and in shat follows, μ is the Haar measure normalized by requiring $\mu(1) = 1$). Uniqueness is seen as follows; It is easy to see that Av_G commutes with any G-morphism $T \in End_G(V)$. Hence, given another projection $P \in End_G(V)$ onto V^G, we have $P = Av_G \circ P = P \circ Av_G = Av_G$. □

Lemma 3.12. *Let $V \in Rep^{fd}(G)$. Then there exists a G-invariant inner product $\langle \cdot, \cdot \rangle$ on V.*

Proof. Basically, we can interpret the space of Hermitian forms on V as $Hom(V, \overline{V})$, and the subspace of G-invariant Hermitian forms as $Hom_G(V, \overline{V}) = Hom(V, \overline{V})^G$. The space of inner products on V is a cone in the space of Hermitian forms. Now we pick any inner product β_0 and consider $\beta := Av_G(\beta_0)$. It will be a G-invariant inner product.

Concretely, let $\langle \cdot, \cdot \rangle'$ be any inner product on V. Define

$$\langle v, w \rangle := \int_G \langle \pi(g) v, \pi(g) w \rangle' d\mu,$$

and check that it is indeed a well-defined G-invariant inner product. □

Claim 3.13 (Maschke)**.** *Let $V \in Rep^{fd}(G)$, and let $W \subset V$ be a subrepresentation. Then there exists a subrepresentation $U \subset V$ such that $V = W \oplus U$.*

Proof. First proof: Let $\langle \cdot, \cdot \rangle$ be a G-invariant inner product on V. Then $W^{\perp}$ is a subrepresentation of V (easy to check), and $V = W \oplus W^{\perp}$.

Second proof: The problem is equivalent to finding a G-morphism $T : (V, \pi) \to (W, \tau)$ satisfying $T \circ i = id_W$ where $i : W \to V$ is the inclusion (then $Ker(T)$ will be the desired U). As a first step, find a linear map $T_0 \in Hom(V, W)$ satisfying $T_0 \circ i = id_W$. Now construct

$$T := Av_G(T_0).$$

Then $T \in Hom_G(V, W)$. In addition,

$$T \circ i = Av_G(T_0) \circ i = Av_G(T_0) \circ Av_G(i) = Av_G(T_0 \circ i) = Av_G(id_W) = id_W.$$

□

Corollary 3.14. *Let $V \in Rep^{fd}(G)$. Then there exist irreducible subrepresentations $V_1, \ldots, V_k \subset V$ such that $V = V_1 \oplus \cdots \oplus V_k$.*

Proof. Use repeatedly Maschke's theorem. □

Corollary 3.15. *Let $V \in Rep^{fd}(G)$. Then we can write $V \cong E_1^{\oplus d_1} \oplus \ldots \oplus E_k^{\oplus d_k}$ where E_i are pairwise non-isomorphic irreducible representations. We have*

$$d_i = \dim Hom_G(E_i, V) = \dim Hom(V, E_i).$$

In particular, d_i doesn't depend on the above decomposition.

Definition 3.16. For $V \in Rep^{fd}(G)$ and irreducible E, the number $\dim Hom_G(E, V)$ is called the **multiplicity** of E appearing in V, and denoted $[V : E]$.

Remark 3.17. We see that two representations V, W are isomorphic if and only if $[V : E] = [W : E]$ for all irreducible E. One can have a more general treatment, undependent on Mashcke's theorem and Schur's lemma, using Jordan-Holder series.

Remark 3.18. Add information about isotypical components. Namely, The subspace $E_i^{\oplus d_i}$ is canonically defined, as the sum of all subrepresentations isomorphic to E_i. There is a unique G-invariant projection onto the isotpypical component (later we will have a formula for it using the character). Any morphism in $Rep^{fd}(G)$ respects the isotypical components, and best would be to formulate that $Rep^{fd}(G)$ is equivalent to the direct sum of copies of $Vect$, one for each irreducible representation isomorphism class.

3.3 Character

Definition 3.19. Let $(V,\pi) \in Rep^{fd}(G)$. The **character**

$$\chi_V \in C(G)$$

is defined by

$$\chi_V(g) := Tr_V(\pi(g)).$$

Definition 3.20. Define the following operations on $C(G)$:

1. $f^*(g) := f(g^{-1})$.
2. $\overline{f}(g) := \overline{f(g)}$.
3. $(f_1 + f_2)(g) := f_1(g) + f_2(g)$.
4. $(f_1 \cdot f_2)(g) := f_1(g)f_2(g)$.
5. $av(f)(g) := \int_G f d\mu$ (considered as a number, or as a constant function on G).
6. $\langle f_1, f_2\rangle = av(f_1 \cdot \overline{f_2})$ (considered as a number, or as a constant function on G).

Proposition 3.21. *The character construction performs the following "representation to function" translations:*

1. *Let V have the trivial G-action. Then $\chi_V = \dim V$.*
2. $\chi_{V^*} = \chi_V^*$.
3. $\chi_{\overline{V}} = \overline{\chi_V}$.
4. $\chi_{V\oplus W} = \chi_V + \chi_W$.
5. $\chi_{V\otimes W} = \chi_V \cdot \chi_W$.
6. $\chi_{Hom(V,W)} = \chi_V^* \cdot \chi_W$.
7. $\chi_{V^G} = av(\chi_V)$.

Proof. Only the last equality is interesting. For $g \in G$, we have $Tr_{V^G}(\pi(g)) = Tr_V(\pi(g) \circ Av_G)$ because $\pi(g) \circ Av_G$ acts as $\pi(g)$ on V^G and as 0 on $Ker(Av_G)$ (which is a complement to V^G in V). Hence

$$Tr_{V^G}(\pi(g)) = Tr_V(\pi(g)\circ Av_G) = Tr_V(\int_G \pi(gh)d\mu(h)) = \int_G Tr_V(\pi(gh))d\mu(h) = \int_G Tr_V(\pi(h))d\mu(h).$$

□

Claim 3.22. *Let $V \in Rep^{fd}(G)$. Then $\chi_V^* = \overline{\chi_V}$.*

Proof. An inner product on V gives an isomorphism $\overline{V} \to V^*$, which is an isomorphism of G-representations if (and only if) the inner product is G-invariant. Thus, since we know that V admits a G-invariant inner product, we get $\overline{V} \cong V^*$ as G-representations, so $\chi_{\overline{V}} = \chi_{V^*}$, so $\overline{\chi_V} = \chi_V^*$. □

Proposition 3.23. *Let $V, W \in Rep^{fd}(G)$. Then*

$$\dim Hom_G(V, W) = \langle \chi_W, \chi_V \rangle.$$

Proof. From all the above we have

$$\dim Hom_G(V, W) = \chi_{Hom_G(V,W)} = av(\chi_{Hom(V,W)}) = av(\chi_V^* \cdot \chi_W) = av(\overline{\chi_V} \cdot \chi_W) = \langle \chi_W, \chi_V \rangle.$$

□

Corollary 3.24 (Orthogonality relations)**.** *Let $E, F \in Rep^{fd}(G)$ be irreducible. Then $\langle \chi_E, \chi_F \rangle$ equals 0 if E and F are non-isomorphic, and 1 if E and F are isomorphic.*

Corollary 3.25. *Let $V, W \in Rep^{fd}(G)$. If $\chi_V = \chi_W$, then V and W are isomorphic.*

Proof. We have $\dim Hom_G(E, V) = \langle \chi_E, \chi_V \rangle$ for every irreducible representation E, hence χ_V determines the multiplicity of E appearing in V. □

We can now give, using the characters, a formula for the projection operators on isotypical components.

Claim 3.26. *Let $E, V \in Rep^{fd}$, with E irreducible. Consider the operator $Pr_E \in End(V)$ defined by*

$$Pr_E(v) := \dim E \cdot \int_G \overline{\chi_E(g)} \pi(g) v d\mu.$$

Then $Pr_E \in End_G(V)$ and it is the projection on the E-isotypical component of V.

Proof. It is easy to check that indeed Pr_E is G-equivariant, using the fact that χ_E is central. Furthermore, Pr_E leaves invariant any subrepresentation (clear from its formula). Let $F \subset V$ be an irreudicble subrepresentation. By Schur's lemma, Pr_E acts by a scalar on F. To find this scalar, we calculate

$$Tr_F(Pr_E) = \dim E \cdot \langle \chi_F, \chi_E \rangle.$$

By the orthogonality relations, we see that Pr_E acts by 1 if $F \cong E$ and by 0 otherwise. □

3.4 Matrix coefficients

Definition 3.27. Let $(V,\pi) \in Rep^{fd}(G)$. The **matrix coefficient map**

$$\mathcal{M}_V : End(V) \to C(G)$$

is defined by

$$\mathcal{M}_V(T)(g) := Tr_V(\pi(g) \circ T).$$

Remark 3.28. Under the isomorphism $V \otimes V^* \cong End(V)$, the matrix coefficient map becomes

$$\mathcal{M}_V(v \otimes \alpha)(g) = \alpha(\pi(g)v);$$

If v is a basis vector and α a dual basis covector, then $\mathcal{M}_V(v \otimes \alpha)$ is indeed just a "matrix coefficient", which explains the terminology.

For the next claim, let us endow $End(V)$ and $C(G)$ with $(G \times G)$-action as follows; $(g,h)T := \pi(h) \circ T \circ \pi(g)^{-1}$ and $((g,h)f)(x) = f(g^{-1}xh)$.

Claim 3.29. *Let $V \in Rep^{fd}(G)$.*

1. $\mathcal{M}_V(Id_V) = \chi_V$.
2. $V \otimes V^* \cong End(V) \xrightarrow{\mathcal{M}_V} C(G)$ *is given by* $v \otimes \alpha \mapsto (g \mapsto \alpha(\pi(g)v))$.
3. $\mathcal{M}_V : End(V) \to C(G)$ *is a morphism of* $(G \times G)$*-representations.*
4. *If V is irreducible, then $\mathcal{M}_V : End(V) \to C(G)$ is injective.*

Proof. To show item 4, we first claim that $End(V)$ is irreducible as a $(G \times G)$-representation. This is since $End(V) \cong V \otimes V^*$ and the following lemma: Let G, H be compact groups and $U \in Rep^{fd}(G), W \in Rep^{fd}(H)$ irreducible. Then $U \otimes W$ is irreducible as a $(G \times H)$-representation. Thus, the map $\mathcal{M}_V$ is either zero or injective, but clearly it is not zero. □

Proposition 3.30. *The matrix coefficient construction performs the following "representation to function" translations:*

1. $\mathcal{M}_{V^*}(T^*) = \mathcal{M}_V(T)^*$.
2. $\mathcal{M}_{\overline{V}}(\overline{T}) = \overline{\mathcal{M}_V(T)}$.
3. $\mathcal{M}_{V \oplus W}(T \oplus S) = \mathcal{M}_V(T) + \mathcal{M}_W(S)$.
4. $\mathcal{M}_{V \otimes W}(T \otimes S) = \mathcal{M}_V(T) \cdot \mathcal{M}_W(S)$.
5. $\mathcal{M}_{Hom(V,W)}(S \circ \cdot \circ T) = \mathcal{M}_V(T)^* \cdot \mathcal{M}_W(S)$.
6. $\mathcal{M}_{V^G}(Av_G \circ T) = av(\mathcal{M}_V(T))$.

Proposition 3.31 (Orthogonality relations)**.** *Let $E, F \in Rep^{fd}(G)$ be irreducible and non-isomorphic, and $T \in End(E), S \in End(F)$. Then*

$$\langle \mathcal{M}_F(S), \mathcal{M}_E(T) \rangle = 0.$$

Proof. We have

$$\langle \mathcal{M}_F(S), \mathcal{M}_E(T)\rangle = \mathcal{M}_{(F\otimes\overline{E})^G}(Av_G \circ (S \otimes \overline{T})).$$

But $\overline{E} \cong E^*$, so $(F \otimes \overline{E})^G \cong (F \otimes E^*)^G = Hom_G(E, F) = 0$. □

Let $E \in Rep^{fd}(G)$ be irreducible. Then E admits a unique up to scalar G-invariant inner product (because there is a unique up to scalar isomorphism between the irreducible representations E^* and $\widetilde{E}$). Thus for $T \in End(E)$ the adjoint operator $T^\circ \in End(E)$ is well-defined. Notice that $\pi(g)^\circ = \pi(g^{-1})$.

Definition 3.32. Let $E \in Rep^{fd}(G)$ be irreducible. The **Hilbert-Schmidt inner product** on $End(E)$ is defined by $\langle T, S\rangle_{HS} := Tr_E(T \circ S^\circ)$.

Claim 3.33. *Let $E \in Rep^{fd}(G)$ be irreducible.*

1. *The Hilbert-Schmidt inner product on $End(E)$ is $(G \times G)$-invariant.*
2. *The map $\mathcal{M}_E : (End(E), \frac{1}{\dim E}\langle,\cdot,\rangle_{HS}) \to (C(G), \langle\cdot,\cdot\rangle)$ is unitary.*

Proof. Since $End(V)$ is irreducible as a $(G \times G)$-representation, and the inner products on $End(V)$ and $C(G)$ are $(G \times G)$-invariant, the map $\mathcal{M}_V$ must be unitary up to a scalar. To find the scalar is an exercise. □

4 The Peter-Weyl theorem

For simplicity, we assume that the group G is separable.

Claim 4.1. *Let $f \in C(G)$. The following are equivalent:*

1. *f is left-G-finite.*
2. *f is right-G-finite.*
3. *f is $(G \times G)$-finite.*
4. *f is in the image of $\mathcal{M}_V$ for some V.*

Proof. (4) $\implies$ (3) is clear since $\mathcal{M}_V$ is $(G \times G)$-equivariant and its domain is finite-dimensional. (3) $\implies$ (2) is clear. Let us show that (2) $\implies$ (4) ((3) $\implies$ (1) $\implies$ (4) is analogous). So, let $f \in C(G)$ be right-G-finite. Let $V \subset C(G)$ be a finite dimensional subrepresentation (w.r.t. the right G-action) which contains f. Denote by $\alpha \in V^*$ the functional $\alpha(h) := h(e)$. Then we have $\mathcal{M}_V(f \otimes \alpha)(g) = \alpha(R_g f) = f(g)$, so $f = \mathcal{M}_V(f \otimes \alpha)$. □

Definition 4.2. Denote by $C(G)^{fin}$ the subspace of $C(G)$ consisting of functions f satisfying the equivalent conditions of the previous claim.

Lemma 4.3. *$C(G)^{fin} \subset C(G)$ is a vector subspace closed under the operations $(f, h) \mapsto f \cdot h, f \mapsto \overline{f}$.*

Denote by $L^2(G)^{fin}$ the subspace of $L^2(G)$ consisting of left-G-finite vectors (we will later see that $L^2(G)^{fin} = C(G)^{fin}$, but for now it is a comfortable notation).

Claim 4.4. *The following are equivalent:*

1. *$C(G)^{fin}$ is dense in $C(G)$.*
2. *$C(G)^{fin}$ is dense in $L^2(G)$.*
3. *$L^2(G)^{fin}$ is dense in $L^2(G)$.*
4. *For every $e \neq g \in G$, there exists an irreducible representation $(\pi, V) \in Rep^{fd}(G)$ such that $\pi(g) \neq id$.*
5. *$C(G)^{fin}$ separates points of G; i.e. for every $g, h \in G$ such that $g \neq h$, there exists $f \in C(G)^{fin}$ such that $f(g) \neq f(h)$.*

Proof.
(1) $\implies$ (2) : Clear, since the map $C(G) \to L^2(G)$ is continuous with dense image.
(2) $\implies$ (3): Clear, since $C(G)^{fin} \subset L^2(G)^{fin}$.
(3) $\implies$ (4): Let $e \neq g \in G$. Then clearly there exist a function $f \in C(G)$ such that $L_g f \neq f$. Thus, clearly g can not act trivially on $L^2(G)^{fin}$ (since then it would act trivially on $L^2(G)$ and hence on f). Pick $\widetilde{f} \in L^2(G)^{fin}$ such that $L_g \widetilde{f} \neq \widetilde{f}$. By definition of $L^2(G)^{fin}$, The vector $\widetilde{f}$ sits in a finite-dimensional subrepresentation $V \subset L^2(G)$ (w.r.t. the left G-action). We get that g acts non-trivially on V, as wanted.
(4) $\implies$ (5): Clear, by considering gh^{-1} and matrix coefficients of a representation on which it acts non-trivially.
(5) $\implies$ (1): $C(G)^{fin} \subset C(G)$ is a vector subspace closed under pointwise multiplication, and pointwise conjugation. Thus, by the Stone-Weierstrass theorem, $C(G)^{fin}$ separates points of G i.f.f. $C(G)^{fin}$ is dense in G.

□

Theorem 4.5 (Peter-Weyl)**.** *The equivalent conditions of the previous theorem are satisfied.*

To prove theorem 4.5, we have some preparations.

Let X, Y be compact spaces, μ a nowhere vanishing measure on X and ν a nowhere vanishing measure on Y. Assume WLOG that $\mu(1) = 1, \nu(1) = 1$. Let $K \in C(Y \times X)$. Consider the formula

$$T_K(f)(x) := \int K(x, y) f(y) d\mu.$$

Lemma 4.6. *T_K well-defines a continuous operator $C(Y) \to C(X)$ of operator norm $\leq ||K||_{L^\infty(X \times Y)}$ and a continuous operator $L^2(Y) \to L^2(X)$ of operator norm $\leq ||K||_{L^2(X \times Y, \mu \times \nu)}$. The later operator is compact, and self-adjoint in case that $K(x, y) = \overline{K(y, x)}$.*

Proof. It is easy to see that we get operators with the claimed bound on norm. The self-adjointness claim is also immediate.

To see that $T_K : L^2(Y) \to L^2(X)$ is compact, one can consider functions K which are linear combinations of functions of the form $K_1(y)K_2(x)$, where $K_1 \in C(Y), K_2 \in C(X)$. By the Stone-Weierstrass theorem, such functions are dense in $C(X \times Y)$. Hence we get that every T_K is a limit (in $C(X \times Y)$, and hence in $L^2(X \times Y)$) of T_K's with K such. But for such K, T_K is of finite rank. □

Let us recall that for a compact self-adjoint operator $T : \mathcal{H} \to \mathcal{H}$ from a (separable) Hilbert space to itself one has the spectral theorem, which says that

$$\mathcal{H} = Ker(T) \oplus \overline{Im(T)}, \quad \overline{Im(T)} = \widehat{\bigoplus_{\lambda}} Ker(T - \lambda \cdot Id),$$

where λ runs over a countable set of non-zero real numbers (for which zero is the only limit point), and each $Ker(T - \lambda \cdot Id)$ is finite-dimensional.

In our case $X = Y = G$. A special class of operators T_K as above is constructed by setting $K(x, y) := k(y^{-1}x)$, where $k \in C(G)$. The resulting $T_K(f) =: f * k$ is known as the **convolution**. It has the special property that it is a G-morphism w.r.t. to the left G-actions: $L_g(f * k) = (L_g f) * k$. Also, if $k^* = \overline{k}$, then T_K is self-adjoint.

A second preparation is the following:

Lemma 4.7. *Let $f \in C(G)$, and $\epsilon > 0$. Let $e \in U \subset G$ be open, such that $U = U^{-1}$ and $|f(x) - f(xy)| \leq \epsilon$ for all $x \in G, y \in U$ (i.e. $||R_y f - f||_{sup} \leq \epsilon$ for all $y \in U$). There exists $u_U \in C(G)$ such that u_U is non-negative, $u_U^* = u_U$, $\int_G u_U d\mu = 1$ and u_U is zero outside U. We have $||f * u_U - f||_{sup} \leq \epsilon$.*

Proof of theorem 4.5. We show that condition (3) is satisfied. Let $f \in L^2(G)$. We want to show that we can approximate f by G-finite elements in $L^2(G)$. First, we can approximate f by continuous functions, hence we can assume that $f \in C(G)$. Now, using an approximation of unity as in the previous lemma, we can approximate f by some $f * u$. Now $f * u$ is in the image of the compact self-adjoint operator $\cdot * u$, and hence can be approximated by sums of elements in non-zero eigenspaces of $\cdot * u$, which are G-finite. □

Corollary 4.8 (Peter-Weyl decomposition)**.** *Let $(E_i, \pi_i)_{i \in I}$ be a representative family of irreducible representations of G. Endow $End(E_i)$ with the inner product $\frac{1}{\dim E_i}\langle \cdot, \cdot \rangle_{HS}$. Then the unitary embeddings $\mathcal{M}_{E_i} : End(E_i) \to L^2(G)$ induce an isomorphism of unitary $(G \times G)$-representations*

$$L^2(G) \cong \widehat{\oplus}_{i \in I} End(E_i)$$

Corollary 4.9. *$L^2(G)^{fin} = C(G)^{fin}$.*

Proof. Let $f \in L^2(G)^{fin}$. We have a finite-dimensional subrepresentation $V \subset L^2(G)^{fin}$ in which f sits. For every irreducible E which does not enter V, the projection of $L^2(G)$ onto $End(E)$ along the Peter-Weyl decomposition induces the zero map on V. Thus we see that f has only finitely many non-zero componenets when decomposed along the Peter-Weyl decomposition.. Thus f lies in $C(G)^{fin}$. $\square$

5 The convolution product

We define an algebra structure on $C(G)$ by:

$$(f_1 * f_2)(x) = \int_G f_1(xy^{-1}) d_2(y) d\mu(y)$$

(this is called the **convolution product**). A "more correct" approach is to consider not $C(G)$ but the isomorphic space $\mathcal{M}^{cont}(G) := C(G) \cdot d\mu$ (of "continuous signed measures"). Then the algebra structure is described by

$$\nu_1 * \nu_2 = m_*(\nu_1 \boxtimes \nu_2)$$

where $\boxtimes : \mathcal{M}(G) \otimes \mathcal{M}(G) \to \mathcal{M}(G \times G)$ is the external product of measures and $m : G \times G \to G$ is the multiplication map. In fact, we obtain an algebra structure on the space of all signed measures $\mathcal{M}(G)$.

The convolution product makes $C(G)$ a Banach algebra w.r.t. the supremum norm, and $\mathcal{M}(G)$ a Banach algebra w.r.t. the L^1-norm (i.e. the standard functional norm on $\mathcal{M}(G)$ considered as dual to $C(G)$).

We have the convenient formulas $\delta_g * (f d\mu) = (L_g f) \cdot d\mu$ and $(f d\mu) * \delta_g = (R_g f) \cdot d\mu$.

5.1 Action on representations

If we have a unitary representation (V, π) of G (or, more generally, just any complete locally convex representation), it extends naturally to a algebra morphism $\pi : \mathcal{M}(G) \to End(V)$, given by

$$\pi(\nu)(v) = \int_G \pi(g) v \cdot d\nu$$

(here, $End(V)$ is the algebra of continuous linear endomorphisms of V).

Let us define the above vector-valued integral. for a continuous function $\phi : G \to V$ we define $\int_G \phi(g) \cdot d\nu$ as the unique vector $w \in V$ such that $\alpha(w) = \int_G \alpha(\phi(g)) \cdot d\nu$ for every continuous functional α on V.

Such a w is unique, if exists (by Hahn-Banach). To show that w exists in case V is a Hilbert space, fix a Hilbert basis (e_i) for V. Denote $c_i = \int_G \langle \phi(g), e_i \rangle d\nu$. Then by Cauchy-Schwartz

$$\sum_i |c_i|^2 \leq C \cdot \int_G ||\phi(g)||^2 d|\nu|.$$

Hence $w = \sum_i c_i e_i$ converges, and we easily see that it is our desired vector.

Back to our algebra morphism $\pi : \mathcal{M}(G) \to End(V)$, it is not difficult to see that $\mathcal{M}(G) \times V \to V$ is continuous.

6 Unitary representations of compact groups

Remark 6.1. Let (H, π) be a unitary representation of G. Then for every subrepresentation $U \subset H$, $U^{\perp}$ is also a subrepresentation of H.

For a unitary representation (H, π) of G, we denote by $H^{fin} \subset H$ the subspace consisting of vectors v for which $(\pi(g)v)_{g \in G}$ span a finite-dimensional subspace.

Lemma 6.2. *Let (H, π) be a unitary representation of G. Then H^{fin} is dense in H.*

Proof. Let $0 \neq v \in H$. We can first approximate v by $\pi(u)v$ for $u \in C(G)$ where u is as in the proof of Peter-Weyl ("approximation of δ_e"). Then, by approximating u by functions from $C(G)^{fin}$, we see that we can approximate v by $\pi(u)v$ for $u \in C(G)^{fin}$. But for such u we have $\pi(u)v \in V^{fin}$, and we are done. □

Claim 6.3. *Let (H, π) be an irreducible unitary representation of G. Then H is finite-dimensional.*

Proof. Since $H \neq 0$, we have $H^{fin} \neq 0$. But then H contains a non-zero finite-dimensional subrepresentation, and so is equal to it (being irreducible). □

Claim 6.4. *Let (H, π) be a unitary representation of G. Then $H = \widehat{\oplus}_{i \in I} E_i$ for some family of irreducible subrepresentations $(E_i)_{i \in I}$.*

Proof. Using Zorn's lemma and taking orthogonal complements, we reduce to showing that if $H \neq 0$, then H contains an irreducible subrepresentation. But since $H^{fin} \neq 0$, H contains a non-zero finite-dimensional subrepresentation, and thus an irreducible finite-dimensional subrepresentation. □

Of course, this situation is radically different from the non-compact one, as we see by considering the regular action of $\mathbb{R}$ on $L^2(\mathbb{R})$.

6.1 The case of $SU(2)$

Set $G = SU(2)$. We consider the subgroup $T \subset G$ of diagonal matrices (it is isomorphic to $\mathbb{C}_1^{\times}$). We denote the characters of T by $\alpha^n(diag(e^{i\theta}, e^{-i\theta})) = e^{in\theta}$ ($n \in \mathbb{Z}$). By linear algebra, every unitary operator is diagnolizable, so every element of G is conjugate to an element in T. Thus, it is plausible that for a central function $f \in C(G)$ we can express the integral $\int_G f \cdot d\mu_G$ in the form $\int_T f|_T \cdot d\nu$ for some measure ν on T. Indeed, we will now use and later prove the following:

Theorem 6.5 (Weyl's integration formula)**.** *Let $f \in C(G)$ be central. Then*

$$\int_G f \cdot d\mu_G = \frac{1}{2} \int_T f|_T \cdot |\alpha - \alpha^{-1}|^2 \cdot d\mu_T = \frac{1}{2\pi} \int_0^{2\pi} f(diag(e^{i\theta}, e^{-i\theta})) \cdot 2sin^2(\theta) \cdot d\theta.$$

Let now $\chi \in C(G)$ be the character of an irreduicble representation. We can write $\chi_T = \sum_{n\in\mathbb{Z}} m_n \cdot \alpha^n$ for $m_n \in \mathbb{Z}_{\geq 0}$, almost all of which are zero. We claim that $m_n = m_{-n}$ for all $n \in \mathbb{Z}$. Indeed, consider the element $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G$. Notice that $wtw^{-1} = t^{-1}$ for $t \in T$. Hence w maps eigenvectors of T with eigencharacter α^n to eigenvectors with eigencharacter α^{-n}.

We will now interpret the constraint $\langle \chi, \chi \rangle = 1$ using Weyl's integration formula. We get:

$$||\chi||_G^2 = av_G(\chi \cdot \overline{\chi}) = \frac{1}{2} av_T(\chi|_T \cdot (\alpha - \alpha^{-1}) \cdot \overline{\chi|_T} \cdot (\alpha^{-1} - \alpha)) = \frac{1}{2} ||\chi|_T \cdot (\alpha - \alpha^{-1})||_T^2$$

so

$$2 = || \sum_n (m_{n-1} - m_{n+1}) \alpha^n ||_T^2 = \sum_n (m_{n-1} - n_{n+1})^2.$$

Taking into account that almost all of the m_n are equal to zero, and that $m_n = m_{-n}$, we can only have $m_n = 1$ for $n \in \{d, d-2, d-4, \ldots, -d\}$ and $m_n = 0$ otherwise, where $d \in \mathbb{Z}_{\geq 0}$.

We now describe some representations $E_d \in Rep^{fd}(G)$ with character as above. Namely, set E_d to be the space of homogenous polynomials of degree d on $\mathbb{C}^2$, with the standard action of $SU(2) \subset GL_2(\mathbb{C})$. Then $E_d = sp\{y^d, y^{d-1}x, \ldots, x^d\}$, and the action of T on $y^{d-i}x^i$ is via α^{d-2i}. So indeed $\chi_{E_d}|_T = \alpha^d + \alpha^{d-2} + \ldots, \alpha^{-d}$.

We deduce that E_d are irreducible (since $||\chi_{E_d}||_G^2 = 1$), and that these are exactly all the irreducible representations of G, up to isomorphism (since we saw that the character of an irreducible representation must coincide with the character of one of the E_d).

7 Lie groups

7.1 Preliminaries

Definition 7.1. Definition of a **smooth manifold**, a **morphism between manifolds**, a **submersion between manifolds**, a **closed submanifold** (a closed subset, whose embedding locally looks like that of a vector subspace).

Given a morphism $\phi : M \to N$ which is a submersion at a point $m \in M$, an important property is that ϕ admits a section locally at $\phi(m)$; i.e., there exists a morphism $s : U \to M$ satisfying $\phi \circ s = i_U$ and $s(\phi(m)) = m$, where $\phi(m) \in U \subset N$ is open and $i_U : U \to N$ is the inclusion.

From this property we get that given a surjective submersion $\phi : M \to N$, to give a morphism from N is the same as to give a morphism from M constant on the fibers of ϕ (this boils down to checking that a map from N is smooth if its composition with ϕ is smooth). In particular ("by Yoneda's lemma"), a bijective submersion is an isomorphism.

Sard's lemma says:

Theorem 7.2. *Let M, N be second-countable non-empty manifolds of dimensions m, n. Let $\phi : M \to N$ be a morphism. Denote by $M_\phi \subset M$ the subset of points at which ϕ is not submersive. Then $\phi(M_\phi) \subset N$ has measure zero (in particular, is not the whole of N). In particular, a surjective morphism must be submersive at some point.*

Definition 7.3. Definition of a **Lie group**, a **morphism between Lie groups**, a **Lie subgroup** (a closed submanifold closed under multiplication and inverse).

Definition 7.4. For a Lie group G, definition of a G**-manifold**.

7.2 Automatic submersiveness results

Claim 7.5. *Let M, N be transitive G-manifolds. Then any surjective morphism of G-manifolds $M \to N$ is submersive.*

Proof. Sard's lemma gives that the morphism is submersive at some point. The transitivity allows to translate this to all points. □

Corollary 7.6. *A bijective morphism between G-manifolds is an isomorphism.*

Notice that a transitive G-manifold M with a point m whose stabilizer is H is unique up to unique isomorphism; This follows from characterizing morphisms from M as morphisms from G which are constant on left H-cosets (via the surjective submersion $G \to M$ given by $g \mapsto gm$). We also have existence:

Claim 7.7. *Let G be a Lie group, and $H \subset G$ a Lie subgroup. Then there exists a transitive G-manifold M with a point $m \in M$ such that $Stab_G(m) = H$. If H is normal in G, then M becomes a Lie group itself (such that $G \to M$ given by $g \mapsto gm$ becomes a group homomorphism).*

Of course, one denotes by G/H such a transitive G-manifold, identifying gH with gm as usual.

Quite importantly, $G \to G/H$ is a smooth fiber bundle with fiber H; In other words, locally over G/H the morphism becomes isomorphic (as a morphism between smooth manifolds) to a projection $H \times U \to U$. This follows quite immediately from $G \to G/H$ being submersive and the fibers being acted upon freely and transitively by H.

7.3 Automatic smoothness results

As corollary of the previous subsection:

Corollary 7.8. *A surjective morphism of Lie groups is submersive. A bijective morphism of Lie groups is an isomorphism.*

Claim 7.9. *Let G be a Lie group, and $H \subset G$ a closed subgroup. Then H a Lie subgroup of G.*

Proof. We will use the exponential map, which we will describe later.

It is enough to show that H is a submanifold of G locally around e.

We define $\mathfrak{h} \subset T_eG$ by $\mathfrak{h} = \{X \in T_eG \mid exp(tX) \in H \ \forall t \in \mathbb{R}\}$. Using the lemma that follows, we see that $\mathfrak{h}$ is a linear subspace of T_eG. Let V be any linear complement to $\mathfrak{h}$ in T_eG. Define a map $\alpha : \mathfrak{h} \times V = T_eG \to G$ by $(X, Y) \mapsto exp(X) \cdot exp(Y)$. Notice that α is a local diffeomorphism around 0. We claim that $\alpha(U) \cap H = \{e\}$ where U is a small neighborhood of 0 in V. If this is true, then the inclusion of H into G looks locally around e, via α, the same as the inclusion of $\mathfrak{h}$ into T_eG - so H is a submanifold of G around e.

Indeed, set $C := \{v \in V \mid \alpha(v) \in H\}$. Then C is a closed subset of V, and is closed under multiplication by scalars in $\mathbb{Z}$. We assume by contradiction that $C \cap U \neq \{0\}$ for every open $0 \in U$. This is an exercise that such C must contain a line through the origin, meaning that V intersects $\mathfrak{h}$ non-trivially, and we get a contradiction. □

Lemma 7.10. *We have*

$$exp(X+Y) = lim_{n\to\infty} \left(exp(\frac{1}{n}X) exp(\frac{1}{n}Y) \right)^n .$$

Proof. We have

$$exp(tX)exp(sY) = exp(tX + sY + \frac{1}{2}ts[X,Y] + O(r^3)).$$

So

$$\left(exp(\frac{1}{n}X) exp(\frac{1}{n}Y) \right)^n = exp(X + Y + \frac{1}{2n}[X,Y] + O(n^{-3})).$$

□

Claim 7.11. *Let G_1, G_2 be Lie groups, and $\phi : G_1 \to G_2$ a continuous group homomorphism. Then ϕ is smooth.*

Proof. Consider $Gr_\phi \subset G_1 \times G_2$, defined by $Gr_\phi = \{(g_1, g_2) \mid g_2 = \phi(g_1)\}$. Since ϕ is continuous, $Gr(\phi)$ is closed in $G_1 \times G_2$. Since ϕ is a group homomorphism, $Gr(\phi)$ is a subgroup of $G_1 \times G_2$. Thus, by claim 7.9, $Gr(\phi)$ is a Lie subgroup of $G_1 \times G_2$. The projection on the first factor gives us a bijective smooth morphism $Gr(\phi) \to G_1$. By corollary 7.8, it is an isomorphism of Lie groups. Thus ϕ is smooth, since it is the composition of the inverse to the projection onto the first factor $Gr(\phi) \to G_1$ with the projection onto the second factor $Gr(\phi) \to G_2$. □

This final claim says the two possible notions of finite-dimensional representations of G - those with continuous matrix coefficients and those with smooth matrix coefficients - coincide. In other words, $C(G)^{fin} \subset C^\infty(G)$ for a compact Lie group G.

7.4 Vector fields

Recall that (smooth) **vector fields** on a manifold M can be described as sections of the tangent bundle, and also as derivation of $C^\infty(M)$. For a vector field $X \in T(M)$, we have its **flow** $F_X(t,m)$, a smooth function $D(X) \to M$ where $D(X)$ (the **domain of definition**) is an open subset of $\mathbb{R} \times M$ intersecting each $\mathbb{R} \times \{m\}$ in an open interval containing zero, with the properties: $F_X(0,m) = m$ for all $m \in M$ and the derivative of $t \mapsto F_X(t,m)$ at t_0 is equal to $X_{F_X(t_0,m)}$. We also denote $F_X^t(m) := F_X(t,m)$.

The Lie bracket of two vector fields X, Y is defined, in the language of derivations, as

$$[X,Y](f) := X(Y(f)) - Y(X(f)) \; \forall f \in C^\infty(M).$$

An equivalent characterization is:

$$[X,Y]|_m = lim_{t\to 0} \frac{1}{t}\left((dF_X^t)^{-1} Y|_{F_X^t(m)} - Y|_m \right).$$

To give another interpretation, recall the lemma:

Lemma 7.12. *Let $\alpha, \beta : M \to N$ be morphisms of manifolds, let $m \in M$, and suppose that $\alpha(m) = \beta(m) =: n$ and $d_m\alpha = d_m\beta$. Then there is a well-defined quadratic homogenous function $Q_{\alpha,\beta} : T_mM \to T_nN$ such that in local charts $\beta - \alpha = Q_{\alpha,\beta} + O(r^3)$. For every function $f \in C^\infty(N)$, we have*

$$f \circ \beta - f \circ \alpha = d_n f \circ Q_{\alpha,\beta} + O(r^3).$$

We apply this lemma to $\alpha, \beta : \mathbb{R}^2 \to M$ given by $\alpha(t,s) = F_X(t, F_Y(s,m))$ and $\beta(t,s) = F_Y(s, F_X(t,m))$, around $0 \in \mathbb{R}^2$. We obtain $Q_{\alpha,\beta}(t,s) = t^2 \cdot ? + ts \cdot Z + s^2 \cdot ?$ where $Z, ? \in T_mM$; But plugging in $t = 0$ or $s = 0$ gives that $? = 0$. We now claim that $Z = [X,Y]_m$. Indeed, for every function $f \in C^\infty(M)$, one one hand we have

$$\frac{\partial^2}{\partial t \partial s}\left[f(\beta(t,s)) - f(\alpha(t,s))\right] = \frac{\partial^2}{\partial t \partial s}\left[d_m f \circ (tsZ)\right] = Z(f)$$

and on the other hand

$$\frac{\partial^2}{\partial t \partial s}\left[f(\beta(t,s)) - f(\alpha(t,s))\right] = \frac{\partial^2}{\partial t \partial s} f(\beta(t,s)) - \frac{\partial^2}{\partial t \partial s} f(\alpha(t,s)) = X(Y(f)) - Y(X(f)).$$

An important property:

Claim 7.13. *Let $X, Y \in T(M)$. Suppose that $[X,Y] = 0$. Then (for small enough s, t depending on m) $F_X(t, F_Y(s,m)) = F_Y(s, F_X(t,m))$.*

Proof. Notice that $[X,Y] = 0$ gives $dF_Y^s(X|_n) = X|_{F_Y^s(n)}$. Fix s, and consider the function $\phi : t \mapsto F_Y(s, F_X(t,m))$. We have $\phi(0) = F_Y(s,m)$. Furthermore, the derivative:

$$\frac{d}{dt}\phi(t) = dF_Y^s(X_{F_X(t,m)}) = X_{F_Y^s(F_X(t,m))} = X_{F_Y(s,F_X(t,m))}.$$

□

7.5 The exponential map

Denote by $T(G)$ the space of vector fields on G. $G \times G$ acts on $T(G)$ as usual, and we denote by $T(G)^G \subset T(G)$ the subspace of left-invariant vector fields.

Lemma 7.14. *The map $T(G)^G \to T_e(G)$ given by sending a vector field to its value at e is an isomorphism.*

Claim 7.15. *For any $X \in T_e(G)$, there exists a unique Lie group morphism $exp_X : \mathbb{R} \to G$ for which the induced linear map $d(exp_X) : \mathbb{R} \cong T_0(\mathbb{R}) \to T_e(G)$ sends $1 \mapsto X$.*

Proof. Denoting by X the corresponding left-invariant vector field by abuse of notation, we notice that $exp_X(t) = F_X(t, e)$, would $F_X(t, e)$ be defined for all $t \in \mathbb{R}$. This is because the formula $F_X(t, e)F_X(s, e) = F_X(s, F_X(t, e)) = F_X(t+s, e)$ holds, because the derivative of the left hand side w.r.t. s is $F_X(t, e)X_{F_X(s,e)} = X_{F_X(t,e)F_X(s,e)}$ (by the left-invariance of X). But now, we easily see that $F_X(t, e)$ exists for all $t \in \mathbb{R}$, because if it exists on some open interval containing $[-\epsilon, \epsilon]$, we extend it to an open interval containing $[-2\epsilon, 2\epsilon]$ by formulas like $F_X(\epsilon + t, e) = F_X(\epsilon, e)F_X(t, e)$...

□

Definition 7.16. We define the **exponential map** $exp : T_e(G) \to G$ by

$$exp(X) := exp_X(1).$$

Lemma 7.17. *The map $exp : T_e(G) \to G$ is smooth.*

Claim 7.18.

1. *$exp(tX) = exp_X(t)$ for every $X \in T_e(G)$ and $t \in \mathbb{R}$.*
2. *$exp(X + Y) = exp(X) \cdot exp(Y)$ if $X, Y \in T_e(G)$ are linearly dependent.*
3. *$exp(0) = e$.*
4. *$exp(nX) = exp(X)^n$ for $X \in T_e(G)$ and $n \in \mathbb{Z}$.*
5. *The differential $d(exp) : T_e(G) \to T_e(G)$ is equal to the identity map.*
6. *There exists an open subset $0 \in U \in T_e(G)$ such that $V := exp(U) \subset G$ is open, and $exp : U \to V$ is a diffeomorphism.*
7. *$exp(T_eG)$ generates G° as a group.*

Proof. The first claim says $exp_{tX}(1) = exp_X(t)$; This is true since $s \mapsto exp_X(st)$ is a Lie group morphism $\mathbb{R} \to G$ with differential tX, hence equal to $s \mapsto exp_{tX}(s)$.

The claims 2,3,4 follow easily from the first claim.

Claim 5: $d(exp)$ evaluated at X is the same as $d(exp \circ \alpha_X)$ evaluated at 1, where $\alpha_X : \mathbb{R} \to T_e(G)$ is given by $\alpha_X(t) = tX$. But $exp \circ \alpha_X = exp_X$, hence $d(exp)$ evaluated at X gives X.

Claim 6 follows from claim 5 by the inverse function theorem.

Claim 7: Obviously $C := exp(T_eG) \subset G^\circ$ (since T_eG is connected). By claim 6, C contains an open neighbourhood U of e. Set $V = U \cap U^{-1}$. Then $\cup_n C^n$ contains $\cup_n V^n$, which is an open subgroup of G°, hence equal to G°. □

Claim 7.19. *Let $\phi : G \to H$ be a morphism of Lie groups. Then $\phi \circ exp_G = exp_H \circ d_e\phi$.*

Proof. □

Claim 7.20. *Let $H \subset G$ be a Lie subgroup. Then for $X \in T_e(G)$, one has $X \in T_e(H)$ if and only if $exp_G(tX) \in H$ for all $t \in \mathbb{R}$.*

Proof. If $X \in T_e(H)$, then $exp_G(tX) \in H$ is clear by the previous claim applied to the inclusion $H \to G$. Conversely, The tangent to the curve $t \mapsto exp_G(tX)$ is X, so if the curve lies in the submanifold H, then X lies in the tangent space T_eH. □

7.6 The adjoint representation

Fix $g \in G$. The map $c_g : G \to G$ given by $x \mapsto gxg^{-1}$ is an automorphism sending e to e, hence its differential at e gives a linear automorphism of $T_e(G)$ which we denote by $Ad(g)$.

Lemma 7.21. *The morphism $G \to Aut(T_e(G))$ given by $g \mapsto Ad(g)$ is a (real) representation (called the **adjoint representation**).*

Claim 7.22. *We have $g \cdot exp(X) \cdot g^{-1} = exp(Ad(g)(X))$ for all $g \in G$ and $X \in T_e(G)$.*

Proof. Since both $g \cdot exp(tX) \cdot g^{-1}$ and $exp(Ad(g)(tX))$ are Lie group morphisms $\mathbb{R} \to G$, it is enough to check that they have the same derivative at $0 \in \mathbb{R}$... □

7.7 The Lie bracket

We will give various characterizations of the Lie bracket

$$[\cdot,\cdot] : T_e(G) \otimes_{\mathbb{R}} T_e(G) \to T_e(G).$$

It has the properties

$$[X, Y] = -[Y, X]$$

and

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]].$$

First definition is using the isomorphism $T_e(G) \cong T(G)^G \cong Der(C^\infty(G))^G$. Here, $Der(C^\infty(G))$ is the space of derivations of the algebra of smooth functions on G, and by $(\cdot)^G$ we mean G-invariants w.r.t. the left action. We define for two derivations D, E a new one given by:

$$[D, E](f) := D(E(f)) - E(D(f)).$$

One verifies that $[D, E]$ is indeed a derviation, and that if D, E are G-invariant, so is $[D, E]$.

Second definition is that $[X, Y]$ is the unique vector such that

$$exp(tX)exp(sY) = exp(tX + sY + \frac{1}{2}ts[X, Y] + O(r^3))$$

(r is the radius on the (t, s)-plane $\mathbb{R}^2$).

Second′ definition is that $[\cdot, \cdot]$ is the unique bilinear form such that

$$exp(X)exp(Y) = exp(X + Y + \frac{1}{2}[X, Y] + O(r^3))$$

(r is the radius on $T_eG \times T_eG$).

Third definition is by taking the differential of $Ad : G \to End(T_eG)$, obtaining a map $d(Ad) : T_eG \to End(T_eG)$ and setting $[X, Y] := d(Ad)(X)(Y)$.

Let us see that the second definition is equivalent to the first. Writing

$$exp(tX)exp(sY) = exp(tX + sY + tsZ + O(r^3))$$

(which is possible since $(t, s) \mapsto exp(tX)(exp(sY)$ and $(t, s) \mapsto exp(tX + sY)$ are identical up to order 1 around $(t, s) = (0, 0)$), by taking inverses we obtain

$$exp(sY)exp(tX) = exp(tX + sY - tsZ + O(r^3))$$

and thus we obtain

$$Q_{\alpha,\beta} = 2tsZ$$

where $\alpha(t, s) = exp(sY)exp(tX)$ and $\beta(t, s) = exp(tX)exp(sY)$. Since $\alpha(t, s) = F_X(t, F_Y(s, e))$ and $\beta(t, s) = F_Y(s, F_X(t, e))$ (where we identify X, Y with the corresponding left-invariant vector fields), we get $2Z = [X, Y]$.

We leave as an exercise to check that second and second′ definitions agree.

Let us see that the third definition is equivalent to the second′. By using $2'$, we have

$$exp(sX)exp(tY)exp(-sX) = exp(tY + st[X, Y]_{2'} + O(r^3)).$$

We obtain $Ad(exp(sX))(Y) = Y + s[X, Y]_{2'} + O(s^2)$ and so $[X, Y]_3 = [X, Y]_{2'}$.

Claim 7.23. *Let $X, Y \in T_eG$. If $[X, Y] = 0$, then $exp(X+Y) = exp(X)exp(Y) = exp(Y)exp(X)$.*

Proof. The expressions $exp(tX)exp(sY)$ and $exp(sY)exp(tX)$ are equal from the corresponding claim for flows of vector fields (applied to the left-invariant vector fields corresponding to X and Y). In particular, $t \mapsto exp(tX)exp(tY)$ is a group homomorphism, with derivative $X + Y$ at zero, hence $exp(t(X + Y)) = exp(tX)exp(tY)$. □

Claim 7.24. *Let $\phi : G \to H$ be a Lie group morphism. Then $d\phi : T_eG \to T_eH$ is a Lie algebra morphism, i.e. it commutes with the Lie bracket.*

7.8 The Lie bracket in matrix groups

For $G = GL(n, F)$, where $F \in \{\mathbb{R}, \mathbb{C}\}$, we have $T_e(G) = M(n, F)$ and $exp(X) = \sum_{n\geq 0} \frac{1}{n!} X^n$. The adjoint representation is given by conjugation: $Ad(g)(X) = gXg^{-1}$ and the Lie bracket is given by $[X, Y] = XY - YX$ (for example, we calcaulte

$$exp(tX)exp(sY) = I + tX + sY + \frac{1}{2}(t^2X^2 + 2tsXY + s^2Y^2) + \ldots$$

and

$$exp(sY)exp(tX) = I + tX + sY + \frac{1}{2}(t^2X^2 + 2tsYX + s^2Y^2) + \ldots$$

so that $exp(tX)exp(sY) - exp(sY)exp(tX) = ts(XY - YX) + \ldots$.

If $H \subset G$ is a closed subgroup, then T_eH consists of the matrices $X \in M(n, F)$ which satisfy $exp(tX) \in H$ for all $t \in \mathbb{R}$. The adjoint represetnation and Lie bracket are just induced by restriction from those for G.

7.9 Some examples

We have the Lie groups $GL(n, \mathbb{R}), GL(n, \mathbb{C})$. We have the compact Lie groups $O(n) \subset GL(n, \mathbb{R})$ and $U(n) \subset GL(n, \mathbb{C})$. Note that this is not immediately clear that $O(n)$ and $U(n)$ are submanifolds, but the non-trivial theorem above states that a closed subgroup is a Lie subgroup.

We also have the Lie groups $SO(n) \subset O(n)$ and $SU(n) \subset U(n)$.

The Lie group $SU(n)$ acts on S^{2n-1}, with stabilizer isomorphic to $SU(n-1)$. In particular $SU(2) \cong S^3$.

The Lie group $SO(n)$ acts on S^{n-1}, with stabilizer isomorphic to $SO(n-1)$. In particular $SO(2) \cong S^1$.

The Lie algebra of $O(n)$ is the subalgebra of $M(n, \mathbb{R})$ consisting of matrices X satisfying $exp(tX^t) = exp(-tX)$ for all $t \in \mathbb{R}$. This is equivalent to $X^t = -X$, i.e. we get the Lie algebra of skew-symmetric matrices.

Similarly, the Lie algebra of $U(n)$ is the Lie algebra of skew-Hermitian matrices (matrices X satisfying $= \overline{X^t} = -X$).

7.10 $SU(2)/\{\pm 1\} \cong SO(3)$

Consider the adjoint representation of $SU(2)$. This is a real three-dimensional representation. The kernel of this representation is ± 1. We obtain a morphism $SU(2)/\{\pm 1\} \to GL(3, \mathbb{R})$, which we can factor via $SU(2)/\{\pm 1\} \to O(3)$ by endowing the representation with a $SU(2)$-invariant inner product. Since $SU(2)$ is connected, this factors via $SU(2)/\{\pm 1\} \to SO(3)$. Since this is an injective morphism between manifolds of the same dimension, it has open image (by the "invariance of domain" theorem), hence it is surjective (because an open subgroup of a connected group must be the whole group). Since a bijective morphism of Lie groups is an isomorphism, we obtain the isomorphism of Lie groups $SU(2)/\{\pm 1\} \cong SO(3)$.

Notice that $diag(SO(2),\{1\}) \subset SO(3)$ is, up to conjugation, the subgroup fixing a vector. Also $T \subset SU(2)$ (the group of diagonal matrices), is the subgroup fixing the vector $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ in the adjoint representation. so we can set the isomorphism $SU(2)/\{\pm 1\} \cong SO(3)$ so that $T/\{\pm 1\}$ corresponds to $diag(SO(2),\{1\})$. The corresponding map $T \to SO(2)$ must be either $diag(e^{i\theta}, e^{-i\theta}) \mapsto \begin{pmatrix} cos(2\theta) & sin(-2\theta) \\ sin(2\theta) & cos(2\theta) \end{pmatrix}$ or the inverse of that. By conjugating by the element $diag(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1) \in O(3)$ those get exchanged.

Since $SU(2)$ has exactly one irreducible representation of each dimension $d \in \mathbb{Z}_{\geq 1}$, and -1 acts trivially exactly on the odd-dimensional ones, we get that $SO(3)$ has exactly one irreducible representation of each dimension $d \in 2\mathbb{Z}_{\geq 0} + 1$, and no more. Also, we see how those representations decompose w.r.t. $diag(SO(2),\{1\})$.

One can also notice that by considering the action of $SU(2)$ on the unit sphere in the adjoint representation, we obtain a fiber bundle $SU(2) \cong S^3 \to S^2$ with fiber $T \cong S^1$, which is the well-known Hopf fibration. It allows to see that $\pi_3(S^2) \cong \pi_3(S^2) \cong \mathbb{Z}$.

Also, we notice that we get that $SO(3)$ is diffeomorphic to $\mathbb{RP}^3$, the projective real 3-space (since $\mathbb{RP}^3$ can be thought of as S^3 modulo antipodality - which is the same as $SU(2)/\{\pm 1\}$).

7.11 The spherical harmonics for $SO(3)$

Let G be a compact group, and $H \subset G$ a closed subgroup. We are interested in the space $C(G/H)^{fin}$. Notice that this is the same as $(C(G)^{fin})^H$. By the Peter-Weyl decomposition, we obtain an isomorphism of G-representations

$$C(G/H)^{fin} \cong \oplus_E E \otimes (E^*)^H,$$

where E runs over representatitves of isomorphism classes of irreducible representations of G. Thus, to know how many times each irreducible representation of G occurs in $C(G/H)^{fin}$, we need to know how many H-invariant functionals each irreducible representation has.

Let us take $G = SO(3)$ and $H = diag(SO(2), 1)$ from above. We have an action of G on $S^2 \subset \mathbb{R}^3$, and the stabilizer of the point $(0,0,1)^t$ is H. So $G/H \cong S^2$. Notice that by above we see that for each irreducible representation E of G, one has $\dim E^H = 1$, and so also $\dim(E^*)^H = 1$ (by replacing E by E^*, which is also an irreducible representation of G).

We deduce that $C(S^3)^{fin}$ decomposes into the direct sum of irreducible representations of G, where each irreducible representation occurs exactly once.

7.12 Lie's theorems

We want to study the functor $Lie : G \mapsto T_e(G)$, from the category of Lie groups to the category of Lie algebras.

Theorem 7.25. *Let G, H be Lie groups.*

1. *If G is connected, $Lie : Hom(G, H) \to Hom(T_e(G), T_e(H))$ is injective.*
2. *If G is connected and simply-connected, $Lie : Hom(G, H) \to Hom(T_e(G), T_e(H))$ is surjective.*
3. *Lie is essentially surjective.*

Proof. To prove 1, notice that for $\phi : G \to H$, the differential $d\phi : T_eG \to T_eH$ determines the ϕ on $exp(T_eG) \subset G$. Since G° is the subgroup of G generated by $exp(T_eG)$, we get that $d\phi$ determines ϕ on G°, which is G in case G is connected.

We omit the proof of 2.

We omit the proof of 3, since it is not important for us. □

7.13 Faithful representations, complexification

Claim 7.26. *Let G be a compact Lie group. Then G is isomorphic to a Lie subgroup of $GL(n, \mathbb{C})$, for some n.*

Proof. We first notice that a compact Lie group satisfies the descending chain condition w.r.t. Lie subgroups (i.e. any descending chain of Lie subgroups must stabilize). This is because a Lie subgroup must either have a smaller dimension, or a fewer connected components. Now, given that we constructed a representation with kernel K, choosing some $e \neq g \in K$, we consider the direct sum of the current representation with a representation on which g acts non-trivially, and obtain a representation with kernel smaller than K. Proceeding like that, by the descending chain condition we will eventually get a faithful representation. □

In other words, the theorem says that compact Lie groups are linear. A standard example of a non-linear Lie group (i.e. a Lie group with is not isomorphic to any closed subgorup of $GL(n, \mathbb{C})$, for no n) is the universal cover of $SL(2, \mathbb{R})$.

To calculate $\pi_1(SL(2, \mathbb{R}))$, we use the action of $SL(2, \mathbb{R})$ on $\mathbb{R}^2 - \{0\}$. This action is transitive, with a stabilizer of some point being $U \subset SL(2, \mathbb{R})$, the subgroup of upper-triangular unipotent matrices. Thus, $SL(2, \mathbb{R})$ is the total space of a fibration over $\mathbb{R}^2 - \{0\}$ with contractible fiber U, hence $\pi_1(SL_2(\mathbb{R})) \cong \pi_1(\mathbb{R}^2 - \{0\}) \cong \mathbb{Z}$. Notice that analogously $\pi_1(SL(2, \mathbb{C})) \cong \pi_1(\mathbb{C}^2 - \{0\}) = 1$. We consider now G - the universal cover of $SL(2, \mathbb{R})$, and the composition $\alpha : G \to SL(2, \mathbb{R}) \to SL(2, \mathbb{C})$ (where the later map is the standard embedding). The corresponding Lie algebra map is the copmlexification $\mathfrak{g} \to \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$. Now, let $G \to H$ be a morphism, where T_eH has a complex structure. Then we can break $T_eG \to T_eH$ via $T_eG \to T_e(SL(2, \mathbb{C})) \to T_eH$ (by the universal property of complexification). By simple-conectedness we can lift those to maps $G \to SL(2, \mathbb{C}) \to H$, and the composition must be the original $G \to H$ by conectedness. Hence, we deduce that $G \to H$ can not be injective (because $G \to SL(2, \mathbb{C})$ isn't). This implies that G has no embedding into a matrix algebra $GL(n, \mathbb{C})$.

8 Maximal tori

8.1 Connected abelian Lie groups

Let G be a connected abelian Lie group.

Claim 8.1. *$exp : (\mathfrak{g}, +) \to (G, \cdot)$ is an epimorphism with discrete kernel.*

Claim 8.2. *Let V be a finite-dimensional vector space over $\mathbb{R}$, and $\Gamma \subset V$ a discrete subgroup. Then there exists a basis $e_1, \ldots, e_n$ of V and an integer $0 \leq k \leq n$ such that $\Gamma = \mathbb{Z}e_1 + \ldots + \mathbb{Z}e_k$. We have $k = n$ if and only if V/Γ is compact.*

Corollary 8.3. *G is isomorphic to $(\mathbb{R}/\mathbb{Z})^k \times \mathbb{R}^{n-k}$ for some $0 \leq k \leq n$.*

Definition 8.4. A **torus** is a compact connected abelian Lie group.

We saw that any torus is isomorphic to $(\mathbb{R}/\mathbb{Z})^k$ for some $k \in \mathbb{Z}_{\geq 0}$.

Definition 8.5. Let G be a Lie group. We say that an element $g \in G$ is a **topological generator** if the subset $g^{\mathbb{Z}} \subset G$ is dense. We say that G is **monogenic** if it admits a topological generator.

Claim 8.6. *A torus is monogenic.*

Proof. Let G be a torus and let us present $G \cong (\mathbb{R}/\mathbb{Z})^k$. We claim that any $g = (a_1, \ldots, a_k)$ for which $1, a_1, \ldots, a_k$ are linearly independent over $\mathbb{Q}$, is a topological generator. This property implies that $\chi(g) \neq 1$ for any non-trivial character $\chi : T \to \mathbb{C}^\times$. We claim now that the following holds (I think this is Weyl's theorem):

$$\int_G f \cdot d\mu = lim_{n\to\infty} \frac{\sum_{i=1}^n f(g^n)}{n}$$

for any $f \in C(G)$. Indeed, by the Peter-Weyl theorem it is immediate to reduce this to $f = \chi$ a character. For $\chi = 1$, this is clear. Thus, we are left to show that for a non-trivial character χ we have

$$lim_{n\to\infty} \frac{\sum_{i=1}^n f(g^n)}{n} = 0.$$

But we have (recalling that $\chi(g) \neq 0$)

$$\frac{\sum_{i=1}^n f(g^n)}{n} = \frac{\chi(g) \cdot (\chi(g)^n - 1)}{(\chi(g) - 1)n},$$

so clearly the desired limit equality holds.

Now, we claim that from this result the density of $g^{\mathbb{Z}} \subset G$ follows. Indeed, would it be not dense, we could find a positive non-zero continuous function f vanishing on $g^{\mathbb{Z}}$. Then we get a contradiction examining the equlity above for such an f. $\square$

Lemma 8.7 (Rigidity of tori)**.** *Let G be a torus, and $\Phi : M \times G \to G$ a smooth map, such that $\Phi_m(g) := \Phi(m, g)$ is an automorphism of G as a Lie group, for every $m \in M$. Then if M is connected, the automorphism Φ_m does not depend on m.*

Proof. Fix a torsion element $g \in G$. For every character χ of G, the image of $m \mapsto \chi(\Phi_m(g))$ lies in the discrete set of roots of unity of order $o(g)$; Hence $m \mapsto \chi(\Phi_m(g))$ is constant. Since the values of all the characters determine the element of g, we get that $m \mapsto \Phi_m(g)$ is constant. Since the subgroup of torsion element is dense in G, we get that $m \mapsto \Phi_m$ is constant. □

8.2 Maximal tori and the root decomposition

Let G be a compact connected Lie group.

Definition 8.8. A maximal torus $T \subset G$ is a closed subgroup, which is a torus, and which is not contained in any closed subgorup which a torus, except itself.

Obviously, maximal tori exist.

The main theorem on maximal tori, which we will prove later, is:

Theorem 8.9 (Main theorem on maximal tori)**.** *Let $T \subset G$ be a maximal torus. Then any element of G is conjugate to an element in T.*

Lemma 8.10. *Let T be a closed connected subgroup of G. Then T is a maximal torus in G if and only if $Lie(T)$ is a maximal abelian subalgebra in $Lie(G)$.*

Proof. Assume first that T is a maximal torus. Suppose that $X \in Lie(G)$ commutes with all the elements in $Lie(T)$. Consider S, the identity component of the closure of the subgroup generated by $T \cup exp(\mathbb{R}X)$. Then S is torus, hence $S = T$. But $Lie(T) + \mathbb{R}X \subset Lie(S)$, so that we get $X \in Lie(T)$.

Assume now that $Lie(T)$ is a maximal abelian subalgebra in $Lie(G)$. Then if T were contained in a bigger torus S, we would have a strict containment of $Lie(T)$ in $Lie(S)$, and thus $Lie(T)$ would not be a maximal abelian subalgebra in $Lie(G)$. □

Definition 8.11. Let $T \subset G$ be a maximal torus. The Weyl group $W(G, T)$ is defined as $N_G(T)/T$.

Remark 8.12. A bit more correct would be to define the Weyl group as $N_G(T)/Z_G(T)$. However, the former definition is more convenient for us now, and later we will see that in fact $Z_G(T) = T$ holds.

Lemma 8.13. *The Weyl group $W(G, T)$ is finite.*

Proof. Considering the map $N_G(T) \times T \to T$, given by $(g, t) \mapsto gtg^{-1}$, we get by rigidity of tori that $N_G(T)^{\circ} \subset C_G(T)$. Thus $C_G(T)$ is of finite index in $N_G(T)$. Moreover, T is of finite index in $C_G(T)$, because $Lie(C_G(T)) = Lie(C_G(\mathfrak{t})) = C_{\mathfrak{g}}(\mathfrak{t})$, so $Lie(C_G(T)) = \mathfrak{t}$ by $\mathfrak{t}$ being a maximal abelian subalgebra of $\mathfrak{g}$. □

We fix a maximal torus $T \subset G$, and consider the adjoint representation of T on $\mathfrak{g}$. We have $\mathfrak{t} = \mathfrak{g}^{T,1}$ (the subspace on which T acts trivially), basically by the above lemma. We thus have an isotypical decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}^{T,\neq 1}$. To work with $\mathfrak{g}^{T,\neq 1}$, it is not necessary, but is convenient for us, to pass to complexification $\mathfrak{g}_{\mathbb{C}}$. Then we get

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\chi \in R(G,T)} \mathfrak{g}_{\mathbb{C}}^{T,\chi},$$

where $R(G,T)$ (the set of **roots**) is the set of non-trivial characters χ of T for which $\mathfrak{g}_{\mathbb{C}}^{T,\chi} \neq 0$. We notice that since $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$, the set $R(G,T)$ is closed under taking inverse and we have $\dim \mathfrak{g}_{\mathbb{C}}^{T,\chi} = \dim \mathfrak{g}_{\mathbb{C}}^{T,\chi^{-1}}$ for all χ. We will later see that $\dim \mathfrak{g}_{\mathbb{C}}^{T,\chi} = 1$ for every $\chi \in R(G,T)$.

8.3 The map $c : G/T \times T \to G$, and a proof of the main theorem on maximal tori

To prove the theorem that every element of G is conjugate to an element of T, we study the map $c : G/T \times T \to G$ given by $c([g],t) = gtg^{-1}$. Notice that the theorem is equivalent to the statement that c is surjective. We will use the following statement from differential topology, the theorem on mapping degrees:

Theorem 8.14. *Let M, N be compact connected oriented manifolds of the same dimension n. Let $\phi : M \to N$ be a morphism of manifolds. Then there exists an integer $deg(f)$, the* ***mapping degree*** *of f, such that for every form $\omega \in \Omega^n(N)$ one has*

$$\int_M \phi^* \omega = deg(f) \cdot \int_N \omega.$$

Moreover, for an regular value $n \in N$ of ϕ, the number $deg(\phi)$ equals the number of preimages of n at which ϕ preserves orientation, minus the number of perimages of n at which ϕ reverses orientation. In particular, $deg(\phi) \neq 0$ implies that ϕ is surjective.

A manifold is oriented if we choose an orientation of each tangent space to the manifold, in a continuous fashion. On an oriented manifold we have a canonical integration of continuous, compactly supported top-forms. Given a morphism $\phi : M \to N$ between oriented manifolds, at each regular point $m \in M$ (i.e. such that $d_m\phi$ is surjective, and hence an isomorphism), we have that $d_m\phi$ is preserves or reverses orientation.

We fix orientations for $\mathfrak{g}$ and $\mathfrak{t}$. Then $\mathfrak{g}/\mathfrak{t}$ gets an induced orientation. By using translation on the left, we obtain from those orientations of $G, T, G/T$. For the last one, one needs to notice that the adjoint action of T on $\mathfrak{g}/\mathfrak{t}$ preserves orientation. Indeed, this follows from T being connected.

Now, the idea is to find a regular value of c, using which it will be easy to calculate the degree $deg(c)$.

Proposition 8.15. *Let t be a generator of T. Then $c^{-1}(t)$ is in bijection, via projecting on the first factor, with $W(G,T)$, and c preserves orientation at each point of $c^{-1}(t)$. In particular, $deg(c) = |W(G,T)|$ and c is thus surjective.*

Proof. If $([g], s) \in c^{-1}(t)$, then $t \in g^{-1}Tg$ so, since t is a generator of T, we have $T \subset g^{-1}Tg$, so $T = g^{-1}Tg$, i.e. $g \in N_G(T)$. From this we see that $c^{-1}(t)$ is in bijection, via projecting on the first factor, with $N_G(T)/T$.

Let us now calculate the differential of c at some point $([g_0], t_0)$.

For this, let us calculate the differential of $\widetilde{c} : G \times T \to G$ given by $\widetilde{c}(g,t) = gtg^{-1}$, at some point (g_0, t_0). We identify tangent spaces to G and T at various points with the tangents space at identity, via the left-invariance. So, the differential at (g_0, t_0) is modeled by the map $\mathfrak{g} \oplus \mathfrak{t} \to \mathfrak{g}$ given by the differential at (e,e) of

$$G \times T \xrightarrow{(g_0 \cdot, t_0 \cdot)} G \times T \xrightarrow{\widetilde{c}} G \xrightarrow{(g_0 t_0 g_0^{-1})^{-1} \cdot} G.$$

This map is calculated to be the same as

$$G \times T \xrightarrow{([t_0^{-1}, pr_1], \widetilde{c})} G \times G \xrightarrow{mult} G \xrightarrow{g_0 \cdot (\cdot) \cdot g_0^{-1}} G.$$

Its differential at (e,e) is

$$\mathfrak{g} \oplus \mathfrak{t} \xrightarrow{((Ad(t_0^{-1}) - id) \circ pr_1, inc_{\mathfrak{g}}^{\mathfrak{t}})} \mathfrak{g} \oplus \mathfrak{g} \xrightarrow{add} \mathfrak{g} \xrightarrow{Ad(g_0)} \mathfrak{g}$$

(where $inc_{\mathfrak{g}}^{\mathfrak{t}}$ is the inclusion). To summarize, we obtain that the differential of $\widetilde{c}$ at the point (g_0, t_0) is modeled by the map $\mathfrak{g} \oplus \mathfrak{t} \to \mathfrak{g}$ given by

$$(X, H) \mapsto Ad(g_0)[Ad(t_0^{-1})X - X + H].$$

Now, to model the differential of c at $([g_0], t_0)$, we identify $T_{[g_0]}(G/T)$ with $T_{[e]}(G/T)$ via left multiplication by g_0. Then we get that the differential of c at $([g_0], t_0)$ is modeled by $\mathfrak{g}/\mathfrak{t} \oplus \mathfrak{t} \to \mathfrak{g}$ given by

$$(X + \mathfrak{t}, H) \mapsto Ad(g_0)[Ad(t_0^{-1})X - X + H].$$

Since G is connected, it is clear that $Ad(g_0)$ is an orientation-preserving automorphism of $\mathfrak{g}$. Also, we identify $\mathfrak{g}/\mathfrak{t}$ with $\mathfrak{g}^{T, \neq 1}$. We see that it is enough for us to study the determinant of the map $\mathfrak{g} \to \mathfrak{g}$ given by

$$X \mapsto Ad(t_0^{-1})X - X$$

for $X \in \mathfrak{g}^{T, \neq 1}$ and

$$X \mapsto X$$

for $X \in \mathfrak{t}$. We can pass to the complexification. Then we get that the determinant is

$$\prod_{\chi \in R} (\chi(t_0)^{-1} - 1)^{\dim \mathfrak{g}^{T,\chi}} = \prod_{\chi \in R/\pm} |\chi(t_0)^{-1} - 1|^{2 \dim \mathfrak{g}^{T,\chi}}.$$

So, if t_0 is a generator of T, so that $\chi(t_0) \neq 1$ for every $\chi \neq 1$, we obtain that c is orientation-preserving at all points of $c^{-1}(t_0)$. ☐

8.3.1 Weyl's integration formula

Let us establish Wey's integration formula. We denote by ω_G ($\omega_T, \omega_{G/T}$) a left G-invariant (T-invariant, G-invariant) non-vanishing top form on G (T , G/T). Then the calculations above show that

$$c^*(\omega_G) = K \cdot \prod_{\chi \in R} (\chi(t)^{-1} - 1)^{\dim \mathfrak{g}_{\mathbb{C}}^{T,\chi}} \cdot (\omega_{G/T} \boxtimes \omega_T),$$

where K is a non-zero constant. Let us denote $\Delta(t) := \prod_{\chi \in R} (\chi(t)^{-1} - 1)^{\dim \mathfrak{g}_{\mathbb{C}}^{T,\chi}}$. We thus obtain, for a function $f \in C(G)$:

$$\int_G f d\mu_G = \int_G f \omega_G = \frac{K}{|W|} \int_{G/T \times T} f(gtg^{-1}) \cdot \Delta(t) \cdot \omega_{G/T} \boxtimes \omega_T = \frac{K}{|W|} \int_T \Delta(t) \left(\int_{G/T} f(gtg^{-1}) d\mu_{G/T} \right) d\mu_T,$$

where $\mu_G, \mu_T, \mu_{G/T}$ are invairiant measures and K is some non-zero constant. We can rewrite this as

$$\int_G f d\mu_G = \frac{K}{|W|} \int_T \Delta(t) \left(\int_G f(gtg^{-1}) d\mu_G \right) d\mu_T,$$

where again K is some non-zero constant, and the Haar measures are normalized such that the total mass of G and T is 1.

Claim 8.16. *In the last formula (with the normalization $\int_T \mu_T = \int_G \mu_G = 1$), we have $K = 1$.*

Proof. complete ☐

Finally, let us summarize, giving Weyl's integration formula:

Corollary 8.17. *Let μ_G, μ_T be Haar measures on G, T, normalized so that the total mass is 1. Then for every $f \in C(G)$, we have:*

$$\int_G f d\mu_G = \frac{1}{|W|} \int_T \Delta(t) \left(\int_G f(gtg^{-1}) d\mu_G \right) d\mu_T.$$

In particular, for a central function $f \in C^c(G)$, we have:

$$\int_G f d\mu_G = \frac{1}{|W|} \int_T \Delta(t) f(t) d\mu_T.$$

8.4 A second proof of the main theorem on maximal tori

This subsection is with errors, need $d_m\alpha - id$ instead of $d_m\alpha$!!

We fix an element $g \in G$. We consider the morphism $\alpha_g : G/T \to G/T$ given by $xT \mapsto gxT$. We notice that the possibility of conjugating g into T is equivalent to α_g having a fixed point.

To count fixed points, we will use the Lefshcetz fixed point theorem (weak form):

Theorem 8.18 (Lefschetz fixed point theorem)**.** *Let M be a compact manifold, and $\alpha : M \to M$ an endomorphism. If α has a finite number of fixed points, then the Lefschetz number*

$$\Lambda_\alpha := \sum_{i \geq 0} (-1)^i Tr(\alpha, H_i(M, \mathbb{Q}))$$

is equal to

$$F_\alpha := \sum_{m \in Fix(\alpha)} i_\alpha(m),$$

where $i_\alpha(m)$ is ± 1, according to the sign of the determinant of $d_m \alpha$.

In particular, if $\Lambda_\alpha \neq 0$, then α has at least one fixed point.

Now the idea is that the Lefschetz number depends only on the homotopy class of α. Notice that in our case, since G is connected, all the maps α_g are homotopic. Hence, it is enough to show that $\Lambda_{\alpha_g} \neq 0$ for some one comfortable g! We again take $g = t_0$ to be a generator of T. Then we see that $Fix(\alpha_{t_0}) = N_G(T)/T$. It is a finite set, and so it is enough to show that $i_{\alpha_{t_0}}(g) = 1$ for every $g \in N_G(T)/T$. By using left translation, we model $d_{gT}\alpha_{t_0}$ as the differential at eT of $G/T \to G/T$ given by $xT \mapsto g^{-1}t_0 gxT$, which is the same as $xT \mapsto g^{-1}t_0 gx(g^{-1}t_0 g)^{-1}T$. Thus, our model map is $\mathfrak{g}/\mathfrak{t} \to \mathfrak{g}/\mathfrak{t}$ given by

$$X + \mathfrak{t} \mapsto Ad(g^{-1}t_0 g)X + \mathfrak{t}.$$

Similarly to before, the determinant of this map is

$$\prod_{\chi \in R(G,T)/\pm 1} |\chi(g^{-1}t_0 g)|^{2 \dim \mathfrak{g}_{\mathbb{C}}^{T,\chi}},$$

which is positive.

To summarize, we see that the Lefschetz number Λ_{α_g} is equal to $|W|$ for every $g \in G$. In particular, the Euler characteristic $\Lambda_{id} = |W|$.

8.5 Corollaries of the main theorem

Corollary 8.19. *Any element of G is contained in a maximal torus.*

Proof. Clear, since if $hgh^{-1} \in T$, then $g \in h^{-1}Th$, and clearly $h^{-1}Th$ is a maximal torus in G. ☐

Corollary 8.20. *Any maximal torus $S \subset G$ is conjugate to T.*

Proof. As we saw before, S is monogenic; i.e. there exists $s \in S$ such that $\overline{s^{\mathbb{Z}}} = S$. Now, if $hsh^{-1} \in T$ for some $h \in G$, then we see that $hSh^{-1} \subset T$. Since hSh^{-1} is a maximal torus, we get $hSh^{-1} = T$. ☐

Definition 8.21. The **rank** of G is the dimension of a maximal torus in G.

Corollary 8.22. *The map $exp : T_eG \to G$ is surjective.*

Proof. G is the union of maximal tori, and the exponential map of a torus is surjective. □

Corollary 8.23. *Let $S \subset G$ be a subtorus. Then $Z_G(S)$ is equal to the union of maximal tori containing S. In particular, $Z_G(T) = T$.*

Proof. Let $g \in Z_G(S)$. Let S_1 be the closure of the subgroup generated by $S \cup \{g\}$. We claim that S_1 is monogenic. If so, S_1 is contained in some maximal torus (since its generator does), and we get what we want. To see that S_1 is monogenic, notice that $S \subset S_1^\circ$, and hence g generates S_1/S_1°, which is a finite group. Hence S_1/S_1° is a cyclic finite group.Denote $m := |S/S_1^\circ|$. Since S_1° is a torus, we can find a generator s of S_1°. We want to find $s_1 \in S$ such that $(gs_1)^m = s$. If this is done, than $\overline{(gs_1)^{\mathbb{Z}}}$ contains $\overline{s^{\mathbb{Z}}} = S_1^\circ$, but also contains gS_1°, so must be the whole of S_1. The equation is equivalent to $s_1^m = sg^{-m}$. Sicne $sg^{-m} \in S_1^\circ$ and S_1° is divisble, the equation is solvable. □

Corollary 8.24. *Let $S \subset G$ be a subtorus. Then $Z_G(S)$ is connected.*

Proof. This is clear, since $Z_G(S)$ is the union of connected subgroups by the previous corollary. □

Corollary 8.25. *Let $X \in Lie(G)$. Then $Z_G(X)$ is connected.*

Proof. Notice that $Z_G(X) = Z_G(\overline{exp(\mathbb{R}X)})$, and $\overline{exp(\mathbb{R}X)}$ is a torus. □

Corollary 8.26. *The center $Z_G(G)$ equals the intersection of all maximal tori.*

Proof. $Z_G(G) \subset \cap_T Z_G(T) = \cap_T T$. On the other hand, $\cap_T T = \cap_T Z_G(T) = Z_G(\cup_T T) = Z_G(G)$. □

Corollary 8.27. *If $t_1, t_2 \in T$ are conjugate in G, then they are also conjugate in $N_G(T)$.*

Proof. Let $gt_1g^{-1} = t_2$. Then T, gTg^{-1} are maximal tori in the connected compact Lie group $Z(t_2)^\circ$. Hence by the main theorem, T, gTg^{-1} are conjugate in $Z(t_2)^\circ$. So let $h \in Z(t_2)^\circ$ be such that $hgT(hg)^{-1} = T$. Then $hg \in N_G(T)$, and $(hg)t_1(hg)^{-1} = t_2$. □

Corollary 8.28. *Restriction gives a well-defined bijection $C(G)^{cent} \to C(T)^W$ (of central continuous functions on G and W-invariant continuous functions on T).*

Proof. By the previous corollary and the main theorem, we have clearly a bijection on the level of functions (not neccesarily continuous). Thus what we need to show is that given a function $f \in C(G)^{cent}$ such that $f|_T$ is continuous, also f is continuous.

Indeed, notice that $c^{-1}f = pr_2^* f|_T$, where $c : G/T \times T \to G$ is the map we studied, and $pr_2 : G/T \times T \to T$ is the projection on the second factor. Now, c is a surjective map between compact spaces, and hence a quotient map. Thus, since $c^{-1}f$ is continuous, so is f. □

8.6 More about the map $c : G/T \times T \to G$

Lemma 8.29. *Let $\phi : M \to N$ be a proper map between manifolds of the same dimension. Then the set of regular values $N^{\phi-reg}$ is open in N, and $\phi : \phi^{-1}(N^{\phi-reg}) \to N^{\phi-reg}$ is a covering map.*

Let us notice that our map $c : G/T \times T \to G$ has the following equivariant structure. We let G act on $G/T \times T$ by acting on the first factor by left regular action. We let G act on G by conjugation. Then c is a G-morphism. Moreover, we let $W(G,T)$ act on $G/T \times T$ by $w([g],t) := ([gw^{-1}], wtw^{-1})$ and on G trivially. Then c is also a $W(G,T)$-morphism. Also, recall that we saw that the differential of c at a point $([g_0], t_0)$ is an isomorphism if and only if $\chi(t_0) \neq 1$ for every $\chi \in R(G,T)$.

Let us denote by $G^{reg} \subset G$ the set of regular values of the map c. Also, denote $T^{reg} = G^{reg} \cap T$.

Claim 8.30. *The subset $G^{reg} \subset G$ is open, with complement of measure zero, and stable under conjugation in G. We have $c^{-1}(G^{reg}) = G/T \times T^{reg}$.*

Proof. That G^{reg} is open follows from the lemma above. That its complement is of measure zero is given by Sard's lemma. It is clear that G^{reg} is stable under conjugation in G, because c is G-equivariant as above. From this stability under conjugation, the last claim is also clear. □

Claim 8.31. *For $g \in G$, TFAE:*

1. *$g \in G^{reg}$.*
2. *For some maximal torus S containing g, we have $\chi(g) \neq 1$ for every $\chi \in R(G,S)$.*
3. *$\dim Z_G(g) = rkG$.*
4. *$Z_G(g)^\circ$ is a maximal torus.*
5. *g is contained in a unique maximal torus.*
6. *For some maximal torus S containing g, we have $Z_G(g) \subset N_G(S)$.*

Proof. (1) $\Leftrightarrow$ (2): By conjugating, we may assume $S = T$ (and so $g = t \in T$). Then elements in $c^{-1}(t)$ have second coordinate conjugate in G to t, and hence conjugate in W to t. Thus, $\chi(t) \neq 1$ for all $\chi \in R(G,T)$ if and only if $\chi(t') \neq 1$ for all $\chi \in R(G,T)$ and all t' - second coordinates of elements in $c^{-1}(t)$. By the calculation when is the differential of c an isomorphism - we are done.

(2) $\Leftrightarrow$ (3): The condition of (2) is equivalent to $Z_{\mathfrak{g}}(g) = Lie(S)$, which is equivalent to $\dim Z_{\mathfrak{g}}(g) = rkG$, and thus equivalent to (3) because $\dim Z_G(g) = \dim Z_{\mathfrak{g}}(g)$.

(3) $\Rightarrow$ (4): If S is a maximal torus containing g, we have $S \subset Z_G(g)^\circ$, and by comparing dimensions we get equality.

(4) $\Rightarrow$ (5): If S is a maximal torus containing g, we have $S \subset Z_G(g)^\circ$, and since $Z_G(g)^\circ$ is a torus we get equality.

(5) $\Rightarrow$ (6): Let S be the maximal torus containing g. We must have $Z_G(g) \subset N_G(S)$, because otherwise, if we take $h \in Z_G(g) - N_G(S)$, we obtain a maximal torus hSh^{-1} containing g and different from S.

(6) $\Rightarrow$ (3): We get $\dim Z_G(g) \leq rkG$, and so $\dim Z_G(g) = rkG$ (because the centralizer of an element always contains a maximal torus, hence always of dimension at leas rkG).

□

Claim 8.32. *$c^{-1}(G^{reg}) \to G^{reg}$ is a $W(G,T)$-covering map.*

Proof. We notice that $W(G,T)$ acts freely on G/T, and hence on $G/T \times T$. Hence, we only need to show that $W(G,T)$ acts transitively on the fibers. Let $g \in G^{reg}$, and $([h],t),([k],s) \in c^{-1}(g)$. It is enough to show that $k^{-1}h \in W(G,T)$. We denote conjugation by $*$. We have $k^{-1}h * t = s$, so t and s are conjugate in G, and hence in $N_G(T)$. So let $w \in N_G(T)$ be such that $w * t = s$. Then $w^{-1}k^{-1}h * t = t$, i.e. $w^{-1}k^{-1}h \in Z_G(t)$. So by the previous claim $w^{-1}k^{-1}h \in N_G(T)$, and thus $k^{-1}h \in N_G(T)$. □

For $\theta \in R(G,T)$, let us denote $U_\theta := Ker(\theta) \subset T$. Then U_θ is of dimension $\dim T - 1$. We have $T^{reg} = T - \cup_{\theta \in R(G,T)} U_\theta$.

Claim 8.33. *G^{reg} is the complement of a subset of codimension ≥ 3, i.e. a subset which lies in the image of a smooth map from a compact manifold of dimension $\leq \dim G - 3$.*

Proof. We consider the maps $G/Z_G(U_\theta) \times U_\theta \to G$ given by $([g],t) \mapsto gtg^{-1}$. Then their images cover $G - G^{reg}$. Notice that $\dim Z_G(U_\theta) = \dim Z_{\mathfrak{g}}(U_\theta) \geq \dim(\mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}^{T,\theta} \oplus \mathfrak{g}_{\mathbb{C}}^{T,\theta^{-1}}) = \dim T + 2$. □

Corollary 8.34. *The map $\pi_i(G^{reg}) \to \pi_i(G)$ (via the inclusion $G^{reg} \to G$) is an isomorphism for $i = 0, 1$ and a surjection for $i = 2$.*

Proof. This is some general property of complements to subsets of codimension ≥ 3. □

Claim 8.35. *We have $\pi_2(G) = 0$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} G/T \times T^{reg} & \longrightarrow & G/T \times T \\ \downarrow & & \downarrow c \\ G^{reg} & \longrightarrow & G \end{array}$$

The bottom arrow is surjective on π_2 as we metioned. The left arrow is surjective on π_2 since it is a covering map (more precisely, we mean that the map from any connected component of the upper space gives a surjection on π_2). Thus, it is enough to show that the map c is zero on π_2. Notice that $\pi_2(T) = 0$, and thus $\pi_2(G/T \times \{e\}) \to \pi_2(G/T \times T)$ is surjective. But the composition $G/T \times \{e\} \to G/T \times T \to G$ is trivial, and we are done. □

Claim 8.36. *We have $\pi_1(G/T) = 0$, and so $\pi_1(T) \to \pi_1(G)$ is surjective.*

Proof. That $\pi_1(T) \to \pi_1(G)$ is surjective follows from $\pi_1(G/T) = 0$ by the long exact sequence of homotopy groups of the fibration $G \to G/T$.

Consider the maps $f_p : G/T \xrightarrow{id \times p} G/T \times T \xrightarrow{c} G$ where $p \in T$. Notice that all these maps are homotopical. Since $\pi_1(G)$ is abelian and f_p are all homotopical, $\pi_1(f_p)$ are all "the same". Thus, to show that $\pi_1(G/T)$ is trivial, it is enough to show that $\pi_1(f_p)$ is trivial for some p, and injective for some other p.

For $p = e$, we have $f_p \equiv e$.

For $p \in T^{reg}$, we get a factorization

$$\pi_1(f_p) : \pi_1(G/T, e) \xrightarrow{id \times p} \pi_1(G/T \times T^{reg}, (e, p)) \xrightarrow{c} \pi_1(G^{reg}, p) \to \pi_1(G, p);$$

Notice that all the maps here are injective (the second since c is a covering map over G^{reg}).

□

8.7 The case of $U(n)$

We fix $G = U(n)$ and T - the diagonal matrices in $U(n)$. We denote by $e = (e_1, \dots, e_n)$ the standard basis for $\mathbb{C}^n$. We recall that $N_G(T)$ consists of matrices permuting the lines $Sp\{e_i\}$. We then have $W(G,T) \cong S_n$.

Notice that we can think of a matrix in T as an ordered list of n numbers (eigenvalues). We then think of $T//W(G,T)$ as multiset of n numbers, or equivalently as a monic polynomial of degree n (whose roots are the multiset of numbers). The equality of classes $G//G \cong T//W(G,T)$ is interpreted as sending a matrix to its characteristic polynomial (or multiset of eigenvalues).

We define a flag in $\mathbb{C}^n$ to be a list $\mathcal{F} = (V_0, V_1, \dots, V_n)$ of subspaces of $\mathbb{C}^n$, such that $\dim V_i = i$ and $V_i \subset V_{i+1}$ for every $0 \leq i \leq n-1$. Given a basis $f = (f_1, \dots, f_n)$ of $\mathbb{C}^n$, we get a flag $\mathcal{F}_f$ for which $V_i = Sp\{f_1, \dots, f_i\}$ (and every flag is of this form for some basis, defined uniquely up to a triangular change-of-basis matrix). We have the standard flag $\mathcal{F}_e$.

The group G acts on the space of flags. The stabilizer of $\mathcal{F}_e$ consists of the unitary upper-triangular matrices, hence is T. Moreover, the action is transitive; This is the Gram-Schmidt algorithm. Thus, we identify G/T with the space of flags.

We now can think of the map $c : G/T \times T \to G$ as sending a pair $(\mathcal{F}, (t_1, \dots, t_n))$ to the unique unitary matrix g which preserves the flag and such that g acts on V_i/V_{i-1} by t_i. Or, we can think of the subset $S \subset G/T \times G$ consisting of pairs $([g], h)$ such that $g^{-1}hg \in T$; then c is isomorphic to the projection onto the second factor $S \to G$. In other words, elements of S are pairs, consisting of a matrix g and a flag it preserves.

The set $G^{reg} \subset G$ consists of the matrices with n distinct eigenvalues. Indeed, T^{reg} is given by the condition $t_i t_j^{-1} \neq 1$ for all $i < j$. A matrix with n distinct eigenvalues has exactly $n!$ flags which it preserves.

8.8 Some examples of maximal tori

In $SU(n)$, the subgroup of diagonal matrices is a maximal torus. More generally, the subgroup of matrices which are diagonal in some orthonormal basis is a maximal torus, and any maximal torus is of that shape. So, the rank of $SU(n)$ is n.

Let's consider $SO(n)$ now. If n is even, consider the subgroup $diag(SO(2), SO(2), \ldots, SO(2))$. It is a maximal torus, so the rank of $SO(n)$ is $n/2$. If n is odd, we consider the subgroup $diag(SO(2), SO(2), \ldots, SO(2), 1)$, which is a maximal torus, and so the rank of $SO(n)$ is $(n-1)/2$ in this case.

Let us find the root space decomposition for $G = U(n)$ (and T - the diagonal matrices). The Lie algebra $Lie(G)$ has a basis iE_{jj}, and $E_{jk} - E_{kj}$ and $iE_{jk} + iE_{jk}$ (for $k > j$). Thus $Lie(T) = \oplus_j \{r \cdot E_{jj}\}_{r \in \mathbb{R}}$ and $V_{jk} := \{zE_{jk} - \overline{z}E_{kj}\}_{z \in \mathbb{C}}$ and we have

$$Lie(G) = Lie(T) \oplus \bigoplus_{k>j} V_{jk}.$$

We compute $Ad(diag(t_l))E_{jk} = t_j t_k^{-1} E_{jk}$, and so $diag(t_l)$ acts on V_{jk} by multiplying the parameter z by $t_j t_k^{-1}$.

Thus, the above decomposition is the isotypical decomposition of $Lie(G)$ as a real T-representation. To decopmose the copmlexification, we write $(V_{jk})_{\mathbb{C}} = U_{jk} \oplus U_{kj}$ where $U_{jk} := Sp\{E_{jk} +\}$

As an example, let us consider the Weyl group $W(G,T)$ where $G = U(n)$ and T is the subgroup of diagonal matrices. A matrix normalizing T must permute the eigenspaces of the operators in T, and thus must be a "permutation matrix up to scalars" (i.e. a matrix, every row of which has exactly one non-zero entry). Thus, $W(G,T) = N_G(T)/T = S_n$.

9 The reflections in the Weyl group

9.1 Connected compact Lie groups of rank 1

Claim 9.1. *Suppose that $rk(G) = 1$, and $G \neq T$. Then $|W(G,T)| = 2$ and* $\dim G = 3$.

Proof. We first show that $|W(G,T)| = 2$. Notice that the torus T admits two automorphisms - id and $-id$, so all we need to show is that $W(G,T) \neq 1$. We fix a G-invariant inner product on $\mathfrak{g}$, and consider a unit vector $H \in \mathfrak{t} \subset \mathfrak{g}$. Denoting by $S(\mathfrak{g})$ the sphere of unit vectors in $\mathfrak{g}$, we consider the map $\phi : G/T \to S(\mathfrak{g})$ given by $gT \mapsto Ad(g)H$. This map ϕ is continuous and injective, hence by the invariance of domain theorem, since G/T and $S(\mathfrak{g})$ are both manifolds of dimension $\dim(G) - 1$, the map ϕ has open image. Since G/T is compact, ϕ

also has closed image. Hence, since $S(\mathfrak{g})$ is connected, ϕ must be surjective (in fact, we see that ϕ is a homeomorphism, and even a diffeomorphism). Hence, in particular, there exists $g \in G$ satisfying $Ad(g)H = -H$. Clearly such g represents a non-trivial element of $N_T(G)/T$.

To show that $\dim G = 3$, we consider the fibration $G \to G/T$ with fiber T, and obtain an exact sequence of homotopy groups $\pi_2(G/T) \to \pi_1(T) \to \pi_1(G)$. Now, take a loop $\gamma : [0,1] \to T$ (based at e). Also, take an element $g \in N_G(T) - T$. Then $g\gamma g^{-1} = \gamma^{-1}$, but on the other hand $g\gamma g^{-1}$ is homotopic to γ, since G is connected (and thus g connectable to e). Thus we obtain that $\gamma = \gamma^{-1}$ in $\pi_1(G)$, so $2\pi_1(T)$ sits in the kernel of the map $\pi_1(T) \to \pi_1(G)$. Hence, we obtain $\pi_2(G/T) \neq 0$. But as $\phi : G/T \cong S(\mathfrak{g})$, we must have then $\dim \mathfrak{g} = 3$ (because S^2 is the only sphere with non-trivial π_2). □

Claim 9.2. *Suppose that $rk(G) = 1$, and $G \neq T$. Then G is isomorphic to $SO(3)$ or $SU(2)$.*

Proof. Considering a G-invariant inner product on $Lie(G)$, we get a morphism $G \to O(3)$. Since G is connected, we get $G \to SO(3)$. The kernel of this morphism is the center of G. The center of G is contained in T, and can not be equal to T, since $Z_G(T) = T$. Thus the center of G is finite. So $G/Z_G(G) \to SO(3)$ is an injective map of manifolds of the same dimension, hence is an open map. Thus the image is an open subgorup, hence also closed, hence all of $SO(3)$. We thus obtain an isomorphism $G/Z_G(G) \cong SO(3)$. So G is a covering group of $SO(3)$, and thus isomorphic to $SO(3)$ or $SU(2)$. □

9.2 The reflections

Claim 9.3. *For $\theta \in R(G,T)$, there is a unique element $1 \neq s_\theta \in W(G,T)$ which is trivial on $U_\theta := Ker(\theta)$.*

Proof. To show the uniqueness of such an element we observe that if we choose a G-invariant inner product on $\mathfrak{g}$, then $Ad(s_\theta) : \mathfrak{t} \to \mathfrak{t}$ is an orthogonal transformation fixing the hyperplane $Ker(d\theta)$, so can only be identity or orthogonal reflection through this hyperplane.

To show existence, denote $G_\theta := C_G(U_\theta)^\circ$.

Notice that T is a maximal torus in G_θ, and U_θ lies in the center of G_θ. We have an embedding $W(G_\theta, T) \to W(G,T)$, whose image consists only of elements which are trivial on U_θ. Thus, it is enough to show that $|W(G_\theta, T)| = 2$.

Next, consider $G'_\theta := G_\theta/U_\theta$. Then T/U_θ is a maximal torus in G'_θ, and we have an isomorphism $W(G'_\theta, T/U_\theta) \cong W(G_\theta, T)$. Thus, we are reduced to showing that $|W(G'_\theta, T/U_\theta)| = 2$. Notice that $G'_\theta \neq T/U_\theta$, because $G_\theta \neq T$ (this follows from θ being a root). Thus, the claim follows from the previous claim. □

As an example, let us consdier $G = U(n)$ and $T = diag$. For the root $\theta_{jk}(t) = t_j t_k^{-1}$, we have $Ker(\theta_{jk}) = \{diag(t_1, \dots, t_n) \mid t_j = t_k\}$. Obviously, the permutation $(jk) \in S_n$ is the sought for $s_{\theta_{jk}}$.

Claim 9.4. *For $\theta \in R(G,T)$ we have $\theta^{-1} \in R(G,T)$ and, moreover, θ, θ^{-1} are the only elements of $R(G,T)$ which are trivial on U_θ°. One has* $\dim \mathfrak{g}_{\mathbb{C}}^{T,\theta} = 1$.

Proof. The fact that $\theta^{-1} \in R(G,T)$ follows by noticing that complex conjugation on $\mathfrak{g}_{\mathbb{C}}$ takes one root space to the other.

Denoting by $R_\theta \subset R(G,T)$ the subset of roots θ' which are trivial on U_θ°, we notice that

$$Z_{\mathfrak{g}}(U_\theta^\circ) = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\theta' \in R_\theta} \mathfrak{g}_{\mathbb{C}}^{T,\theta'},$$

and thus $\dim Z_G(U_\theta^\circ) = \dim T + \sum_{\theta' \in R_\theta} \dim \mathfrak{g}_{\mathbb{C}}^{T,\theta'}$. From this we see that

$$\dim Z_G(U_\theta^\circ)^\circ / U_\theta^\circ = 1 + \sum_{\theta' \in R_\theta} \dim \mathfrak{g}_{\mathbb{C}}^{T,\theta'}.$$

So the group $Z_G(U_\theta^\circ)^\circ / U_\theta^\circ$ has rank 1 and dimension at least 3, so by the claim we had, it must have dimension 3. Thus $|R_\theta| = 2$, i.e. $R_\theta = \{\theta, \theta^{-1}\}$, and $\dim \mathfrak{g}_{\mathbb{C}}^{T,\theta} = 1$. □

Notice that in the example of $SU(2)$, indeed we have $U_{\theta_{12}} = \{\pm 1\}$, so that U_θ's can be not connected.

10 The real root system

10.1 Weights

Let T be a torus. The map $exp : \mathfrak{t} \to T$ is a surjective homomorphism. Denote the kernel of exp by $\mathfrak{t}_{\mathbb{Z}}$. The subgroup $\mathfrak{t}_{\mathbb{Z}}$ is a lattice in $\mathfrak{t}$ (the **coweight lattice**). Denote by $\mathfrak{t}_{\mathbb{Z}}^* \subset \mathfrak{t}^*$ the dual lattice to $\mathfrak{t}_{\mathbb{Z}}$ (the **weight lattice**).

For $\lambda \in \mathfrak{t}^*$, we denote by $e(\lambda)$ the homomorphism $\mathfrak{t} \to \mathbb{C}_1^\times$ given by $e(\lambda)(H) := e^{2\pi i \lambda(H)}$. Then $e(\lambda)$ is trivial on $\mathfrak{t}_{\mathbb{Z}}$ if and only if $\lambda \in \mathfrak{t}_{\mathbb{Z}}^*$. On the other hand, elements of $X^*(T)$, that is homomorphisms $T \to \mathbb{C}_1^\times$, are identified with homomorphisms $\mathfrak{t} \to \mathbb{C}_1^\times$ trivial on $\mathfrak{t}_{\mathbb{Z}}$. We obtain an isomorphism $\mathfrak{t}_{\mathbb{Z}}^* \cong X^*(T)$, given by sending $\lambda \in \mathfrak{t}_{\mathbb{Z}}^*$ to the unique $\chi \in X^*(T)$ satisfying $\chi \circ exp = e(\lambda)$.

10.2 Roots

Denote

$$R = \{\alpha \in \mathfrak{t}_{\mathbb{Z}}^* \mid e(\alpha) \in R(G,T)\}$$

(we might call those "**real roots**" to distinguish from the roots in $R(G,T)$).

Remark 10.1. R is a finite set of non-zero vectors, and $R = -R$.

Claim 10.2 ("the root system is reduced")**.** *Given $\alpha \in R$, one has $\mathbb{R}\alpha \cap R = \{\alpha, -\alpha\}$.*

Proof. $\mathbb{R}\alpha \cap R$ coincides with the set of real roots β for which $e(\beta)$ is trivial on $U_{e(\alpha)}^\circ$, and we saw that $e(\beta)$ and $e(-\beta)$ are the only roots trivial on U_θ°. □

10.3 The Weyl group

The group $W = W(G,T)$ acts on $\mathfrak{t}$ via the adjoint action. We have also the corresponding action on $\mathfrak{t}^*$.

Remark 10.3. W sends R into R.

It is convenient to introduce an auxiliary W-invariant inner product $\kappa(\cdot,\cdot)$ on $\mathfrak{t}$. Then the elements of W are all orthogonal transformations w.r.t. $\kappa(\cdot,\cdot)$. We denote also by $\kappa(\cdot,\cdot)$ the induced inner product on $\mathfrak{t}^*$. The elements of W act on $\mathfrak{t}^*$ by orthogonal transformations as well.

Recall the elements $s_\alpha := s_{e(\alpha)} \in W$ for $\alpha \in R$. Since $s_\alpha \neq e$ and s_α fixes $\mathfrak{t}_\alpha := Ker(\alpha)$, we see that s_α is the orthogonal reflection through $\mathfrak{t}_\alpha$.

We denote $\mathfrak{t}^{reg} := \mathfrak{t} - \cup_{\alpha\in R}\mathfrak{t}_\alpha$.

Lemma 10.4. *Let $H \in \mathfrak{t}$. Then $H \in \mathfrak{t}^{reg}$ if and only if $Stab_W(H) = \{e\}$.*

Proof. If $H \in \mathfrak{t}_\alpha$, then $s_\alpha \in Stab_W(H)$. If $H \in \mathfrak{t}^{reg}$, then $Z_{\mathfrak{g}}(H) = \mathfrak{t}$, so $Z_G(H)$, being connected (we saw that the centralizer of any element in the Lie algebra is connected), is equal to T, meaning $Stab_W(H) = \{e\}$. □

Remark 10.5. We can contrast the above lemma with the situation with W acting on T. There we also had $T^{reg} = T - \cup_{\alpha\in R}Ker(e(\alpha))$, but there might be elements in T^{reg} with non-trivial stabilizer in W. For example, take $G = SO(3)$ and $T = diag(SO(2), 1)$. Then $diag(-1,-1,1) \in T$ is regular, but the element $diag(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, -1)$ (which is a representative of the non-trivial element in the Weyl group) stabilizes it. The centralizer of $diag(-1,-1,-1)$ is $N_G(T)$.

Lemma 10.6. *For $w \in W$ and $\alpha \in R$, we have $ws_\alpha w^{-1} = s_{w\alpha}$.*

Proof. The element $ws_\alpha w^{-1}$ is an orthogonal reflection fixing $\mathfrak{t}_{w\alpha}$, hence equal to $s_{w\alpha}$. □

10.4 Weyl chambers

Denote $\mathfrak{t}_\alpha := Ker(\alpha)$, and define a **Weyl chamber** in $\mathfrak{t}$ to be a connected component of $\mathfrak{t}^{reg} := \mathfrak{t} - \cup_{\alpha\in R}\mathfrak{t}_\alpha$. The Weyl chambers are open, convex and conical, and are just the non-empty subsets of the form

$$U = \{H \in \mathfrak{t} \mid \epsilon_\alpha\langle H, \alpha\rangle > 0 \ \forall \alpha \in R\}$$

for various combinations of $\epsilon_\alpha = \pm 1$.

Claim 10.7.

1. *W acts freely on the set of Weyl chambers.*

2. *The subgroup of W generated by $\{s_\alpha\}_{\alpha\in R}$ acts transitively on the set of Weyl chambers.*

3. W is generated by $\{s_\alpha\}_{\alpha \in R}$*.*

Proof. Part 3 follows formally from parts 1 and 2. Part 1: Suppose that $w \in W$ fixes some Weyl chamber U. Take any $H \in U$ and consider $H' = \frac{1}{o(w)} \sum_{0 \le i < o(w)} w^i H$ (here $o(w)$ denoted the order of w as element of the group W). Then $H' \in U$ (because U is convex), and $wH' = H'$. Thus $w = e$ by the above lemma. Part 2: Let H, H' sit in two Weyl chambers. We want to show that there exists $w \in \langle s_\alpha \rangle_{\alpha \in R}$ such that wH' sits in the same Weyl chamber as H. Let $w \in \langle s_\alpha \rangle_{\alpha \in R}$ be such that the distance between wH' and H is minimal. If wH' and H sit in different Weyl chambers, then $\kappa(\alpha, wH')$ and $\kappa(\alpha, H)$ have a different sign for some $\alpha \in R$. But then $s_\alpha(wH')$ is closer to H than wH', contradicting the choice of w. $\square$

10.5 Coroots

Since s_α is a reflection of $\mathfrak{t}$, it is also a reflection of $\mathfrak{t}^*$. Thus, we have a unique $\alpha^\vee \in \mathfrak{t}$ satisfying

$$s_\alpha(\lambda) = \lambda - \langle \alpha^\vee, \lambda \rangle \alpha$$

for all $\lambda \in \mathfrak{t}^*$. Notice that we have $\langle \alpha^\vee, \alpha \rangle = 2$. The elements $\alpha^\vee \in \mathfrak{t}^*$ are called **coroots**.

Claim 10.8. *For any* $\alpha \in R$ *we have* $\alpha^\vee \in \mathfrak{t}_\mathbb{Z}$*.*

Proof. Denote $H := \alpha^\vee/2$. We have $\langle H, \alpha \rangle = 1$. Thus $e(\alpha)(exp(H)) = e^{2\pi i \alpha(H)} = 1$. In other words, $exp(H) \in U_{e(\alpha)}$ and hence $s_\alpha(exp(H)) = exp(H)$. But on the other hand $s_\alpha(exp(H)) = exp(s_\alpha(H)) = exp(-H)$. Hence we obtain $exp(H) = exp(-H)$ and so $exp(\alpha^\vee) = exp(2H) = 1$, i.e. $\alpha^\vee \in \mathfrak{t}_\mathbb{Z}$. $\square$

Corollary 10.9. *For any* $\alpha, \beta \in R$ *we have* $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$*, and so* $s_\alpha(\beta) - \beta \in \mathbb{Z} \cdot \alpha$*.*

Using the form $\kappa(\cdot, \cdot)$, we have

$$s_\alpha(\lambda) = \lambda - \frac{2\kappa(\alpha, \lambda)}{\kappa(\alpha, \alpha)} \alpha,$$

and we deduce that

$$\langle \alpha^\vee, \lambda \rangle = \frac{2\kappa(\alpha, \lambda)}{\kappa(\alpha, \alpha)}$$

for all $\lambda \in \mathfrak{t}^*$, i.e. $\alpha^\vee$ corresponds to $\frac{2}{\kappa(\alpha,\alpha)}\alpha$ under the isomorphism $\mathfrak{t} \cong \mathfrak{t}^*$ given by κ.

10.6 Positive and simple roots

Fix a Weyl chamber U in $\mathfrak{t}$, which we call the **fundamental Weyl chamber**. Say that $\alpha \in R$ is a **positive root** if α satisfies $\langle H, \alpha \rangle > 0$ for all $H \in U$ (equivalently, for some $H \in U$). Denote by $R^+ \subset R$ the subset of positive roots.

Notice that $R = R^+ \cup -R^+$ (any root is either a positive root or the minus of a positive root). Notice that

$$U = \{H \in \mathfrak{t} \mid \langle H, \alpha \rangle > 0 \ \forall \alpha \in R^+\}.$$

Say that $\alpha \in R^+$ is a **simple root**, if α can not be written as a sum of two positive roots. Denote by $S \subset R^+$ the subset of simple roots.

Lemma 10.10. *Let $\alpha, \beta \in S$ be different simple roots. Then $\kappa(\alpha, \beta) \leq 0$.*

Proof. Write $s_\alpha(\beta) = \beta - n\alpha$ and $s_\beta(\alpha) = \alpha - m\beta$, where $n = \langle \alpha^\vee, \beta \rangle = \frac{2\kappa(\alpha,\beta)}{\kappa(\alpha,\alpha)}$ and $m = \langle \beta^\vee, \alpha \rangle = \frac{2\kappa(\beta,\alpha)}{\kappa(\beta,\beta)}$. We saw that $n, m \in \mathbb{Z}$, and we see by Cauchy-Schwartz that $|nm| < 4$ (this is a strict inequality since α, β are not propotional). Assuming by contradiction that $\kappa(\alpha, \beta) > 1$ (so $n, m \neq 0$) we obtain $1 \in \{n, m\}$. Then either $\alpha - \beta$ or $\beta - \alpha$ is a root, in which case we obtain that either α or β can be written as a sum of two positive roots - in contradiction to them being simple roots. ☐

Corollary 10.11. *The set of simple roots is linearly independent.*

Proof. This is a general fact - a set of non-zero vectors in a real inner product space lying on one side of a hyperplane and forming obtuse angles (i.e. the inner products are nonpositive) is linearly independent. ☐

Claim 10.12. *Any root can be written as a sum of simple roots, either with all non-negative integer coefficients, or with all non-positive integer coefficients.*

Proof. Given a positive root, start decomposing it into sums of positive roots. The process must end because $\langle H, \cdot \rangle$ decreases. At the end, we obtain a decomposition of our positive root into a sum of simple roots. ☐

Notice that

$$U = \{H \in \mathfrak{t} \mid \langle H, \alpha \rangle > 0 \ \forall \alpha \in S\}.$$

Claim 10.13. *The set $\{s_\alpha\}_{\alpha \in S}$ generates W.*

Proof. The proof is the same as for R^+ instead of S: We show that $\langle s_\alpha \rangle_{\alpha \in S}$ acts transitively on the set of Weyl chambers. The same proof is possible, thanks to the formula for U above. ☐

10.7 The length function

Define $\ell : W \to \mathbb{Z}_{\geq 0}$, by setting $\ell(w)$ to be the minimal possible length of an expression of w as a product of simple reflections s_α, $\alpha \in S$.

Lemma 10.14. *For $\alpha \in S$ and $\beta \in R^+$ such that $\alpha \neq \beta$, we have $s_\alpha(\beta) \in R^+$.*

Proof. Write $\beta = \sum_{\gamma \in S} n_\gamma \gamma$ (where $n_\gamma \in \mathbb{Z}_{\geq 0}$). Notice that since $\alpha \neq \beta$, there must be $\alpha \neq \gamma' \in S$ with $n_{\gamma'} \neq 0$.

We have

$$s_\alpha(\beta) = \beta - \langle \alpha^\vee, \beta \rangle \alpha = (n_\alpha - \langle \alpha^\vee, \beta \rangle)\alpha + \sum_{\alpha \neq \gamma \in S} n_\gamma \gamma.$$

Thus, since γ' appears in $s_\alpha(\beta)$ with a positive coefficient, all simple roots must appear with a positive coefficient, and $s_\alpha(\beta) \in R^+$. □

Remark 10.15. In the special case when $\beta \in S$, we obtain $\langle \alpha^\vee, \beta \rangle < 0$. Indeed, $s_\alpha(\beta) = \beta - \langle \alpha^\vee, \beta \rangle \alpha$, and again we use the fact that a root is either a positive integral combination of simple roots, or a negative integral combination of simple roots.

So, for the so-called **Cartan matrix** $S \times S \to \mathbb{Z}$ given by $(\alpha, \beta) \mapsto \langle \alpha^\vee, \beta \rangle$, the diagonal entries are -2, and the off-diagonal entries are negative.

Claim 10.16. *For $w \in W$, we have*

$$\ell(w) = |\{\beta \in R^+ \mid w(\beta) \in -R^+\}|.$$

Proof. For $w = e$ the claim is true, and for w a simple reflection the claim is true by the above lemma. Assume by induction that the claim is true for w of length $l-1$, and let w be of length l. Write $w = s_\alpha v$ where $\ell(v) = l-1$. We want to show that $\alpha \in vR^+$, then the claim becomes quite clear. Let us suppose that $\alpha \in -vR^+$, and denote $\beta := -v^{-1}(\alpha)$ (so $\beta \in R^+$ and $v\beta = -\alpha \in -R^+$). Then we can break the decomposition of v into $l-1$ simple reflections as $v = gs_\gamma h$, where $s_\gamma h$ is the first sub-word to send β into a negative root. Then this means that $h\beta = \gamma$. We get $g\gamma = \alpha$. But this means $gs_\gamma g^{-1} = s_\alpha$, so that $v = gs_\gamma h = s_\alpha gh$ and hence $w = s_\alpha v = gh$, and we get a contradiction to $\ell(w) = l$. □

Claim 10.17. *Let U be a Weyl chamber. Then every W-orbit in $\mathfrak{t}$ intersects $Cl(U)$ at a unique point (here $Cl(\cdot)$ denotes the closure in the topology).*

Proof. Let $H \in \mathfrak{t}^*$. That WH intersects $Cl(U)$ is shown as in the proof of part 2 of claim ...; Namely, we take arbitrary $H' \in U$ and consider $w \in W$ for which the distance between wH and H' is minimal. Then $wH \in Cl(U)$.

To show uniqueness, Assume that $H, H' \in Cl(U)$ and $H' = wH$ for some $w \in W$. We want to show that $H' = H$. We do it by induction on $\ell(w)$ (the case $\ell(w) = 0$ is obvious). Assume that $\ell(w) > 0$. Then for some $\alpha \in S$, we have $w\alpha \in -R^+$. Then $\langle H, \alpha \rangle = \langle H', w\alpha \rangle$ is both non-negative and non-positive. So $\langle H, \alpha \rangle = 0$, i.e. $H \in \mathfrak{t}_\alpha$. Thus $H' = (ws_\alpha)H$. Notice that $\ell(ws_\alpha) = \ell(w) - 1$, so that we are done by induction. □

Remark 10.18. In other words, we see that the closure of a Weyl chamber is a **fundamental domain** for the action of W on $\mathfrak{t}$.

Claim 10.19. *There is a unique element $w_0 \in W$ satisfying $w_0 R^+ = -R^+$ (or, equivalently, $\ell(w_0) = |R^+|$.*

Proof. Uniqueness: Given two such elements w_0, w_0', we have $w_0^{-1} w_0' R^+ = R^+$, so $\ell(w_0^{-1} w_0' = 0$ and so $w_0^{-1} w_0' = e$, i.e. $w_0 = w_0'$.

Existence: Given an element $w \in W$ of length $l < |R^+|$, notice that $-S$ is not contained in wR^+, because otherwise also $-R^+$ would be contained in wR^+ (since positive roots are sums of simple roots), meaning that $l = |R^+|$. Thus, picking $\alpha \in S$ satisfying $w^{-1}\alpha \in R^+$, we have $\ell(s_\alpha w) = \ell(w) + 1$. Continuing in this way, we will eventually obtain an element of length $|R^+|$. □

The element w_0 is called "the longest element in the Weyl group".

Claim 10.20. *We have $w_0^2 = 1$, and $\ell(w_0 w) = \ell(w_0) - \ell(w)$ for every $w \in W$.*

Proof. $w_0^2 R^+ = R^+$, so $\ell(w_0^2) = 0$, so $w_0^2 = e$. The claim about lengths is clear. □

10.8 The center and the fundamental group

In $\mathfrak{t}^*$, we have the **weight lattice** $\mathfrak{t}^*_{\mathbb{Z}}$, and the **root lattice** $\mathbb{Z}R$ - the $\mathbb{Z}$-span of R. In $\mathfrak{t}$, we have the **coweight lattice** $\mathfrak{t}_{\mathbb{Z}}$, and the **coroot lattice** $\mathbb{Z}R^\vee$ - the $\mathbb{Z}$-span of the set $R^\vee$ of coroots. Notice that $\mathbb{Z}R \subset \mathfrak{t}^*_{\mathbb{Z}}$ and $\mathbb{Z}R^\vee \subset \mathfrak{t}_{\mathbb{Z}}$.

Claim 10.21. *One has a canonical isomorphism $Z(G) \cong (\mathbb{Z}R)^{dual}/\mathfrak{t}_{\mathbb{Z}}$. In particular, $Z(G)$ is finite if and only if R spans $\mathfrak{t}^*$, and in that case $Z(G) \cong Hom(\mathfrak{t}^*_{\mathbb{Z}}/\mathbb{Z}R, \mu)$ (where $\mu \subset \mathbb{C}_1^\times$ is the subgroup of roots of unity).*

Proof. We have $T \cong \mathfrak{t}/\mathfrak{t}_{\mathbb{Z}}$ via the exponential map. Also, we have $Z(G) \subset T$ and $Z(G) = \cap_{\alpha \in R} Ker(e(\alpha))$, and those are represented by elements in $\mathfrak{t}$ on which the α's takes integral values, i.e. $(\mathbb{Z}R)^{dual}$. □

Claim 10.22. *One has a canonical isomorphism $\pi_1(G) \cong \mathfrak{t}_{\mathbb{Z}}/\mathbb{Z}R^\vee$.*

Proof. Let us start by constructing a homomorphism $\mathfrak{t}_{\mathbb{Z}} \to \pi_1(G)$. Namely, the fiber sequence $\mathfrak{t}_{\mathbb{Z}} \to \mathfrak{t} \to T$ shows that $\pi_1(T) \cong \mathfrak{t}_{\mathbb{Z}}$, and we use $\pi_1(T) \to \pi_1(G)$ (induced by the embedding).

This homomorphism is surjective, as shown before.

Let us show that $\mathbb{Z}R^\vee$ is in the kernel of the homomorphism. Given $\alpha^\vee \in R^\vee$, consider the path $t \mapsto exp(t\alpha^\vee)$ ($t \in [0,1]$). We need to show that this path is homotopic to the trivial path in G. Indeed, recall that $exp(\frac{1}{2}\alpha^\vee) \in U_{e(\alpha)}$, i.e. $s_\alpha exp(\frac{1}{2}\alpha^\vee) s_\alpha^{-1} = exp(\frac{1}{2}\alpha^\vee)$. On the other hand, $s_\alpha exp(t\alpha^\vee) s_\alpha^{-1} = exp(-t\alpha^\vee)$ for all $t \in \mathbb{R}$. Thus, we obtain $s_\alpha exp((\frac{1}{2}+t)\alpha^\vee) s_\alpha^{-1} = exp((\frac{1}{2}-t)\alpha^\vee)$. Since G is connected, the left hand side of the previous eqation, as a path $[\frac{1}{2}, 1] \to G$, is homotopic, with end points fixed, to the path $exp((\frac{1}{2}+t)\alpha^\vee)$. We get that our original path performs at $[\frac{1}{2}, 1]$ a way homotopic to the reverse of the way it performs at $[0, \frac{1}{2}]$. Thus, our path is homotopic to the trivial path.

We are left to show that the kernel of the homomorphism is contained in $\mathbb{Z}R^\vee$. We skip this for now.

□

Corollary 10.23. *The group $Z(G)$ is finite if and only if the group $\pi_1(G)$ is finite.*

10.8.1 Semisimplicity

Claim 10.24. *Let G be a connected compact Lie group. TFAE:*

1. *G has no non-trivial abelian connected normal subgroups.*
2. *$Z(G)$ is finite.*
3. *$Z(\mathfrak{g}) = 0$.*
4. *$\pi_1(G)$ is finite.*
5. *The universal cover of G is compact.*
6. *The Killing form $B(X,Y) := Tr(ad(X) \circ ad(Y))$ on $\mathfrak{g}$ is negative-definite.*
7. *The Killing form $B(X,Y) := Tr(ad(X) \circ ad(Y))$ on $\mathfrak{g}$ is non-degenerate.*

Proof. To show the equivalence of (1) and (2), it is enough to show that every abelian connected normal subgroup lies in the center. Indeed, such a (closed) subgorup is torus, which is contained in every maximal torus (since for every maximal torus, some conjugate of the torus is contained in it), and hence contained in the center (which we saw is the intersection of all maximal tori).

Since $Lie(Z(G)) = Z(\mathfrak{g})$, the equivalence of (2) and (3) is clear.

We already stated the equivalence of (2) and (4).

The equivalence of (4) and (5) is clear by covering theory.

We note now that the Killing form $B(X,Y)$ is negative-semidefinite. Indeed, let us fix a G-invariant inner product on $\mathfrak{g}$. Then $ad(X)$ are skew-Hermitian w.r.t. that inner product. Thus we see in a diagonal basis that $Tr(ad(X)^2) \leq 0$ (as a sum of squares of purely imaginary numbers).

Thus, the equivalence of (6) and (7) is clear.

We also have $X \in Z(\mathfrak{g})$ i.f.f. $ad(X) = 0$ i.f.f. $B(X,X) = Tr(ad(X)^2) = 0$, so $Z(\mathfrak{g}) = Ker(B)$, and thus the equivalence of (3) and (6) is clear.

□

Remark 10.25. A connected compact Lie group is called **semi-simple** if it satisfies the equivalent conditions of the above claim.

Remark 10.26. It is also true that if G is a connected Lie group, such that its Killing form is negative-definite, then G is compact (with finite center).

10.9 The root datum

The data of $(\mathfrak{t}_{\mathbb{Z}}^*, \mathfrak{t}_{\mathbb{Z}}, R, R^\vee)$, together with the duality between $\mathfrak{t}_{\mathbb{Z}}^*$ and $\mathfrak{t}_{\mathbb{Z}}$, and the bijection between R and $R^\vee$, is called a **root datum**. When defined axiomatically, it should satisfy some axioms, of course (more or less, one should have $\langle \alpha^\vee, \alpha \rangle = 2$, the map $\lambda \mapsto \lambda - \langle \alpha^\vee, \lambda \rangle \alpha$ should preserve R and the map $H \mapsto H - \langle H, \alpha \rangle \alpha^\vee$ should preserve $R^\vee$).

It is a theorem (I think!) that the association to a compact connected Lie group of its root datum gives a bijection on isomorphism classes.

Notice that on the root datum side, there is a natural "involution" - one can swap $\mathfrak{t}_{\mathbb{Z}}^*$ and $\mathfrak{t}_{\mathbb{Z}}$, and R and $R^\vee$. Thus, one gets an "involution" on the isomorphism classes of compact connected Lie groups! This is called the "Langlands dual group". Notice how indirect this is.

10.10 Examples

10.10.1 $SU(2)$

Consider $G = SU(2)$ (and T the usual diagonal subgroup). Then $\mathfrak{t} = Sp\{H\}$, where $H = diag(i, -i)$. We have $\mathfrak{t}_{\mathbb{Z}} = \mathbb{Z} \cdot 2\pi H$. Elements $\lambda \in \mathfrak{t}^*$ are determined by the value $\lambda(H)$. We have $\lambda \in \mathfrak{t}_{\mathbb{Z}}^*$ if and only if $\lambda(H) \in \frac{1}{2\pi}\mathbb{Z}$. The Weyl group is $W = \{e, w_0\}$, where $w_0(H) = -H$. Taking into account

$$\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & z \\ -\overline{z} & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 2iz \\ -\overline{2iz} & 0 \end{pmatrix},$$

and recalling that $[H, X] = 2\pi i \alpha(H) X$ for $X \in \mathfrak{g}_{\mathbb{C}}^{T, e(\alpha)}$, we have two roots $\alpha, -\alpha$, where $\alpha(H) = \frac{1}{\pi}$. The condition $\langle \alpha^\vee, \alpha \rangle = 2$ implies $\alpha^\vee = 2\pi H$.

Notice that $|\mathfrak{t}_{\mathbb{Z}}^* / \mathbb{Z}\alpha| = 2$ and indeed $|Z(G)| = 2$. Notice that $|\mathfrak{t}_{\mathbb{Z}} / \mathbb{Z}\alpha^\vee| = 1$ and indeed $|\pi_1(G)| = 1$ (i.e. G is simply-connected).

10.10.2 $SO(3)$

Consider $G = SO(3)$ (and $T = diag(SO(2), 1)$). Actually, let us think of G as $\widetilde{G}/\pm 1$, and of T as $\widetilde{T}/\pm 1$, where $\widetilde{G} = SU(2)$ and $\widetilde{T}$ is the diagonal subgroup in $\widetilde{G}$.

Thus, we have the same $\mathfrak{t}$ as before, the same roots, and in fact the same W. What changes is the integral lattices. Notice that $\mathfrak{t}_{\mathbb{Z}} = \frac{1}{2}\widetilde{\mathfrak{t}_{\mathbb{Z}}}$. Thus, $\mathfrak{t}_{\mathbb{Z}}^* = 2\widetilde{\mathfrak{t}_{\mathbb{Z}}^*}$. Thus, the situation "reverses"; Now $|\mathfrak{t}_{\mathbb{Z}}^* / \mathbb{Z}\alpha| = 1$ and $|Z(G)| = 1$, while $|\mathfrak{t}_{\mathbb{Z}} / \mathbb{Z}\alpha^\vee| = 2$, and $|\pi_1(G)| = 2$.

Indeed, $SU(2)$ and $SO(3)$ are "Langlands dual" to each other.

10.10.3 $U(n)$ and $SU(n)$

Consider $G = U(n)$ (and T the usual diagonal subgroup). Then $\mathfrak{t}$ consists of diagonal matrices with purely imaginary components - we identify $\mathbb{R}^n \cong \mathfrak{t}$ by $(x_1, \ldots, x_n) \mapsto diag(ix_1, \ldots, ix_n)$. The Weyl group $W = S_n$ acts on $\mathfrak{t}$ by permuting components. As a W-invariant inner product, one may take the standard inner product on $\mathbb{R}^n$. From the formula

$$[diag(ix_1, \ldots, ix_n), zE_{jk} - \overline{z}E_{kj}] = i(x_j - x_k)zE_{jk} - \overline{i(x_j - x_k)z}E_{kj},$$

we see that the roots are $\alpha_{jk}(x_1, \ldots, x_n) = \frac{x_j - x_k}{2\pi}$.

For $SU(n)$ the situation is similar, except that $\mathfrak{t}$ is identified with $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \ldots + x_n = 0\}$.

10.10.4 *SU*(3)

Let us consider $SU(3)$ as an example of the previous subsection. Then an orthonormal basis (up to scalar) for $\mathfrak{t}^*$ is

$$e_1 = (1, -1, 0), e_2 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}),$$

and in terms of this basis (up to scalar) we have

$$\alpha_{12} = e_1, \ \alpha_{23} = -\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2.$$

Thus, one imagines

$$\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{21}, \alpha_{31}, \alpha_{32}$$

sitting on the unit circle as the 6-th roots of unity.

We declare $\alpha_{12}, \alpha_{23}, \alpha_{13}$ to be the positive roots (and thus α_{12}, α_{23} the simple roots). The Weyl group, which is isomorphic to S_3, is generated by two simple reflections s_{12}, s_{23}.

For example, notice that s_{13} sends

$$\alpha_{12} \mapsto -\alpha_{23}, \alpha_{23} \mapsto -\alpha_{12}, \alpha_{13} \mapsto -\alpha_{13}.$$

Thus, $\ell(s_{13}) = 3$, and thus s_{13} is the longest element in the Weyl group.

10.11 Dominant, regular, integral weights

Define

$$U^\vee = \{\lambda \in \mathfrak{t}^* \mid \langle \alpha^\vee, \lambda \rangle > 0 \ \forall \alpha \in S\}.$$

Using the identification $\kappa(\cdot, \cdot)$, which identifies U and $U^\vee$ (or more abstractly by realizing that $R^\vee$ is a root system by itself...), we see that every W-orbit in $\mathfrak{t}^*$ intersects $Cl(U^\vee)$ at a unique point.

We will call elements $\lambda \in \mathfrak{t}^*$ **weights**. We will say that a weight λ is: **dominant** if $\lambda \in Cl(U^\vee)$, **regular** if $\langle \alpha^\vee, \lambda \rangle \neq 0$ for every $\alpha \in S$ (or, equivalently, $\alpha \in R^+$ or even $\alpha \in R$), and **integral** if $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$.

Let us remark that some define a weight to be what we call an integral weight, because only those are relevant for finite-dimensional representations of compact groups. Also, some define a dominant weight to be a weight λ satisfying $\langle \alpha^\vee, \lambda \rangle \notin \mathbb{Z}_{\leq -1}$ for every $\alpha \in R$ (instead of our $\langle \alpha^\vee, \lambda \rangle \geq 0$), which makes more sense in some contexts (anyhow, for integral weights those coincide).

11 Weight theory

11.1 Representation of the Lie algebra

Let $\mathfrak{g}$ be a (real/complex) Lie algebra. A **representation** of $\mathfrak{g}$ on a complex vector space V is an $\mathbb{R}$-linear/$\mathbb{C}$-linear map $\pi : \mathfrak{g} \to End_{\mathbb{C}}(V)$ satisfying $\pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$ for all $X, Y \in \mathfrak{g}$.

Let G be a Lie group, and $(V,\pi) \in Rep^{fd}(G)$. Then we get an induced representation of $\mathfrak{g} := Lie(G)$ on V by differentiating $\pi : G \to Aut(V)$, thus obtaining $d\pi : \mathfrak{g} \to End(V)$. For example, from the adjoint representation Ad of G on $\mathfrak{g}$ we obtain the adjoint representation ad of $\mathfrak{g}$ on $\mathfrak{g}$, given by $\pi(X)(Y) = [X,Y]$.

Thus, we can think that we have a functor $Rep^{fd}(G) \to Rep^{fd}(\mathfrak{g}) \cong Rep^{fd}(\mathfrak{g}_{\mathbb{C}})$. This functor is trivially faithful. If G is connected, it is full. If G is simply-connected, it is essentially surjective. Also, if G is connected, given a representation $V \in Rep^{fd}(G)$, a subspace $W \subset V$ is a G-submodule if and only if it is a $\mathfrak{g}$-submodule.

11.2 PBW theorem, Casimir element

11.2.1 The PBW theorem

Let $X_1, \ldots, X_n$ be a basis for the Lie algebra $\mathfrak{g}$. Let V be a representation of $\mathfrak{g}$, and $v \in V$. Then we claim that W, the span of the vectors of the form $X_n^{m_n} \ldots X_1^{m_1} v$, where $m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}$, is a $\mathfrak{g}$-submodule of V (so, in fact, the $\mathfrak{g}$-submodule generated by v).

We need to show that $X_a X_n^{m_n} \ldots X_1^{m_1} v \in W$ for every $1 \leq a \leq n$. We do this by induction on $m_1 + \ldots + m_n$ and then reverse induction on a.

Looking at the maximal $1 \leq b \leq n$ for which $m_b \neq 0$, if $b \leq a$ then clearly we have the desired. Otherwise, we have

$$X_a X_b^{m_b} \ldots X_1^{m_1} v = X_b X_a X_b^{m_b - 1} \ldots X_1^{m_1} v + [X_a, X_b] X_b^{m_b - 1} \ldots X_1^{m_1} v.$$

Then the first expression on the right is in W by the reverse induction on a, and the second expression on the right is in W by the induction on $m_1 + \ldots + m_n$.

maybe some slight mistake with induction here

11.2.2 The universal enveloping algebra

One can be more precise than as above. One has the **universal enveloping algebra** $U(\mathfrak{g})$, which is an associative algebra (with unit), with a map of Lie algebras $i : \mathfrak{g} \to U(\mathfrak{g})$, universal (complete...). The PBW theorem says that given a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$, the elements $i(X_1)^{m_1} \ldots (X_n)^{m_n}$ form a basis for $U(\mathfrak{g})$, where $(m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n$ (here, we abuse notation and write X instead of $i(X)$).

In a more canonical way, the PBW theorem says the following. Let us denote by $U(\mathfrak{g})^{\leq i}$ the subspace of $U(\mathfrak{g})$ spanned by elements of the form $Y_1 \ldots Y_i$.

Thus, $U(\mathfrak{g})^{\leq 0} = \mathbb{C} \cdot 1$, and it is easy to see using the universal property that $\cup_i U(\mathfrak{g})^{\leq i} = U(\mathfrak{g})$. Also, $U(\mathfrak{g})^{\leq i} \cdot U(\mathfrak{g})^{\leq j} \subset U(\mathfrak{g})^{\leq i+j}$. Thus, we can form the associated graded algebra $grU(\mathfrak{g})$. This is a commutative algebra. Indeed, notice that $Y_1 \dots Y_k Y_{k+1} \dots Y_i \in Y_1 \dots Y_{k+1} Y_k \dots Y_i + U(\mathfrak{g})^{\leq i-1}$... Hence, the linear map $\mathfrak{g} \to gr^1 U(\mathfrak{g})$ induces a map of commutative algebras $S(\mathfrak{g}) \to grU(\mathfrak{g})$. The PBW theorem states that this is an isomorphism.

11.2.3

Notice that from the definition of $U(\mathfrak{g})$, one has an equivalence, between representations of $\mathfrak{g}$ and modules over $U(\mathfrak{g})$.

11.2.4 The Casimir element

The Casimir element is an interesting element C in the center of $U(\mathfrak{g})$. As such, it acts by scalar on every irreducible representation of G, by Schur's lemma. This gives sometimes an easy way to see that two irreducible representations are not isomorphic - simply C acts by different scalars on them.

Notice that G acts on $U(\mathfrak{g})$, by extending the adjoint action on $\mathfrak{g}$. One sees that an element Z of $U(\mathfrak{g})$ lies in the center, i.f.f. $[X, Z] = 0$ for all $X \in \mathfrak{g}$, if and only if (given G connected) ${}^g Z = Z$ for all $g \in G$.

We consider a non-degenerate G-invariant form B on $\mathfrak{g}$. This form gives a G-equivaraint identification $\mathfrak{g} \cong \mathfrak{g}^*$. Then we can consider

$$id \in End(\mathfrak{g}) \cong \mathfrak{g}^* \otimes \mathfrak{g} \cong \mathfrak{g} \otimes \mathfrak{g} \to U(\mathfrak{g}),$$

where all maps are G-equivariant, and id is G-invariant. Thus, the image of id under this chain of maps, which we call C (or maybe more precisely C_B) is a G-invariant element of $U(\mathfrak{g})$, so an element in the center of $U(\mathfrak{g})$.

If G is semisimple, we can take B to be the Killing form.

Let us calculate C for $SU(2)$ (taking B to be the Killing form). We have the basis $H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ for $Lie(SU(2)) = \mathfrak{su}(2)$, and the relations are

$$[H, X] = 2Y, [H, Y] = -2X, [X, Y] = 2H.$$

Thus the Killing form B is given, in the H, X, Y basis, by the matrix

$$\begin{pmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix}.$$

It follows that the dual basis to H, X, Y is H, X, Y up to scalar. Thus C (up to scalar) is given by $H^2 + X^2 + Y^2$.

Let us consider the basis H_0, X_0, Y_0 for the complexification of $\mathfrak{su}(2)$, given by

$$H_0 = iH, X_0 = \frac{1}{2}(iX - Y), Y_0 = \frac{1}{2}(iX + Y).$$

Then we have

$$[H_0, X_0] = 2X_0, [H_0, Y_0] = -2Y_0, [X_0, Y_0] = H_0.$$

The Casimir has then form (up to scalar; The previous one is minus this one) $H_0^2 + 2X_0Y_0 + 2Y_0X_0 = H_0^2 + 2H_0 + 4Y_0X_0$. When we later talk about highest weight, one can mention the general idea - we see that $C = H_0^2 + 2H_0 + 4Y_0X_0$ acts on a highest weight vector of weight λ via $(2\pi i\lambda(H_0))^2 + 2(2\pi i\lambda(H_0)) = c^2+2c$ where we write $c = 2\pi i\lambda(H_0)$. Thus, representations with highest weights c_1, c_2 can be isomorphic only if $c_2 = c_1$ or $c_2 = -2 - c_1 = -(c_1 + 1) - 1$. Actually, this is quite useless for dominant highest weights, so only is valuable if we consider non-finite-dimensional representations

11.3 Weights

Let $V \in Rep^{fd}(G)$.

Definition 11.1.

1. A **weight vector** $v \in V$ is a non-zero eigenvector for T (equivalently, a non-zero eigenvector of $\mathfrak{t}$).

2. The **weight** of a weight vector $v \in V$ is the element $\lambda \in \mathfrak{t}_{\mathbb{Z}}^*$ such that T acts on v via $e(\lambda)$ (equivalently, $\mathfrak{t}$ acts on v via $2\pi i\lambda$).

3. The **weight space of weight** λ is the subspace $V^\lambda \subset V$ consisting of weight vectors of weight λ and 0.

We have $V = \oplus_{\lambda\in\mathfrak{t}_{\mathbb{Z}}^*} V^\lambda$.

Let us denote by fc_V the formal sum $\sum_{\lambda\in\mathfrak{t}_{\mathbb{Z}}^*} \dim V^\lambda \cdot e(\lambda) \in \mathbb{Z}[e(\mathfrak{t}_{\mathbb{Z}}^*)]$ (here, $\mathbb{Z}[e(\mathfrak{t}_{\mathbb{Z}}^*)]$ is the group algebra of $\mathfrak{t}_{\mathbb{Z}}^*$, where we write $e(\lambda)$ instead of λ to become multiplicative...).

Notice that the information of fc_V is equivalent to the information of $\chi_V|_T$, which in its turn is equivalent to the information of χ_V. In other words, two representations with the same fc are isomorphic.

Also, notice that fc_V is W-invariant, since application of w sets an isomorphism between V^λ and $V^{w\lambda}$.

11.4 The subalgebras $\mathfrak{n}, \mathfrak{n}^-$

Lemma 11.2. *One has* $[\mathfrak{g}_{\mathbb{C}}^{\theta_1}, \mathfrak{g}_{\mathbb{C}}^{\theta_2}] \subset \mathfrak{g}_{\mathbb{C}}^{\theta_1\cdot\theta_2}$.

Proof. If $X \in \mathfrak{g}_{\mathbb{C}}^{\theta_1}$ and $Y \in \mathfrak{g}_{\mathbb{C}}^{\theta_2}$, we have

$$Ad(t)([X, Y]) = [Ad(t)(X), Ad(t)(Y)] = [\theta_1(t)X, \theta_2(t)Y] = \theta_1(t)\theta_2(t)[X, Y].$$

□

Write $\mathfrak{n} := \oplus_{\alpha \in R^+} \mathfrak{g}_{\mathbb{C}}^{e(\alpha)}$ and $\mathfrak{n}^- = \oplus_{\alpha \in R^+} \mathfrak{g}_{\mathbb{C}}^{e(-\alpha)}$. Then

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}^- \oplus \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}$$

and $\mathfrak{n}, \mathfrak{n}^-$ are nilpotent Lie subalgebras of $\mathfrak{g}_{\mathbb{C}}$.

Lemma 11.3. *Let $V \in Rep^{fd}(G)$. Then $\mathfrak{g}_{\mathbb{C}}^{e(\alpha)} V^{\lambda} \subset V^{\lambda+\alpha}$.*

Proof. Let $X \in \mathfrak{g}_{\mathbb{C}}^{e(\alpha)}$ and $v \in V^{\lambda}$. Then

$$tXv = Ad(t)(X)tv = e(\lambda)(t)e(\alpha)(t)Xv.$$

□

11.5 Highest weight vectors

Let $V \in Rep^{fd}(G)$.

Definition 11.4. A **highest weight vector** $v \in V$ is a weight vector satisfying $\mathfrak{n}v = 0$.

Lemma 11.5. *Let $0 \neq V \in Rep^{fd}(G)$. Then V admits a highest weight vector.*

Proof. The set of weights of V is finite, and application of elements of the various $\mathfrak{g}_{\mathbb{C}}^{\alpha}$, for $\alpha \in R^+$, increases $\langle H, \cdot \rangle$, where H is an element of the fundamental Weyl chamber. □

Lemma 11.6. *Let $V \in Rep^{fd}(G)$, and let $v \in V$ be a highest weight vector with highest weight λ. Then the submodule W generated by v is spanned by vectors of the form $Y_1 \dots Y_m v$, where $Y_1, \dots, Y_m \in \mathfrak{n}^-$. The weights appearing in W lie in $\lambda - \mathbb{Z}_{\geq 0} R^+$. The weight λ appears in W with multiplicity 1.*

Proof. We take a basis for $\mathfrak{g}_{\mathbb{C}}$ such that first comes $\mathfrak{n}$, then $\mathfrak{t}_{\mathbb{C}}$, then $\mathfrak{n}^-$. By the PBW theorem we readily see the desired. □

Lemma 11.7. *Let $V \in Rep^{fd}(G)$ be irreducible. Then V admits a unique up to scalar highest weight vector.*

Proof. Let v be a highest weight vector in V. Then V is generated by v since V is irreducible. If w is another highest weight vector of V, then V is generated also by w. From the previous lemma we get that $wt(v) = wt(w) - \mathbb{Z}_{\geq 0} R^+$ and $wt(w) = wt(v) - \mathbb{Z}_{\geq 0} R^+$. This implies $wt(v) = wt(w)$, and by the above lemma v, w are proportional. □

Definition 11.8. Let $V \in Rep^{fd}(G)$ be irreducible. The weight of the highest weight vector in V is called the **highest weight of** V.

Lemma 11.9. *Let $V, W \in Rep^{fd}(G)$ be irreducible and non-isomorphic. Then their highest weights are different.*

Proof. Suppose that the highest weights of V, W are equal (call them λ). Let $v \in V, w \in W$ be highest weight vectors. Consider $(v, w) \in V \oplus W$. This is a highest weight vector in $V \oplus W$. Let $E \subset V \oplus W$ be the submodule generated by (v, w). Consider the projection $p : E \to V$. Since $p(v, w) = v \neq 0$ and V is irreducible, we get that p is surjective. Consider the kernel $Ker(p) \subset W$. Then $w \notin Ker(p)$, because E contains only (v, w) as a vector of weight λ, up to scalar (and $(0, w)$ is not proportional to (v, w)). Thus, since W is irreducible, we get $Ker(p) = 0$. Hence $p : E \to V$ is an isomorphism. Similarly we get that E is isomorphic to W, and hence V and W are isomorphic. □

Claim 11.10. *Let $V \in Rep^{fd}(G)$ be irreducible. Then its highest weight $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$ is dominant, i.e. $\langle \alpha^\vee, \lambda \rangle \geq 0$ for all $\alpha \in R^+$.*

Proof. We note two things about the set of weights of V: It is W-invariant, and lies in $\lambda - \mathbb{Z}_{\geq 0} \cdot R^+$. If λ is not dominant, then $\langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}_{\leq -1}$ for some $\alpha \in R^+$. Then we obtain that $s_\alpha(\lambda) = \lambda + \mathbb{Z}_{\geq 1} \cdot \alpha$ lies in $\lambda - \mathbb{Z}_{\geq 0} \cdot R^+$, which is clearly impossible. □

To sum up, associating to an irreducible representation its highest weight gives an injection of the set of isomorphism classes of irreducible representations into the set of dominant integral weights. Our goal is to show that this is in fact a surjection, and to give more concrete information about the character of an irreducible representation with a given highest weight.

11.6 Another description

An aesthetic drawback of the highest weight parametrization of irreducible representations is that it depends on a choice of a fundamental Weyl chamber.

11.6.1 Some partial orders

Fix a fundamental Weyl chamber U in $\mathfrak{t}$ again. We have the corresponding dual cone U^* in $\mathfrak{t}^*$ defined by

$$U^* = \{\lambda \in \mathfrak{t}^* \mid \langle U, \lambda \rangle > 0\}.$$

It is easy to see that

$$U^* = \mathbb{R}_{>0} S,$$

and thus

$$Cl(U^*) = \mathbb{R}_{\geq 0} S = \mathbb{R}_{\geq 0} R^+.$$

Let us define the partial order on $\mathfrak{t}^*$, by declaring $\lambda \leq \mu$ if $\mu - \lambda \in Cl(U^*)$. Let us define another partial order on $\mathfrak{t}^*$, be declaring $\lambda \leq' \mu$ if $\mu - \lambda \in \mathbb{Z} R^+$.

Since $Cl(U^*) = \mathbb{R}_{\geq 0} R^+$, we have $\lambda \leq' \mu \implies \lambda \leq \mu$.

Let us define a partial order on the set of W-orbits in $\mathfrak{t}^*$, by declaring $W\lambda \preceq W\mu$ if $W\lambda \subset Conv(W\mu)$, where $Conv(S)$ denotes the convex hull of S.

To see that this is indeed a partial order, assume that $W\lambda \preceq W\mu \preceq W\lambda$ (we want to see that $W\lambda = W\mu$). We will use a W-invariant inner product $\kappa(\cdot, \cdot)$, and the resulting norm $||\cdot||$. From the above relations we have $||\lambda|| \leq ||\mu|| \leq ||\lambda||$, and hence $||\lambda|| = ||\mu||$. Since $Conv(W\lambda) \cap \{\omega \mid ||\omega|| = ||\lambda||\} = W\lambda$, we obtain $\mu \in W\lambda$, i.e. $W\mu = W\lambda$ as desired.

11.6.2 "Highest orbit"

Lemma 11.11. *Let $\lambda \in \mathfrak{t}^*$. Then λ is dominant if and only if $w\lambda \leq \lambda$ for all $w \in W$.*

Proof. Assume $w\lambda \leq \lambda$ for all $w \in W$. Then in particular, for $\alpha \in R^+$, we have $s_\alpha\lambda \leq \lambda$, and recalling $s_\alpha\lambda = \lambda - \langle\alpha^\vee, \lambda\rangle\alpha$, we deduce $\langle\alpha^\vee, \lambda\rangle \geq 0$.

Conversely, assume that $\langle\alpha^\vee, \lambda\rangle \geq 0$ for all $\alpha \in R^+$. Let $w \in W$, and use induction on $\ell(w)$ to show that $w\lambda \leq \lambda$. For $\ell(w) = 0$ the statement is clear, while for $\ell(w) = 1$ it is clear from the condition.

If now $\ell(s_\alpha w) > \ell(w)$, we have $w^{-1}\alpha \in R^+$. Thus, for every $H \in U$:

$$\langle H, s_\alpha w\lambda - \lambda\rangle = \langle H, w\lambda - \lambda\rangle - \langle\alpha^\vee, w\lambda\rangle\langle H, \alpha\rangle \leq \langle H, w\lambda - \lambda\rangle \leq 0$$

because $\langle\alpha^\vee, w\lambda\rangle = \langle(w^{-1}\alpha)^\vee, \lambda\rangle$. □

Although we don't need it in the following, it is in general also important to know:

Lemma 11.12. *Let $\lambda \in \mathfrak{t}^*$. Then $\langle\alpha^\vee, \lambda\rangle \notin \mathbb{Z}_{\leq -1}$ for all $\alpha \in S$ if and only if $w\lambda \not>' \lambda$ for all $w \in W$.*

Proof. □

Lemma 11.13. *Let $\lambda, \mu \in \mathfrak{t}^*$ be dominant. Then $\mu \leq \lambda$ if and only if $W\mu \preceq W\lambda$.*

Proof. Suppose that $W\mu \preceq W\lambda$. Then $\mu = \sum c_w w\lambda$ for some $c_w \geq 0$, such that $\sum c_w = 1$. Then $\mu = \sum c_w w\lambda \leq \sum c_w\lambda = \lambda$.

Assume now that $W\mu \not\preceq W\lambda$. Then there exists $H \in \mathfrak{t}$ such that $\langle H, \mu\rangle > \langle H, w\lambda\rangle$ for all $w \in W$ (because disjoint compact convex sets ($\{\mu\}$ and $Conv(W\lambda)$) can always be separated by an affine hyperplane)). Taking $w \in W$ such that $w^{-1}H \in Cl(U)$, we obtain $\langle w^{-1}H, w^{-1}\mu\rangle > \langle w^{-1}H, \lambda\rangle$. Slightly moving H, we might assume that actually $w^{-1}H \in U$. Thus, we obtain $w^{-1}\mu \not\leq \lambda$. Thus $\mu \not\leq \lambda$ (because $w^{-1}\mu \leq \mu$ by the previous lemma). □

Claim 11.14. *Let $V \in Rep^{fd}(G)$ be irreducible. Then there exists exactly one W-orbit $\mathcal{O} \subset supp(fc_V)$, such that $supp(fc_V) \subset Conv(\mathcal{O})$. For $\mu \in \mathcal{O}$, $e(\mu)$ appears with multiplicity 1 in fc_V.*

Proof. Let $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$ be the (dominant integral) highest weight of V. We claim that $\mathcal{O} := W\lambda$ is the desired W-orbit.

Let $\mu \in supp(fc_V)$. We want to show that $W\mu \preceq W\lambda$. Indeed, by replacing μ by an element in its W-orbit, we can assume that μ is dominant. From lemma 11.6, we have $\mu \leq' \lambda$. Hence $\mu \leq \lambda$, and hence by the previous lemma $W\mu \preceq W\lambda$.

The uniqueness of $\mathcal{O}$ follows immediately from $\preceq$ being a partial order.

$\square$

Claim 11.15. *Let $V, W \in Rep^{fd}(G)$ be irreducible. Then if $\mathcal{O}_V = \mathcal{O}_W$ (the orbits as in the previous claim), $V \cong W$.*

Proof. Notice that the highest weight of V is recoverable as the unique dominant element in $\mathcal{O}_V$. Hence this claim follows from the claim that the highest weight determine the irreducible representation. $\square$

To sum up, associating to an irreducible representation its "highest orbit" gives an injection of the set of isomorphism classes of irreducible representations into the set of W-orbits in $\mathfrak{t}_{\mathbb{Z}}^*$. Again, we will show that this is in fact a surjection.

12 Weyl's character formula

12.1 The representation rings

Let us consider the group algebras $\widetilde{R}(T) := \mathbb{Z}[(e(\lambda))_{\lambda \in \mathfrak{t}^*}]$ and it's subgroup algebra $R(T) := \mathbb{Z}[(e(\lambda))_{\lambda \in \mathfrak{t}_{\mathbb{Z}}^*}]$. Thus, we defined for every $V \in Rep^{fd}(G)$ the formal character $fc_V \in R(T)$.

Notice that we have a natural homomorphism $\mathbb{C} \otimes_{\mathbb{Z}} \widetilde{R}(T) \to C(\mathfrak{t})$. We claim that it is an injection. In other words, the functions $(e(\lambda))_{\lambda \in \mathfrak{t}^*}$ in $C(\mathfrak{t})$ are linearly independent. This is routine to check, by induction on the dimension of $\mathfrak{t}$. As a result, it is easy to check that if a function in $\mathbb{C} \otimes \widetilde{R}$ is invariant under translation by $\mathfrak{t}_{\mathbb{Z}}$ (i.e. descends to a function in $C(T)$), then it in fact lies in $\mathbb{C} \otimes_{\mathbb{Z}} R(T)$.

Notice that $\mathbb{C} \otimes_{\mathbb{Z}} R(T)$, identified with a subalgebra of $C(T)$, is exactly $C(T)^{fin}$.

The standard inner product on $C(T)$ gives us an inner product on $R(T)$, given by $\langle e(\lambda_1), e(\lambda_2) \rangle = \delta_{\lambda_1, \lambda_2}$ for $\lambda_1, \lambda_2 \in \mathfrak{t}_{\mathbb{Z}}^*$. We extend this inner product to $\widetilde{R}(T)$, by the same formula (but this time $\lambda_1, \lambda_2 \in \mathfrak{t}^*$).

We also have:

Lemma 12.1. *The ring $\widetilde{R}(T)$ is an integral domain (and hence also $R(T)$).*

Proof. More generally, we prove that the group algebra of a finite-dimensional vector space V, $A[V]$, is an integral domain, if the ring of coefficients A is an integral domain. This we do by induction on the dimension of V. If $\dim V = 1$, this is easy by considering "highest coefficients". To perform induction, break $V = V_1 \oplus V_2$, and then $A[V] \cong A[V_1][V_2]$, so we can perform induction. $\square$

12.2 Antisymmetrization

We have the sign homomorphism $sgn : W \to \{\pm 1\}$ - the unique homomorphism sending simple reflections s_α to -1. One can construct it as $sgn(w) := (-1)^{\ell(w)}$; or, as $sgn(w) = det(w; \mathfrak{t}^*)$ (the determinant of w acting on $\mathfrak{t}^*$).

In a representation E of W we have the vectors $e \in E$ satisfying $wv = v$ for all $w \in W$, which for our current purposes we will call symmetric vectors (and denote by E^{sym} the subspace of such), and the vectors $e \in E$ satisfying $wv = sgn(w)v$ for all $w \in W$, which we will call antisymmetric vectors (and denote by E^{asym} the subspace of such). We also have, given $e \in E$, the antisymmetrization

$$\Lambda(e) := \sum_{w \in W} sgn(w) \cdot we \in E^{asym}.$$

12.3 The functions $\Lambda(\lambda)$

For $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$, we have

$$\Lambda(\lambda) := \sum_{w \in W} sgn(w) \cdot we(\lambda) \in R(T)^{asym}.$$

Claim 12.2.

1. *The elements $\Lambda(\lambda)$ form a $\mathbb{Z}$-basis for $R(T)^{asym}$, as λ runs over dominant regular elements of $\mathfrak{t}^*_{\mathbb{Z}}$.*
2. *For $\lambda_1, \lambda_2 \in \mathfrak{t}^*_{\mathbb{Z}}$ one has*
$$\langle \Lambda(\lambda_1), \Lambda(\lambda_2) \rangle = |W| \cdot \delta_{\lambda_1, \lambda_2}.$$

Proof.

1. Obviously all $\Lambda(\lambda)$, as λ runs over dominant regular elements of $\mathfrak{t}^*_{\mathbb{Z}}$ are linearly independent, since for a regular dominant λ, $w\lambda$ are not regular dominant for every $w \neq e$. So, we only need to show that every $f \in R(T)^{asym}$ can be written as an integral linear combination of our elements.

   Write $f = \sum_{\lambda \in \mathfrak{t}^*_{\mathbb{Z}}} n_\lambda \cdot e(\lambda)$ (a finite sum with integer coefficients). The antisymmetry gives $n_{w\lambda} = sgn(w) n_\lambda$ for all $w \in W, \lambda \in \mathfrak{t}^*_{\mathbb{Z}}$. We claim that $n_\lambda = 0$ for singular λ. Indeed, if λ is singular, then $s_\alpha \lambda = \lambda$ for some $\alpha \in R$, but then $s_\alpha f = -f$ gives $n_\lambda = -n_\lambda$ so $n_\lambda = 0$.

   We can now write
$$f = \sum_{\lambda \in \mathfrak{t}^*_{\mathbb{Z}}} n_\lambda \cdot e(\lambda) = \sum_{\lambda \in \mathfrak{t}^*_{\mathbb{Z}} \text{ dom reg}} \sum_{w \in W} n_{w\lambda} \cdot e(w\lambda) =$$
$$= \sum_{\lambda \in \mathfrak{t}^*_{\mathbb{Z}} \text{ dom reg}} n_\lambda \sum_{w \in W} sgn(w) \cdot e(w\lambda) = \sum_{\lambda \in \mathfrak{t}^*_{\mathbb{Z}} \text{ dom reg}} n_\lambda \cdot \Lambda(\lambda).$$
2. This is clear.

□

12.4 The function δ

Recall the element $\Delta \in R(T)^{sym}$ from Weyl's integration formula, given by

$$\Delta = \prod_{\alpha \in R} (e(\alpha) - 1).$$

We describe an element $\delta \in \widetilde{R}(T)^{asym}$, which satisfies $\delta \cdot \overline{\delta} = \Delta$ (here $\bar{\cdot}$ is the linear map sending $e(\lambda)$ to $e(-\lambda)$). Sometimes $\delta \in R(T)$, so represents an actual function on T, and sometimes not.

For this, set

$$\rho := \frac{1}{2} \sum_{\alpha \in R^+} \alpha \in \mathfrak{t}^*,$$

and then

$$\delta := e(\rho) \prod_{\alpha \in R^+} (1 - e(-\alpha)) = e(-\rho) \prod_{\alpha \in R^+} (e(\alpha) - 1) = \prod_{\alpha \in R^+} (e(\alpha/2) - e(-\alpha/2)) \in \widetilde{R}(T).$$

Clearly $\delta \cdot \overline{\delta} = \Delta$, and we claim $\delta \in R(T)^{asym}$. Indeed, we want to check $s_\alpha \delta = -\delta$ for $\alpha \in S$ (this is enough since $\langle s_\alpha \rangle_{\alpha \in S} = W$). For this, we recall that $s_\alpha(\alpha) = -\alpha$ and $s_\alpha(\beta) \in R^+$ for $\alpha \neq \beta \in R^+$.

12.5 Weyl's character formula

Theorem 12.3. *Let $V \in Rep^{fd}(G)$ be an irreducible representation with highest weight $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$. Then $fc_V = \frac{\Lambda(\lambda + \rho)}{\delta}$.*

Proof. Notice that $fc_V \cdot \delta \in \widetilde{R}(T)^{asym}$. Thus, we can write

$$fc_V \cdot \delta = \sum_{\mu \in \mathfrak{t}^*_{\mathbb{Z}} \text{ dom reg}} n_\mu \Lambda(\mu)$$

(a finite sum with integer coefficients).

Weyl's integration formula gives

$$1 = \langle \chi_V, \chi_V \rangle = ... = \frac{1}{|W|} \langle fc_V \cdot \delta, fc_V \cdot \delta \rangle = \sum_{\mu \in \mathfrak{t}^*_{\mathbb{Z}} \text{ dom reg}} n_\mu^2.$$

Thus, n_μ is non-zero for exactly one μ, and for this μ we have $n_\mu = \pm 1$. To find μ and the sign, recall that

$$fc_V = e(\lambda) + \text{smaller},$$

where "smaller" means a linear combination with integer coefficients of $e(\lambda')$'s, where $\lambda' \in \lambda - \mathbb{Z}R^+$, $\lambda' \neq \lambda$. Thus, from observing the definition of δ, we get

$$fc_V \cdot \delta = e(\lambda + \rho) + \text{smaller}.$$

From this we see that $\mu = \lambda + \rho$ and $n_\mu = 1$. $\square$

12.6 The main existence theorem

Theorem 12.4. *Let $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$ be dominant. Then there exists an irreducible representation $V \in Rep^{fd}(G)$ with highest weight λ.*

The theorem will follow from the following proposition:

Proposition 12.5. *Let $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$. Then $\frac{\Lambda(\lambda+\rho)}{\delta}$ exists in $C(T)$, in the sense that it exists in $C(\mathfrak{t})$ and is invariant under translation by $\mathfrak{t}_{\mathbb{Z}}$, so descends to T.*

Proof of theorem 12.4 given proposition 12.5. Let λ be dominant, and let $f \in C(T)$ be the function as in the statement of the proposition. Notice that $f \in C(T)^{sym}$, and thus there exists a continuous function $\widetilde{f} \in C(G)^{cent}$ which restricts to f. For an irreducible $V \in Rep^{fd}(G)$ with highest weight $\mu \neq \lambda$ we have

$$\langle \chi_V, \widetilde{f} \rangle_G = \frac{1}{|W|} \int_T \frac{\Lambda(\mu+\rho)}{\delta} \frac{\overline{\Lambda(\lambda+\rho)}}{\delta} \Delta = \frac{1}{|W|} \int_T \Lambda(\mu+\rho) \overline{\Lambda(\lambda+\rho)} = \frac{1}{|W|} \langle \Lambda(\mu+\rho), \Lambda(\lambda+\rho) \rangle = 0.$$

Now, since $\widetilde{f} \neq 0$, it can not be orthogonal to all characters by the Peter-Weyl theorem, and thus there must be an irreducible representation with highest weight λ. ☐

In order to prove proposition 12.5, we have some lemmas:

Lemma 12.6.

1. *Let f be an analytic function on $\mathfrak{t}$, vanishing on the vanishing locus of $e(\lambda) - 1$, for some $0 \neq \lambda \in \mathfrak{t}^*$. Then $\frac{f}{e(\lambda)-1}$ exists as an analytic function on $\mathfrak{t}$.*

2. *Let f be an analytic function on $\mathfrak{t}$, vanishing on the vanishing loci of $e(\alpha)-1$, for all $\alpha \in R^+$. Then $\frac{f}{\prod_{\alpha \in R^+}(e(\alpha)-1)}$ exists as an analytic function on $\mathfrak{t}$.*

3. *Let $\lambda \in \mathfrak{t}^*$ be such that $\langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}$ for all $\alpha \in R$. Then $\Lambda(\lambda)$ vanishes on the vanishing loci of $e(\alpha) - 1$, for all $\alpha \in R^+$.*

4. *Let $\lambda \in \mathfrak{t}^*_{\mathbb{Z}}$. Then the function $\frac{\Lambda(\lambda+\rho)}{\delta}$ is invariant under translation by $\mathfrak{t}_{\mathbb{Z}}$.*

Proof.

1. This is standard, by developing into Taylor series.

2. We get this by applying recursively the previous item, while noticing that the complement of the union of vanishing loci of $e(\beta) - 1$, for $\beta \neq \alpha$, is dense in the vanishing locus of $e(\alpha) - 1$.

3. Llet $H \in \mathfrak{t}$ such that $e(\alpha) - 1$ vanishes on H for some $\alpha \in R^+$. One has $\Lambda(\lambda)(wH) = sgn(w) \cdot \Lambda(\lambda)(H)$. Hence, if

$$\Lambda(\lambda)(H) = -\Lambda(\lambda)(s_\alpha H) = -\Lambda(\lambda)(H - \langle H, \alpha\rangle \alpha^\vee) = -\Lambda(\lambda)(H).$$

The reason for the last equality is that $e(w\lambda)(-\langle H, \alpha\rangle\alpha^\vee) = 1$. This is since $\langle H, \alpha\rangle \in \mathbb{Z}$ (because $e(\alpha) - 1$ is zero on H), and $(w\lambda)(\alpha^\vee) \in \mathbb{Z}$ (by the given on λ). Thus, we get $\Lambda(\lambda)(H) = 0$.

4. Up to the factor $\prod_{\alpha \in R^+}(e(\alpha)-1)$, which is invariant under $\mathfrak{t}_\mathbb{Z}$, our function is

$$\sum_{w \in W} sgn(w) \cdot e(w(\lambda + \rho) + \rho).$$

Thus, it is enough to show that $w\rho + \rho \in \mathfrak{t}^*_\mathbb{Z}$ for all $w \in W$. Since $2\rho \in \mathfrak{t}^*_\mathbb{Z}$, it is enough to show that $w\rho - \rho \in \mathfrak{t}^*_\mathbb{Z}$ for all $w \in W$. Notice that if we know this for w_1, w_2, then from $w_1 w_2 \rho - \rho = w_1(w_2\rho - \rho) + (w_1\rho - \rho)$ we know this also for $w_1 w_2$. Hence, it is enough to check for $w = s_\alpha$, $\alpha \in S$. Then $s_\alpha \rho - \rho = -\alpha \in \mathfrak{t}^*_\mathbb{Z}$.

□

12.7 Some formulas

If we consider the trivial representation $V^0 \in Rep^{fd}(G)$, then Weyl's character formula yields **Weyl's denominator formula**

$$\delta = \Lambda(\rho) = \sum_{w \in W} sgn(w) \cdot e(w\rho).$$

So we can rewrite Weyl's character formula, for an irreducible representation $V \in Rep^{fd}(G)$, as:

$$fc_V = \frac{\Lambda(\lambda + \rho)}{\Lambda(\lambda)} = \frac{\sum_{w \in W} sgn(w) \cdot e(w(\lambda + \rho))}{\sum_{w \in W} sgn(w) \cdot e(w\rho)}.$$

We have **Weyl's dimension formula**:

$$\dim V^\lambda = \frac{\prod_{\alpha \in R^+} \langle \alpha^\vee, \lambda + \rho\rangle}{\prod_{\alpha \in R^+} \langle \alpha^\vee, \rho\rangle} = \frac{\prod_{\alpha \in R^+} \kappa(\lambda + \rho, \alpha)}{\prod_{\alpha \in R^+} \kappa(\rho, \alpha)}.$$

To show this, denoting by $\widetilde{\kappa} : \mathfrak{t}^* \cong \mathfrak{t}$ the isomorphism induced by the inner product κ, we have:

$$\Lambda(\lambda + \rho)(t \cdot \widetilde{\kappa}(\rho)) = \sum_{w \in W} sgn(w) \cdot e^{2\pi i \cdot \kappa(t\rho, w(\lambda + \rho))} = \Lambda(\rho)(t \cdot \widetilde{\kappa}(\lambda + \rho)),$$

and using Weyl's denominator formula we continue:

$$= e^{2\pi i \kappa(t(\lambda + \rho), \rho)} \prod_{\alpha \in R^+} (1 - e^{-2\pi i \kappa(t(\lambda + \rho), \alpha)}),$$

and developing into a power series in t we continue:

$$= (2\pi i)^{|R^+|} \left[\prod_{\alpha \in R^+} \kappa(\lambda + \rho, \alpha) \right] \cdot t^{|R^+|} + O(t^{|R^+|+1}).$$

From this the formula clearly follows.

12.8 Example

Let us consider $G = SU(3)$. We have simple roots $\alpha_1 := (\sqrt{2}, 0), \alpha_2 := (\sqrt{2}cos(2\pi/3), \sqrt{2}sin(2\pi/3))$ and another positive root $\gamma := \alpha_1 + \alpha_2$. Since $SU(3)$ is simply connected, the integral weights are those satisfying $\kappa(\alpha_1, \lambda), \kappa(\alpha_2, \lambda) \in \mathbb{Z}$. Dominant weights are those satisfying $\kappa(\alpha_1, \lambda), \kappa(\alpha_2, \lambda) \geq 0$. We have $\rho = \gamma$. We have $\kappa(\alpha_i, \alpha_i) = 2$ and $\kappa(\alpha_1, \alpha_2) = -1$. We have the two fundamental weights ω_1, ω_2 given by $\kappa(\alpha_i, \omega_j) = \delta_{ij}$. Those are dominant and integral, and in fact freely generate the monoid of dominant and integral weights. We can now calcaulte $\dim V^{n_1\omega_1 + n_2\omega_2}$. We obtain

$$\dim V^{n_1\omega_1 + n_2\omega_2} = (1 + n)(1 + m)(1 + \frac{n + m}{2}).$$