NOTES FOR LECTURE ABOUT SIMPSON'S "HIGGS BUNDLES AND LOCAL SYSTEMS"

ALEXANDER YOM DIN

1. Notations and conventions

1.1. I try to digest part of Simpson's paper "Higgs bundles and local systems" for a learning seminar at IAS, 2019-2020. The notes are a bit disorganized.

1.2. We fix X - a smooth projective variety over $\mathbb{C}$.

We denote by $\mathcal{A}^\bullet(X)$ the coconnective commutative dg-algebra of smooth differential forms on X. We have the differential

$$d : \mathcal{A}^\bullet(X) \to \mathcal{A}(X)^{\bullet+1}$$

which decomposes $d = d' + d''$ into the $(1,0)$ and $(0,1)$-parts. More generally, we will use $(-)'$ for the $(1,0)$-part and $(-)''$ for the $(0,1)$-part.

By a **bundle** on X we mean a smooth (C^∞) vector bundle. Bundles form a category $Bun(X)$.

Given $\mathcal{V} \in Bun(X)$, we denote by $\mathcal{A}^0(X)$ the space of smooth sections of $\mathcal{V}$ and

$$\mathcal{A}^\bullet(\mathcal{V}) := \mathcal{A}^0(\mathcal{V}) \underset{\mathcal{A}^0(X)}{\otimes} \mathcal{A}^\bullet(X).$$

2. Simpson's correspondence

2.1. **Connections.** Let

$$\delta : \mathcal{A}^\bullet(X) \to \mathcal{A}^{\bullet+1}(X)$$

be a derivation, i.e.

$$\delta(\omega_1\omega_2) = \delta(\omega_1)\omega_2 + (-1)^{\deg(\omega_1)}\omega_1\delta(\omega_2) \quad \forall \omega_1, \omega_2 \in \mathcal{A}^\bullet(X),$$

such that additionally $\delta^2 = 0$. A **δ-connection** on a vector bundle $\mathcal{V}$ on X is a

$$\nabla : \mathcal{A}^\bullet(\mathcal{V}) \to \mathcal{A}^{\bullet+1}(\mathcal{V})$$

for which

$$\nabla(s\omega) = \nabla(s)\omega + (-1)^{\deg(s)}s\delta(\omega) \quad \forall s \in \mathcal{A}^\bullet(\mathcal{V}),\ \omega \in \mathcal{A}^\bullet(X).$$

Notice that ∇^2 is $\mathcal{A}^\bullet(X)$-linear. We say that ∇ is **flat** if $\nabla^2 = 0$.

2.2. Flat bundles.

2.2.1. Let us denote by

$$Bun_{flt}(X)$$

the category of bundles $\mathcal{V}$ on X equipped with a flat d-connection ("**flat bundles**"). Morphisms are morphisms of bundles which commute with the connection.

2.2.2. Recall that we have:

$$\left\{ \begin{matrix} \text{representations of} \\ \pi_1(X, x_0) \end{matrix} \right\} \overset{\approx}{\longleftrightarrow} \left\{ \begin{matrix} \text{local systems} \\ \text{on } X \end{matrix} \right\} \overset{\approx}{\longleftrightarrow} Bun_{flt}(X).$$

Here local systems are sheaves locally isomorphic to the constant sheaf with fiber $\mathbb{C}^n$ for some $n \in \mathbb{Z}_{\geqslant 0}$. The equivalence from the third to the second is by passing to the subsheaf of sections killed by the connection. Representations are finite-dimensional representations, and the equivalence from the second to the first is by the construction of monodromy.

2.2.3.

Remark 2.1. Thus, $Bun_{flt}(X)$ uses the differential geometry of X but is equivalent to a topological notion.

2.3. **Higgs bundles.**

2.3.1. Let us denote by

$$Bun_{Hgs}(X)$$

the category of bundles $\mathcal{V}$ on X equipped with a flat d''-connection ("**Higgs bundles**"). Morphisms are morphisms of bundles which commute with the connection.

2.3.2. Decomposing the d''-connection ∇ on $\mathcal{V}$ into $(1,0)$ and $(0,1)$ parts,

$$\nabla = \nabla' + \nabla'',$$

we have that ∇'' is a d''-connection, ∇' is a 0-connection, and the flatness condition $\nabla^2 = 0$ is rewritten as the system of conditions

$$\begin{cases} (\nabla'')^2 = 0 \\ [\nabla', \nabla''] = \nabla'\nabla'' + \nabla''\nabla' = 0 \\ (\nabla')^2 = 0 \end{cases}.$$

2.3.3. The first condition is, by the Newlander-Nirenberg condition, exactly what is needed for ∇'' to come from an holomorphic structure on the bundle $\mathcal{V}$ (and ∇'' determines uniquely that holomorphic structure - the holomorphic sections are the sections killed by ∇''). The second condition is then interpreted as saying that ∇' is holomorphic, i.e. corresponds to a holomorphic 1-form with values in the endomorphism bundle of $\mathcal{V}$ viewed as a holormorphic bundle with the use of ∇''.

2.3.4. We therefore have

$$\left\{ \begin{matrix} \text{holomorphic bundle with a holomorphic} \\ \text{1-form with values in the endomorphism bundle} \\ \text{which squares to 0} \end{matrix} \right\} \overset{\approx}{\longleftrightarrow} Bun_{Hgs}(X).$$

2.3.5.

Remark 2.2. Thus, incorporating GAGA, $Bun_{Hgs}(X)$ uses the complex structure of X, but can be formulated as an algebro-geometric notion.

Remark 2.3. Perhaps in a different direction of emphasis, also $Bun_{flt}(X)$, using GAGA, can be reformulated as an algebro-geometric notion.

3. The Simpson correspondence

3.1. **Pre-flat correspondence.**

3.1.1. Let K be an Hermitian metric on $\mathcal{V}$. Using K, one constructs a bijection

$$\begin{array}{c}\{d\text{-connections on } \mathcal{V}\} \\ \updownarrow \\ \{d''\text{-connections on } \mathcal{V}\}\end{array}.$$

3.1.2. Let ∇ be a d-connection on $\mathcal{V}$. Decompose into $(1,0)$ and $(0,1)$-parts

$$\nabla = \nabla' + \nabla''.$$

Let $\widetilde{\nabla}'$ (resp. $\widetilde{\nabla}''$) be the unique d'-connection with $(1,0)$-values (resp. d''-connection with $(0,1)$-values) such that $\widetilde{\nabla}' + \nabla''$ (resp. $\nabla' + \widetilde{\nabla}''$) preserves K. Then the corresponding d''-connection we consider is

$$D := \frac{1}{2}\left(\nabla' - \widetilde{\nabla}'\right) + \frac{1}{2}\left(\nabla'' + \widetilde{\nabla}''\right).$$

3.1.3. A calculation shows that in this correspondence $\nabla \underset{K}{\longleftrightarrow} D$, if $D^2 = 0$ then also $(\nabla - D)^2 = 0$ (here $\nabla - D$ is a d'-connection).

3.2. **Harmonic bundles.**

3.2.1. A **harmonic bundle** is a vector bundle $\mathcal{V}$ equipped with:

- A flat d-connection ∇_{flt}
- A flat d''-connection ∇_{Hgs}

such that there exists a Hermitian metric K on $\mathcal{V}$ for which

$$\nabla_{flt} \underset{K}{\longleftrightarrow} \nabla_{Hgs}.$$

3.2.2. It is also convenient then to consider the d'-connection

$$\nabla_{coHgs} := \nabla_{flt} - \nabla_{Hgs}.$$

By above, we also have $\nabla^2_{coHgs} = 0$, and therefore we find that

$$[\nabla_{Hgs}, \nabla_{coHgs}] = \nabla_{Hgs}\nabla_{coHgs} + \nabla_{coHgs}\nabla_{Hgs} = 0.$$

3.2.3. We will have some preliminaries before discussing what are morphisms of harmonic bundles.

3.3. **Hodge theory - first dive.**

3.3.1. Let $\mathcal{V}$ be a harmonic bundle. Fixing a Kahler structure on X, we obtain an inner product on each space $A^k(\mathcal{V})$ for $k \in \mathbb{Z}_{\geqslant 0}$. We therefore can talk about formal adjoints of differential operators between these spaces.

3.3.2. Denote by

$$\Lambda : A^\bullet(X) \to A^{\bullet-2}(X)$$

the formal adjoint of multiplication by ω.

Proposition 3.1 (Kahler relations)**.** *One has*

$$\nabla^*_{coHgs} = i[\Lambda, \nabla_{Hgs}], \;\; \nabla^*_{Hgs} = -i[\Lambda, \nabla_{coHgs}].$$

Remark 3.2. In fact these relations hold also in the pre-flat correspondence, when ∇_{flt} is not assumed to be flat, but ∇_{Hgs} is. Simpson mentions that this is a motivation for the pre-flat correspondence.

3.3.3. The Laplacian of a flat δ-connection

$$\nabla : \mathcal{A}^\bullet(\mathcal{V}) \to \mathcal{A}^{\bullet+1}(\mathcal{V})$$

is defined as

$$\Delta := [\nabla, \nabla^*] = \nabla\nabla^* + \nabla^*\nabla : A^\bullet(\mathcal{V}) \to A^\bullet(\mathcal{V}).$$

An $\alpha \in A^\bullet(\mathcal{V})$ is called **∇-harmonic** if $\Delta\alpha = 0$. This is equivalent to $\nabla\alpha = 0$ and $\nabla^*\alpha = 0$.

3.3.4. We therefore have Laplacians

$$\Delta_{flt} := \Delta_{\nabla_{flt}},\ \Delta_{Hgs} := \Delta_{\nabla_{Hgs}}, \Delta_{coHgs} := \Delta_{\nabla_{coHgs}}.$$

Proposition 3.3. *One has*

$$[\nabla_{coHgs}, \nabla^*_{Hgs}] = 0,\ [\nabla_{Hgs}, \nabla^*_{coHgs}] = 0,$$
$$\Delta_{coHgs} = \Delta_{Hgs},$$
$$\Delta_{flt} = 2\Delta_{coHgs} = 2\Delta_{Hgs}.$$

Proof. This is an elementary algebraic calculation using the Kahler relations and the relation $[\nabla_{coHgs}, \nabla_{Hgs}] = 0$. □

3.3.5. In particular, the notion of harmonic form is the same for $\nabla_{flt}, \nabla_{Hgs}, \nabla_{coHgs}$.

3.3.6. By elliptic theory, one obtains:

Proposition 3.4. *We have*

$$A^\bullet(\mathcal{V}) = Im(\Delta) \oplus Ker(\Delta),$$

and $Ker(\Delta)$ is finite-dimensional, for Δ any of the three Laplacians above.

Corollary 3.5 (Hodge decomposition)**.** *We have*

$$A^\bullet(\mathcal{V}) = Im(\nabla) \oplus Im(\nabla^*) \oplus Ker(\Delta).$$

for ∇ any of the three connections above (and Δ the corresponding Laplacian).

Proof. Clearly $Im(\Delta) \subset Im(\nabla) + Im(\nabla^*)$ so $A^\bullet(\mathcal{V}) = Im(\nabla) + Im(\nabla^*) + Ker(\Delta)$. If $\nabla\alpha + \nabla^*\beta + \gamma = 0$ (where $\gamma \in Ker(\Delta)$) then applying ∇ we obtain $\nabla\nabla^*\beta = 0$ and therefore (by a standard inner product trick) $\nabla^*\beta = 0$. Similarly $\nabla\alpha = 0$. And we are left with $\gamma = 0$. □

3.4. **The category of harmonic bundles.**

3.4.1.

Lemma 3.6. *Let $\mathcal{V}$ be a harmonic bundle. Let $s \in A^0(\mathcal{V})$. Then $\nabla_{flt}s = 0$ if and only if $\nabla_{Hgs}s = 0$.*

Proof. Notice that

$$\nabla_{flt}s = 0 \iff \Delta_{flt}s = 0 \iff \Delta_{Hgs}s = 0 \iff \nabla_{Hgs}s = 0.$$

The middle equivalence is by Proposition **??** while the left one, say, is because $\Delta_{flt}s = \nabla^*_{flt}\nabla_{flt}s$, and so if $\Delta_{flt}s = 0$ then

$$0 = \langle\nabla^*_{flt}\nabla_{flt}s, s\rangle = \langle\nabla_{flt}s, \nabla_{flt}s\rangle = ||\nabla_{flt}s||^2$$

and thus $\nabla_{flt}s = 0$. □

3.4.2.

Lemma 3.7. *Let $\mathcal{V}_1, \mathcal{V}_2$ be harmonic bundles. Let $T : \mathcal{V}_1 \to \mathcal{V}_2$ be a morphism of bundles. Then $T \circ \nabla_{flt} = \nabla_{flt} \circ T$ if and only if $T \circ \nabla_{Hgs} = \nabla_{Hgs} \circ T$.*

Proof. There is the standard monoidal formalism, by which we have a harmonic bundle

$$\underline{Hom}(\mathcal{V}_1, \mathcal{V}_2),$$

and a morphism of bundles $\mathcal{V}_1 \to \mathcal{V}_2$ is interpreted as a section of $\underline{Hom}(\mathcal{V}_1, \mathcal{V}_2)$. Furthermore, the morphism will commute with $\nabla_?$ (? standing for flt or Hgs) if and only if the corresponding section is killed by $\nabla_?$. Therefore the current Lemma follows from Lemma **??**. □

3.4.3. In view of Lemma **??**, we can define the category $Bun_{hrm}(X)$ of harmonic bundles, whose objects are harmonic bundles, and morphisms are morphisms of bundles commuting with either ∇_{flt} or ∇_{Hgs}, which is the same.

3.5. **The correspondence.** We have fully faithful "forgetful" functors

$$\begin{array}{ccc} & Bun_{hrm}(X) & \\ \swarrow & & \searrow \\ Bun_{flt}(X) & & Bun_{Hgs}(X) \end{array}.$$

The main theorem is then:

Theorem 3.8 (Simpson Correspondence)**.**

(1) *The essential image of the left functor consists of semisimple flat bundles, i.e. those which can be written as a direct sum of irreducible ones.*
(2) *The essential image of the right functor consists of slope* 0 *polystable Higgs bundles - Higgs bundles which can be written as a direct sum of stable Higgs bundles with all rational Chern classes trivial (in particular, those have slope* 0*).*

The proof is via hard analysis.

3.6. **The rank 1 example.**

3.6.1. Let us suppose for simplicity that $H^2(X, \underline{\mathbb{Z}})$ has no torsion (for example, $\dim_{\mathbb{C}} X = 1$). Fix $x_0 \in X$ and abbreviate $\pi_1 := \pi_1(X, x_0)$. Also, by the Higgs field of a Higgs bundle we will understand **minus** the $(1,0)$-part.

3.6.2. The set of isomorphism classes of local systems of rank 1 is in bijection with

$$Hom(\pi_1, \mathbb{C}^\times) \cong H^1(X, \underline{\mathbb{C}^\times}) \cong H^1(X, \underline{\mathbb{C}^\times_1}) \times H^1(X, \underline{\mathbb{R}^\times_{>0}}) \cong$$

$$\cong \frac{H^1(X, \underline{\mathbb{R}})}{H^1(X, \underline{\mathbb{Z}})} \times H^1(X, \underline{\mathbb{R}}) \cong \frac{Hrm^1_{\mathbb{R}}(X)}{H^1(X, \underline{\mathbb{Z}})} \times Hrm^1_{\mathbb{R}}(X).$$

Here taking quotient by $H^1(X, \underline{\mathbb{Z}})$ is in the sense of taking quotient by the image of this. The second factor was by treated by taking logarithm and the first factor was treated using the short exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{R}} \longrightarrow \underline{\mathbb{C}^\times_1} \longrightarrow 0 .$$

Then both factors were treated by considering harmonic representatives.

Given (ω_1, ω_2) on the right, the corresponding homomorphism $\chi : \pi_1 \to \mathbb{C}^\times$ is seen to be given by

$$\chi(\gamma) = e^{2\pi i \int_\gamma \omega_1 + \int_\gamma \omega_2}.$$

3.6.3. The Higgs field on a Higgs bundle of rank 1 is simply an holomorphic 1-form (there is no dependence on the bundle, as the endomorphism bundle of a bundle of rank 1 is the trivial bundle of rank 1). Therefore the set of isomorphism classes of Higgs bundles of rank 1 (with the Chern vanishing conditions) is in bijection with

$$Pic^0(X) \times H^0(X, \Omega^1) \cong \frac{H^1(X, \mathcal{O})}{H^1(X, \underline{\mathbb{Z}})} \times H^0(X, \Omega^1) \cong \frac{Hrm^{0,1}(X)}{H^1(X, \underline{\mathbb{Z}})} \times Hrm^{1,0}(X).$$

The first factor was treated using the short exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^\times \longrightarrow 0 \,.$$

Then both factors were treated by considering harmonic representatitves.

3.6.4. Let us consider a flat bundle of rank 1 corresponding to a $\mathbb{C}_1^\times$-local system with monodromy $\chi : \pi_1 \to \mathbb{C}_1^\times$. It admits an invariant Hermitian metric, w.r.t. which the d''-connection corresponding to the d-connection ∇_{flt} is simply ∇''_{flt}, in particular it is flat. Therefore the corresponding Higgs bundle is the holomorphic bundle where holomorphic sections are holomorphic functions multiplied by flat sections, and the Higgs field is trivial. Therefore, the map providing the correspondence is $H^1(X, \underline{\mathbb{C}_1^\times}) \to H^1(X, \mathcal{O}^\times)$ induced by the injection of sheaves $\underline{\mathbb{C}_1^\times} \to \mathcal{O}^\times$. In terms of the descriptions above, this map is seen to correspond to the projection onto $(0,1)$-type $Hrm^1_{\mathbb{R}}(X) \to Hrm^{0,1}(X)$. From this we see that if $\omega \in Hrm^{0,1}(X)$ is the form representing the resulting Higgs bundle in our above decomposition, we have

$$\chi(\gamma) = e^{2\pi i 2\Re \int_\gamma \omega}.$$

3.6.5. Let us now consider a trivial holomorphic bundle of rank 1 together with a Higgs field $\omega \in H^0(X, \Omega^1)$ and the standard Hermitian metric. One calculates that the Higgs connection $f \mapsto -\omega f + d'' f$ corresponds to the flat connection $f \mapsto -(\omega + \bar{\omega}) f + df$. Therefore the corresponding monodromy $\chi : \pi_1 \to \mathbb{C}^\times$ is given by $\chi(\gamma) = e^{2\Re \int_\gamma \omega}$.

3.6.6. Let us summarize:

$$\pi_0 Bun_{Hgs}(X)^{polyst,1} \cong \frac{Hrm^{0,1}(X)}{H^1(X, \underline{\mathbb{Z}})} \times H^0(X, \Omega^1),$$

and the monodromy $\chi : \pi_1 \to \mathbb{C}^\times$ of the flat connection corresponding to (ω, θ) in this decomposition is

$$\chi(\gamma) = e^{2\pi i \cdot 2\Re(\int_\gamma \omega) + 2\Re(\int_\gamma \theta)}$$

4. Hodge theory for an harmonic bundle

4.1. Let $\mathcal{V}$ be a harmonic bundle. Let us denote

$$A^\bullet_=(\mathcal{V}) := Ker(\nabla_{coHgs}) \subset A^\bullet(\mathcal{V}).$$

One immediately checks that ∇_{flt} and ∇_{Hgs} preserve $A^{\bullet}_{=}(\mathcal{V})$ (and of course are equal on it). We therefore have complexes and morphisms of complexes

$$(A^{\bullet}_{=}(\mathcal{V}), \nabla_{flt} = \nabla_{Hgs}) \tag{4.1}$$
$$(A^{\bullet}(\mathcal{V}), \nabla_{flt}) \hookleftarrow (A^{\bullet}_{=}(\mathcal{V}), \nabla_{flt} = \nabla_{Hgs}) \hookrightarrow (A^{\bullet}(\mathcal{V}), \nabla_{Hgs})$$

Proposition 4.1. *The above morphisms are quasi-isomorphisms.*

Proof. We want to show that harmonic forms represent the cohomology of

$$(A^{\bullet}_{=}(\mathcal{V}), \nabla_{flt} = \nabla_{Hgs}).$$

So let α be harmonic and assume that $\alpha = \nabla_{flt}\beta = \nabla_{Hgs}\beta$. Then by ∇_{flt}-Hodge decomposition, $\alpha = 0$.

Conversely, let α be such that $\nabla_{flt}\alpha = 0$ and $\nabla_{Hgs}\alpha = 0$. We want to show that α can be written as a harmonic form plus a form of the shape $\nabla_{flt}\beta = \nabla_{Hgs}\beta$. By the ∇_{Hgs}-Hodge decomposition, we can write α as a harmonic form plus a form in the image of ∇_{Hgs}. Substracting the harmonic form, we can assume that α itself lies in the image of ∇_{Hgs}. Then the next Lemma shows that α lies in the image of $\nabla_{Hgs}\nabla_{coHgs}$. Writing $\alpha = \nabla_{Hgs}\nabla_{coHgs}\gamma$, we notice that it is clear that also $\alpha = \nabla_{flt}\nabla_{coHgs}\gamma$, so $\beta = \nabla_{coHgs}\gamma$ gives the desired. □

Lemma 4.2.

$$\mathrm{Ker}(\nabla_{coHgs}) \cap \mathrm{Im}(\nabla_{Hgs}) \subset \mathrm{Im}(\nabla_{Hgs}\nabla_{coHgs}).$$

Proof. Let us abbreviate $D := \nabla_{Hgs}, E := \nabla_{coHgs}$. Let α belong to the left hand side. We write $\alpha = D\beta$, and by E-Hodge decomposing β we can assume $\beta = E^*\gamma$. By E-Hodge decomposing γ, we can assume that $E\gamma = 0$. We have:

$$\Delta_D D\gamma = DD^*D\gamma = D\Delta_D\gamma = D\Delta_E\gamma =$$

$$= DEE^*\gamma = DE\beta = -ED\beta = -E\alpha = 0.$$

Thus, $D\gamma$ is harmonic and D-exact, and hence by the D-Hodge decomposition we have

$$D\gamma = 0.$$

Now the Kahler relation $E^* = i[\Lambda, D]$ gives

$$\alpha = DE^*\gamma = iD\Lambda D\gamma - iDD\Lambda\gamma = 0.$$

□

5. Extending to extensions

5.1. dg-categories.

5.1.1. A **dg-category** $\mathcal{C}$ is a category enriched in complexes. That is, for any two objects $M_1, M_2 \in \mathcal{C}$ we are given a complex of morphisms $Hom(M_1, M_2)$, and so on. We will write

$$Hom^i(M_1, M_2)$$

for the i-th component of $Hom(M_1, M_2)$,

$$Z^i Hom(M_1, M_2) \subset Hom^i(M_1, M_2)$$

for the closed elements, and

$$H^i Hom(M_1, M_2)$$

for the i-th cohomology of $Hom(M_1, M_2)$.

5.1.2. We have three categories attached to the dg-category $\mathcal{C}$:

$$Hom^0\mathcal{C}, Z^0\mathcal{C}, H^0\mathcal{C},$$

where objects are the same as objects of $\mathcal{C}$, and morphisms are derived from the complex of morphisms by the way suggested in the notation. We call $Z^0\mathcal{C}$ the **underlying category** of $\mathcal{C}$ and $H^0\mathcal{C}$ the **homotopy category** of $\mathcal{C}$.

5.1.3. We have the notion of dg-functors. A dg-functor will be called strictly fully faithful (resp. dg-fully faithful) if it induces isomorphism (resp. quasi-isomorphisms) on Hom-complexes. A dg-functor F will be called strictly essentially surjective (resp. dg-essentially surjective) if Z^0F is essentially surjective (resp. H^0F is essentially surjective). A dg-functor will be called a strict equivalence (resp. dg-equivalence) if it is strictly fully faithful and strictly essentially surjective (resp. dg-fully faithful and dg-essentially surjective).

5.1.4. Ignoring set-theoretic nuiance, a very important distinction between the theory of categories and the theory of dg-categories is that an equivalence of categories admits an inverse, while an equivalence of dg-categories does not necessarily. Thus, we might typically get dg-equivalences

$$\mathcal{C}_1 \leftarrow \mathcal{C} \rightarrow \mathcal{C}_2$$

and then we want to think of $\mathcal{C}_1$ and $\mathcal{C}_2$ as dg-equivalent under that, although there might be no actual dg-functor $\mathcal{C}_1 \to \mathcal{C}_2$ which realizes this dg-equivalence. Anyhow, applying H^0 we obtain equivalences of categories $H^0\mathcal{C}_1 \leftarrow H^0\mathcal{C} \to H^0\mathcal{C}_2$, and we can find an equivalence of categories $H^0\mathcal{C}_1 \to H^0\mathcal{C}_2$ realizing this (which will be unique up to a unique isomorphism if things are formulated correctly).

5.2. **dg-enhancements in our case.**

5.2.1. Our three categories

$$Bun_?(X)$$

for $? \in \{flt, Hgs, hrm\}$ can be extended to dg-categories

$$Bun_?^{dg}(X)$$

(this means that we find a dg-category whose underlying category is our category). Namely, this is based on on the copmlexes

$$(A^\bullet(-), \nabla_{flt}),\ (A^\bullet(-), \nabla_{Hgs}),\ (A^\bullet_=(-), \nabla_{flt} = \nabla_{Hgs}) \tag{5.1}$$

above. For $\mathcal{V}, \mathcal{W} \in Bun_?(X)$, we define the Hom-complex between $\mathcal{V}$ and $\mathcal{W}$ in $Bun_?^{dg}(X)$ to be what we get by evaluating (**??**) on $\underline{Hom}(\mathcal{V}, \mathcal{W})$.

5.2.2. Using (**??**), we have dg-functors

$$Bun_{flt}^{dg}(X) \longleftarrow Bun_{hrm}^{dg}(X) \longrightarrow Bun_{Hgs}^{dg}(X).$$

By Proposition **??**, these are dg-fully faithful.

5.2.3. We therefore obtain a dg-equivalence

$$Bun_{flt}^{dg}(X)^{ss} \xleftarrow{\approx} Bun_{hrm}^{dg}(X) \xrightarrow{\approx} Bun_{Hgs}^{dg}(X)^{polyst}.$$

We would like now to see how to extend this to a dg-equivalence

$$Bun_{flt}^{dg}(X) \overset{\approx}{\longleftrightarrow} Bun_{Hgs}^{dg}(X)^{semist},$$

where a Higgs bundle is semistable if it admits a filtration with subquotients being polystable Higgs bundles (with the Chern vanishing as before).

6. Passing to extensions

6.1. Semidirect extensions.

6.1.1. A **semidirect extension** in $\mathcal{C}$ is the data of two closed morphisms

$$M_1 \xrightarrow{\alpha} M \xrightarrow{\beta} M_2,$$

for which $\beta \circ \alpha = 0$, such that there exist non-closed morphisms

$$\alpha^* \in Hom^0(M, M_1),\ \beta^* \in Hom^0(M_2, M)$$

such that

$$\alpha^* \circ \beta^* = 0,\ \alpha^* \circ \alpha = 1,\ \beta \circ \beta^* = 1,\ \alpha \circ \alpha^* + \beta^* \circ \beta = 1.$$

In other words, in $Hom^0\mathcal{C}$ our data becomes part of a direct sum diagram. An isomorphism between two semidirect extensions

$$M_1 \xrightarrow{\alpha} M \xrightarrow{\beta} M_2$$

and

$$M_1 \xrightarrow{\alpha'} M' \xrightarrow{\beta'} M_2$$

is a closed isomorphism γ fitting in a commutative diagram

$$\begin{array}{ccccc} M_1 & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M_2 \\ \| & & \downarrow{\scriptstyle \gamma} & & \| \\ M_1 & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & M_2 \end{array}.$$

6.1.2. Given a semidirect extension and fixing α^* and β^* as above, one checks that the element

$$\alpha^* \circ d(\beta^*) \in Hom^1(M_2, M_1)$$

is closed, and its cohomology class does not depend on the choice of α^* and β^*. So we obtain an element of $H^1 Hom(N, L)$ in this way. One checks then that we obtain in this way an injection from the set of isomorphism classes of semidirect extensions to $H^1 Hom(M_2, M_1)$.

6.1.3. We say that $\mathcal{C}$ is **strictly extension-saturated** if the above injection is a bijection for every $M_1, M_2 \in \mathcal{C}$.

6.1.4. The dg-category of complexes is strictly extension-saturated. Indeed, given complexes M_1, M_2 and $\delta \in Z^1 Hom(M_2, M_1)$, we can define on $M_1 \underline{\oplus} M_2$ the differential

$$\begin{pmatrix} d & \delta \\ 0 & d \end{pmatrix}$$

and this gives a semidirect extension as desired. More generally, we can take complexes $M_1, \ldots, M_n$ and a matrix

$$\Delta = \begin{pmatrix} 0 & \delta_{21} & \cdots & \delta_{n1} \\ \vdots & \ddots & & \vdots \\ & & \ddots & \delta_{n,n-1} \\ 0 & \cdots & & 0 \end{pmatrix}$$

where $\delta_{ij} \in Hom^1(M_i, M_j)$, such that

$$(d \cdot \mathrm{Id} + \Delta)^2 = 0,$$

and then consider $M_1 \underline{\oplus} \ldots \underline{\oplus} M_n$ with the differential $d \cdot \mathrm{Id} + \Delta$. This will be a complex which can be obtained by a series of semidirect extensions:

$$M_1 \to M_{12} \to M_2,$$

$$M_{12} \to M_{123} \to M_3,$$

$$\vdots$$

$$M_{1 \cdots n-1} \to M_{1 \cdots n} \to M_n.$$

Notice that the condition $(d \cdot \mathrm{Id} + \Delta)^2 = 0$ can be written as

$$d(\Delta) + \Delta^2 = 0,$$

the **Maurer-Cartan equation** (here $d(\Delta)$ is the entrywise application of d to Δ).

6.1.5. Inspired by the above construction in the dg-category of complexes, we can formally add semidirect extensions to any dg-category $\mathcal{C}$, as follows. We construct a new dg-category $\mathcal{C}^{sde}$. An object of $\mathcal{C}^{sde}$ consists of a series $M_1, \ldots, M_n$ of objects in $\mathcal{C}$, and a matrix

$$\Delta = \begin{pmatrix} 0 & \delta_{21} & \cdots & \delta_{n1} \\ \vdots & \ddots & & \vdots \\ & & \ddots & \delta_{n,n-1} \\ 0 & \cdots & & 0 \end{pmatrix}$$

where $\delta_{ij} \in Hom^1(M_i, M_j)$, such that

$$d(\Delta) + \Delta^2 = 0.$$

One can see what the complexes of morphisms should be by looking at the example of complexes: ...

6.1.6. Notice that we have a strictly fully-faithful dg-functor $\mathcal{C} \to \mathcal{C}^{sde}$. If $\mathcal{C} \to \mathcal{D}$ is a dg-functor, we have a naturally constructed $\mathcal{C}^{sde} \to \mathcal{D}^{sde}$ and we have naturally

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{C}^{sde} & \longrightarrow & \mathcal{D}^{sde} \end{array}$$

(the diagram 2-commutes).

6.1.7. It is easy to check that $\mathcal{C}^{sde}$ is strictly extension-saturated. Additionally, $\mathcal{C}$ is strictly extension-saturated if and only if the strictly fully-faithful dg-functor $\mathcal{C} \to \mathcal{C}^{sde}$ is a strict equivalence (i.e. the corresponding $Z^0\mathcal{C} \to Z^0\mathcal{C}^{sde}$ is essentially surjective). One can say that $\mathcal{C}$ is **extension-saturated** if the weaker condition, that the dg-functor is dg-essentially surjective (i.e. $H^0\mathcal{C} \to H^0\mathcal{C}^{sde}$ is essentially surjective), is satisfied.

6.1.8. Suppose that $\mathcal{C} \to \mathcal{D}$ is a strictly fully-faithful dg-functor, with $\mathcal{D}$ strictly extension-saturated and also such that every object of $\mathcal{D}$ can be obtained from objects in the essential image of our functor via successive semidirect extensions. Then we claim that the corresponding $\mathcal{C}^{sde} \to \mathcal{D}^{sde}$ is a strict equivalence. Since $\mathcal{D} \to \mathcal{D}^{sde}$ is also a strict equivalence, we obtain a strict equivalence between $\mathcal{C}^{sde}$ and $\mathcal{D}$.

One checks first that given a dg-functor $F : \mathcal{C}_1 \to \mathcal{C}_2$, a semidirect extension $M_1 \to M \to M_2$ in $\mathcal{C}_1$ and an object $N \in \mathcal{C}_1$, if the functor F is strictly fully faithful on (N, M_1) and (N, M_2) then it is also strictly fully faithful on (N, M): We will have a commutative diagram in the dg-category of complexes

$$\begin{array}{ccccc} Hom(N, M_1) & \longrightarrow & Hom(N, M) & \longrightarrow & Hom(N, M_2) \\ \downarrow & & \downarrow & & \downarrow \\ Hom(FN, FM_1) & \longrightarrow & Hom(FN, FM) & \longrightarrow & Hom(FN, FM_2) \end{array}$$

with rows being semidirect extensions, and then it is easy to see that since the extremal vertical arrows are isomorphisms, so is the middle one. Similarly, one checks that if F is strictly full faithful on (M_1, N) and (M_2, N) then it is also strictly fully faithful on (M, N).

Similarly, given a dg-functor $F : \mathcal{C}_1 \to \mathcal{C}_2$, if objects $M_1, M_2 \in \mathcal{C}_2$ are in the strict essential image of F and $M_1 \to M \to M_2$ is a semidirect extension, then M lies in the strict essential image of F.

We ask ourselves now on which pairs of objects our functor $\mathcal{C}^{sde} \to \mathcal{D}^{sde}$ is strictly fully faithful. If both objects are in $\mathcal{C}$, then it is so by assumption. Since every object in $\mathcal{C}^{sde}$ can be obtained from objects of $\mathcal{C}$ by repeated semidirect extension, the above remark shows that the functor will be strictly fully faithful on all pairs of objects.

Now we ask oursevles which objects of $\mathcal{D}^{sde}$ lie in the strict essential image of our functor $F : \mathcal{C}^{sde} \to \mathcal{D}^{sde}$. Suppose that $N_1, N_2 \in \mathcal{D}^{sde}$ do and that $N_1 \to M \to N_2$ is a semidirect extension. Fix $FM_1 \cong N_1, FM_2 \cong N_2$. Corresponding to the semidirect extension we have a class in $H^1 Hom(N_2, N_1)$. Since F is already known to be strictly fully faithful, we have a corresponding class in $H^1 Hom(M_1, M_2)$, to which corresponds a semidirect extension $M_1 \to M \to M_2$ in $\mathcal{C}^{sde}$. Then $FM_1 \to FM \to FM_2$ is a semidirect extension in $\mathcal{D}^{sde}$, with the same class as $N_1 \to N \to N_2$, and therefore $FM \cong N$. In other words we showed that the strict essential image is closed under semidirect extensions. Since it contains the strict essential image of $\mathcal{C}$, it contains the whole of $\mathcal{D}$ by assumption, and then the whole of $\mathcal{D}^{sde}$.

6.1.9. Suppose that $\mathcal{C} \to \mathcal{D}$ is a dg-equivalence. Then $\mathcal{C}^{sde} \to \mathcal{D}^{sde}$ is a dg-equivalence. We omit the proof for now.

6.2. **Our case.**

6.2.1. We claim that $Bun^{dg}_{flt}(X)$ and $Bun^{dg}_{Hgs}(X)$ are strictly extension-saturated. Indeed, let us be given bundles with connections $\mathcal{V}$ and $\mathcal{W}$ (in any of the cases), and an element $T \in Z^1 Hom(\mathcal{W}, \mathcal{V})$. Then T can be interpreted as a $\mathcal{A}^0(X)$-linear map

$$\mathcal{A}^0(\mathcal{W}) \to \mathcal{A}^1(\mathcal{V}),$$

so by extension an $\mathcal{A}^\bullet(X)$-linear map

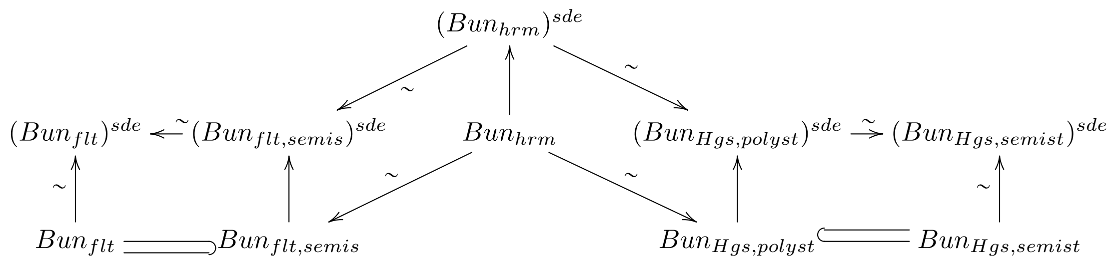
$$\mathcal{A}^\bullet(\mathcal{W}) \to \mathcal{A}^{\bullet+1}(\mathcal{V}),$$

satisfying $[\nabla, T] = \nabla \circ T + T \circ \nabla = 0$. We can then define on $\mathcal{V} \oplus \mathcal{W}$ the connection

$$\nabla(v, w) = (\nabla v + Tw, \nabla w).$$

This gives a semidirect extension as desired. Notice that $Bun^{dg}_{Hgs,semist}(X)$ is basically defined as the full dg-subcategory of $Bun^{dg}_{Hgs}(X)$ consisting of objects which can be obtained via a successive semidirect extension from object in $Bun^{dg}_{Hgs,polyst}(X)$, so it is also strictly extension-saturated.

6.2.2. We now obtain the following diagram of dg-categories and dg-functors (here we drop the dg-superscript, and the (X), for aesthetics):



Hence, we obtain an equivalence of dg-categories

$$Bun_{flt}^{dg}(X) \approx Bun_{Hgs,semist}^{dg}(X).$$

Passing to H^0, we obtain an equivalence of categories

$$Bun_{flt}(X) \approx Bun_{Hgs,semist}(X).$$

7. The non-abelian Hodge structure

7.1. **Tannakian formalism.**

7.1.1. We have seen the most probably amazing transformation

$$Rep(\pi_1) \approx LS(X) \approx Bun_{flt}(X) \approx Bun_{Hgs}(X)^{semist},$$

and want to figure out whether this tells something interesting about π_1.

7.1.2. So first natural step is to see whether π_1 can be resconstructed from $Rep(\pi_1)$, or say a discrete group Γ from $Rep(\Gamma)$. A usual double-commutator idea is that $Rep(\Gamma)$ is defined as things on which Γ acts, so maybe Γ can be recovered as the things which act on the things in $Rep(\Gamma)$. Namely, given $\gamma \in \Gamma$, for every $V \in Rep(\Gamma)$, denoting by $\underline{V}$ the underlying vector space (forgetting the Γ-action), we have an invertible operator

$$\gamma : \underline{V} \to \underline{V}.$$

Moreover, for every morphism $T : V \to W$ in $Rep(\Gamma)$, the following diagram commutes:

$$\begin{array}{ccc} \underline{V} & \xrightarrow{T} & \underline{W} \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\ \underline{V} & \xrightarrow{T} & \underline{W} \end{array} .$$

Also, there is compatability with the tensor product. for every $V, W \in Rep(\Gamma)$:

$$\begin{array}{ccc} \underline{V \otimes W} & \longleftrightarrow & \underline{V} \otimes \underline{W} \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma\otimes\gamma} \\ \underline{V \otimes W} & \longleftrightarrow & \underline{V} \otimes \underline{W} \end{array} .$$

7.1.3. So let us now consider the group Γ^{proalg} consisting of families of invertible operators $(\gamma_V)_{V \in Rep(\Gamma)}$ with commutation conditions as above. More categorically, this can be expressed as the group of tensor automorphisms of the forgetful functor $Rep(\Gamma) \to Vect$. We clearly have a homomorphism $\Gamma \to \Gamma^{proalg}$.

7.1.4. For $W \in Rep(\Gamma)$, we can consider a group Γ_W^{proalg} similar to the above, but where we run only over $V \in Rep(\Gamma)$ which are subquotients of direct sums of tensor products of copies of W and $W^\vee$. One can show that this group is a Zariski closed subgroup of $GL(\underline{W})$ (via the obvious map $\Gamma_W^{proalg} \to GL(\underline{W})$). We have

$$\Gamma^{proalg} = \lim \Gamma_W^{proalg},$$

and this gives Γ^{proalg} the structure of a **pro-algebraic group** - the cofiltered limit of affine algebraic groups.

7.1.5. Moreover, given a homomorphism $\Gamma \to G$ where G is an affine algebraic group, we can embed $G \to GL(W)$ for some vector space W, making W a representation of Γ. Then the image of $\Gamma \to \Gamma^{proalg} \to GL(W)$ lies in G, so $\Gamma \to G$ factors via $\Gamma \to \Gamma^{proalg}$. One can see that this gives a bijection, for every affine algebraic group G:

$$Hom(\Gamma^{proalg}, G) \xrightarrow{\sim} Hom(\Gamma, G)$$

where on the left we have abstract group homomorphisms, and on the right we have group homomorphsms which factor as $\Gamma^{proalg} \to \Gamma_W^{proalg} \to G$ for some W, where the second homomorphism is algebraic. This explains that Γ^{proalg} has a universal property of being the **pro-algebraic completion** of Γ.

7.1.6. Let us consider for example the case when Γ is commutative. Then every homomorphism from Γ to an affine algebraic group is written canonicaly as the product of a homomorphism with image consisting of unipotent elements and a homomorphism with image consisting of semisimple elements. Then one deduces a canonical decomposition

$$\Gamma^{proalg} = \Gamma^{prou} \times \Gamma^{pross}.$$

Γ^{prou} is the commutative pro-algebraic group for which

$$Hom(\Gamma^{prou}, \mathbb{G}_a) \cong Hom(\Gamma, \mathbb{G}_a).$$

We can think of $Hom(\Gamma, \mathbb{G}_a)$ as a $\mathbb{C}$-vector space, so the filtered colimit of finite-dimensional $\mathbb{C}$-vector spaces, and so Γ^{prou} is described as a cofiltered limit, which can be thought of as the linearly topologized $\mathbb{C}$-vector space which is the continuous dual of the discrete vector space $Hom(\Gamma, \mathbb{G}_a)$. Similarly, Γ^{pross} is the pro-diagnolizable group for which

$$Hom(\Gamma^{pross}, \mathbb{G}_m) \cong Hom(\Gamma, \mathbb{G}_m).$$

I will skip the further particular form, since it will be glued from a pro-torus and a profinite group, and I got confused about the gluing (a source says it is a direct product, and I could not figure out why).

7.2. **The Hodge action.**

7.2.1. Suppose that we are given a symmetric monoidal auto-equivalence $F : Rep(\Gamma) \to Rep(\Gamma)$, together with a functorial monoidal isomorphism $\underline{F(V)} \cong \underline{V}$.

Then we can construct an automorphism of Γ^{proalg} as follows. Given $(\gamma_V) \in \Gamma^{proalg}$, we define a new element (δ_V) by considering

$$\begin{array}{ccc} \underline{V} & \xrightarrow{\delta_V} & \underline{V} \\ \updownarrow & & \updownarrow \\ \underline{F(V)} & \xrightarrow{\gamma_{F(V)}} & \underline{F(V)} \end{array}$$

7.2.2. In our case, notice that the equivalences

$$Rep(\pi_1) \approx LS(X) \approx Bun_{flt}(X) \approx Bun_{Hgs}(X)^{semist}$$

are compatible with the "fiber functor" - the forgetful functor for $Rep(\pi_1)$, and the functors of fiber at x_0 for the other categories. Therefore π_1^{proalg} can be recovered from all of them. Given $c \in \mathbb{C}^\times$, we have an auto-equivalence of $Bun_{Hgs}(X)^{semist}$ given by sending $(\mathcal{V}, \nabla'_{Hgs} + \nabla''_{Hgs})$ to $(\mathcal{V}, c\nabla'_{Hgs} + \nabla''_{Hgs})$. This is compatible with the fiber functor, and so yields an automorphism of π_1^{proalg}. In this way we obtain an action of $\mathbb{C}^\times$ (viewed as a discrete group for now) on π_1^{proalg}. This is Simpson's Hodge structure on the pro-algebraic completion of the fundamental group.

7.3. **Example.**

7.3.1. Fix $x_0 \in X$ and let as assume that $\Gamma := \pi_1(X, x_0)$ is commutative (recall that it is also known that Γ is finitely presented, so a finitely generated abelian group). Let us again assume for simplicity that $H^2(X, \underline{\mathbb{Z}})$ has no torsion. We want to calculate the action of $\mathbb{C}^\times$ on Γ^{proalg}. From the description above, we see that it is enough to calculate the action of $\mathbb{C}^\times$ on $Hom(\Gamma, \mathbb{C}^\times)$ and the action of $\mathbb{C}^\times$ on $Hom(\Gamma, \mathbb{C})$.

7.3.2. Let us describe the action of $\mathbb{C}^\times$ on $Hom(\Gamma, \mathbb{C}^\times)$. Recall that we had:

$$Hom(\Gamma, \mathbb{C}^\times) \cong \frac{Hrm^{0,1}(X)}{H^1(X, \underline{\mathbb{Z}})} \times H^0(X, \Omega^1)$$

where the bijection from right to left is given by

$$(\omega, \theta) \mapsto \left(\gamma \mapsto e^{2\pi i \cdot 2\Re(\int_\gamma \omega) + 2\Re(\int_\gamma \theta)}\right).$$

Via this bijection, the action is by scaling the second factor.

7.3.3. Let us now describe the action of $\mathbb{C}^\times$ on

$$Hom(\Gamma, \mathbb{C}) \cong H^1(X, \underline{\mathbb{C}}).$$

Fix $\mu : \Gamma \to \mathbb{C}$. We embed

$$\mathbb{G}_a \cong \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subset GL_2,$$

and obtain a two-dimensional representation of Γ which is an extension of two trivial representations. The corresponding flat bundle is then an extension of two trivial bundles, and thus can be expressed via a connection

$$\begin{pmatrix} d & \omega \\ 0 & d \end{pmatrix}.$$

Here ω is a closed 1-form, which can also be thought of as representing an element in $H^1 Hom(Triv, Triv)$ in $Bun^{dg}_{flt}(X)$. One calculates that $\mu(\gamma) = -\int_\gamma \omega$. We can choose ω to be harmonic, and decompose into $(1,0)$ and $(0,1)$-parts $\omega = \omega' + \omega''$. Since the Hom-complexes in $Bun^{dg}_{flt}(X)$ and $Bun^{dg}_{Hgs}(X)$ were defined to correspond precisely by harmonic representatives, the Higgs bundle corresponding to our flat bundle is the extension of two trivial ones, with connection

$$\begin{pmatrix} d'' & \omega \\ 0 & d'' \end{pmatrix}.$$

Then application of $c \in \mathbb{C}^\times$ yields

$$\begin{pmatrix} d'' & c\omega' + \omega'' \\ 0 & d'' \end{pmatrix},$$

and since $c\omega' + \omega''$ is harmonic, we can go back to flat bundles and consider the flat bundle given by

$$\begin{pmatrix} d & c\omega' + \omega'' \\ 0 & d \end{pmatrix}.$$

Thus, we see that the action of $\mathbb{C}^\times$ on

$$Hom(\Gamma, \mathbb{C}) \cong H^1(X, \underline{\mathbb{C}})$$

is, decomposing

$$H^1(X, \underline{\mathbb{C}}) \cong H^{1,0}(X) \oplus H^{0,1}(X),$$

the tautological homotethy action on the $(1,0)$-part and the trivial action on the $(0,1)$-part.