Lecture notes for a Weil II learning seminar - following [1, section I.3]

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Contents

1 Conventions, reminders, etc.

1.1

We will follow [1, section I.3].

Imprecisions: We consider morphisms into $G(\bar{\mathbb{Q}}_\ell)$, say they are continuous and so on, when we really mean that the morphism factors through and is continuous into some $G(E)$, where $E/\mathbb{Q}_\ell$ is a finite extension. We even write $\bar{\mathbb{Z}}_\ell^\times$ at some point, when we again mean $\mathcal{O}_E^\times$, etc.

1.2

Throughout, we fix the following. k_1 will denote a finite field, and k/k_1 an algebraic closure. We write $q := |k_1|$. We fix a prime ℓ relatively prime to q, and an algebraic closure $\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$. We also fix an identification $\tau : \bar{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$.

By X_1, Y_1 etc. we mean normal geometrically connected algebraic schemes over k_1, and denote $X := k \otimes_{k_1} X_1$. By $k_1 \subset k_n \subset k$ we denote the unique extension field of k_1 inside k of degree n. We denote $X_n := k_n \otimes_{k_1} X_1$.

Given a closed point $x \in X$, we might think of it as a morphism $Spec(k) \to X_1$ over $Spec(k_1)$. Its image is a closed point $v \in X_1$, we denote by $k_1(x)$ the corresponding residue field, and $q_x := |k_1(x)|$.

1.3

Let $x \in X$ be a closed point. One has a short exact sequence

$$1 \to \pi_1(X, x) \to \pi_1(X_1, x) \to Gal(k/k_1)^{op} \to 1$$

of profinite groups. One also has, corresponding to $Spec(k_1(x)) \to X_1$, a morphism $Gal(k/k_1(x))^{op} \to \pi_1(X_1, x)$. We denote by $Fr_x \in \pi_1(X_1, x)$ the image under this morphism of the geometric Frobenius. Notice that under the thrumorphism $Gal(k/k_1(x))^{op} \to \pi_1(X_1, x) \to Gal(k/k_1)^{op}$ the geometric Frobenius in $Gal(k/k_1(x))^{op}$ goes to the $[k_1(x) : k_1]$-th power of the geometric Frobenius in $Gal(k/k_1)^{op}$.

One denotes by $W(k/k_1) \subset Gal(k/k_1)^{op}$ the subgroup generated by the geometric Frobenius, and by $W(X_1, x) \subset \pi_1(X_1, x)$ the corresponding subgroup, i.e. one now has

$$1 \to \pi_1(X, x) \to W(X_1, x) \to W(k/k_1) \to 1.$$

Then, as noted above, $Fr_x \in W(X_1, x)$. Note, however, that the topology that we take on $W(k/k_1)$ is the discrete one, and not the one induced on it as a subgroup of $Gal(k/k_1)^{op}$! Accordingly we define the topology of $W(X_1, x)$.

Identifying $W(k/k_1)$ with $\mathbb{Z}$ by sending the geometric Frobenius to 1, the resulting map $W(X_1, x) \to \mathbb{Z}$ we call the degree map, and denote *deg*.

1.4

All sheaves are assumed to be constructible $\bar{\mathbb{Q}}_\ell$-sheaves (in fact, we will only deal with smooth sheaves in this lecture). Given a Weil sheaf $\mathcal{F}_1$ on X_1, we will denote by $\mathcal{F}$ the corresponding sheaf on X. Given a closed point $x \in X$, we have an equivalence of categories between smooth Weil sheaves on X_1 and continuous finite-dimensional representations of $W(X_1, x)$ over $\bar{\mathbb{Q}}_\ell$, which we will utilize a lot; It is given by sending $\mathcal{F}_1$ to the geometric fiber $(\mathcal{F}_1)|_x$.

1.5

For $\lambda \in \bar{\mathbb{Q}}_\ell^\times$, let us denote $w_\tau^q(\lambda) = \log_{q^{1/2}} |\tau(\lambda)|$.

Recall that for a real number $\beta \in \mathbb{R}$, a Weil sheaf $\mathcal{F}_1$ on X_1 is said to be **punctually τ-pure of weight** β if for every $x \in X$, all the eigenvalues λ of $Fr_x \in W(X_1, x)$ on $(\mathcal{F}_1)|_x$ satisfy $w_\tau^{q_x}(\lambda) = \beta$.

A Weil sheaf $\mathcal{F}_1$ on X_1 is said to be **punctually τ-pure** if it is punctually τ-pure of weight β for some β. Such a β is then uniquely determined (unless $\mathcal{F}_1 = 0$) and we set $w(\mathcal{F}_1) := \beta$.

Note that given a morphism $\pi : Y_1 \to X_1$ and a Weil sheaf $\mathcal{F}_1$ on X_1 which is punctually τ-pure of weight β, the Weil sheaf $\pi^*\mathcal{F}_1$ on Y_1 is punctually τ-pure of weight β as well.

Note that, for $m \in \mathbb{Z}_{\geq 1}$, if $\mathcal{F}_1^{\otimes m}$ is punctually τ-pure of weight β then $\mathcal{F}_1$ is punctually τ-pure of weight β/m.

2 Purity of smooth Weil sheaves of rank 1

Definition 2.1. We will say that an abelian topological group Γ is **almost pro-p** if it is an extension of a finite group by a pro-p-group.

Remark 2.2. The extension of a finite group by an almost pro-p group is again an almost pro-p group. Also, almost pro-p-groups are stable under quotients and finite products.

Remark 2.3. Let Γ be an almost pro-p group. Then any character $\chi : \Gamma \to \bar{\mathbb{Q}}_\ell^\times$ is of finite order (i.e. there exists $m \in \mathbb{Z}_{\geq 1}$ such that $\chi^m = 1$). Indeed, since an almost pro-p group is compact, the image of χ lands in $\bar{\mathbb{Z}}_\ell^\times$, which is an almost pro-ℓ group, and thus the claim easily follows noticing that there are no non-trivial morphisms from a pro-p-group to a pro-ℓ-group.

We denote by $\pi_1(X)^{ab}$ the abelianization of $\pi_1(X, x)$ (and similarly for $\pi_1(X_1)^{ab}, W(X_1)^{ab}$), the point being that these do not depend on the choice of x.

Claim 2.4. *The image of $\pi_1(X)^{ab} \to W(X_1)^{ab}$ is an almost pro-p group.*

Proof. Let us first assume that X_1 is a curve so that, since it is normal, it is a smooth curve. By class field theory, see remark 2.7, it is enough to show that $\frac{\mathbb{A}_0^\times}{F^\times \cdot \prod_{v \in X_1} \mathcal{O}_v^\times}$ is an almost pro-p-group. Notice that it admits a surjection onto $\frac{\mathbb{A}_0^\times}{F^\times \cdot \prod_{v \in \overline{X_1}} \mathcal{O}_v^\times}$, which is a finite group (in bijection with the group of k_1-points of the Jacobian of X_1). The kernel of the surjection admits itself a surjection from $\prod_{v \in \overline{X_1} - X_1} \mathcal{O}_v^\times$. Thus we are reduced to showing that the latter is an almost pro-p-group, and thus that some $\mathcal{O}_v^\times$ is an almost pro-p-group. But $\mathcal{O}_v^\times$ admits a surjection onto the finite group $k(v)^\times$, and the kernel is a pro-p-group.

The general case we will not prove in detail. Notice that if we find $\pi : Y_1 \to X_1$ such that $\pi_1(Y)^{ab} \to \pi_1(X)^{ab}$ is surjective, then the claim for Y_1 implies that for X_1. Hence, by passing to an open dense subscheme, we can assume that X_1 is quasi-projective. Then one passes to a generic-enough linear section of X_1 of dimension 1. Et cetera.

$\square$

Theorem 2.5. *Let $\mathcal{F}_1$ be a smooth Weil sheaf of rank 1 on X_1. Then $\mathcal{F}$ has finite order, i.e. there exists $m \in \mathbb{Z}_{\geq 1}$ such that $\mathcal{F}^{\otimes m} \cong \bar{\mathbb{Q}}_\ell$.*

Proof. This follows from claim 2.4 and remark 2.3. □

Corollary 2.6. *Let $\mathcal{F}_1$ be a smooth Weil sheaf of rank 1 on X_1. Then $\mathcal{F}_1$ is punctually τ-pure.*

Proof. By a remark in subsection 1.5, it is enough to check that $\mathcal{F}_1^{\otimes m}$ is punctually τ-pure, for some m. Thus, by the theorem, we can assume that $\mathcal{F} \cong \bar{\mathbb{Q}}_\ell$. Then, $\mathcal{F}_1$ is the pullback of a smooth Weil sheaf of rank 1 on $Spec(k_1)$. Hence, by a remark in subsection 1.5, it is enough to check that a smooth Weil sheaf of rank 1 on $Spec(k_1)$ is punctually τ-pure, which is clear. □

Let us recall the class field theory we have used in claim 2.4:

Remark 2.7. Let us recall some class field theory. Assume that X_1 is a curve, and denote by F and $\mathbb{A}$ the field of rational functions on X_1 and the adeles of the compactification of X_1. The main claim of class field theory is that one has a unique continuous morphism

$$r : \mathbb{A}^\times \to \pi_1(X_1)^{ab}$$

with the following two properties:

1. $r|_{F^\times} = 1$.
2. for $v \in X_1$, one has $r(f) = Fr_v^{ord_v(f)}$ where $f \in F_v^\times$.

Moreover, the morphism r has dense image.

Then one can formally deduce the following. One has a commutative diagram:

$$\begin{array}{ccc} \mathbb{A}^\times & \xrightarrow{r} & \pi_1(X_1)^{ab} \\ {\scriptstyle ord}\downarrow & & \downarrow \\ \mathbb{Z} & \xrightarrow[1 \mapsto Fr]{} & Gal(k_1) \end{array}$$

and one obtains an induced continuous surjective morphism $\mathbb{A}^\times \to W(X_1)^{ab}$, which upon restricting to ideles of degree 0 gives a continuous surjective morphism

$$r : \frac{\mathbb{A}_0^\times}{F^\times \cdot \prod_{v \in X_1} \mathcal{O}_v^\times} \to Im\left(\pi_1(X)^{ab} \to W(X_1)^{ab}\right).$$

3 Determinant weights

Definition 3.1 (Determinant weights)**.** Let $\mathcal{F}_1$ be a smooth Weil sheaf on X_1. We define a finite subset $dw_\tau(\mathcal{F}_1) \subset \mathbb{R}$ of **determinant τ-weights** of $\mathcal{F}_1$ as follows.

1. If $\mathcal{F}_1$ has rank 1, recall that $\mathcal{F}_1$ is punctually τ-pure, and define $dw_\tau(\mathcal{F}_1) := \{w_\tau(\mathcal{F}_1)\}$.
2. If $\mathcal{F}_1$ is irreducible, denoting by d the rank of $\mathcal{F}_1$, we denote $dw_\tau(\mathcal{F}_1) := dw_\tau(\wedge^d \mathcal{F}_1)/d$.
3. In the general case, we set $dw_\tau(\mathcal{F}_1)$ to be the union of the singletons $dw_\tau(\mathcal{G}_1)$ where $\mathcal{G}_1$ runs over irreducible constituents of $\mathcal{F}_1$.

Remark 3.2. Let us verbalize the advantage of determinant τ-weights. Given a smooth Weil sheaf $\mathcal{F}_1$ on X_1, to understand τ-weights of $\mathcal{F}_1$, we need to vary the closed point $x \in X$, but to understand determinant τ-weights, we can do with any fixed x. Suppose for simplicity that $\mathcal{F}_1$ is irreducible and consider, for $x \in X$, the corresponding $W(X_1, x)$-representation $V = (\mathcal{F}_1)|_x$. The τ-weights of V are $w_\tau^{q_x}(\lambda)$ as λ runs over eigenvalues of Fr_x, while the determinant τ-weight is $w_\tau^{q_x}(det(Fr_x))/\dim V$. The τ-weights recover the determinant τ-weight (the latter is the arithmetic average of the former), but they might change when we change x, while the determinant τ-weight does not.

Remark 3.3. For a short exact sequence of smooth Weil sheaves on X_1:

$$0 \to \mathcal{F}_1 \to \mathcal{G}_1 \to \mathcal{H}_1 \to 0$$

one has

$$dw_\tau(\mathcal{G}_1) = dw_\tau(\mathcal{F}_1) \cup dw_\tau(\mathcal{H}_1).$$

The main result we want to establish is the following one:

Claim 3.4.

1. *Let $\mathcal{F}_1, \mathcal{F}'_1$ be smooth Weil sheaves on X_1. Then*

$$dw_\tau(\mathcal{F}_1 \otimes \mathcal{F}'_1) = dw_\tau(\mathcal{F}_1) + dw_\tau(\mathcal{F}'_1).$$

2. *Let $\mathcal{F}_1$ be a smooth Weil sheaf on X_1. Denote by $w_1, \ldots, w_m$ the elements of $dw_\tau(\mathcal{F}_1)$, and denote by d_i the sum of ranks of irreducible consistuents $\mathcal{G}_1$ of $\mathcal{F}_1$ for which $dw_\tau(\mathcal{G}_1) = \{w_i\}$. Then*

$$dw_\tau(\bigwedge^r \mathcal{F}_1) = \coprod_{\substack{0 \le r_i \le d_i \\ r_1 + \ldots + r_m = r}} \{r_1 w_1 + \ldots + r_m w_m\}$$

3. *Let $\mathcal{F}_1$ be a smooth Weil sheaf on X_1, and let $\pi : Y_1 \to X_1$ be a dominant morphism. Then*

$$dw_\tau(\pi^*(\mathcal{F}_1)) = dw_\tau(\mathcal{F}_1).$$

Proof. We will proof this in section 5 below. □

4 Catching determinant weights using central elements

Let G_1 be an affine algebraic group over $\bar{\mathbb{Q}}_\ell$ and let $\rho : W(X_1, x) \to G_1$ be a morphism with dense image. We denote by $G \subset G_1$ the Zariski closure of $\rho(\pi_1(X, x))$. Let us say that ρ is **good** with data $(z, m) \in Z(G_1) \times \mathbb{Z}_{\neq 0}$ if G° is semisimple and $z \equiv_G \rho(Fr_x)^m$.

Lemma 4.1. *Let $\theta : G_1 \to G_1'$ be a surjective algebraic morphism. If $\rho : W(X_1, x) \to G_1$ is good with data (z, m), then $\theta \circ \rho : W(X_1, x) \to G_1'$ is good with data $(\theta(z), m)$.*

Proof. Clear. □

Let V be a finite-dimensional representation of $W(X_1, x)$ over $\bar{\mathbb{Q}}_\ell$ - write $\rho : W(X_1, x) \to GL(V)$. We denote by $G_1 \subset GL(V)$ the Zariski closure of $\rho(W(X_1, x))$ (and call it the **monodromy group**). We denote by $G \subset G_1$ the Zariski closure of $\rho(\pi_1(X, x))$ (and call it the **geometric monodromy group**). We say that V is **good** with data (z, m), if the corresponding $\rho : W(X_1, x) \to G_1$ is so.

Claim 4.2. *Suppose that V is good, with data (z, m). Then the set $dw_\tau(V)$ is equal to the set of numbers $w_\tau^{q_x}(\lambda)/m$ as λ runs over the eigenvalues of z on V.*

Proof. By the previous lemma, we can assume that V is an irreducible $W(X_1, x)$-representation. Then $dw_\tau(V)$ consists simply of $w_\tau^{q_x}(det(Fr_x))/\dim V$. Notice that the image of det on G consists of roots of unity, because G° is semisimple. Hence $w_\tau^{q_x}(det(z)) = w_\tau^{q_x}(det(Fr_x^m)det(G)) = w_\tau^{q_x}(det(Fr_x)) \cdot m$. Since V is irreducible, by Schur's lemma, z is a multiple of the identity by some scalar λ. Hence $w_\tau^{q_x}(det(z)) = w_\tau^{q_x}(\lambda) \cdot \dim V$. Thus, we get $w_\tau^{q_x}(\lambda) = dw_\tau(V)/m$. □

We will see in the next section:

Theorem 4.3 (Grothendieck)**.** *Suppose that V is a $W(X_1, x)$-representation which is semisimple as a $\pi_1(X, x)$-representation. Then V is good.*

5 Proof of claim 3.4

5.1 Proof of part 1

Claim 5.1. *Let V, V' be two representations. Suppose that $V \oplus V'$ is good with data $(\widetilde{z}, m)$; we can write $\widetilde{z} = (z, z') \in \widetilde{G}_1 \subset G_1 \times G_1'$ (here $\widetilde{G}_1$ stands for the monodromy group of $V \oplus V'$). Then $V, V', V \otimes V'$ are all good, with respective data $(z, m), (z', m), (z \otimes z', m)$.*

Proof. We have the representations $V \oplus V', V, V', V \otimes V'$. The ρ's for the three last ones factor via the ρ for the first one in an obvious way, from which in view of lemma 4.1 we obtain the desired outcome. □

We can then deduce the following claim, which translates to the desired claim 3.4 from the previous section:

Claim 5.2. *Suppose that V, V' are $W(X_1, x)$-representations. Then*

$$dw_\tau(V \otimes V') = dw_\tau(V) + dw_\tau(V').$$

Proof. We can reduce immediately to V, V' being irreducible, so that in particular we can assume that $V \oplus V'$ is semisimple. Then from claim 5.1 and claim 4.2 we obtain the result. □

5.2 Proof of part 2

Proof. Step 1: Suppose first that $\mathcal{F}_1$ is irreducible. If the rank of $\mathcal{F}_1$ is smaller than r, then the claim is clear, so we assume that it is not. By theorem we have that $\mathcal{F}_1$ is good, with some data (z, m). Then by lemma 4.1, we see that $\bigwedge^r \mathcal{F}_1$ is good, with data $(\wedge^r z, m)$. Thus the determinant τ-weight of $\mathcal{F}_1$ is $w_\tau^{q_x}(\lambda)/m$, while the determinant weight of $\bigwedge^r \mathcal{F}_1$ is $w_\tau^{q_x}(\lambda^r)/m$, so the latter is r-times the former, so that the claim follows.

Step 2: Let us denote by $\mathcal{G}_1^1, \ldots, \mathcal{G}_1^p$ the irreducible consistuents of $\mathcal{F}_1$. One has a filtration of $\bigwedge^r \mathcal{F}_1$, whose subquotients are:

$$0 \leq s_i \leq rk(\mathcal{G}_1^i),\ s_1 + \ldots + s_p = r : \quad \bigwedge^{s_1} \mathcal{G}_1^1 \otimes \ldots \otimes \bigwedge^{s_p} \mathcal{G}_1^p,$$

so that by step 1 and also part 1 of the claim, we obtain

$$dw_\tau(\bigwedge^r \mathcal{F}_1) = \coprod_{\substack{0 \leq s_i \leq rk(\mathcal{G}_1^i) \\ s_1 + \ldots + s_m = r}} \{s_1 dw_\tau(\mathcal{G}_1^1) + \ldots + s_m dw_\tau(\mathcal{G}_1^p)\},$$

which upon grouping irreducible consistuents of the same detereminant τ-weight gives the desired claim. □

5.3 Proof of part 3

Proof. We choose a point $y \in Y$ over $x \in X$, and denote by H_1, H the corresponding monodromy groups for $\pi^* \mathcal{F}_1$. Let us note that the image of $\pi_1(Y, y) \to \pi_1(X, x)$ has finite index in $\pi_1(X, x)$ by claim 7.4.

Step 1: We can assume that the image of $\pi_1(Y, y) \to \pi_1(X, x)$ is normal in $\pi_1(X, x)$.

Consder the biggest normal subgroup $\Gamma' \subset \pi_1(X, x)$ which is contained in the image of $\pi_1(Y, y) \to \pi_1(X, x)$. It is a closed subgroup of finite index in $\pi_1(X, x)$, and it is also normal in $W(X_1, x)$ (fixing an element $\alpha \in W(Y_1, y)$ of degree 1 and its image β in $W(X_1, x)$, we see that β normalizes the image of $\pi_1(Y, y) \to \pi_1(X, x)$ and since α normalizes $\pi_1(Y, y)$ and hence β normalizes Γ'). Now denote by Γ the preimage of Γ' in $\pi_1(Y, y)$. Then Γ is a closed subgroup of finite index in $\pi_1(Y,y)$, and it is normal in $W(Y_1,y)$ (again because it is normalized by α). Then, using lemma 7.3, we consider a geometrically connected finite etale cover $(\widetilde{Y}_1,\widetilde{y}) \to (Y_1,y)$ such that $\pi_1(\widetilde{Y},\widetilde{y}) = \Gamma$. Then if know the claim in the normal case, we see that the determinant τ-weights of $\mathcal{F}_1$ are the same as of the pullback to $\widetilde{Y}_1$, and the determinant τ-weights of $\pi^*(\mathcal{F}_1)$ are the same as of the pullback to $\widetilde{Y}_1$, so that we obtain that the determinants τ-weights of $\mathcal{F}_1$ and $\pi^*(\mathcal{F}_1)$ are the same.

Step 2: We can assume that $\mathcal{F}_1$ is irreducible.

Step 3: We now prove the claim. Since $V = (\mathcal{F}_1)|_x$ is irreducible as a $W(X_1,x)$-representation, it is semisimple as a $\pi_1(X,x)$-representation, and hence also semisimple as a $\pi_1(Y,y)$-representation by remark 7.1. Then by theorem 5.2 both G° and H° are semisimple. Moreover, we can find $(z,m) \in Z(H_1) \times \mathbb{Z}_{\neq 0}$ such that $z \equiv_H \rho(Fr_y)^m$. Notice now that conjugation by z preserves G, and is trivial on a normal subgroup of finite index H. Thus it is trivial on G°, and hence by part 2 of lemma 7.2 we see that some power of z centralizes G. We can thus assume, by replacing z with this power, that z itself centralizes G. Since z centralizes also H_1, so centralizes some element of degree 1 in H_1, and hence in G_1, we see that z centralizes the whole G_1, i.e. $z \in Z(G_1)$. We have $z \equiv_H \rho(Fr_y)^m$ and also $z \equiv_G \rho(Fr_x)^{m[k_1(y):k_1(x)]}$. Hence $dw_\tau(\pi^*\mathcal{F}_1)$ consists of $w_\tau^{q_y}(\lambda)/m$ where λ runs over eigenvalues of z, while $dw_\tau(\mathcal{F}_1)$ consists of $w_\tau^{q_x}(\lambda)/(m \cdot [k_1(y):k_1(x)])$ where λ runs over eignevalues of z, which are the same, because $q_y = q_x^{[k_1(y):k_1(x)]}$.

□

6 Proof of theorem 5.2

Our goal in this section is to prove theorem 5.2. We fix a $W(X_1,x)$-representation V (denote also $\rho : W(X_1,x) \to GL(V)$). Recall that G_1 (resp. G) denotes the Zariski closure of $\rho(W(X_1,x))$ (resp. $\rho(\pi_1(X,x))$) inside $GL(V)$.

Claim 6.1. *Suppose that V is semisimple as a $\pi_1(X,x)$-representation. Then G° is semisimple.*

Proof.

Step 1: We first reduce to the case when $G = G^\circ$.

For this, we consider the subgroup $\Gamma \subset \pi_1(X,x)$ consisting of elements β for which $\rho(\beta) \in G^\circ$. Then Γ is a closed subgroup of finite index in $\pi_1(X,x)$, which is normal in $W(X_1,x)$. Then by lemma 7.3 we can find $\pi : (Y_1,y) \to (X_1,x)$ such that $\pi_1(Y,y) = \Gamma$. Then, denoting by $\mathcal{F}_1$ the smooth Weil sheaf on X_0 corresponding to the representation of $W(X_1,x)$ on V, we have that $\pi^*\mathcal{F}_1$ is a smooth Weil sheaf for which the corresponding representation of $\Gamma = \pi_1(Y,y)$ (where $y \in Y$ is a suitable point) is simply the representation of Γ on V, so that the corresponding geometric monodromy group is G° (notice also that the representation of Γ on V is semisimple by remark 7.1).

Step 2: The group G is reductive.

Indeed, suppose by contradiction that the unipotent radical $U \subset G$ is not trivial. Then the eigenspace $V^{U,1}$ is not zero, and it is also not V because V is faithful as a G-representation. Moreover, $V^{U,1}$ has no U-complement in V. Now, since U is normal in G, we see that $V^{U,1}$ is preserved by G. Since $V^{U,1}$ has no U-complement in V, it also has no G-complement in V, contradicting the assumption that V is semisimple as a G-representation.

Step 3: Let us consider an element $\alpha \in W(X_1, x)$ for which $deg(\alpha) = 1$. We reduce to the case when there exists $g \in G$ such that $\rho(\alpha)h\rho(\alpha)^{-1} = ghg^{-1}$ for all $h \in G$.

Consider the automorphism ι_α of G given by conjugation by $\rho(\alpha)$. Note that $Z(G)$ acts diagnolizably on V (as it is a diagonalizable group), and $\rho(\alpha)$ permutes the different eigenspaces of the $Z(G)$-action on V, so for some $m \in \mathbb{Z}_{\geq 1}$, the operator $\rho(\alpha^m)$ preserves these eigenspaces, and thus $\rho(\alpha^m)z\rho(\alpha^{-m})$ acts on V the same as z does for every $z \in Z(G)$, hence since V is a faithful G-module, we get that $\rho(\alpha^m)$ centralizes $Z(G)$. In other words, ι_α^m acts trivially on $Z(G)$. By claim 7.4, we get that for some $n \in \mathbb{Z}_{\geq 1}$, the automorphism $\iota_\alpha^{mn} = \iota_{\alpha^{mn}}$ is inner. Pulling back $\mathcal{F}_1$ to X_{mn} allows us to assume that ι_α is itself inner, as desired.

Step 4: We can now finally show that G is semisimple.

Consider the morphism $W(X_1, x) \to G$, which on $\pi_1(X, x)$ is equal to ρ, and on α is equal to g (where $g \in G$ is an element whose existence is asserted in step 3). This morphism is continuous. If G would not be semisimple, we would have a non-trivial algebraic morphism $G \to \bar{\mathbb{Q}}_\ell^\times$. Composing with our $W(X_1, x) \to G$, we obtain a character $W(X_1, x) \to \bar{\mathbb{Q}}_\ell^\times$ whose restriction to $\pi_1(X, x)$ has Zariski dense image. Since then this restriction to $\pi_1(X, x)$ is clearly not of finite order, we obtain a contradiction, in view of theorem 2.5.

□

Theorem 6.2. *Suppose that V is semisimple as a $\pi_1(X, x)$-representation. Then there exists $z \in Z(G_1)$ and $m \in \mathbb{Z}_{\neq 0}$ such that $z \equiv_G \rho(Fr_x)^m$.*

Proof.

Step 1: Let us fix some element $\alpha \in W(X_1, x)$ of degree 1. We will show first that there exists $z \in G_1$ and $m \in \mathbb{Z}_{\neq 0}$ such that z centralizes G° and $\rho(\alpha)$, and $z \equiv_G \rho(Fr_x)^m$.

Consider the automorphism of G° given by conjugation by $\rho(\alpha)$. By claim 7.4, any automorphism of G° can be written as the product of an inner automorphism and an automorphism of finite order, and so we can find $g \in G^\circ$ and $m \in \mathbb{Z}_{\geq 1}$ such that $(g\rho(\alpha))^m$ centralizes G°. Set $z := g\rho(\alpha)$. Then $z \equiv_G \rho(\alpha)$ and z^m centralizes G° and $\rho(\alpha)$ (the latter since $\rho(\alpha) \in G^\circ z$). Clearly taking a yet bigger power of z will make it comparable to a non-zero integer power of $\rho(Fr_x)$ modulo G (since $\rho(Fr_x)$ is comparable to some non-zero integer power of $\rho(\alpha)$ modulo G). Substituting the big enough power of z for z, we obtain the desired.

Step 2: We will now show that there exists $n \in \mathbb{Z}_{\neq 0}$ such that z^n centralizes G (where z is as in step 1). Then clearly we will be done (substituting z^n for z and mn for m).

Indeed, this follows from part 2 of lemma 7.2.

□

7 Inventory of auxiliary claims

Remark 7.1. Let us remark that if Γ' is a normal subgroup in Γ, and V is a semisimple finite-dimensional Γ-representation, then it is semisimple as a Γ'-representation. Indeed, notice that the maximal semisimple Γ'-subrepresentation $V' \subset V$ is invariant under Γ, hence admits a Γ-invariant complement, which hence must be zero (otherwise this complement would contain an irreducible Γ'-subrepresentation, contradicting the definition of V').

Lemma 7.2.

1. *Let G be a group and $G^\circ \subset G$ a normal subgroup. Assume also that $Z(G^\circ)$ is finite. Let θ be an automorphism of G which is trivial on G° and on G/G°. Then, denoting $r := |Z(G^\circ)|$, one has $\theta^r = id$.*
2. *In addition to the previous assumptions, assume that G° has finite index in G. Let θ be an automorphism of G which is trivial on G°. Then θ has finite order.*

Proof.

1. Fix $g \in G$ and consider the map $\phi_g : \mathbb{Z} \to G$ given by
$$\phi_g(n) := \theta^n(g)g^{-1}.$$
We want to show that $\phi_g(r) = 1$. Notice the formula
$$\phi_g(n+m) = \theta^m(\phi_g(n))\phi_g(m).$$
Since we can also notice $Im(\phi_g) \subset G^\circ$, the formula reduces to
$$\phi_g(n+m) = \phi_g(n)\phi_g(m),$$
i.e. ϕ_g is a group homomorphism. Next, notice that we have the formulas
$$\phi_{hg}(n) = h\phi_g(n)h^{-1}, \quad \phi_{gh}(n) = \phi_g(n) \quad (g \in G, h \in G^\circ)$$
which together imply that $Im(\phi_g) \subset Z(G^\circ)$. Hence, clearly now $\phi_g(r) = \phi_g(1)^r = 1$.
2. Some power of θ will be trivial on the quotient G/G°, since it is finite. Then we reduce to the previous item.

□

Lemma 7.3. *Let $\Gamma \subset \pi_1(X, x)$ be a closed subgroup of finite index, which is normal in $W(X_1, x)$. Then there exists a finite etale cover $\pi : (Y_1, y) \to (X_1, x)$ with $\pi_1(Y, y) = \Gamma$ (and Y_1 is in our class, so geomterically connected).*

Proof. Fix an element $\alpha \in \pi_1(X_1, x)$ of degree 1, and consider the subgroup $\Gamma_1 \subset \pi_1(X_1, x)$ given by $\Gamma_1 := \overline{\Gamma \cdot \alpha^{\mathbb{Z}}}$. Then Γ_1 is a closed subgroup of finite index in $\pi_1(X_1, x)$, whose projection onto $Gal(k/k_1)^{op}$ is surjective, and whose intersection with $\pi_1(X, x)$ is Γ. Let now $\pi : Y_1 \to X_1$ be the connected finite etale cover corresponding to $\Gamma_1 \subset \pi_1(X_1, x)$. Then Y_1 is still geometrically connected because $\pi_1(X, x) \cdot \Gamma_1 = \pi_1(X_1, x)$. □

Claim 7.4. *Let G be a redcutive algebraic group. Choose a Torel $T \subset B \subset G$ and choose for every simple root α an element $1 \neq u_\alpha \in U_\alpha$. Consider the subgroup $A \subset Aut(G)$ consisting of automorphisms which preserves the subsets T, B and $\{u_\alpha\}$ (but don't necessarily have to fix elementwise these subsets). Then $Aut(G) = Inn(G) \rtimes A$. Moreover, there is natural isomorphism of A with the group of automorphisms of the corresponding based root data. In particular, the subgroup of A consisting of elements fixing elementwise $Z(G)$ is finite and so if G is semisimple then A is finite.*

Claim 7.5. *Let $\pi : Y_0 \to X_0$ be dominant. Then the image of $\pi_1(Y, y) \to \pi_1(X, x)$ has finite index in $\pi_1(X, x)$.*

Proof. (I didn't figure out completely a proof - problem in step 4 and also didn't consider carefully the difference of X and X_1)

Step 0: Since open dense embeddings are surjective on π_1, it is enough to check this locally somewhere.

Step 1: This holds for a finite etale cover, by the basic theory of the etale fundamental group.

Step 2: This holds for an etale map, because an etale map is quasi-finite and hence locally somewhere finite, reducing to the previous case.

Step 3: This holds for a smooth map, because a smooth map can be locally decomposed into an etale map and a smooth projection onto a factor - the first dealt with in the previous step, and the second admiiting a right inverse and hence is surjective on π_1.

Step 4: This holds for a dominant map, because it is locally smooth. (This is true only in characteristic 0! So need some other ingredient still) □

References

[1] Kiehl, Weissauer; *Weil Conjectures, Perverse Sheaves and ladic Fourier Transform*

[2] Deligne; *La conjecture de Weil: II*