Kahler analogue of Weil Conjectures (after Serre)

Sasha Yom Din

June 23, 2014

1 Remarks and notations

There might be errors, inaccuracies, and unpleasancies in the following text. I will be happy if you let me know about it.

2 Counting fixed points of powers

Let X be a compact smooth connected oriented manifold. Let $\phi : X \to X$ be a smooth map. We suppose the following (technical) condition: For every $n \geq 1$, ϕ^n has finitely many fixed points, all of which are simple (the determinant of the map $Id - d(\phi^n)$ on the tangent space to each fixed point of ϕ^n is positive). Note that if X is a complex manifold and ϕ a holomorphic map, then the condition on the determinant above is equivalent to the condition that the complex differential map $d(\phi^n)$ hasn't 1 as an eigenvalue.

We denote by $\mathcal{O}$ the set of finite orbits of ϕ. We denote by b_n the number of fixed points of ϕ^n. We denote by a_n the number of formal non-negative integer combinations $\sum m_i O_i$, $O_i \in \mathcal{O}$, such that $\sum m_i |O_i| = n$. We define:

$$z(t) = \sum_{n \geq 0} a_n t^n = \prod_{O \in \mathcal{O}} \frac{1}{1 - t^{|O|}}$$

and

$$w(t) = \sum_{n \geq 1} b_n t^n.$$

Then $w(t) = t\frac{d}{dt} log z(t)$ and $z(t) = exp(\int \frac{w(t)}{t})$.

2.1 First claim - how zeta looks

Let us denote by $c_{k,j}$ the eigenvalues (counted with multiplicities) of ϕ^* acting on $H^k(X, \mathbb{C})$. Let us also denote $p_k(t) := \prod_j (1 - c_{k,j} t)$ (which is $det(Id - t\phi^*, H^k(X, \mathbb{C}))$, a slightly normalized characteristic polynomial).

Theorem 2.1 (Rationality)**.**

$$z(t) = \frac{\prod_{k\ odd} p_k(t)}{\prod_{k\ even} p_k(t)}.$$

A trivial manipulation (passing from $z(t)$ to $w(t)$) shows that this formula is equivalent to the following one:

$$b_n = \sum_k (-1)^k \sum_j (c_{k,j})^n$$

which can be rewritten:

$$b_n = \sum_k (-1)^k Tr((\phi^*)^n, H^k(X, \mathbb{C})),$$

and this is just the Lefschetz fixed point formula.

2.2 Second claim - duality

We now assume that the dimension d of X is even, and that the degree Q of ϕ is positive (the degree is the scalar by which ϕ^* acts on $H^d(X, \mathbb{C})$).

Theorem 2.2 (Functional equation)**.**

$$z((Qt)^{-1}) = \epsilon \cdot (-1)^\chi \cdot t^\chi \cdot Q^{\chi/2} \cdot z(t)$$

where $\chi = \sum_k (-1)^k dim H^k(X, \mathbb{C})$ *is the Euler characteristic and* ϵ *is the sign of the determinant of* ϕ^* *acting on* $H^{d/2}(X, \mathbb{C})$*.*

Indeed, it is immedaite, using Poincare duality, to see that the multiset $c_{d-k,j}$ is equal to the multiset $\frac{Q}{c_{k,j}}$. After this, the theorem is a matter of simple manipulations.

2.3 Third claim - the roots

We now suppose that X is a Kahler manifold, with Kahler form ω. Also, we suppose that $\phi^*[\omega] = q[\omega]$, where $[\omega]$ denotes the cohomology class of ω and $q > 0$ is some positive number.

Theorem 2.3 (Riemann hypothesis)**.** *All the* $c_{k,j}$ *are algebraic integers, all of whose complex conjugates have an absolute value* $q^{k/2}$*.*

Notice that once we show that the $c_{k,j}$ have absolute value $q^{k/2}$, the rest is clear (since ϕ^* acts already on the $\mathbb{Z}$-form $H^k(X, \mathbb{Z})$ and the conjugates of eigenvaules are other eigenvaules).

We will prove this theorem after some preliminaries on Hodge theory.

3 Hodge theory - Results

Let X be a compact smooth complex manifold of complex dimension n. Assume that X is equipped with a Kahler class c.

3.1 Since X is a compact oriented manifold

The cohomology ring $H^{\cdot} := H^{\cdot}(X, \mathbb{C})$ is a super-commutative unital finite-dimensional graded ring, with $H^k = 0$ if $k < 0$ or $k > 2n$. There is an isomorphism $\int : H^{2n} \to \mathbb{C}$.

Theorem 3.1 (Poincare). *The pairing $H^k \otimes_{\mathbb{C}} H^{2n-k} \to H^{2n} \xrightarrow{\int} \mathbb{C}$ is perfect.*

Remark 3.2. All the structure above has a real form, and so we can speak about real operators, conjugation, etc.

Remark 3.3. If $f : X \to Y$ is a smooth map, $f^* : H^{\cdot}(Y, \mathbb{C}) \to H^{\cdot}(X, \mathbb{C})$ is a real algebra homomorphism.

3.2 Since X is Kahler

Theorem 3.4 (Hodge decomposition). *There is a canonical decomposition $H^k = \oplus H^{p,q}$, where p, q run over non-negative integers, with sum k. Also, $\overline{H^{p,q}} = H^{q,p}$.*

Remark 3.5. We have $H^{p,q} = 0$ if $p > n$ or $q > n$.

Remark 3.6. The multiplication respects this decomposition: $H^{p_1,q_1} \cdot H^{p_2,q_2} \subset H^{p_1+p_2,q_1+q_2}$.

Remark 3.7. If $f : X \to Y$ is a holomorphic map, f^* respects the decomposition: $f^* H^{p,q}(Y, \mathbb{C}) \subset H^{p,q}(X, \mathbb{C})$.

3.3 Since X has a chosen Kahler class

The class c belongs to $H^{1,1}$. We define $L : H^{\cdot} \to H^{\cdot+2}$ as $L(x) = cx$. It is a real operator.

Theorem 3.8 (Hard Lefschetz). *For $0 \leq k \leq n$,*

$$L^k : H^{n-k} \to H^{n+k}$$

is an isomorphism.

Definition 3.9. For $0 \leq k \leq n$, we define H^{n-k}_{pr} to be the kernel of $L^{k+1} : H^{n-k} \to H^{n+k+2}$.

Corollary 3.10. For $0 \leq k \leq n$, we have a decomposition:

$$H^k = \oplus_{i \geq 0} L^i H^{k-2i}_{pr}.$$

Remark 3.11. Since L shifts degree by $(1,1)$, it is clear that:

$$L^i H_{pr}^{k-2i} = \oplus_{p,q} \left(L^i H_{pr}^{k-2i} \cap H^{p,q} \right).$$

Definition 3.12. For $0 \leq k \leq n$, we define a Hermitian form on H^{n-k} as follows:

$$Q(x,y) = i^k \int L^k(x) \cdot \bar{y}$$

.

Remark 3.13. The coefficient i^k makes the form symmetric (in the complex sense - swapping becomes conjugation).

Remark 3.14. It is clear that the form Q is non-degenerate.

Theorem 3.15 (Hodge-Riemann bilinear relations)**.** *For $0 \leq k \leq n$, the Hermitian form Q restricted to $L^i H_{pr}^k \cap H^{p,q}$ is, up to a scalar (which depends on i, p, q, n), positive definite.*

Remark 3.16. We can also define a Lefschetz decomposition and a form Q, with the same results, for H^{n+k} $(0 \leq k \leq n)$; One can just transport everything from H^{n-k} to H^{n+k} via L^k (So that Q is a L-invariant form on $H^{\cdot}$).

4 Proof of the theorem on absolute values of roots

We have the operator ϕ^* on $H^k(X,\mathbb{C})$. Notice that it is an algebra homomorphism of $H^{\cdot}(X,\mathbb{C})$. We normalize it $T := q^{-k/2}\phi^*$, so that $T([\omega]) = [\omega]$. So, T commutes with L. Also, it acts as 1 on $H^{2n}(X,\mathbb{C})$ (since $T([\omega^n]) = [\omega^n]$). Finally, T is a real operator. We conclude from all this that T preserves the form Q, and the Lefschetz and Hodge decompositions (the later also takes into account that ϕ is holomorphic). But then it is clear that T is unitary w.r.t. a form which we can define on $H^k(X,\mathbb{C})$ (by choosing different multiples of Q on the different $L^i H_{pr}^{k-2i}(X,\mathbb{C}) \cap H^{p,q}(X,\mathbb{C})$). In particular, all the eigenvalues of T are of absolute value 1. This means that ϕ^* has all eigenvalues on $H^k(X,\mathbb{C})$ of absolute value $q^{k/2}$.

5 Weil conjectures for varieties over finite fields

Let X_1 be a smooth, projective and geometrically connected variety over a finite field k_1. Let d denote the dimension of X_1. Let $q = |k_1|$. Let us denote by k_n an extension field of k_1 of degree n. Let b_n denote the number of elements in $X_1(k_n)$ (the number of k_n-rational points of X_1). We then form the generating series $w(t) = \sum_{n \geq 1} a_n t^n$ as before, and set $z(t) = exp(\int \frac{w(t)}{t})$ as before.

5.1 First claim - how zeta looks

Theorem 5.1 (Rationality)**.** *One can construct (canonically) polynomials* $p_k(t) \in \mathbb{Z}[t]$*, for* $0 \leq k \leq 2d$*, so that:*

$$z(t) = \frac{\prod_{k\ odd} p_k(t)}{\prod_{k\ even} p_k(t)}.$$

Also, all $p_k(t)$ *have free coefficient* 1*,* $p_0(t) = 1 - t$ *and* $p_{2n}(t) = 1 - q^n t$*.*

Remark 5.2. Moreover, if X_1 is the reduction modulo a maximal ideal of a smooth projective variety $\tilde{X}$ over a number field, then $deg(p_k(t))$ is equal to $dimH^k(\tilde{X}(\mathbb{C}), \mathbb{C})$!

5.2 Second claim - duality

Theorem 5.3 (Functional equation)**.**

$$z((q^d t)^{-1}) = \pm t^{\chi} \cdot q^{d\chi/2} \cdot z(t)$$

where χ *is the Euler characteristic of* X *(which can be defined as the self-intersection of the diagonal of* $X \times X$*, where* X *is the base change of* X_1 *to an algebraic closure of* k_1*).*

Remark 5.4. One can be as precise about the sign in the above functional equation as in the Kahler case, once one has the language of l-adic cohomology.

5.3 Third claim - the roots

Theorem 5.5 (Riemann hypothesis)**.** *The roots of* $p_k(t)$*, when considered as complex numbers, have absolute value* $q^{k/2}$*.*

5.4 The facts above have a proof

To prove the facts above, one first considers X - the base change of X_1 to an algebraic closure of k_1. Then one notices that there is the Frobenius morphism $F : X \to X$, and that the fixed points of F^n are in bijection with $X_1(k_n)$. Grothendieck developed a theory of l-adic cohomology, and proved things like Poincare duality and Lefschetz fixed point formula. This gives the first two claims. For the third claim, Grothendieck envisioned the so-called "standard conjectures", from which it will follow (similarly to the Kahler case above). But these are still conjectures, and meanwhile Deligne proved the third claim (it is not easy at all).

A Hodge theory - Riemannian manifolds

Let X be a compact smooth connected manifold.

A.1 Forms and de-Rham cohomology

Denote by A^k the space of smooth complex-valued k-forms on X (these are non-zero for $0 \leq k \leq dim(X)$). Recall that we have the exterior differential $d : A^k \to A^{k+1}$, and that $d^2 = 0$. Recall $H^k(X, \mathbb{C}) := Ker(d)/Im(d)$.

A.2 Hodge theorem

Let us now fix an orientation and a Riemannian metric on X. These induce an inner product on the spaces A^k. The operator $d : A^k \to A^{k+1}$ admits a (unique) formal adjoint; That is, an operator $d^* : A^{k+1} \to A^k$ which satisfies $(d\alpha_1, \alpha_2) = (\alpha_1, d^*\alpha_2)$ for all $\alpha_1 \in A^k, \alpha_2 \in A^{k+1}$. Let us also recall that one usually denotes $\Delta = dd^* + d^*d$ (the Laplacian).

Would A^k be finite-dimensional, since Δ is formally self-adjoint, we would obviously have $A^k = Im(\Delta) \oplus Ker(\Delta)$. Nevertheless, it is true in our case as well:

Theorem A.1. *$A^k = Im(\Delta) \oplus Ker(\Delta)$.*

This theorem is "not easy", it requires Sobolev analysis.

Corollary A.2. *$A^k = Im(d^*) \oplus Im(d) \oplus (Ker(d^*) \cap Ker(d))$.*

Definition A.3. A form $\alpha \in A^k$ is called harmonic if $d(\alpha) = 0$ and $d^*(\alpha) = 0$. We denote tha space of harmonic k-forms by $A^k_h \subset A^k$.

So, the previous corollary can be written $A^k = Im(d^*) \oplus Im(d) \oplus A^k_h$.

Corollary A.4. *A^k_h projects isomorphically onto $H^k(X, \mathbb{C})$.*

One can think of it as a beautiful section map: a space classifying "real" things is usually presented as a space classifying "rigidified" things, up to something (many times - there is more than one possible presentation). Then, we might wonder if among all "rigidified" things representing the same "real" thing there is a most efficient one, in some sense. The elementary example on which the current instance is based is: In a finite dimensional inner product space, a quotient space admits a subspace as a model - the orthogonal complement. It picks in any coset the element which is closest to the origin. Indeed:

Claim A.5. *The following are equivalent for a form $\alpha \in A^k$:*

1. *$d(\alpha) = 0$ and $d^*(\alpha) = 0$.*
2. *$\Delta(\alpha) = 0$.*
3. *$d(\alpha) = 0$ and α has the smallest norm among elements of its cohomology class $\alpha + Im(d)$.*

B Hodge theory - Complex manifolds

Let X be a complex smooth connected manifold of (complex) dimension n. Recall that X is then canonically oriented (to wit - canonically after we choose a square-root of 1 in $\mathbb{C}$, which we assume we did, and call it i).

B.1 Types of forms and Dolbeault cohomology

In that case, we have a decomposition $A^k = \oplus A^{p,q}$ where $p, q \geq 0$ and $p+q = k$. To describe it, it is enough to do so in the "punctual" case of a complex finite-dimensional vector space V and the space of k-forms $W := Hom_{\mathbb{R}}(\bigwedge_{\mathbb{R}}^k V, \mathbb{C})$. We then can describe a subspace $W^{p,q}$ as the subspace of maps $\phi : \bigwedge_{\mathbb{R}}^k V \to \mathbb{C}$ which satisfy $\phi(cv_1, \ldots, cv_k) = c^{p-q}\phi(v_1, \ldots, v_k)$ for all $c \in \mathbb{C}$ of absolute value 1.

Notice that for $k = 1$ we get just $W = W^{1,0} \oplus W^{0,1}$ where $W^{1,0}$ is the space of complex-linear maps and $W^{0,1}$ is the space of complex-antilinear maps. We can describe then $W^{p,q}$ in general as the space of k-forms which can be obtained as a sum of products of p complex-linear 1-forms and q complex-antilinear 1-forms.

We proceed to notice that $d : A^k \to A^{k+1}$ maps $A^{p,q}$ into $A^{p+1,q} + A^{p,q+1}$ (we can see this by noticing that any k-form can be written in local complex coordinates $z_1, \ldots, z_n$ as a sum of products of a function, 1-forms dz_i and 1-forms $d\bar{z}_i$).

One denotes then $d = \partial + \bar{\partial}$, where $\partial : A^{p,q} \to A^{p+1,q}$ and $\bar{\partial} : A^{p,q} \to A^{p,q+1}$. The relation $d^2 = 0$ translates into the three relations $\partial^2 = 0$, $\bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$. In what follows we write claims and definitions for ∂, and all have the obvious analogs for $\bar{\partial}$.

We also write $H_{\partial}^{p,q}(X, \mathbb{C}) = Ker(\partial)/Im(\partial)$. So, to contrast, $H^0(X, \mathbb{C})$ is the space of locally constant functions on X, while $H_{\bar{\partial}}^{0,0}(X, \mathbb{C})$ is the space of holomorphic functions on X. As another elucidation, $H^{\cdot}(X, \mathbb{C})$ calculates the sheaf cohomology of the constant sheaf $\mathbb{C}$, while $H_{\bar{\partial}}^{p,\cdot}$ calculates the sheaf cohomoloy of the sheaf Ω^p of holomorphic p-forms on X.

B.2 Hodge theorem

Let us now fix an Hermitian metric on X. It induces a Riemannian metric (its real part). So, we again get inner products on the spaces A^k. One has that the different $A^{p,q}$ are orthogonal to each other. Thus the formal adjoint ∂^* of ∂ maps $A^{p,q}$ into $A^{p-1,q}$. Again, one writes $\Delta_{\partial} = \partial\partial^* + \partial^*\partial$. Similarly to before:

Theorem B.1. $A^{p,q} = Im(\Delta_{\partial}) \oplus Ker(\Delta_{\partial})$.

Definition B.2. A form $\alpha \in A^{p,q}$ is called ∂-harmonic if $\Delta_{\partial}\alpha = 0$. We denote the space of ∂-harmonic (p,q)-forms by $A_{\partial-h}^{p,q}$.

Corollary B.3. $A_{\partial}^{p,q}$ projects isomorphically onto $H_{\partial}^{p,q}(X, \mathbb{C})$.

C Kahler manifolds

Let X be a complex smooth connected manifold of (complex) dimension n. Suppose that we are given a Hermitian metric on X. Before we considered the Riemannian metric which it induces - its real part. But we can also consider its imaginary part; It can be considered as a real form $\omega \in A^{1,1}$. X is called Kahler if $d(\omega) = 0$. We assume this in the current section.

C.1 Identities

We introduce an operator $L : A^k \to A^{k+2}$, defined as $\omega \wedge \cdot$. We note that L maps $A^{p,q}$ into $A^{p+1,q+1}$. We denote by Λ its formal adjoint.

Claim C.1. We have:

$$[\Lambda, \partial] = i\bar{\partial}^*, [\Lambda, \bar{\partial}] = -i\partial^*$$

We will not prove this claim; It is proved first on $\mathbb{C}^n$, and then reduced to this case.

Corollary C.2. We have (commutators are written in the "super"-sense):

1. $[\partial, \bar{\partial}^*] = 0$.
2. $\Delta_\partial = \Delta_{\bar{\partial}}$.
3. $\Delta = \Delta_\partial + \Delta_{\bar{\partial}} = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$.
4. $[L, \Delta] = 0$.

Corollary C.3. Δ commutes with projection onto (p,q)-types. In particular, the (p,q)-components of an harmonic form, are harmonic (So that $A_h^k = \oplus A_h^{p,q}$).

C.2 Hodge theorem

Let us denote by $H^{p,q}(X, \mathbb{C})$ the subspace of $H^k(X, \mathbb{C})$ which consists of classes which can be represented by a closed form of type (p,q).

Theorem C.4. $H^k(X, \mathbb{C}) = \oplus H^{p,q}(X, \mathbb{C})$.

Proof. Indeed, from $A_h^k = \oplus A_h^{p,q}$ and a previous Hodge theorem, it is clear that $H^k(X, \mathbb{C}) = \sum H^{p,q}(X, \mathbb{C})$. For the independence it is enough to show that a closed (p,q)-form α is cohomologous to a harmonic (p,q)-form. Indeed, we can write $\alpha = \Delta(\beta) + \gamma$, where γ is harmonic. Projecting this equality to the (p,q)-type (and recalling that Δ preserves types in our Kahler case), we get an equality $\alpha = \Delta(\beta_1) + \gamma_1$, with γ_1 an harmonic (p,q)-form and β_1 a (p,q)-form. Since α and γ_1 are closed, we see that $\Delta(\beta_1) = dd^*\beta_1$, and hence α is cohomologous to γ_1.

□

Remark C.5. It is clear that we also have the property $H^{p,q}(X, \mathbb{C}) = \overline{H^{q,p}(X, \mathbb{C})}$.

D Lefschetz theory

We continue with our X, a complex smooth connected manifold of (complex) dimension n, which is equipped with a Kahler metric inducing a Kahler form ω.

D.1 Lefschetz theorem

We define $P : A^k \to A^k$ to be the operator of multiplying by $(n-k)$.

Lemma D.1. $[L, \Lambda] = P$, $[P, L] = 2L$, $[P, \Lambda] = -2\Lambda$.

The two last equalities are trivial. The first one is a computation purely in linear algebra.

So now, from the theory of locally finite representations of sl_2 (or from simple explicit calculations) we get:

Theorem D.2. *For $0 \leq k \leq n$, the map $L^{n-k} : A^k \to A^{2n-k}$ is an isomorphism.*

Since L commutes with d and with Δ , this theorem enables us to deduce the following one (using the fact that harmonic forms represent cohomology classes bijectively):

Theorem D.3. *For $0 \leq k \leq n$, the map $L^{n-k} : H^k(X, \mathbb{C}) \to H^{2n-k}(X, \mathbb{C})$ is an isomorphism.*

D.2 Lefschetz decomposition

Define, for $0 \leq k \leq n$:

$$A^k_{pr}(X, \mathbb{C}) = Ker(L^{n-k+1}), H^k_{pr}(X, \mathbb{C}) = Ker(L^{n-k+1}).$$

Then the following theorem follows easily from the Lefschetz theorem:

Theorem D.4. *For $0 \leq k \leq n$:*

$$A^k(X, \mathbb{C}) = \oplus_{i \geq 0} L^i A^{k-2i}_{pr}(X, \mathbb{C}).$$

and

$$H^k(X, \mathbb{C}) = \oplus_{i \geq 0} L^i H^{k-2i}_{pr}(X, \mathbb{C}).$$

Notice that this decomposition respects the Hodge decomposition into types.

D.3 Hodge-Riemann bilinear relations

Lemma D.5. *For $0 \leq k \leq n$, let $\alpha \in A^{p,q} \cap A^k_{pr}(X, \mathbb{C})$. Then $L^{n-k}\alpha = c \cdot *\alpha$ where c is a non-zero number which depends only on p, q, n (and ommited by laziness).*

This lemma is a computation purely in linear algebra.

Let us now define an Hermitian pairing on $H^k(X, \mathbb{C})$:

$$Q(x, y) = i^k \int L^{n-k} x \wedge \bar{y}.$$

The coefficient i^k is put there to make the pairing Hermitian for odd k (and not anti-Hermitian). This pairing is non-degenerate by Poinacre duality and Lefschetz theorem. One easily sees that the Hodge decomposition is orthogonal w.r.t. this pairing. One also easily sees that the Lefschetz decomposition is orthogonal w.r.t. this pairing. The lemma above shows:

Theorem D.6. *The restriction of $Q(x, y)$ to every $H^{p,q}(X, \mathbb{C}) \cap L^i H^{k-2i}_{pr}(X, \mathbb{C})$ is positive definite, up to some non-zero scalar.*

References

[1] Voisin and Schneps, Hodge theory and complex algebraic geometry

[2] http://www.math.lsa.umich.edu/ mityab/beilinson/DanREU07.pdf