SEMINAR TALK ABOUT HOWE-MOORE THEOREM

SASHA YOM DIN

1. The theorem

Let G be a locally compact group, $\mathcal{H}$ an unitary representation of G. We will say that $\mathcal{H}$ is C_0, if every matrix coefficient decays to zero at infinity (becomes small when we exit compact sets).

Let us note that if $\mathcal{H}$ contains a finite-dimensional subrepresentation, it can not be C_0. This is because the determinant function on this subspace would be of absolute value 1 on one hand, but on the other hand it is expressable as a polynomial in matrix coefficients.

In particular, a C_0 representation can not contain G-invariant vectors.

The Howe-Moore theorem states:

Theorem 1.1. *Let $G = SL(n, \mathbb{R})$ (or, more generally, a simple Lie group with finite center). Then any unitary representation of G without invariant vectors is C_0.*

2. Some lemmas

We first note a useful reformulation: Suppose that $\mathcal{H}$ is an unitary representation of G, and as $g_n \to \infty$, not all matrix coefficients tend to zero. Then for some $u, w \in \mathcal{H}$, $(g_n u, w) \not\to 0$, and so we can extract a subsequence (call it g_n again) so that $(g_n u, w)$ stays uniformly away from zero. Then by compactness of the unit ball in the weak topology, we can extract a subsequence (call it g_n again) so that $g_n u \xrightarrow{w} v$ for some $v \in \mathcal{H}$, $v \neq 0$. We will use it later.

Mautner's lemma is the following:

Lemma 2.1. *Let G be a locally compact group, $\mathcal{H}$ an unitary representation of G. Let a_k be a series of elements of G, $n \in G$, $v, u \in \mathcal{H}$. Suppose that $a_k v \xrightarrow{w} u$, and $a_k^{-1} n a_k \to 1$. Then $nu = u$.*

Proof. For any $w \in \mathcal{H}$:

$$(nu - u, w) = lim(na_k v - a_k v, w) = lim(a_k^{-1} n a_k v - v, a_k^{-1} w)$$

But $lim||a_k^{-1} n a_k v - v|| = 0$, while $||a_k^{-1} w||$ is bounded, so by Cauchy-Schwartz our limit is zero. So $nu - u = 0$. $\square$

Another lemma which we will need is the following:

Lemma 2.2. *Suppose that $G = KAK$, where K is a compact subgroup, and A is any subgroup. Let $\mathcal{H}$ be an unitary representation of G. Then it is C_0 i.f.f. all the matrix coefficients, restricted to A, vanish at infinity.*

Proof. Suppose that all matrix coefficients, restricted to A, vanish at infinity, but $g_n \to \infty$ and $(g_n u, v) \not\to 0$ for $g_n \in G$ and some $u, v \in \mathcal{H}$.

We can extract a subsequence of g_n (call it g_n again) so that $(g_n u, v)$ stays uniformly away from zero. Write now $g_n = k_n a_n k'_n$ with $k_n, k'_n \in K, a_n \in A$. We can extract a subsequence of g_n (call it g_n again) so that $k_n \to k, k'_n \to k'$, for some $k, k' \in K$. Then still $(g_n u, v)$ stays uniformly away from zero, hence in particular $(g_n u, v) \not\to 0$. On the other hand, it is clear that $a_n \to \infty$, so that:

$$(g_n u, v) = (k_n a_n k'_n u, v) - (k a_n k'_n u, v) + (k a_n k'_n u, v) - (k a_n k' u, v) + (k a_n k' u, v) =$$

$$= (a_n k'_n u, k_n^{-1} v - k^{-1} v) + (k'_n u - k' u, (k a_n)^{-1} v) + (a_n k' u, k^{-1} v)$$

The first and second terms converge to zero by Cauchy-Scwhartz, the last one by assumption. Contradiction.

□

3. Cartan decomposition

Let $G = SL(n, \mathbb{R})$. Denote by B (N) the subgroup of (unipotent) upper-triangular matrices. Denote by A^+ the subgroup of diagonal matrices with positive entries on the diagonal. Denote by K the subgroup of orthogonal matrices.

Lemma 3.1. *We have $G = KA^+K$.*

Proof. Let $g \in G$. gg^t is positive, hence by spectral theory it has a positive square root $\sqrt{gg^t}$. Writing $g = \sqrt{gg^t}k$, we calculate explicitly $kk^t = 1$, i.e. $k \in K$. Thus, we can express any element as the product of a positive one by a orthogonal one (polar decomposition). Furthermore, the spectral theory again says that a positive element we can express as sas^t for $s \in K, a \in A^+$ (if $det(s) = -1$, we can change the situation by multiplying by the element $diag(-1, 1, \ldots, 1)$). All together, any element lies in KA^+K. □

4. The case of SL(2,R)

Let $G = SL(2, \mathbb{R})$.

Lemma 4.1. *Let $\mathcal{H}$ be a unitary representation of G. Suppose that $v \in \mathcal{H}$ is N-invariant. Then it is G-invariant.*

Proof. Write $\phi(g) = (gv, v)$. This is a continuous function on G.

Note the easy equivalences, for some subgroup $H \subset G$:

- ϕ is constant on H.
- v is H-invariant.
- ϕ is H-bi-invariant.

So our ϕ is N-bi-invariant. Thus we can interpret it as N-invariant function on G/N, which can be thought of as the real plane without the origin (this is since G acts on this plane, and N is the stabilizer of $(1,0)^t$). The N-orbits are the lines $y = a$ $(a \in \mathbb{R} - \{0\})$, and the points of the x-axis. Thus, our ϕ is constant on the lines $y = a$, and so from continuity is also constant on the x-axis. But the x-axis is the B-orbit of $(1,0)^t$, so we get that ϕ is constant on B. From the remark above, ϕ is B-bi-invariant.

Now, in the same manner, we interpret ϕ as a B-invariant function on G/B, which can be thought of as the real projective line. Since this line has an open dense B-orbit, we get that ϕ is constant on the whole projective line, so we get that ϕ is constant on G. By the remark above, v is G-invariant.

□

Now we can prove the special case of Howe-Moore theorem, when $G = SL(2,\mathbb{R})$. We introduce the character $\alpha(a) = a_{1,1}^2$. Then $a\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} a^{-1} = \begin{pmatrix} 1 & \alpha(a)x \\ 0 & 1 \end{pmatrix}$.

Theorem 4.2. *The Howe-Moore theorem holds for $G = SL(2,\mathbb{R})$.*

Proof. Let $\mathcal{H}$ be an unitary representation of G, and assume that $\mathcal{H}$ is not C_0. From $G = KA^+K$ and the relevant lemmas, we can find $a_n \in A^+$, $a_n \to \infty$, and $v, u \in \mathcal{H}$, $u \neq 0$, such that $a_n v \xrightarrow{w} u$. Since $a_n \to \infty$, we can find a subsequence (call it a_n again) so that $\alpha(a_n)$ converges to zero or to infinity, suppose to infinity (the zero case is analogous). Then $a_n^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} a_n \to 1$, and so by Mautner's lemma u is N-invariant. Hence by the previous lemma, u is G-invariant. □

5. The case of $SL(n,\mathbb{R})$

Let $G = SL(n,\mathbb{R})$. For $1 \leq i < j \leq n$, we write $E_{i,j}(x)$ ($E^-_{i,j}(x)$) for the matrix with x in the (i,j) ((j,i)) place, 1's on the diagonal, and 0 everywhere else (where x is real). We write $H_{i,j}(t)$ for the diagonal matrix with t in the i place, t^{-1} in the j place, and 1's everywhere else (where t is real non-zero). We have the subgroup $G_{i,j}$, isomorphic to $SL(2,\mathbb{R})$, containing $E_{i,j}(x)$, $E^-_{i,j}(x)$, $H_{i,j}(t)$, and the corresponding $W_{i,j} = E_{i,j}(1) - E^{-1}_{i,j}(1)$. We also write $\alpha_{i,j}(a) = a_{i,i}a^{-1}_{j,j}$ (character of A), so that $aE_{i,j}(x)a^{-1} = E_{i,j}(\alpha_{i,j}(a)x)$.

By Gauss elimination, G is generated by the $G_{i,j}$'s.

Lemma 5.1. *Let $\mathcal{H}$ be an unitary representation of G, and suppose that for some (i_0, j_0), we have a $E_{i_0,j_0}(x)$-invariant vector $v \in \mathcal{H}$. Then v is G-invariant.*

Proof. By the $SL(2,\mathbb{R})$-lemma that we saw, v is G_{i_0,j_0}-invariant. For any $j_0 \neq j > i_0$, $H_{i_0,j_0}(t^{-1})E_{i_0,j}(x)H_{i_0,j_0}(t) = E_{i_0,j}(t^{-1}x)$, so by Mautner's lemma, $E_{i_0,j}(x)$ fixes v. By $SL(2,\mathbb{R})$-lemma, $G_{i_0,j}$ fixes v. In the same way, if $i_0 \neq i < j_0$, G_{i,j_0} fixes v. Thus we conclude that all $G_{i,j}$ fix v, so G fixes v. □

Theorem 5.2. *The Howe-Moore theorem holds for $G = SL(n,\mathbb{R})$.*

Proof. Let $\mathcal{H}$ be an unitary representation of G, and assume that $\mathcal{H}$ is not C_0. From $G = KA^+K$ and the relevant lemmas, we can find $a_n \in A^+$, $a_n \to \infty$, and $v, u \in \mathcal{H}$, $u \neq 0$, such that $a_n v \xrightarrow{w} u$. Since $a_n \to \infty$, we can find a subsequence (call it a_n again) so that for some $1 \leq i \leq n-1$, $\alpha_{i,i+1}(a_n)$ converges to zero or to infinity, suppose to infinity (the zero case is analogous). Then $a_n^{-1}E_{i,i+1}(x)a_n \to 1$, and so by Mautner's lemma u is $E_{i,i+1}(x)$-invariant. Hence by the previous lemma, u is G-invariant. □

6. Application to property (T)

Claim 6.1. Suppose that the Lie group G satisfies:

- Every unitary representation of G which has no non-zero G-invariant vectors is C_0.
- G contains a copy of $SL_n(\mathbb{R}) \ltimes \mathbb{R}^n$ $(n \geq 2)$.

Then G has property (T).

Proof. Let $\mathcal{H}$ be a unitary representation of G, which has almost invariant vectors. Then $\mathcal{H}$ has almost invariant vectors as a representation of $SL_n(\mathbb{R}) \ltimes \mathbb{R}^n$. From the relative property (T) of $(SL_n(\mathbb{R}) \ltimes \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^n$ has an invariant vector in $\mathcal{H}$. As $\mathbb{R}^n$ is not compact, this prevents $\mathcal{H}$ to be C_0. Hence it has a G-invariant vector. $\square$

Corollary 6.2. $SL(n, \mathbb{R})$ has property (T), where $n \geq 3$.

6.1. **the real rank.** Now we want to define the real rank of a reductive linear Lie group. We suppose that G is embedded in $GL(n, \mathbb{R})$, and the definition will not depend eventually on this embedding (although we will not show it). We define a real torus to be a closed connected Lie subgroup of G, which can be conjugated inside $GL(n, \mathbb{R})$ to sit in the diagonal. Equivalent to this conjugation property is the requirement of this subgroup to be abelian, and every element of it to be diagnolizable. The real rank of G is defined as the dimension of a maximal real torus.

Example: The real rank of $SL(n, \mathbb{R})$ is $n-1$. Indeed, the connected component of the diagonal subgroup of $SL(n, \mathbb{R})$ is clearly an $n-1$-dimensional real torus. Since any real torus will have an embedding into this diagonal subgroup, we see that the real rank is $n-1$.

Example: The real rank of a compact group is 0. Indeed, a compact subgroup of the group of diagonal matrices must be finite (a subgroup of a product of $\{\pm 1\}$).

Example: The real rank of $SO(p, q)$ is $min(p, q)$. Recall that $SO(p, q)$ is the group of transformations of $V = \mathbb{R}^{p+q}$ preserving the (say) standard symmetric bilinear form of index (p, q) $((x, y) = x_1 y_1 + \ldots + x_p y_p - x_{p+1} y_{p+1} - \ldots - x_q y_q)$. We will show that the real rank coincides with the maximal possible dimension of an isotropic subspace of V (i.e. a subspace such that the restriction of the form to it vanishes).

Let us recall first that indeed, the dimension of a maximal isotropic subspace is $min(p, q)$. If $U \subset V$ is an isotropic subspace, with basis $u_1, \ldots, u_m$, from linear algebra we can find an isotropic subspace $W \subset V$, with basis $w_1, \ldots, w_m$, such that $(u_i, w_j) = \delta_{i,j}$. Then $U + W$ is unisotropic, and so we can take its orthogonal complement $Z \subset V$. From linear algebra, $U + W$ is a sum of hyperbolic planes, so that we have at least m pluses and m minuses in our form. Thus, $m \leq min(p, q)$. Conversely, it is very easy to write our space as an orthogonal sum of $min(p, q)$ hyperbolic planes and a definite space, showing the converse.

Now, suppose that $U \subset V$ is an isotropic subspace, with basis $u_1, \ldots, u_m$, and W, etc. as in the previous paragraph. Then if we consider transformations which are identity on Z, and act by scalars on the u_i's and w_i's, with the scalar acting on u_i the inverse of the scalar acting on w_i, we get an m-dimensional torus (taking the connected component).

Conversely, let T be a real torus. Consider a basis $v_1, \ldots, v_n$ of V, which diagnolizes T, say with eigencharacters χ_i. We can order the v_i so that for any two of the first k characters (possibly coinciding), one is not the inverse of the other, and the later ones are already inverses of some of the first k, or of themselves. Then for any $i, j \leq k$, we have that $(v_i, v_j) = (tv_i, tv_j) = \chi_i(t)\chi_j(t)(v_i, v_j)$ for any $t \in T$, and since $\chi_i \neq \chi_j^{-1}$, we conclude $(v_i, v_j) = 0$. Thus $U = span\{v_1, \ldots, v_k\}$ is isotropic. On the other hand, the map $T \to \mathbb{R}^k$ defined by $t \mapsto (\chi_1(t), \ldots, \chi_k(t))$ has clearly a finite kernel, thus $dim(T) \leq k$.

6.2. **continuation.** Now, we have the following technical claim:

Claim 6.3. A simply-connected algebraic Lie group of real rank ≥ 2, has inside it a copy of $SL(2, \mathbb{R}) \ltimes \mathbb{R}^2$ or of $SL(3, \mathbb{R}) \ltimes \mathbb{R}^3$.

The Howe-Morre theorem can be proved for any simple linear Lie group, and thus we conclude:

Theorem 6.4. *A simple simply-connected algebraic Lie group of real rank ≥ 2 has property (T).*