Constructible Sheaves and Exit Paths (after Lurie)

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1 Remarks and notations

This text just follows Lurie's "Higher Algebra", appendix A ("Constructible Sheaves and Exit Paths").

There might be errors, inaccuracies, and unpleasancies in the following text. I will be happy if you let me know about it.

In this text, proofs are "proofs".

We denote by $(\cdot, \cdot)$ the mapping spaces in different ∞-categories, and by 1 the final object.

2 Presentable ∞-categories

Presentable ∞-categories are, in particular, cocomplete and complete.

A cocontinuous functor between presentable ∞-categories has a right adjoint.

An accessible continuous functor between presentable ∞-categories has a left adjoint. Accessibility is a technical condition (it means that the functor commutes with κ-filtered colimits for some regular cardinal κ); It suffices for us to know that a functor which admits a left or right adjoint is accessible, and a composition of accessible functors is accessible.

3 ∞-Topoi

3.1 ∞-Topoi and geomteric morphisms

An ∞-topos is an ∞-category with extra properties. It is, in particular, presentable. Also, pull-backs commute with small colimits (if $x \to y$ is an arrow in an ∞-topos, then $\cdot \times_y x$ commutes with small colimits).

A geomteric morphism $\sigma : \mathcal{X} \to \mathcal{Y}$ between ∞-topoi is, by definition, an adjoint pair $\sigma^* : \mathcal{X} \rightleftarrows \mathcal{Y} : \sigma_*$, such that σ^* commutes with finite limits.

3.2 The ∞-topos of spaces

We denote by $\mathcal{S}$ the ∞-topos of spaces (the coherent nerve of the simplicial category of Kan complexes). It is final in the category of ∞-topoi - every ∞-topos admits a (up to homotopy) unique geometric morphism to $\mathcal{S}$.

Let $\mathcal{X}$ be an ∞-topos and $\pi : \mathcal{X} \to \mathcal{S}$ the geometric morphism to $\mathcal{S}$ (we will always denote it by π in what follows). One might call π_* the global sections fucntor; it can be identified with $(1, \cdot)$ (since $\pi_*(x) = (1, \pi_* x) = (\pi^* 1, x) = (1, x)$). We might call π^* the constant object functor; it can be identified with $s \mapsto colim_s 1$ (sending a Kan complex s to the colimit over s, thought of as an indexing ∞-category, of the constant diagarm with value 1).

3.3 Essential geometric morphisms

A geometric morphism $\pi : \mathcal{X} \to \mathcal{Y}$ is called essential, if π^* admits a left adjoint. We denote then this left adjoint by $\pi_!$.

3.4 Etale geometric morphisms

Let $\mathcal{X}$ be an ∞-topos. Let $u \in \mathcal{X}$. We will consider the over-∞-category $\mathcal{X}_{/u}$. Let us recall that the forgetful functor $\mathcal{X}_{/u} \to \mathcal{X}$ reflects equivalences and commutes with colimits.

The over-category $\mathcal{X}_{/u}$ is an ∞-topos. We have an essential geometric morphism $j : \mathcal{X}_{/u} \to \mathcal{X}$. j^* is given by $j^*(x) = x \times u$ (equipped with the second projection). $j_!$ is just the forgetful functor. j_* seems not to be describable by a "simple formula".

If $\sigma : \mathcal{X} \to \mathcal{Y}$ is a geometric morphisms of ∞-topoi and $u \in \mathcal{Y}$, then we have a pullback square (in an appropriately understood category of ∞-topoi)

$$\begin{array}{ccc} \mathcal{X}_{/\sigma^* u} & \longrightarrow & \mathcal{Y}_{/u} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

Let us also note that if we consider $j : \mathcal{X}_{/u} \to \mathcal{X}$, then we can identify $(u, x) = (\pi \circ j)_* j^* x$ (a generalization of $(1, x) = \pi_* x$).

3.5 Coverings

Let $\mathcal{X}$ be an ∞-topos. Let $(u_\alpha \to 1)$ be a family of morphisms. Such a family is called a covering if $\coprod u_\alpha \to 1$ is an effective epimorphism ($a \to b$ is called an effective epimorphism if b is the coequalizer of the Cech diagram $\ldots \to a \times_b a \times_b a \to a \times_b a \to a$).

4 Shape

4.1

Let $\mathcal{X}$ be an ∞-topos. We will call $\mathcal{X}$ of constant shape if $\pi_* \circ \pi^*$ is corepresentable. The corepresenting object is then called the shape of $\mathcal{X}$.

We will call $\mathcal{X}$ locally of constant shape if $\mathcal{X}_{/u}$ is of constant shape for every $u \in \mathcal{X}$.

Claim 4.1. Let $\mathcal{X}$ be an ∞-topos. Then $\mathcal{X}$ is locally of constant shape if and only if π is essential.

Proof. Let $u \in \mathcal{X}$ and denote as usual $j : \mathcal{X}_{/u} \to \mathcal{X}$ and $\pi : \mathcal{X} \to \mathcal{S}$. Also, denote $\rho = \pi \circ j$ (it is the unique geometric morphism $\mathcal{X}_{/u} \to \mathcal{S}$). The sought for $\pi_! u$ should satisfy $(\pi_! u, s) = (u, \pi^* s) = (pi \circ j)_* j^*(\pi^* s) = \rho_* \rho^* s$. So, we see that $\pi_!$ exists if and only if $\rho_* \rho^*$ is corepresentable for every $u \in \mathcal{X}$, and that $\pi_!(u)$ is then the shape of $\mathcal{X}_{/u}$.

□

So, $\pi_! 1$ is the shape of $\mathcal{X}$. It can also be called the fundamental ∞-groupoid of $\mathcal{X}$.

There is a projection formula:

Claim 4.2. Let $\mathcal{X}$ be an ∞-topos locally of constant shape. Then for any $s \in \mathcal{S}$, $x \in \mathcal{X}$ and arrows $s \to t$ and $\pi_! x \to t$,, the natural arrow:

$$\pi_!(x \times_{\pi^* t} \pi^* s) \to (\pi_! x) \times_t s$$

is an equivalence.

Proof. The first step of devissage is to reduce to t being equal to 1. Write $t = colim(t_\alpha)$, with t_α contractible (i.e final in $\mathcal{S}$). Then $x = colim(x \times_{\pi^* t} \pi^* t_\alpha)$ (as objects in $\mathcal{X}_{/\pi^* t}$). Since both sides are cocontinuous in x, this reduces us to show the claim for a situation where $\pi_! x \to t$ has a factorization $\pi_! x \to t_1 \to t$, where t_1 is contractible. But then we can set $s_1 = s \times_t t_1$ and then the left side equals $\pi_!(x \times_{\pi^* t_1} \pi^* s_1)$ and the right side equals $\pi_! x \times_{t_1} s_1$, so that we can assume that t is contractible.

The second step of devissage is to reduce to s being equal to 1. Indeed, both sides are cocontinuous in s.

Now, when $s = t = 1$, the claim is trivial.

□

5 Locally Constant Objects

Definition 5.1. Let $\mathcal{X}$ be an ∞-topos. An object $x \in \mathcal{X}$ is called constant, if it is in the image of π^*. An object $x \in \mathcal{X}$ is called locally constant, if there exists a covering $(u_\alpha \to 1)$ such that $(j_\alpha)^* x$ is a constant object in $\mathcal{X}_{/u_\alpha}$, for every α (where j_α denotes the usual geometric morphism $\mathcal{X}_{/u_\alpha} \to \mathcal{X}$).

Of course, pullbacks under geometric morphisms of locally constant objects are locally constant.

Lemma 5.2. *Let $\mathcal{X}$ be an ∞-topos, $t \in \mathcal{S}$, and $\psi : \mathcal{X} \to \mathcal{S}_{/t}$ some geometric morphism. Then the image of ψ^* consists of locally constant objects.*

Proof. Let us first fix an object $k \to t$ in $\mathcal{S}_{/t}$, with k being contractible (i.e. a final object in $\mathcal{S}$). Then if we consider the diagram

$$\begin{array}{ccc} \mathcal{X}_{/\psi^* k} & \longrightarrow & \mathcal{S}_{/k} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{S}_{/t} \end{array}$$

and notice that $\mathcal{S}_{/k}$ is equivalent to $\mathcal{S}$, we realize that objects in the image of ψ^* become constant when pullbacked to $\mathcal{X}_{/\psi^* k}$. All what is left to do is to find a covering of t by contractible objects; One can take the covering by simplices.

□

Lemma 5.3. *Let $\mathcal{X}, \mathcal{Y}$ be ∞-topoi, and $\psi : \mathcal{X} \to \mathcal{Y}$ an essential geometric morphism such that $\psi_!(1) = 1$ and ψ^* is fully faithful. Then every locally constant object in $\mathcal{X}$ is in the image of ψ^*.*

Proof. Is this true? I tried to extract it from the text, but did not check details.

Let $x \in \mathcal{X}$ be a locally constant object. Note that if x is actually constant, then it belongs to the inverse image under **any** geomteric morphism (because the geomteric morphism into $\mathcal{S}$ factors through any ∞-topos). We can find a relation $colim(v_\beta) = 1$ in $\mathcal{X}$, so that $(j_\beta)^* x$ is a constant object in $\mathcal{X}_{/v_\beta}$ (we achieve it by considering a covering of 1 on which x is constant, and take the Cech nerve). Note that then $colim(\psi_! v_\beta) = 1$. I am not sure what exactly happens now; $\mathcal{X}$ is the limit of $\mathcal{X}_{/v_\beta}$, $\mathcal{Y}$ is the limit of $\mathcal{Y}_{/\psi_! v_\beta}$, and the fully-faithful functor ψ^* is the limit of fully-faithful functors $(\psi_\beta)^*$. Thus it is enough to check for every β, where it is trivial since the object is already constant.

□

Let $\mathcal{X}$ be an ∞-topos of locally constant shape. Then the functor $\pi_! : \mathcal{X} \to \mathcal{S}$ induces a functor $\psi_! : \mathcal{X} = \mathcal{X}_{/1} \to \mathcal{S}_{/\pi_! 1}$. $\psi_!$ admits a right adjoint ψ^*, described by $\psi^*(s \to \pi_! 1) = (\pi^* s) \times_{\pi^* \pi_! 1} 1$. We see that ψ^* commutes with colimits, and hence admits a right adjoint ψ_*. Summarazing, we get an essential geometric morphism $\psi : \mathcal{X} \to \mathcal{S}_{/\pi_! 1}$.

Theorem 5.4. *In the above assumptions and notation, ψ^* is fully faithful, and its image consists exactly of locally constant objects.*

Proof. Let us show first that ψ^* is fully faithful. For this, we will show that $\psi_! \psi^* \to id$ is an equivalence, on every object. Indeed, if for an object $s \to \pi_! 1$, if we apply $\psi_! \psi^*$ to it we get $\pi_!(\pi^* s \times_{\pi^* \pi_! 1} 1) \to \pi_! 1$, and by the projection formula from above, it is equivalent to $s \to \pi_! 1$.

The second claim follows from the two lemmas above.

□

6 Topological spaces

6.1 Sheaves on topological spaces

Let X be a topological space. We have the partially ordered set $U(X)$ of open subsets of X. We denote by $PSh(X)$ the category of functors $U(X)^{op} \to \mathcal{S}$. It has a full subcategory $Sh(X)$, consisting of objects $p \in PSh(X)$ which satisfy "descent". This just means that we take the biggest full subcategory in which for a covering (U_α) of X in the point-set-topological sense, the family $(U_\alpha \to X)$ will be a covering in the ∞-topos sense.

$Sh(X)$ is an ∞-topos. Lurie notes that $Sh(X)$ differs from the more common version - the one extracted from the local model structure. The later takes into account hypercovers. In general, the later is the hypercompletion of the former. I do not know for which class of spaces they coincide.

A morphism of topological spaces $X \to Y$ gives rise to a geomteric morphism $Sh(X) \to Sh(Y)$. If U is an open subset of X, then $j : Sh(U) \to Sh(X)$ identifies with $Sh(X)_{/j_!1} \to Sh(X)$.

6.2 Shape for topological spaces

Claim 6.1. Let X be a paracompact topological space. Then, considering the geometric morphism $\pi : Sh(X) \to \mathcal{S}$, we have $\pi_*\pi^*(s) = Map(X, |s|)$. Here, $Map(Y, Z)$ is the Kan complex whose n-simplices are continuous maps $Y \times |\Delta^n| \to Z$.

Proof. Not from this appendix, but from "Higher topos theory".

□

Corollary 6.2. Let X be a paracompact topological space. Then $Sh(X)$ has constant shape if and only if there exists a Kan complex k and a morphism $X \to |k|$ such that for every Kan complex s, the map $(k, s) = Map(|k|, |s|) \to Map(X, |s|)$ is an equivalence.

For example, if X is a paracompact topological space which has the homotopy type of a CW-complex, then $Sh(X)$ has constant shape, being $Sing(X)$. Indeed, the weak equivalence $|Sing(X)| \to X$ is then an homotopy equialence, thus we obtain a map $X \to |Sing(X)|$.

Definition 6.3. Let X be a paracompact topological space. We will say that X has singular shape if for every CW-complex Y, the map $Map(X, Y) \to Map(|Sing(X)|, Y)$ is an equivalence.

Thus, the remark above shows that a paracompact topological space which has the homotopy type of a CW-complex has singular shape.

Definition 6.4. Let X be a paracompact topological space. We will say that X is locally of singular shape if for every open subspace $U \subset X$, U has singular shape.

The lemma A.4.14 says that if we cover a paracompact topological space X by open subspaces, such that every finite intersection of them is of singular shape, then X is of singular shape. This shows, for example, that topological manifolds are locally of singular shape.

6.3 Description of the Galois correspondence

Let X be a paracompact topological space locally of singular shape. Then the ∞-topos $\mathcal{S}_{/Sing(X)}$ is equivalent to the ∞-topos of locally constant sheaves on X. For $s \in \mathcal{S}_{/Sing(X)}$, the corresponding sheaf can informally be described as having sections $(Sing(U), s)_{Sing(X)}$ on an open subset U (it is a mapping space in the ∞-category $\mathcal{S}_{/Sing(X)}$).

We can perefer a different model for $\mathcal{S}_{/Sing(X)}$ (I hope that what I tell here is correct). Note that in the current model $\mathcal{S}_{/t}$, objects are spaces together with a morphism to t. Morphisms are, morally, morphisms which commute with the structure map to t **up to homotopy**. Instead, we can consider only fibrations over t, but then consider only morphisms which commute with the structure map **on the nose**. I.e., let us consider the simplicial category of Kan fibrations over t, and denote it by $\mathcal{S}^{simp}_{/t}$.

In such terms, the association to an space over $Sing(X)$ of a sheaf on X becomes more concrete. To a Kan fibration $s \to Sing(X)$ we associate the sheaf $U \mapsto Hom_{\mathcal{S}^{simp}_{/Sing(X)}}(Sing(U), s)$. So our sheaf is an actual 1-functor, so to speak.

7 Constructible sheaves

7.1 Stratifications

Let A be a partially ordered set. We can regard A as a topological space by declaring a set $U \subset A$ to be open if $x \in U$ and $x \leq y$ imply $y \in U$.

Definition 7.1. An A-stratification of a topological space X is a morphism of topological spaces $X \to A$. $X \to A$ is then called a stratified topological spce. The strata X_a are the inverse images of singletons $\{a\} \in A$. There is an obvious notion of a morphism of stratified spaces (which includes a morphism of the partially ordered sets and a morphism of the spaces, commuting appropriately).

Definition 7.2. Let $X \to A$ be a stratified topological space. It is said to be conically stratified if...

7.2 Constructible sheaves

Definition 7.3. Let $X \to A$ be a stratified topological space. A sheaf $p \in Sh(X)$ is called A-constructible if $(i_a)^*p$ is locally constant, for all $i_a : Sh(X_a) \to$

$Sh(X)$ (the geometric morphism induced by the inclusion $X_a \to X$). We denote by $Sh^A(X)$ the full ∞-subcategory of $Sh(X)$ consisting of A-constructible sheaves.

7.3 Exit path category

Let us stratify $|\Delta^n| = \{(t_0, \ldots, t_n)|t_0 + \ldots + t_n = 1, t_i \geq 0\}$ by $\{0, \ldots, n\}$, by sending each vector to the index of the last non-zero entry.

Definition 7.4. Let $X \to A$ be a stratified topological space. We define a simplicial subset $Sing^A(X) \subset Sing(X)$ as follows. An n-simplex $|\Delta^n| \to X$ will belong to $Sing^A(X)$ if and only if this morphism extends to a morphism of stratified spaces.

Claim 7.5. Let $X \to A$ be a conically stratified topological space. Then $Sing^A(X)$ is an ∞-category.

7.4 The theorem

Let $X \to A$ be a conically stratified topological space. Suppose that A is finite (Lurie deals more generally with A which satisfies ascending chain condition). Suppose that X is a paracompact topological space locally of singular shape.

Then the ∞-category $Sh^A(X)$ is equivalent to the ∞-category $\mathcal{S}_{/Sing^A(X)}$. Here, for an ∞-category t, we denote by $\mathcal{S}_{/t}$ the ∞-category which classifies functors from t to $\mathcal{S}$. So, a possible model for $\mathcal{S}_{/t}$ is the coherent nerve of the simplicial category of left fibrations to t.

References

[1] Lurie, Higher Algebra, appendix A

[2] Lurie, Higher Topos Theory