PERVERSE SHEAVES ON THE STRATIFIED LINE

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1. Generalities

We deal with complex algebraic varieties X. Recall that there is the notion of an algebraic smooth stratification S of X (to be called just stratification) - S is a partition of X into locally closed smooth connected non-empty subvarieties ("strata"), such that the closure of each stratum is union of strata. A sheaf on X is called constructible w.r.t S if its restriction (i.e. "upper star") to each stratum is a locally constant (i.e. a "local system"). A sheaf on X is called constructible if there exists an algebraic smooth stratification w.r.t which it is constructible.

We have the basic object - the bounded derived category of sheaves on X with constructible cohomologies, denoted $D(X)$. It is a triangulated subcategory of the bounded derived category of sheaves on X. If S is a stratification of X , we have the full subcategory $D_S(X) \subset D(X)$, consisting only of those complexes whose cohomology is smooth when restricted to strata in S. This is a triangulated subcategory.

If $f : X \to Y$ is an algebraic morphism, we have triangulated functors $f_!, f_* : D(X) \to D(Y)$ and $f^!, f^* : D(Y) \to D(X)$. f_* is right adjoint to f^*, and $f_!$ is left adjoint to $f^!$. We have a morphism of functors $f_! \to f_*$, which is an isomorphism if f is proper. We also have a functor $Hom : D(X)^{op} \times D(X) \to D(X)$.

We define $D_X = \pi^! C$ to be the dualizing complex on X. Here C is the constant sheaf on the point and $\pi : X \to pt$ is the projection from X to the point. We define the duality functor $\mathbb{D} : D(X)^{op} \to D(X)$ by $\mathbb{D} = Hom(\cdot, D_X)$. Then wee have an isomorphism $\mathbb{D} \circ \mathbb{D} = \mathrm{Id}$. We also have isomorphisms, for a morphism $f : X \to Y$, $\mathbb{D} \circ f_* \circ \mathbb{D} = f_!$ and $\mathbb{D} \circ f^* \circ \mathbb{D} = f^!$. If S is a stratification of X, $\mathbb{D}$ preserves $D_S(X)$.

If X is smooth, of pure (complex) dimension d, $D_X[-2d]$ is a local system. If $\mathcal{L}$ is a local system on such an X, $(\mathbb{D}\mathcal{L})[-2d]$ is a local system.

If f is an open embedding, we have $f^! = f^*$ and isomorphisms (via adjunction) $f^! f_! = \mathrm{Id}, f^* f_* = \mathrm{Id}$. If g is a closed embedding, we have (since g is proper) $g_! = g_*$, and isomorphisms (via adjunction) $g^! g_! = \mathrm{Id}, g^* g_* = \mathrm{Id}$.

If g is a closed embedding and f the emebedding of its open complement, we have $f^* g_* = 0$, and thus by adjunction also $g^! f_* = 0$ and $g^* f_! = 0$. Also, we have distinguished triangles $g_! g^! \to \mathrm{Id} \to f_* f^* \to$ and $f_! f^! \to \mathrm{Id} \to g_* g^* \to$, where the first two arrows are via adjunction. The third arrow is then uniquely determined, since $Hom(f_! \cdot, g_* \cdot) = Hom(g_! \cdot, f_* \cdot) = 0$.

2. Gluing

2.1. **Triangulated setup.** Let us be given D, D_U, D_Z - three triangulated categories, and exact functors $i_\bullet : D_Z \to D$, $j^\bullet : D \to D_U$ (also denoted $i_\bullet = i_* = i_!$ and $j^\bullet = j^* = j^!$). We assume that $i_\bullet$ and $j^\bullet$ admit exact left and right adjoints (adjunctions denoted $(i^*, i_\bullet, i^!)$ and $(j_!, j^\bullet, j_*)$). We also assume $j^\bullet i_\bullet = 0$.

We assume that there are distinguished triangles $i_!i^! \to \mathrm{Id} \to j_*j^* \to$ and $j_!j^! \to \mathrm{Id} \to i_*i^* \to$, where the first two arrows are via adjunction. The third arrow is uniquely determined (since $i^*j_! = i^!j_* = 0$ (follows from $j^\bullet i_\bullet$ by adjunction) and thus $Hom(i_!\cdot, j_*\cdot) = Hom(j_!\cdot, i_*\cdot) = 0$). Finally, we assume that the adjunction morphisms $\mathrm{Id} \to i^!i_\bullet$, $i^*i_\bullet \to \mathrm{Id}$, $\mathrm{Id} \to j^\bullet j_!$ and $j^\bullet j_* \to \mathrm{Id}$ are isomorphisms.

This is a "short exact sequence" of triangulated categories.

We have a morphism of functors $i^! \to i^*$; It is the one that after composition with (the fully faithful functor) $i_\bullet$ becomes the composition $i_\bullet i^! \to \mathrm{Id} \to i_\bullet i^*$. We also have a morphism of functors $j_! \to j_*$; It is the only one that after precomposition with $j^\bullet$ becomes the composition $j_!j^\bullet \to \mathrm{Id} \to j_*j^\bullet$.

2.2. **t-structure.** Now, suppose that we are given t-structures on D_U and D_Z. We define a t-structure on D as follows: $\mathcal{F} \in D$ is in $D^{\leq 0}$ if $j^*\mathcal{F}$ and $i^*\mathcal{F}$ are in corresponding D^0. Similarly, $\mathcal{F} \in D$ is in $D^{\geq 0}$ if $j^!\mathcal{F}$ and $i^!\mathcal{F}$ are in corresponding $D^{\geq 0}$.

Claim 2.1. This is indeed a t-structure.

Proof. All checkings are easy except the axiom about existence of "decomposition" into negative and positive parts. Let us show this. So, fix $\mathcal{F} \in D$. We have the morphism $\tau_{\leq 0}j^!\mathcal{F} \to j^!\mathcal{F}$, and thus we get a morphism $j_!\tau_{\leq 0}j^!\mathcal{F} \to \mathcal{F}$. Complete it to $j_!\tau_{\leq 0}j^!\mathcal{F} \to \mathcal{F} \to \mathcal{G} \to$. In the same way, construct $i_!\tau_{\leq 0}i^!\mathcal{G} \to \mathcal{G} \to \mathcal{H} \to$. Finally, construct $\mathcal{K} \to \mathcal{F} \to \mathcal{H} \to$. Then it is easy to see that $\mathcal{H} \in D^{\geq 1}$. To see that $\mathcal{K} \in D^{\leq 0}$, use octahedron axiom for composition $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ to get $j_!\tau_{\leq 0}j^!\mathcal{F} \to \mathcal{K} \to i_!\tau_{\leq 0}i^!\mathcal{G} \to$, and then the assertion is easy. □

We note that if the t-structures in D_Z and D_U are non-degenerate, so is our t-structure on D.

We denote by P_Z, P, P_U the corresponding hearts.

2.3. **Hearts.** The functors $i_\bullet$ and $j^\bullet$ are clearly t-exact, while $i^!, j_*$ are left t-exact and $i^*, j_!$ are right t-exact.

As usual, precomposibg with the inclusion $P \to D$ and composing with $H^0 : D \to P$, we get functors and adjunctions between hearts: $({}^pi^*, i_\bullet, {}^pi^!)$ and $({}^pj_!, j^\bullet, {}^pj_*)$. We have also the relation $j^\bullet i_\bullet = 0$. The following sequences are exact: $0 \to i_\bullet {}^pi^! \to \mathrm{Id} \to {}^pj_*j^\bullet$ and ${}^pj_!j^\bullet \to \mathrm{Id} \to i_\bullet i^* \to 0$. The following adjunction morphisms are isomorphisms: $\mathrm{Id} \to {}^pi^!i_\bullet$, ${}^pi^*i_\bullet \to \mathrm{Id}$, $\mathrm{Id} \to j^\bullet {}^pj_!$ and $j^\bullet {}^pj_* \to \mathrm{Id}$.

So we get a "short exact sequence" of abelian categories $P_Z \to P \to P_U$. Namely, $i_\bullet$ is fully faithful, and its image is a Serre subcategory; P_U is the Serre quotient of P by P_Z.

2.4. **Extensions.** An $\mathcal{F} \in P$ is called an extension of $\mathcal{G} \in P_U$, if $j^\bullet\mathcal{F} = \mathcal{G}$. We have clearly the extensions ${}^pj_!\mathcal{G}$ and ${}^pj_*\mathcal{G}$.

Note that $j_!\mathcal{G} \to j_*\mathcal{G}$ factors through $j_!\mathcal{G} \to {}^pj_!\mathcal{G} \to {}^pj_*\mathcal{G} \to j_*\mathcal{G}$. The image of ${}^pj_!\mathcal{G} \to {}^pj_*\mathcal{G}$ we denote by $j_{!*}\mathcal{G}$; It is the minimal extension functor.

We have cohmological characterization of these three extensions. We first state a lemma.

Lemma 2.2. *Let $\mathcal{G} \in P_U$, and let $k \in \mathbb{Z}$. Then there exists, up to a unique isomorphism, a unique extension $\mathcal{F}$ of $\mathcal{G}$ which satisfies: $i^*\mathcal{F} \in D^{\leq k-1}$ and $i^!\mathcal{F} \in D^{\geq k+1}$ (let us call it, for briefness, the k-extension).*

Proof. As for existence, construct truncation $i^*j_*\mathcal{G} \to \tau_{\geq r} i^* j_* \mathcal{G}$ and then by adjunction $j_*\mathcal{G} \to i_\bullet \tau_{\geq r} i^* j_* \mathcal{G}$. The cocone of this morphism is seen to satisfy the properties. Uniqueness is similar; If $\mathcal{F}$ is such an object, we have a morphism $\mathcal{F} \to j_*\mathcal{G}$, and then it is not hard to show that $\mathcal{F}$ must be the cocone of the above mentioned morphism... □

Claim 2.3. For $\mathcal{G} \in P_U$, ${}^p j_! \mathcal{G}$ is its -1-extension, ${}^p j_* \mathcal{G}$ is its 1-extension, and $j_{!*}\mathcal{G}$ is its 0-extension.

Proof. The proofs are some easy exact triangle chasings... □

We also have the following characterizations (we say that an object is supported on Z, if its $j^\bullet$ is 0 or, equivalently, it is $i_\bullet$ of something):

Claim 2.4. For $\mathcal{G} \in P_U$: ${}^p j_* \mathcal{G}$ has no subobjects supported on Z, and it is the "biggest" extension with this property (any other embeds into it); ${}^p j_! \mathcal{G}$ has no quotients supported on Z, and it is the "biggest" extension with this property (any other is a quotient of it); $j_{!*}\mathcal{G}$ has no subobjects and no quotients supported on Z, and this characterizes it up to a unique isomorphism.

Now, we determine the simple objects in P:

Claim 2.5. The simple objects in P are: $i_\bullet \mathcal{G}$ for simple $\mathcal{G} \in P_Z$, and $j_{!*}\mathcal{G}$ for simple $\mathcal{G} \in P_U$.

Finally, we have:

Claim 2.6. If P_U and P_Z have finite length (i.e. any object has finite length), then so does P.

Proof. The proof is not difficult, by induction on the length of restriction to U. □

3. Notations for baby case

X denotes the complex projective line. $i : Z \to X$ denotes the closed inclusion of the origin point, and $j : U \to X$ denotes the open complement to Z. We get a stratification of U,Z by themselves, and of X by U and Z. We change notation and write $D(Z), D(U), D(X)$ for the derived categories with cohomologies constructible w.r.t. these stratifications.

4. Perversity and the perverse t-structure

Let $p = (p_0, p_1) \in \mathbb{Z}^2$. This data is called "perversity".

We define a t-structure on $D(Z)$ as follows: $D^{p,\leq 0}(Z) = D^{\leq p_0}(Z)$ and $D^{p,\geq 0}(Z) = D^{\geq p_0}(Z)$. It is clear that this is a t-structure, with heart equivalent to f.d. vector spaces (but "sitting" in degree p_0).

We define a t-structure on $D(U)$ as follows: $D^{p,\leq 0}(U) = D^{\leq p_1}(Z)$ and $D^{p,\geq 0}(U) = D^{\geq p_1}(U)$. It is clear that this is a t-structure, with heart equivalent to local systems on U (but "sitting" in degree p_1). In our case, local systems are equivalent to f.d. vector spaces.

Our $D = D(X), D_U = D(U), D_Z = D(Z)$ satisfy the formalism of gluing that we handled before. Thus, we have a t-structure on $D(X)$, glued from the ones on $D(U)$ and $D(Z)$.

We denote by $P^p(X)$ the heart of $D(X)$ w.r.t. the t-structure associated to p. This is the abelian category of p-perverse sheaves. For example, the trivial perversity $(0,0)$ yields the usual t-structure, with heart usual constructible sheaves.

Now let us observe what the duality $\mathbb{D}$ does to these t-structures. Write $p^* = (-p_0, -2-p_1)$ (the "dual" perversity).

Claim 4.1. We have $\mathbb{D} : D^{p,\leq 0} \to D^{p^*,\geq 0}$, and $\mathbb{D} : D^{p,\geq 0} \to D^{p^*,\leq 0}$.

In particular, $\mathbb{D} : P^p(X) \to P^{p^*}(X)$. Note that there is a (unique) auto-dual perversity; $p = (0,-1)$. This is the most important one, and we write $P(X)$ perverse sheaves w.r.t. this perversity.

5. Some calculations

I have done some calculations. I consider the perversity $p = (0,-n)$. Note that this will describe all situations, since $(a+r, b+r)$ and (a,b) are isomorphic by a shift. I write C_U for the constant sheaf on U, shifted by n, and C_Z for the constant sheaf on Z.

	-2	-1	0	1	2	3	4
j$_!C_U = {}^pj_!C_U$	X	X	X	X			
${}^pj_!C_U = j_{!*}C_U$	X	X	X			X	X
${}^pj_!C_U = j_{!*}C_U = {}^pj_*C_U$	X	X				X	X
j$_{!*}C_U = {}^pj_*C_U$	X	X			X	X	X
${}^pj_*C_U = j_*C_U$				X	X	X	X
dim $\mathrm{Ext}^1(j_{!*}C_U, i_\bullet C_Z)$	0	0	0	1	1	0	0
dim $\mathrm{Ext}^1(i_\bullet C_Z, j_{!*}C_U)$	0	0	1	1	0	0	0

Now, let us describe the categories $P^p(X)$ for the different perversities. In all cases, we have two irredcible objects $j_{!*}C_U$ and $i_\bullet C_Z$. Ext^1 between one of this irreducibles with itself is 0. The projective cover of $j_{!*}C_U$ is ${}^pj_!C_U$.

For $n \geq 3$ or $n \leq -1$, $P^p(X)$ is semi-simple, and so everything is clear.

For $n = 0$: The object that we get as an extension, using a non-zero class in $Ext^1(i_\bullet C_Z, j_!C_U)$, is a projective cover of $i_\bullet C_Z$ (in fact, this object is just the constant sheaf on X). We can compute everything, and get that our category $P^p(X)$ is equivalent to the category of representations of the quiver $\circ \to \circ$.

For $n = 2$: $i_\bullet C_Z$ is projective. We get the same quiver description as for $n = 0$.

For $n = 1$: The object that we get as an extension, using anon-zero class in $Ext^1(i_\bullet C_Z, j_!C_U)$, is a projective cover of $i_\bullet C_Z$. We can compute everything, and get that our category $P^p(X)$ is equivalent to the category of representations of the quiver $\circ \rightleftarrows \circ$, with the composition of the arrows in one direction being zero (only in one direction).